Dynamical behavior of a stochastic forward-backward algorithm using random monotone operators

Pascal Bianchi, Walid Hachem

Dynamical behavior of a stochastic forward-backward algorithm using random monotone operators

Pascal Bianchi * Walid Hachem†

August 12, 2015

Abstract

The purpose of this paper is to study the dynamical behavior of the sequence (x_n) produced by the forward-backward algorithm

$$
\begin{aligned}
y_{n+1} &\in B(u_{n+1}, x_n), \\
x_{n+1} &= (I + \gamma_{n+1} A(u_{n+1}, \cdot))^{-1}(x_n - \gamma_{n+1} y_{n+1}),
\end{aligned}
$$

where $A(\xi) = A(\xi, \cdot)$ and $B(\xi) = B(\xi, \cdot)$ are two functions valued in the set of maximal monotone operators on $\mathbb{R}^N$, (u_n) is a sequence of independent and identically distributed random variables, and (γ_n) is a sequence of vanishing step sizes. Following the approach of the recent paper [16], we define the operators $\mathcal{A}(x) = \mathbb{E}[A(u_1, x)]$ and $\mathcal{B}(x) = \mathbb{E}[B(u_1, x)]$, where the expectations are the set-valued Aumann integrals with respect to the law of u_1, and assume that the monotone operator $\mathcal{A} + \mathcal{B}$ is maximal (sufficient conditions for maximality are provided). It is shown that with probability one, the interpolated process obtained from the iterates x_n is an asymptotic pseudo trajectory in the sense of Benaïm and Hirsch of the differential inclusion $\dot{z}(t) \in -(\mathcal{A} + \mathcal{B})(z(t))$. The convergence of the empirical means of the x_n's towards a zero of $\mathcal{A} + \mathcal{B}$ follows, as well as the convergence of the sequence (x_n) itself to such a zero under a demipositivity assumption. These results find applications in a wide range of optimization or variational inequality problems in random environments.

Keywords : Dynamical systems, Random maximal monotone operators, Stochastic forward-backward algorithm, Stochastic proximal point algorithm.

AMS subject classification : 47H05, 47N10, 62L20, 34A60.

1 Introduction

1.1 The setting

In the fields of convex analysis and monotone operator theory, the forward-backward splitting algorithm [26, 25] is one of the most studied algorithms for finding iteratively a zero of a sum of two maximal monotone operators. As is well known, a set-valued operator $\mathsf{A} : \mathbb{R}^N \to 2^{\mathbb{R}^N}$

*CNRS LTCI; Télécom ParisTech. e-mail: `pascal.bianchi@telecom-paristech.fr`

†CNRS LTCI; Télécom ParisTech. e-mail: `walid.hachem@telecom-paristech.fr`

where N is some positive integer is said monotone if $\forall(x,y) \in \mathrm{gr}(\mathsf{A})$, $\forall(x',y') \in \mathrm{gr}(\mathsf{A})$, $\langle y-y', x-x'\rangle \geq 0$ where $\mathrm{gr}(\mathsf{A})$ stands for the graph of A. A non empty monotone operator is said maximal if its graph is a maximal element in the inclusion ordering. A typical maximal monotone operator is the subdifferential of a function belonging to Γ_0, the family of proper lower semicontinuous convex functions on $\mathbb{R}^N$. A splitting algorithm for minimizing the sum of two functions in Γ_0, or more generally, for finding a zero of a sum of two maximal monotone operators, is an algorithm that involves each of the two operators separately. Denote by $\mathcal{M}$ the set of maximal monotone operators on $\mathbb{R}^N$, and let $\mathrm{dom}(\mathsf{A}) = \{x \in \mathbb{R}^N : \mathsf{A}(x) \neq \emptyset\}$ be the domain of the operator A. Given $\mathsf{A}, \mathsf{B} \in \mathcal{M}$ where B is assumed single-valued and where $\mathrm{dom}(\mathsf{B}) = \mathbb{R}^N$, the forward-backward algorithm reads

