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On a limit of perturbed conservation laws with diffusion and non-positive dispersion

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Abstract. We consider a conservation law perturbed by a linear diffusion and non-positive dispersion $u_t + f(u)_x = \varepsilon u_{xx} - \delta(|u_{xx}|^n)_x$. We prove the convergence of the previous solution to the entropy weak solution of the hyperbolic conservation law $u_t + f(u)_x = 0$, in both cases $n = 1$ and $n = 2$.

Keywords. diffusion, dispersion, KdV equation, Burgers equation, hyperbolic conservation laws, entropy measure-valued solutions.

MS Codes. 35G25, 35L65, 35Q53, 76B15.

1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad (1.2)$$

where ε and δ are small parameters, and g is a non positive function, and we focus on the specific form

$$g(v) = -|v|^n,$$

where $n \geq 1$. Note that if $n = 1$, g is lipschitz, if $1 < n < 2$, g is C^1 and for $n \geq 2$, g is C^2 .

When $\delta = 0$ we reduce to the viscous (generalized) Burgers equation and the approximate solutions $u^{\varepsilon,0}$ converge to the entropy solution of the hyperbolic equation (called the *vanishing viscosity method*, see, e.g., Whitham [21] or Kružkov [9])

$$u_t + f(u)_x = 0 \quad (1.3)$$

$$u(x, 0) = u_0(x). \quad (1.4)$$

On the other hand, when $\varepsilon = 0$, if we consider the flux function $f(u) = u^2$ and the linear dispersion δu_{xxx} we obtain the Korteweg-de Vries equation. The approximate solutions $u^{0,\delta}$ do not converge in a strong topology (see Lax-Levermore [11]). So, as parameters ε and δ vanish, we are concerned with singular limits and to ensure convergence it is necessary to be in the dominant dissipation regime.

The pioneer study of these singular limits was given by Schonbek [16] about the (generalized) Korteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta u_{xxx}.$$

In the case of a convex flux function $f(u)$, she proved the convergence of the solutions of this perturbed equation to the entropy solution of (1.3), when both ε and δ tend to zero, *at least* under the condition $\delta = O(\varepsilon^3)$ (depending on the behavior of the flux f). Also, according to Perthame-Ryzhik [15], the sharp condition should be $\delta = o(\varepsilon)$. LeFloch-Natalini [13] proved the convergence in the case of a nonlinear viscosity function β and linear capillarity

$$u_t + f(u)_x = \varepsilon \beta(u_x)_x - \delta u_{xxx}.$$

Then, Correia-LeFloch [5] improved the estimates in Schonbek [16] and LeFloch-Natalini [13] and for the first time treated the multidimensional equation, but still in the case of a nonlinear viscosity function and linear capillarity. In fact, the dominant dissipation regime is also assured by the nonlinear viscosity. In our case, we consider the reverse situation.

In general for $\varepsilon = 0$, like for the Korteweg-de Vries equation, the divergent behaviour is expected, as we are considering "pure-dispersive equations". But, Brenier-Levy [3] considered the fully nonlinear equation

$$u_t + f(u)_x = -\delta(u_{xx}^2)_x$$

as a nonlinear generalization of the Korteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions that differs completely from the linear case. In particular, Brenier and Levy [3] conjectured that for strictly convex flux functions f and for the following perturbed problem

$$u_t + f(u)_x = -\delta g(u_{xx})_x - \varepsilon u_{xxx},$$

we have convergence when ε and δ tend to zero under the condition $\varepsilon = o(\delta)$.

The paper is organized as follows. In Section 2, we present the main results of convergence. Section 3 deals with the uniform estimates needed for convergence. Finally, Section 4 is devoted to proving the convergence to the entropy solution of the hyperbolic equation, when both ε and δ go to zero.

2 Main Results

Two main convergence results are presented. The first one is concerned with $g(v) = -|v|$ (i.e. $n = 1$) while the second one is devoted to the case $g(v) = -v^2$ (i.e. $n = 2$).

2.1 Case f convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$

In this case, we prove the following result.

Theorem 2.1 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^2)$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

Then, setting $u = u_{\varepsilon,\delta}$ the solution of (1.1) – (1.2), the family solutions $(u_{\varepsilon,\delta})$ converges to the entropy solution of (3) – (4).

In the case of the linear dispersion, i.e. $g(u_{xx}) = u_{xx}$ treated in [16], Schonbek gets for a general flux satisfying $f''(u) \leq C$, a convergence with rate $\delta = O(\varepsilon^3)$. Also, when $f''(u) \leq C(1 + |u|)$, the author obtain the convergence with the rate $\delta = O(\varepsilon^4)$. The case of $g(u_{xx}) = -|u_{xx}|$ seems giving a weakly dispersive effects than a classical linear dispersion.

2.2 Case f convex, $\delta > 0$ and $g(u_{xx}) = -u_{xx}^2$

Here function g is regular, and we obtain:

Theorem 2.2 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2.$$

Then, setting $u = u_{\varepsilon, \delta}$ the solution of (1.1) – (1.2), the family solutions $(u_{\varepsilon, \delta})$ converges to the entropy solution of (3) – (4).

The dispersion here is strongly nonlinear but regular, which provides the well-posedness [1, 2]. We can see that comparing to the results in [16], the rate are quite similar when the flux is convex and satisfies $f''(u) \leq C$ ($\delta = o(\varepsilon^{5/2})$ whereas $\delta = O(\varepsilon^3)$ in [16]).

3 A priori estimates

Assume that η is a regular function and φ a function defined by $\varphi' = \eta' f'$, and let us multiply (1.1) by $\eta'(u)$. We obtain

$$\begin{aligned} \eta(u)_t + \varphi(u)_x &= \varepsilon (\eta'(u) u_x)_x - \varepsilon \eta''(u) u_x^2 \\ &\quad + \delta (\eta'(u) g(u_{xx}))_x - \delta \eta''(u) u_x g(u_{xx}). \end{aligned} \quad (3.1)$$

Integrating over $(0, t) \times \mathbb{R}$ with $\eta(u) = |u|^{\alpha+1}$, the conservative terms vanish and we obtain the following lemma.

Lemma 3.1 *Let $\alpha \geq 1$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be any dispersion function. Each solution of (1.1) satisfies for $t \in [0, T]$*

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^{\alpha+1} dx &+ (\alpha + 1) \alpha \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds \\ &+ (\alpha + 1) \alpha \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x g(u_{xx}) dx ds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1} dx. \end{aligned} \quad (3.2)$$

Usually, taking $\alpha = 1$ in (3.2), we deduce the a priori L^2 first energy estimates.

Let us introduce the functions $\mathcal{G}$, and G defined by $\mathcal{G}'' = G' = g$, i.e,

$$G(u) = -\frac{1}{n+1} |u|^n u, \text{ and } \mathcal{G} = -\frac{1}{(n+1)(n+2)} |u|^{n+2}.$$

Using the multiplier $(q+2)(|u_x|^q u_x)_x$ to (1.1), we have

$$\begin{aligned} ((q+2) u_t |u_x|^q u_x)_x - (|u_x|^{q+2})_t &= -(q+2)(q+1) |u_x|^q u_{xx} f'(u) u_x \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + \delta (q+2)(q+1) |u_x|^q u_{xx} g'(u_{xx}) u_{xxx} \\ &= -(q+1) (|u_x|^{q+2})_x f'(u) \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + \delta (q+2)(q+1) n |u_x|^q G(u_{xx})_x, \end{aligned}$$

and we get the estimate

$$\begin{aligned}
((q+2)u_t|u_x|^q u_x)_x - (|u_x|^{q+2})_t &= -((q+1)|u_x|^{q+2}f'(u))_x \\
&+ (q+1)|u_x|^{q+2}u_x f''(u) \\
&+ \varepsilon(q+2)(q+1)|u_x|^q u_{xx}^2 \\
&+ (\delta(q+2)(q+1)n|u_x|^q G(u_{xx}))_x \\
&- \delta(q+2)(q+1)q(n+2)n|u_x|^{q-2}u_x \mathcal{G}(u_{xx}).
\end{aligned} \tag{3.3}$$

Similarly, using the multiplier $(q+2)(u_x^{q+1})_x$ to (1.1), we can write

$$\begin{aligned}
((q+2)u_t u_x^{q+1})_x - (u_x^{q+2})_t &= -(q+2)(q+1)u_x^{q+1}f'(u)u_{xx} \\
&+ \varepsilon(q+2)(q+1)u_x^q u_{xx}^2 \\
&+ (\delta(q+2)(q+1)n u_x^q G(u_{xx}))_x \\
&- \delta(q+2)(q+1)q(n+2)n u_x^{q-1} \mathcal{G}(u_{xx}),
\end{aligned}$$

thus

$$\begin{aligned}
((q+2)u_t u_x^{q+1})_x - (u_x^{q+2})_t &= -((q+1)u_x^{q+2}f'(u))_x \\
&+ (q+1)u_x^{q+3}f''(u) \\
&+ \varepsilon(q+2)(q+1)u_x^q u_{xx}^2 \\
&+ (\delta(q+2)(q+1)n u_x^q G(u_{xx}))_x \\
&- \delta(q+2)(q+1)q(n+2)n u_x^{q-1} \mathcal{G}(u_{xx}).
\end{aligned} \tag{3.4}$$

Integrating (3.3) and (3.4) over $\mathbb{R} \times [0, t]$ provides

$$\begin{aligned}
&\int_{\mathbb{R}} |u_x(t)|^{q+2} dx + \varepsilon(q+2)(q+1) \int_0^t \int_{\mathbb{R}} |u_x|^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} |u'_0|^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q+2} f''(u) dx ds \\
&\quad + \delta(q+2)(q+1)q(n+2)n \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q-2} \mathcal{G}(u_{xx}) dx ds,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\int_{\mathbb{R}} u_x(t)^{q+2} dx + \varepsilon(q+2)(q+1) \int_0^t \int_{\mathbb{R}} u_x^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} (u'_0)^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x^{q+3} f''(u) dx ds \\
&\quad + \delta(q+2)(q+1)q(n+2)n \int_0^t \int_{\mathbb{R}} u_x^{q-1} \mathcal{G}(u_{xx}) dx ds.
\end{aligned} \tag{3.6}$$

We define now the sets, for $t \in [0, T]$

$$\mathcal{U}_t^+ = \{x \in \mathbb{R}; u_x(x, t) > 0\},$$

and

$$\mathcal{U}_t^- = \{x \in \mathbb{R}; u_x(x, t) < 0\}.$$

Adding (3.6) to (3.5) for q odd, we obtain:

Lemma 3.2 *Let q be an odd number. Then, each solution of (1.1) satisfies for $t \in [0, T]$*

$$\begin{aligned}
& \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\
& + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |\mathcal{G}(u_{xx})| dx ds \\
& + (q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \\
& = \int_{\mathcal{U}_0^+} |u'_0|^{q+2} dx,
\end{aligned} \tag{3.7}$$

where the last left hand-side term can be replaced by

$$- (q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+1} f'(u) u_{xx} dx ds. \tag{3.8}$$

Now, the combination of Lemmas 3.1 and 3.2 gives the following estimate

Proposition 3.3 *Let $\varepsilon, \delta > 0$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function. The solution $u = u_{\varepsilon, \delta}$ of (1.1) – (1.2) satisfies the uniform estimate*

$$\int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} |u_x| |u_{xx}|^n dx ds \leq C_0, \tag{3.9}$$

for all $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$.

Proof. When $g(u) = -|u|^n$, equality (3.2) writes

$$\begin{aligned}
& \int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \alpha(\alpha+1) \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds = \\
& \|u_0\|_{\alpha+1}^{\alpha+1} + \alpha(\alpha+1) \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds.
\end{aligned} \tag{3.10}$$

Also, when f is convex, we can rewrite (3.7) for $q \geq 1$ odd as

$$\begin{aligned}
& \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\
& + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |u_{xx}|^{n+2} dx ds + \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \leq C_0
\end{aligned} \tag{3.11}$$

However, using the Young inequality, we get

$$\begin{aligned}
& \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \\
& = \int_0^t \int_{\mathcal{U}_s^+} \left(\frac{1}{c t^{\frac{\alpha-1}{\alpha+1}}} |u|^{\alpha-1} \right) \left(c t^{\frac{\alpha-1}{\alpha+1}} |u_x| \right) (\delta |u_{xx}|^n) dx ds \\
& \leq \left(\frac{1}{t c^{\frac{\alpha+1}{\alpha-1}}} \right) \left(\frac{\alpha-1}{\alpha+1} \right) \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds + \frac{c^k t^{\frac{k}{\alpha+1}}}{k} \int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds \\
& + \frac{n}{n+2} \delta^{1+\frac{2}{n}} \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds,
\end{aligned} \tag{3.12}$$

where c and k are two constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1),$$

and

$$\frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k = \frac{(n+2)(\alpha+1)}{(4+n) - n\alpha},$$

and if $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$, we get $k \geq 3$.

Now, q is chosen odd such that $2+q \geq k$ to obtain

$$|u_x|^k \leq |u_x|^3 + |u_x|^{q+2}.$$

Using (3.11) with $q = 1$ and $q \geq k-2$ odd, we obtain

$$\int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds \leq C_0. \quad (3.13)$$

Now, integrating (3.10) over $[0, t]$, we get

$$\begin{aligned} \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds &\leq \int_0^t \int_{\mathbb{R}} |u|^{\alpha+1} dx ds \\ &\leq t C_0 + \alpha(\alpha+1) t \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds. \end{aligned} \quad (3.14)$$

Thus, injecting (3.13) and (3.14) in (3.12), it comes

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0 + \frac{1}{4} \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \quad (3.15)$$

and we obtain

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0. \quad (3.16)$$

Finally, injecting (3.16) in (3.10) we obtain (3.9). $\square$

3.1 Case f convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$

We are concerned here with the equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta |u_{xx}|_x. \quad (3.17)$$

Proposition 3.4 *Let $\varepsilon > 0$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function, such that*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

Then, the solution $u = u_{\varepsilon, \delta}$ of (3.17) satisfies the estimate

$$\int_{\mathbb{R}} u_x(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C + \frac{C}{\delta}, \quad (3.18)$$

where $C > 0$ is a constant independent of ε and δ .

In addition, if $\delta = O(\varepsilon^2)$, the estimate (3.9) with $\alpha = 1$ is

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds \leq C. \quad (3.19)$$

Proof. On the one hand, (3.5) is rewritten with $q = 0$ as

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \\ = \int_{\mathbb{R}} (u'_0)^2 dx + 2 \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds. \end{aligned} \quad (3.20)$$

Since f satisfies $f''(u) \leq C(1 + |u|^\beta)$, with $0 \leq \beta < 3$. Then

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}),$$

where C is a generic constant. Thus,

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}| dx ds \\ &\quad + C \int_0^t \int_{\mathbb{R}} |u|^{\beta+1} |u_x| |u_{xx}| dx ds. \end{aligned} \quad (3.21)$$

Applying (3.9) with $n = 1$, $\alpha = 2$ and $\alpha = \beta + 2 < 5$, we get

$$\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| \leq \frac{C}{\delta}. \quad (3.22)$$

Finally, injecting (3.22) in (3.20) provides (3.18).