$$x_{n+1} = (I + \gamma\mathsf{A})^{-1}(x_n - \gamma\mathsf{B}(x_n))$$

where I is the identity operator, γ is a real positive step, and $(\cdot)^{-1}$ is the inverse operator defined by the fact that $(x,y) \in \mathrm{gr}(A^{-1}) \Leftrightarrow (y,x) \in \mathrm{gr}(A)$ for an operator A. This algorithm involves a forward step $(I - \gamma\mathsf{B})(x_n)$ followed by a backward step that consists in applying to the output of the former the *resolvent* $(I + \gamma\mathsf{A})^{-1}$, also called the *proximity operator* when A is the subdifferential of a function in Γ_0 (it is well known that $(I + \gamma\mathsf{A})^{-1}$ is a single valued operator with domain $\mathbb{R}^N$ since $\mathsf{A} \in \mathcal{M}$ [17, 10]). Let $Z(\mathsf{A}) = \{x \in \mathbb{R}^N : 0 \in \mathsf{A}(x)\}$ be the set of zeros of the operator A. Assuming a so-called cocoercivity assumption on B and a condition on γ, the forward-backward algorithm is known to converge to an element of $Z(\mathsf{A} + \mathsf{B})$ provided the latter set is nonempty [10].

The purpose of this paper is to study a version of the forward-backward algorithm where at each iteration n, operators A and B are replaced with some operators *randomly chosen* amongst some collections $(A(\xi))_{\xi\in\Xi}$ and $(B(\xi))_{\xi\in\Xi}$ respectively where $(\Xi, \mathscr{T})$ is a measurable space (measurability issues are made precise below). Let $(u_n)_{n\in\mathbb{N}^*}$ be a sequence of Ξ-valued independent and identically distributed (iid) random variables defined on some probability space and denote by μ the probability distribution of u_1. Let $(\gamma_n)_{n\in\mathbb{N}^*}$ be a sequence of deterministic positive steps in $\ell^2 \setminus \ell^1$. Assume that $\mathrm{dom}(B(\xi)) = \mathbb{R}^N$ for μ-almost all $\xi \in \Xi$. Denote by $A(\xi, x)$ and $B(\xi, x)$ the respective images of x under the operators $A(\xi)$ and $B(\xi)$. Starting with an arbitrary $x_0 \in \mathbb{R}^N$, the aim of this paper is to study the iterations

$$\begin{aligned} y_{n+1} &\in B(u_{n+1}, x_n) \\ x_{n+1} &= (I + \gamma_{n+1}A(u_{n+1}, \cdot))^{-1}(x_n - \gamma_{n+1}y_{n+1}). \end{aligned} \tag{1}$$

Notice that contrary to the standard forward-backward algorithm recalled above, the steps γ_n are now made converge to zero to alleviate the noise effect due to the randomness of (u_n), and the operator $B(\xi)$ is no longer assumed single valued.

Our purpose is to study the dynamical behavior of the sequences (x_n) so defined. A central role will be played by the operators

$$\mathcal{A} = \int A(\xi)\,\mu(d\xi) \quad \text{and} \quad \mathcal{B} = \int B(\xi)\,\mu(d\xi)$$

where these set-valued integrals who were introduced in the recent paper [16] are to be recognized as *Aumann integrals* [7, 6]. One can immediately check that the operators $\mathcal{A}$, $\mathcal{B}$ and consequently $\mathcal{A}+\mathcal{B}$, are monotone. Assuming that $\mathcal{A}+\mathcal{B}$ is moreover maximal (verifiable maximality conditions for this operator are provided below) and writing $\mathcal{D} = \mathrm{dom}(\mathcal{A}+\mathcal{B})$, it is a standard fact from monotone operator theory that for every $z_0 \in \mathcal{D}$, the differential inclusion $\dot{z}(t) \in -(\mathcal{A}+\mathcal{B})(z(t))$ admits a unique absolutely continous solution $z : \mathbb{R}_+ \to \mathbb{R}^N$ such that $z(0) = z_0$, and the map $\Phi(z_0, t) = z(t)$ thus obtained can be extended to a *semiflow* from $\overline{\mathcal{D}} \times \mathbb{R}_+$ to $\overline{\mathcal{D}}$ [17, 5]. With this at hand, we define the affine interpolated process obtained from the sequence (x_n) as follows. Let $\tau_n = \sum_1^n \gamma_k$ for $n \in \mathbb{N}$. Notice that $\tau_n \to_n \infty$ since $(\gamma_n) \notin \ell^1$. For any $t \in [\tau_n, \tau_{n+1})$, set