On the other hand, estimate (3.2), with $\alpha = 1$ is written as

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds = \|u_0\|_2^2 + 2\delta \int_0^t \int_{\mathbb{R}} u_x |u_{xx}| dx ds. \quad (3.23)$$

Then

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \leq C_0 + 2\delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds. \quad (3.24)$$

Now, if $\delta \leq K\varepsilon^2$

$$\begin{aligned} \delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds &\leq \frac{\sqrt{\delta}}{\sqrt{K}\varepsilon} \int_0^t \int_{\mathbb{R}} (\sqrt{\varepsilon} |u_x|) (\sqrt{K\delta\varepsilon} |u_{xx}|) dx ds \\ &\leq \frac{\sqrt{\delta}}{2\sqrt{K}\varepsilon} \left(\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + K\delta\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right) \\ &\leq \frac{1}{2} \left(\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + K\delta\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right). \end{aligned} \quad (3.25)$$

Using (3.18) and injecting (3.25) in (3.24) give (3.19). $\square$

Proposition 3.5 *Let $\varepsilon > 0, \delta = o(\varepsilon^2)$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

Then, the solution $u = u_{\varepsilon, \delta}$ satisfies

- (a) $\{\varepsilon u_x^2\}$ is bounded in $L^1((0, t) \times \mathbb{R})$.
- (b) $\{\varepsilon u_x\} \rightarrow 0$ as $\varepsilon \rightarrow 0$, in $L^2((0, t) \times \mathbb{R})$.
- (c) $\{\delta u_x |u_{xx}|\} \rightarrow 0$ as $\varepsilon \rightarrow 0$, in $L^1((0, t) \times \mathbb{R})$.
- (d) $\{\delta |u_{xx}|\} \rightarrow 0$ as $\varepsilon \rightarrow 0$, in $L^2((0, t) \times \mathbb{R})$.

Proof. The statements (a), (b) are obtained thanks to (3.19).

Now, in the same manner we obtained (3.25), we have

$$\delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds \leq \frac{\sqrt{\delta}}{2\varepsilon} \left(\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right),$$

and using (3.18) and (3.19) it gets

$$\delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds \leq C \frac{\sqrt{\delta}}{\varepsilon}, \quad (3.26)$$

which gives (c) as soon as $\delta = o(\varepsilon^2)$.

Finally, (d) is obtained thanks to (3.18) since,

$$\delta^2 \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq \frac{\delta}{\varepsilon} \left(\delta \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right) \leq C_0 \frac{\delta}{\varepsilon}. \quad (3.27)$$

□

3.2 Case f convex, $\delta > 0$ and $g(u_{xx}) = -u_{xx}^2$

We are concerned here with the equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta (u_{xx}^2)_x. \quad (3.28)$$

It is noteworthy that the initial value problem associated with (3.28) is well-posed [1, 2].

Proposition 3.6 *Let $\varepsilon > 0$, $\delta = O(\varepsilon)$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function. Then, the solution $u = u_{\varepsilon, \delta}$ of (3.28) satisfies the estimate (3.9) with $\alpha = 1$, i.e.,*

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u_x| u_{xx}^2 dx ds \leq C_0. \quad (3.29)$$

If in addition

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2,$$

then the solution $u = u_{\varepsilon, \delta}$ checks

$$\int_{\mathbb{R}} u_x(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C_0 + C_0 \delta^{-1/2} \varepsilon^{-1/4}. \quad (3.30)$$

Proof. On the one hand, (3.2) with $n = 2$, $\alpha = 1$ is written

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds = \|u_0\|_2^2 + 2\delta \int_0^t \int_{\mathbb{R}} u_x u_{xx}^2 dx ds, \quad (3.31)$$

and, from (3.7) with $q = 1$, we get

$$\varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds \leq C_0. \quad (3.32)$$

Thus, if $\delta \leq (k\varepsilon)$, (3.32) and (3.31) give (3.29).

On the other hand, assuming $f''(u) \leq C(1 + |u|^\beta)$, with $0 \leq \beta < 1/2$, we have

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}).$$

Then, to estimate the last term in (3.20), we proceed as in the case $n = 1$:

$$\begin{aligned}
\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}| dx ds + C \int_0^t \int_{\mathbb{R}} |u|^{\beta+1} |u_x| |u_{xx}| dx ds \\
&\leq C \delta^{-1/2} \varepsilon^{-1/4} \int_0^t \int_{\mathbb{R}} \left(\delta^{1/2} |u|^{1/2} |u_x|^{1/2} |u_{xx}| \right) \left(|u|^{1/2} \right) \left(\varepsilon^{1/4} |u_x|^{1/2} \right) dx ds \\
&\quad + C \delta^{-1/2} \varepsilon^{-1/4} \int_0^t \int_{\mathbb{R}} \left(\delta^{1/2} |u|^{\beta+1/2} |u_x|^{1/2} |u_{xx}| \right) \left(|u|^{1/2} \right) \left(\varepsilon^{1/4} |u_x|^{1/2} \right) dx ds.
\end{aligned}$$

Using the Young inequality, it comes

$$\begin{aligned}
\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \delta^{-1/2} \varepsilon^{-1/4} \left(\frac{\delta}{2} \int_0^t \int_{\mathbb{R}} |u|^{2\beta+1} |u_x| u_{xx}^2 dx ds \right. \\
&\quad \left. + \frac{\delta}{2} \int_0^t \int_{\mathbb{R}} |u| |u_x| u_{xx}^2 dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^2 dx ds + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right). \quad (3.33)
\end{aligned}$$

Inequality (3.9) with $n = 2$, $\alpha = 2$ and $\alpha = 2\beta + 2 < 3$ is written as

$$\delta \int_0^t \int_{\mathbb{R}} |u| |u_x| u_{xx}^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{2\beta+1} |u_x| u_{xx}^2 dx ds \leq C_0. \quad (3.34)$$

Now, injecting (3.34) and (3.29) in (3.33), we obtain

$$\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| \leq C \delta^{-1/2} \varepsilon^{-1/4}. \quad (3.35)$$