$$x(t) = x_n + \frac{x_{n+1} - x_n}{\gamma_{n+1}}(t - \tau_n).$$

Now, borrowing a concept introduced by Benaïm and Hirsch in the field of stochastic approximation, we show that under some conditions, the interpolated process is almost surely (a.s.) a bounded *Asymptotic Pseudo Trajectory* (APT) [12, 13] for the semiflow Φ.

The convergence of the algorithm towards an element of the set of zeros $\mathcal{Z} = Z(\mathcal{A}+\mathcal{B})$ is of obvious interest. In this regards, the above APT property yields two important corollaries. Using a result of [14], the sequence of empirical means $(\bar{x}_n)$ given by

$$\bar{x}_n = \frac{\sum_{k=1}^n \gamma_k x_k}{\sum_{k=1}^n \gamma_k}$$

is shown to converge a.s. to a (random) element of $\mathcal{Z}$. Yet, the sequence x_n itself is not in general guaranteed to converge (a simple counterexample can be found in [17, 30]). Nevertheless, it is known that any solution $z(t)$ of the differential inclusion converges to an element of $\mathcal{Z}$ provided that the operator $\mathcal{A}+\mathcal{B}$ is *demipositive* [18]. When this condition holds, the interpolated process $x(t)$ (and hence the sequence (x_n)) inherits from $z(t)$ the convergence towards a point of $\mathcal{Z}$, as a consequence of the above APT property. Verifiable conditions for the demipositivity of $\mathcal{A}+\mathcal{B}$ can be easily devised.

1.2 Application examples

We provide herein some application examples of Algorithm (1) without insisting for the moment on the assumptions.

Example 1. Let $g : \Xi \times \mathbb{R}^N \to (-\infty, \infty]$ be such that $g(\xi, \cdot) \in \Gamma_0$ for μ-almost any $\xi \in \Xi$. Let $G(x) = \int g(\xi, x)\mu(d\xi)$ and consider the minimization problem $\min_{x \in \mathbb{R}^N} G(x)$ which is assumed to have a solution. Writing $A(\xi) = \partial_x g(\xi, \cdot)$ and $B(\xi) = 0$ and using some assumptions making licit the interchange between the integration and the subdifferentiation, we are led to finding a zero of the mean operator $\mathcal{A} = \int A(\xi)\mu(d\xi)$. The algorithm boils down to the random *proximal point algorithm* $x_{n+1} = \mathrm{prox}_{\gamma_{n+1} g(u_{n+1}, \cdot)}(x_n)$ where $\mathrm{prox}_f(x) = (I + \partial f)^{-1}(x)$ is the proximity operator associated to the function $f \in \Gamma_0$. Instances of this algorithm can be the following:

1. Consider a distributed multiagent system. Set $\Xi = \{1, \ldots, m\}$ where the positive integer m represents the number of agents in a network, write $G(x) = \sum_{i=1}^m g(i, x)$, and assume that Agent i has access to its private function $g(i, \cdot)$ only. Up to an irrelevant m^{-1} factor, $G(x)$ coincides with the integral $\int g(\xi, x)\mu(d\xi)$ where μ is taken as the uniform distribution on $\{1, \ldots, m\}$. The random proximal point algorithm goes as follows: at iteration $n + 1$, an agent wakes up at random according to μ, applies the proximity operator to its private function and hands out the result to the next active agent. This is the so called incremental proximal algorithm [15].