Finally, injecting (3.35) in (3.20) we get the required estimate (3.30). $\square$

Proposition 3.7 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2.$$

Then, the solution $u = u_{\varepsilon, \delta}$ of (3.28) satisfies

- (a) $\{\varepsilon u_x^2\}$ is bounded in $L^1((0, t) \times \mathbb{R})$.
- (b) $\{\varepsilon u_x\} \rightarrow 0$ as $\varepsilon \rightarrow 0$, in $L^2((0, t) \times \mathbb{R})$.
- (c) $\{\delta u_x^- u_{xx}^2\}$, as $u_x^- = \max(0, -u_x)$, is bounded in $L^1((0, t) \times \mathbb{R})$.
- (d) $\{\delta u_x^+ u_{xx}^2\} \rightarrow 0$, as $u_x^+ = \max(0, u_x)$ when $\varepsilon \rightarrow 0$ in $L^1((0, t) \times \mathbb{R})$.
- (e) $\{\delta u_{xx}^2\} \rightarrow 0$ as $\varepsilon \rightarrow 0$, in $L^1((0, t) \times \mathbb{R})$.

Proof. The statements (a), (b) and (c) are obtained from (3.29).

Now, (d) is obtained from (3.7) with $q = 1$ since

$$\delta \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds \leq \frac{\delta}{\varepsilon} \left(\varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds \right) \leq C \frac{\delta}{\varepsilon}. \quad (3.36)$$

Finally, (3.30) provides (e) since

$$\delta \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq \delta^{\frac{1}{2}} \varepsilon^{-5/4} (\delta^{\frac{1}{2}} \varepsilon^{5/4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds) \leq C \sqrt{\delta \varepsilon^{-5/2}}. \quad (3.37)$$

$\square$

4 Convergence Proof

We now define the measure-valued solutions to the first order Cauchy problem (1.3)-(1.4) as DiPerna [7].

Definition 4.1 Assume that $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R})$ satisfies the growth condition

$$|f(u)| \leq \mathcal{O}(|u|^m) \text{ as } |u| \rightarrow \infty, \quad \text{for some } m \in [0, q]. \quad (4.1)$$

A Young measure ν is called an entropy measure-valued (e.m.-v.) solution to (1.3)-(1.4) if

$$\langle \nu, |u - k| \rangle_t + \langle \nu, \text{sgn}(u - k)(f(u) - f(k)) \rangle_x \leq 0, \quad \text{for all } k \in \mathbb{R}, \quad (4.2)$$

in the sense of distributions on $(0, T) \times \mathbb{R}$, and

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds = 0, \quad \text{for all compact set } K \subseteq \mathbb{R}. \quad (4.3)$$

A representation theorem of Young's measures associated with a sequence of uniformly bounded functions of L^q is used to link the structure of measure and the strong convergence [16].

Lemma 4.2 Let $\{u_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $L^\infty((0, T); L^q(\mathbb{R}))$. Then there exists a subsequence denoted by $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ and a weakly- $\star$ measurable mapping $\nu : \mathbb{R} \times (0, T) \rightarrow \text{Prob}(\mathbb{R})$ such that, for all functions $h \in \mathcal{C}(\mathbb{R})$ satisfying (4.1), $\langle \nu_{(x,t)}, h \rangle$ belongs to $L^\infty((0, T); L_{loc}^{q/m}(\mathbb{R}))$ and the following limit representation holds:

$$\iint_{\mathbb{R} \times (0, T)} \langle \nu_{(x,t)}, h \rangle \phi(x, t) dx dt = \lim_{n \rightarrow \infty} \iint_{\mathbb{R} \times (0, T)} h(\tilde{u}_n(x, t)) \phi(x, t) dx dt, \quad (4.4)$$

for all $\phi \in L^1(\mathbb{R} \times (0, T)) \cap L^\infty(\mathbb{R} \times (0, T))$.

Conversely, given ν , there exists a sequence $\{u_n\}$ satisfying the same conditions as above and such that (4.4) holds for any h satisfying (4.1).

Proof of the main results. We begin proving (4.2) by using Proposition 3.3, resp. Proposition 3.5, for $n = 1$, resp. $n = 2$, and we apply the Young measure representation theorem in the suitable L^q space (4.4) to show that ν satisfies (4.2). Also, we use a standard regularization of $\text{sgn}(u - k)(f(u) - f(k))$ and $|u - k|$ ($k \in \mathbb{R}$), since it is sufficient to show that there exists a bounded measure $\mu \leq 0$ such that

$$\eta(u)_t + q(u)_x \longrightarrow \mu \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)) \quad (4.5)$$

for an arbitrary convex function η (we assume that η' and η'' are bounded on $\mathbb{R}$).

Now, to prove (4.5), we rewrite the formulae (3.1) in the form

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4, \quad (4.6)$$

where,

$$\begin{aligned} \mu_1 : &= \varepsilon (\eta'(u) u_x)_x; \\ \mu_2 : &= -\varepsilon \eta''(u) u_x^2; \\ \mu_3 : &= \delta (\eta'(u) g(u_{xx}))_x; \\ \mu_4 : &= -\delta \eta''(u) u_x g(u_{xx}). \end{aligned}$$

We distinguish the case $n = 1$ from $n = 2$.

Case n = 1: $g(u_{xx}) = -|u_{xx}|$, f convex and $\delta = o(\varepsilon^2)$. We have

$$| \langle \mu_1, \theta \rangle | \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) u_x| dx ds \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x u_x| dx ds \leq C \|\theta_x\|_{L^2} \|\varepsilon u_x\|_{L^2}, \quad (4.7)$$

$$| \langle \mu_2, \theta \rangle | \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^2| dx ds \leq C \|\theta\|_{L^\infty} \|\varepsilon u_x^2\|_{L^1}. \quad (4.8)$$

Since η is a convex function, we notice for a non negative function θ

$$\langle \mu_2, \theta \rangle = -\varepsilon \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^2 dx ds \leq 0. \quad (4.9)$$

In the same way, we have

$$\begin{aligned} | \langle \mu_3, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) |u_{xx}|| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta_x |u_{xx}|| dx ds \\ &\leq C \|\theta_x\|_{L^2} \|\delta |u_{xx}|\|_{L^2}, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} | \langle \mu_4, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x |u_{xx}|| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x |u_{xx}|| dx ds \\ &\leq C \|\theta\|_{L^\infty} \|\delta u_x |u_{xx}|\|_{L^1}. \end{aligned} \quad (4.11)$$

Combining (4.7), (4.8), (4.9), (4.10) and (4.11), with (a), (b), (c), (d) in Proposition 3.3, gives (4.5) where μ is non positive bounded measure.

Case n = 2: $g(u_{xx}) = -u_{xx}^2$, f convex and $\delta = o(\varepsilon^{5/2})$. Estimates (4.7), (4.8), (4.9) remain true. Concerning μ_3 , we have

$$\begin{aligned} | \langle \mu_3, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) u_{xx}^2| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta_x u_{xx}^2| dx ds \\ &\leq C \|\theta_x\|_{L^\infty} \|\delta u_{xx}^2\|_{L^1}. \end{aligned} \quad (4.12)$$

Now, μ_4 is split as

$$\mu_4 = \mu_{41} + \mu_{42},$$

with

$$\begin{aligned} \mu_{41} &:= \delta \eta''(u) u_x^+ u_{xx}^2; \\ \mu_{42} &:= -\delta \eta''(u) u_x^- u_{xx}^2, \end{aligned}$$

where $u_x^+ = \max(0, u_x)$ and $u_x^- = \max(0, -u_x)$. Then we have

$$\begin{aligned} | \langle \mu_{41}, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^+ u_{xx}^2| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x^+ u_{xx}^2| dx ds \\ &\leq C \|\theta\|_{L^\infty} \|\delta u_x^+ u_{xx}^2\|_{L^1}, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} | \langle \mu_{42}, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^- u_{xx}^2| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x^- u_{xx}^2| dx ds \\ &\leq C \|\theta\|_{L^\infty} \|\delta u_x^- u_{xx}^2\|_{L^1}. \end{aligned} \quad (4.14)$$

Again, since $\eta'' \geq 0$, it gets for a non negative function θ

$$\langle \mu_{42}, \theta \rangle = -\delta \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^- u_{xx}^2 dx ds \leq 0. \quad (4.15)$$

Finally, from inequalities (4.7), (4.8), (4.9), (4.12), (4.13), (4.14) and (4.15), combined with (a), (b), (c) and (d) in Proposition 3.5, we obtain (4.5) where μ is non positive bounded measure.

Now we will prove (4.3). We follow arguments of DiPerna [7] and Szepessy [18]: we have to check that, for each compact set K of $\mathbb{R}$,

$$\lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds = \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds = 0.$$

By Jensen's inequality

$$\frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds \leq m(K)^{1/2} \left(\frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds \right)^{1/2},$$

where $m(K)$ denotes the Lebesgue measure of K . Then we will establish that

$$\lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds = 0.$$

Let $K_i \subset K_{i+1}$ ($i = 0, 1, \dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i \geq 0} K_i = \mathbb{R}$. Using the identity $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$, we get for all $i = 0, 1, \dots$

$$\begin{aligned} & \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(\cdot, s) - u_0)^2 dx ds \\ & \leq \frac{1}{t} \int_0^t \left(\int_{K_i} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{K_i} u_0^2 dx - 2 \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right) ds \\ & \leq \int_{\mathbb{R} \setminus K_i} u_0^2 dx + \frac{1}{t} \int_0^t \left(\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \right) \\ & \quad + \frac{2}{t} \int_0^t \left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| ds. \end{aligned} \quad (4.16)$$

For the first term of the right hand side, we clearly have

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R} \setminus K_i} u_0^2 dx = 0.$$

Now, injecting (3.26) in (3.23), in the case $n = 1$, and respectively (3.36) in (3.31), in the case $n = 2$, we obtain

$$\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \leq C \frac{\sqrt{\delta}}{\varepsilon},$$

and respectively,

$$\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \leq C \frac{\delta}{\varepsilon}.$$

In both cases, the right hand side of these inequalities tends to zero when $\varepsilon \rightarrow 0$.