2. In a centralized setting, assume that the distribution μ is unknown and that an observer seeks to minimize G on-line, based on the sole knowledge of the sequence (u_n). The random proximal point algorithm can be seen here as the proximal analogue of the well-known stochastic (sub)gradient algorithm, where implicit steps are performed instead of explicit ones. Indeed, the former reads $x_{n+1} = x_n - \gamma_{n+1}\partial_x g(u_{n+1}, x_{n+1})$ while in the latter, $\partial_x g(u_{n+1}, x_{n+1})$ is replaced with $\partial_x g(u_{n+1}, x_n)$. It is interesting to note that in the special case where $g(\xi, .)$ is quadratic (say, $g(\xi, x) = \|\xi_1 x - \xi_2\|^2$ where $\xi_1 \in \mathbb{R}^{m\times N}$, $\xi_2 \in \mathbb{R}^m$ and $\xi = (\xi_1, \xi_2)$) the random proximal point algorithm boils down to the so-called *normalized least mean square* which is well-known in signal processing applications.

3. Given a positive integer m, let $C_1, \ldots, C_m$ be closed convex sets in $\mathbb{R}^N$, and let $f \in \Gamma_0$. Consider the problem

$$\min_{x\in C} f(x), \quad C = \bigcap_{i=1}^{m} C_i$$

where a minimizer is assumed to exist. Assume nevertheless that the projection operator onto C is difficult to implement while the projection on any of the C_i is easy (think of *e.g.* half spaces). Here we set $\Xi = \{0, 1, \ldots, m\}$, $g(0, \cdot) = f$, $g(\xi, \cdot) = \iota_{C_\xi}$ for $1 \le \xi \le m$ where ι_C is the indicator function of the set C, and $\mu = \sum_{i=0}^m \alpha_i \delta_i$ where all the α_i are positive and δ_i is the Dirac measure at i. Under mild assumptions on the set $C_1, \ldots, C_m$, the optimization problem is equivalent to finding a zero of the mean operator

$$\mathcal{A} = \alpha_0 \partial f + N_{C_1} + \cdots + N_{C_m}$$

where for every x, $N_{C_i}(x)$ is the normal cone of C_i at x. In practice, according to the outcome of u_{n+1}, either the operator $\operatorname{prox}_{\gamma_{n+1} f}$ is applied to the current iterate, or a simple projection onto one of the C_i is performed.
A refinement consists in assuming that the function f is itself an expectation with respect to an unknown probability law as in Case 2 above. We can then replace the operator $\operatorname{prox}_{\gamma_{n+1} f}$ with a randomized version as in Example 1.2.

The above examples are instances of the following general case:

Example 2. Given a probability space $(\mathsf{X}, \mathscr{X}, \nu)$, let the functions $f : \mathsf{X} \times \mathbb{R}^N \to (-\infty, \infty)$ and $g : \mathsf{X} \times \mathbb{R}^N \to (-\infty, \infty]$ satisfy $f(\eta, \cdot), g(\eta, \cdot) \in \Gamma_0$ for ν-almost all $\eta \in \mathsf{X}$. Consider the sum $F(x) + G(x)$ where $F(x) = \int f(\eta, x)\nu(d\eta)$ and $G(x) = \int g(\eta, x)\nu(d\eta)$. Considering the sets $C_1, \ldots, C_m$ of the preceding example, we aim at solving the problem

$$\min_{x\in C} F(x) + G(x), \quad C = \bigcap_{i=1}^{m} C_i$$

where the minimum is assumed to exist. It is also assumed that the $\operatorname{prox}_{g(\xi,\cdot)}$ operator can be easily implemented, while the functions $f(\xi, \cdot)$ are better suited for operations involving only the subgradients. Define on $\{0, 1, \ldots, m\}$ the probability distribution $\zeta = \sum_{i=0}^m \alpha_i \delta_i$ where all α_i are positive. On the space $\mathsf{X} \times \{0, \ldots, m\}$ equipped with the probability $\mu = \nu \otimes \zeta$, let $\xi = (\eta, i)$, and define the random operators

$$A(\xi) = \begin{cases} \alpha_0^{-1} \partial_x g(\eta, \cdot) & \text{if } i = 0, \\ N_{C_i} & \text{otherwise} \end{cases} \quad \text{and} \quad B(\xi) = \partial_x f(\eta, \cdot). \tag{2}$$

Then the minimization problem introduced above amounts to finding a zero of the operator $\mathcal{A} + \mathcal{B} = \partial F + \partial G + \sum_{i=1}^{m} N_{C_i}$. Given a sequence $(u_n = (v_n, I_n))$ of Ξ-valued iid random variables with probability law μ, the algorithm (1) reads

$$y_{n+1} \in \partial f(x_n, v_{n+1}),$$
$$x_{n+1} = \begin{cases} \operatorname{prox}_{\alpha_0^{-1}\gamma_{n+1} g(v_{n+1},\cdot)}(x_n - \gamma_{n+1} y_{n+1}) & \text{if } I_{n+1} = 0, \\ \operatorname{proj}_{C_{I_{n+1}}}(x_n - \gamma_{n+1} y_{n+1}) & \text{otherwise,} \end{cases}$$

where $\operatorname{proj}_{C_i}(\cdot)$ is the projection operator onto C_i.

The algorithm (1) can be also used to solve a variational inequality problem:

Example 3. Let $C = \cap C_i$ be as in Example 1.3. Consider the problem of finding $x_\star \in C$ that solves the variational inequality

$$\forall x \in C, \ \langle F(x_\star), x - x_\star \rangle \geq 0$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is a monotone single-valued operator on $\mathbb{R}^N$ [40][22]. Since the projection on C is difficult, one can use the simple stochastic algorithm $x_{n+1} = \operatorname{proj}_{C_{u_{n+1}}}(x_n - \gamma_{n+1} F(x_n))$ where the random variables u_n are distributed on the set $\{1, \ldots, m\}$. The variant where F is itself an expectation can also be considered.

1.3 About the literature

The problem of minimizing an objective function in a noisy environment has given birth to a very rich literature in the field of the stochastic approximation [11, 23]. In the framework of this paper, most of this literature studies the evolution of the projected stochastic gradient or subgradient algorithm where the projection is made on a fixed constraint set.

In the case where the constraint set has a complicated structure, an incremental minimization algorithm with random constraint updates has been proposed in [27], who seeks to minimize a deterministic convex function f on a finite intersection of closed convex constraint sets. The algorithm developed in [27] consists in a subgradient step over the objective f followed by an update step towards a randomly chosen constraint set. Along the same principle, a distributed algorithm involving an additional consensus step has been proposed in [24]. Random iterations involving proximal and subgradient operators in the spirit of Example 2 were considered in [15] and in [39]. In [39], the functions $g(\xi, .)$ are supposed to have a full domain, satisfy the inequality $\|g(\xi, x) - g(\xi, y)\| \leq L(\|x - y\| + 1)$ for some constant L which does not depend on ξ and, finally, are such that $\int \|g(\xi, x)\|^2 \mu(d\xi) \leq L(1 + \|x\|^2)$. In the present paper, such conditions are not needed. An other work of the same authors which is also close to ours is [40], where among other things, the problem described in Example 3 was considered in the case where $F(x) = \int f(\xi, x)\mu(d\xi)$. In [40], it is assumed that F is strongly monotone and that the stochastic Lipschitz property $\int \|f(\xi, x) - f(\xi, y)\|^2 \mu(d\xi) \leq C\|x - y\|^2$ holds, where C is a positive constant. In our work, the strong monotonicity of F is not needed, and the Lipschitz property is essentially replaced with the condition $\|f(\xi, x)\|^2 \leq M(\xi)(1 + \|x\|)$ where $M(\xi)$ satisfies a moment condition.

Regarding the convergence rate analysis, let us mention [3, 35] which investigate the performance of the algorithm $x_{n+1} = \operatorname{prox}_{\gamma_{n+1} g}(x_n - \gamma_{n+1} H_{n+1})$ where H_{n+1} is a noisy estimate of the gradient $\nabla f(x_n)$. The same algorithm is addressed in [36] where the proximity operator is replaced by the resolvent of a fixed maximal monotone operator, and H_{n+1} is replaced by a noisy version of a (single-valued) cocoercive operator evaluated at x_n. The paper [37] addresses the statistical analysis of the empirical means of the estimates obtained from the random proximal point algorithm of Example 1.