To estimate the last term in the inequality (4.16), we choose $\{\theta_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\lim_{n \rightarrow \infty} \theta_n = u_0 \quad \text{in } L^2(\mathbb{R}).$$

Then, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| &\leq \int_{K_i} |u_0 - \theta_n| |u^{\varepsilon, \delta}(\cdot, s) - u_0| dx \\
&\quad + \left| \int_{K_i} \theta_n (u_0^{\varepsilon, \delta} - u_0) + \int_{K_i} \theta_n (u^{\varepsilon, \delta}(\cdot, s) - u_0^{\varepsilon, \delta}) dx \right| \\
&\leq \|u_0 - \theta_n\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \\
&\quad + \|\theta_n\|_{L^2(\mathbb{R})} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} + \left| \int_0^s \int_{K_i} \theta_n \partial_s u^{\varepsilon, \delta} dx d\tau \right|.
\end{aligned}$$

In view of (3.19) for $n = 1$ and respectively (3.29) for $n = 2$, we have

$$\|u_0 - \theta_n\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \leq (\|u_0\|_{L^2(\mathbb{R})} + C) \|u_0 - \theta_n\|_{L^2(\mathbb{R})},$$

which tends to zero when $n \rightarrow \infty$, since $\lim_{\varepsilon \rightarrow 0+} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} = 0$. Finally, it remains to prove that

$$\lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n \partial_s u^{\varepsilon, \delta} dx d\tau \right| ds = 0.$$

We have, by (1.1),

$$\begin{aligned}
\left| \int_0^s \int_{K_i} \theta_n \partial_s u^{\varepsilon, \delta} dx d\tau \right| &= \left| \int_0^s \int_{K_i} \theta_n (-f(u^{\varepsilon, \delta})_x + \varepsilon u_{xx}^{\varepsilon, \delta} + \delta g(u_{xx}^{\varepsilon, \delta})_x) dx d\tau \right| \\
&\leq \int_0^s \int_{K_i} |(\theta_n)_x f(u^{\varepsilon, \delta})| dx d\tau + \varepsilon \int_0^s \int_{K_i} |(\theta_n)_x u_x^{\varepsilon, \delta}| dx d\tau \\
&\quad + \delta \int_0^s \int_{K_i} |(\theta_n)_x g(u_{xx}^{\varepsilon, \delta})| dx d\tau \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

To compute each quantity I_1 , I_2 and I_3 , we distinguish the cases $n = 1$ from $n = 2$.

Case $n = 1$: $g(u_{xx}) = -|u_{xx}|$, f convex and $\delta = o(\varepsilon^2)$. Since f is such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

where $\beta < 3$. Thus,

$$|f(u)| \leq C(1 + |u|^m),$$

where $m < 5$. Then, Proposition 3.1 implies

$$\int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \leq \int_0^s \int_{\mathbb{R}} |u^{\varepsilon, \delta}|^m dx d\tau \leq C s, \quad (4.17)$$

and

$$\begin{aligned}
I_1 &= \int_0^s \int_{K_i} |(\theta_n)_x| |f(u^{\varepsilon, \delta})| dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_n)_x| dx d\tau + C \int_0^s \int_{K_i} |(\theta_n)_x| |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_n)_x| dx d\tau + C \|(\theta_n)_x\|_{L^\infty(\mathbb{R})} \int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C s \|(\theta_n)_x\|_{L^1(\mathbb{R})} + C s \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}.
\end{aligned} \quad (4.18)$$

Thanks to (3.19), we can write

$$\begin{aligned}
I_2 &= \varepsilon \int_0^s \int_{K_i} |(\theta_n)_x| |u_x^{\varepsilon, \delta}| \, dx \, d\tau \\
&\leq \left(\varepsilon \int_0^s \int_{K_i} |(\theta_n)_x|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left(\varepsilon \int_0^s \int_{K_i} |u_x^{\varepsilon, \delta}|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
&\leq C \varepsilon^{\frac{1}{2}} s^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{4.19}$$

Finally for I_3 , using (3.27), we get

$$\begin{aligned}
I_3 &= \delta \int_0^s \int_{K_i} |(\theta_n)_x g(u_{xx}^{\varepsilon, \delta})| \, dx \, d\tau = \int_0^s \int_{K_i} \delta |(\theta_n)_x| |u_{xx}^{\varepsilon, \delta}| \, dx \, d\tau \\
&\leq \left(\int_0^s \int_{K_i} |(\theta_n)_x|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left(\delta^2 \int_0^s \int_{K_i} |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
&\leq s^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \left(\delta^2 \int_0^s \int_{K_i} |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
&\leq s^{\frac{1}{2}} \left(\frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})}
\end{aligned} \tag{4.20}$$

Now, from (4.18), (4.19) and (4.20), we deduce

$$\begin{aligned}
&\lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n u_s^{\varepsilon, \delta} \, dx \, d\tau \right| ds \\
&\leq \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{C}{t} \left(t^2 (\|(\theta_n)_x\|_{L^1(\mathbb{R})} + \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \right. \\
&\quad \left. + t^{\frac{3}{2}} \left(\frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \right) \\
&\leq \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} C \left(t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + t^{\frac{1}{2}} \left(\frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \right),
\end{aligned}$$

and since $\delta = o(\varepsilon^2)$, we obtain the desired conclusion, and Theorem 2.1 is proved.

Case n = 2: $g(u_{xx}) = -u_{xx}^2$, f convex and $\delta = o(\varepsilon^{5/2})$. Here, f is such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

where $\beta < 1/2$, thus,

$$|f(u)| \leq C(1 + |u|^m),$$

where $m < 5/2$.

Estimates I_1 and I_2 are obtained in the same manner as $n = 1$ using (3.29) instead of (3.18). From (3.37) we obtain

$$\begin{aligned}
I_3 &= \delta \int_0^s \int_{K_i} |(\theta_n)_x g(u_{xx}^{\varepsilon, \delta})| \, dx \, d\tau = \int_0^s \int_{K_i} \delta |(\theta_n)_x| |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \\
&\leq \|(\theta_n)_x\|_{L^\infty(\mathbb{R})} \left(\delta \int_0^s \int_{K_i} |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \right) \\
&\leq \sqrt{\delta \varepsilon^{-5/2}} \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}
\end{aligned} \tag{4.21}$$

Finally, using (4.18), (4.19) and (4.21), we obtain

$$\begin{aligned}
& \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n u_s^{\varepsilon, \delta} dx d\tau \right| ds \\
& \leq \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{C}{t} \left(t^2 (\|(\theta_n)_x\|_{L^1(\mathbb{R})} + \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \right. \\
& \quad \left. + t \sqrt{\delta \varepsilon^{-5/2}} \|(\theta_n)_x\|_{L^\infty(\mathbb{R})} \right) \\
& \leq \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} C \left(t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\delta \varepsilon^{-5/2}} \right),
\end{aligned}$$

and since $\delta = o(\varepsilon^{5/2})$, we obtain the desired conclusion, and Theorem 2.2 is proved. $\square$

References

- [1] Bedjaoui N., Correia J.M.C., Mammeri Y. Well-posedness of the generalized Korteweg-de Vries-Burgers equation with nonlinear dispersion and nonlinear dissipation. *Int. J Pure Math.* Vol 2 (2015), 38-46.
- [2] Bedjaoui N., Correia J.M.C., Mammeri Y. On vanishing dissipative-dispersive perturbations of hyperbolic conservation laws. *Proceedings of the 5th International Conference on Fluid Mechanics and Heat & Mass Transfer* (2014) Lisbon (Portugal).
- [3] Brenier Y., Levy D. Dissipative Behavior of Some Fully Non-Linear KdV-Type Equations. *Physica D* 137, 2000, pp. 277-294.
- [4] Burgers J.M. A Mathematical Model Illustrating the Theory of Turbulence. *Adv. Appl. Mech.*, 1; 1948, p.171.
- [5] Correia J.M.C., LeFloch P.G. Nonlinear Diffusive-Dispersive Limits for Multidimensional Conservation Laws, in *Nonlinear Partial Differential Equations*. World Scientific, Singapore, 1998.
- [6] Correia J.M.C., LeFloch P.G. Nonlinear Hyperbolic Conservation Laws, in *Nonlinear Evolution Equations and their Applications*. Proc. Luso-Chinese Symposium, World Scientific, Singapore, 1999.
- [7] DiPerna R.J. Measure-Valued Solutions to Conservation Laws. *Arch. Rat. Mech. Anal.*, 88; 1985, p.223.
- [8] Hopf E. The Partial Differential Equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 3; 1950, p.201.
- [9] Kruřkov S.N. First order quasilinear equations in several independent variables. *Mat. Sb.*, 81; 1970, p.285; *Math. USSR Sb.*, 10; 1970, p.217.
- [10] Landau L.D., Lifschitz E.M. *Fluid Mechanics*. Addison Wesley, Reading, MA, 1971.
- [11] Lax P.D., Levermore C.D. The small dispersion limit of the Korteweg-de Vries equation. *Comm. Pure Appl. Math.*, 36; 1983, I, p.253, II, p.571, III, p.809.
- [12] LeFloch P.G. Hyperbolic Systems of Conservation Laws. *ETH Lect. Notes Ser., Birkhäuser-Verlag*, Basel, 2002.
- [13] P. G. LeFloch and R. Natalini. Conservation laws with vanishing nonlinear diffusion and dispersion, *Nonlinear Analysis* 36, 1999, pp. 213–230
- [14] Marcati P.A., Natalini R. Convergence of the Pseudoviscosity Approximation for Conservation Laws. *Nonlin. Anal. T.M.A.*, 23; 1994, p.621.

- [15] B. Perthame and L. Ryzhik, Moderate Dispersion in Conservation Laws with Convex Fluxes, *Comm. Math. Sci.* 5(2), 2007, pp. 473–484.
- [16] Schonbek M. Convergence of Solutions to Nonlinear Dispersive Equations. *Comm. Part. Diff. Equa.*, 7; 1982, p.959.
- [17] M. Slemrod, From Boltzmann to Euler: Hilbert’s 6th problem revisited, *Computers and Mathematics with Applications* 65, 2013, pp. 1497-1501.
- [18] Szepessy A. An Existence Result for Scalar Conservation Laws using Measure-Valued Solutions. *Comm. Part. Diff. Equa.*, 14; 1989, p.1329.
- [19] Tartar L. The Compensated Compactness Method Applied to Systems of Conservation Laws, in *J. Ball, editor, Systems of Nonlinear Partial Differential Equations (Oxford, 1982)*, vol.111 of *NATO ASI Sc. Inst. Ser. C: Math. and Phys. Sc.*, p.263. D. Reidel publ. Co., Dordrecht-Boston, Mass., 1983.
- [20] von Neumann J., Richtmyer R.D. A Method for the Numerical Calculation of Hydrodynamical Shocks. *J. Appl. Phys.*, 21; 1950, p.380.
- [21] Whitham G.B. *Linear and Nonlinear Waves*. Pure & Appl. Math., Wiley-Interscience Publ., New York, 1974.