This paper follows the line of thought of the recent paper [16], who studies the behavior of the random proximal iterates $x_{n+1} = (I + \gamma_{n+1} A(u_{n+1}))^{-1}(x_n)$ in a Hilbert space and establishes the convergence of the empirical means $\bar{x}_n$ towards a zero of the mean operator $\mathcal{A}(x) = \int A(\xi, x)\, \mu(d\xi)$. In the present paper, the proximal point algorithm is replaced with the more general forward-backward algorithm. Thanks to the dynamical approach developed here, the convergences of both $(\bar{x}_n)$ and possibly (x_n) are established.

Finally, it is worth noting that apart from the APT of Benaïm and Hirsch [12], many authors introduced alternative concepts to analyze the asymptotic behavior of perturbed solutions to evolution systems (see [2] and references therein). An important one is the notion of *pseudo-orbit* of [1, 2] which has been shown useful to analyze certain perturbed solution to differential inclusions of the form (3). The pseudo-orbit property is however more demanding than the APT property and is in general harder to verify. Fortunately, the concept of APT is proved here sufficient to guarantee that the interpolated process $x(t)$ almost surely inherits both the ergodic and non-ergodic convergence properties of the orbits of Φ.

1.4 Paper organization

Section 2 is devoted to the exact problem description and to the statements of the results. Theorem 2.1 shows that under proper assumptions, the interpolated process $x(t)$ is a.s. an APT for the differential inclusion $\dot{z}(t) \in (\mathcal{A} + \mathcal{B})(z(t))$. The consequences of this theorem in terms of convergence of $(\bar{x}_n)$ or (x_n) towards a zero of $\mathcal{A} + \mathcal{B}$ follow. These results are followed by a proposition devoted to the maximality of $\mathcal{A} + \mathcal{B}$. Application examples are then provided in Section 3 with an emphasis on Example 2 above. Section 4 is devoted to the proofs.

2 Problem statement and results

2.1 Set-valued functions and set-valued integrals

Let $(\Xi, \mathscr{T}, \mu)$ be a probability space where $\mathscr{T}$ is μ-complete. Consider the space $\mathbb{R}^N$ equipped with its Borel field $\mathscr{B}(\mathbb{R}^N)$, and let F be a function from Ξ to $2^{\mathbb{R}^N}$ such that $F(\xi)$ is a closed set for any $\xi \in \Xi$. The set-valued function F is said *measurable* if $\{\xi : F(\xi) \cap H \neq \emptyset\} \in \mathscr{T}$ for any set $H \in \mathscr{B}(\mathbb{R}^N)$. This is known to be equivalent to asserting that the domain $\operatorname{dom}(F) = \{\xi \in \Xi : F(\xi) \neq \emptyset\}$ of F belongs to $\mathscr{T}$ and that there exists a sequence of measurable functions $\varphi_n : \operatorname{dom}(F) \to \mathbb{R}^N$ such that $F(\xi) = \overline{\{\varphi_n(\xi)\}}$ for all $\xi \in \operatorname{dom}(F)$ [19, Chap. 3] [20].

Assume now that F is measurable and that $\mu(\operatorname{dom}(F)) = 1$. For $1 \le p < \infty$, denote by $\mathcal{L}^p(\Xi, \mathscr{T}, \mu; \mathbb{R}^N)$ the Banach space of measurable functions $\varphi : \Xi \to \mathbb{R}^N$ such that $\int \|\varphi\|^p d\mu < \infty$, and let

$$\mathcal{S}_F^p = \{\varphi \in \mathcal{L}^p(\Xi, \mathscr{T}, \mu; \mathbb{R}^N) \,:\, \varphi(\xi) \in F(\xi)\ \mu - \text{a.e.}\}.$$

If $\mathcal{S}_F^1 \neq \emptyset$, the function F is said integrable. The *Aumann integral* [7, 6] of F is the set

$$\int F d\mu = \left\{ \int_\Xi \varphi d\mu \,:\, \varphi \in \mathcal{S}_F^1 \right\}.$$

2.2 Random maximal monotone operators

Consider a function $A : \Xi \to \mathcal{M}$. Note that the graph $\operatorname{gr}(A(\xi))$ of any element $A(\xi)$ is a closed subset of $\mathbb{R}^N \times \mathbb{R}^N$ [17]. Assume that the function $\xi \mapsto \operatorname{gr}(A(\xi))$ is measurable as a closed set-valued function from Ξ to $2^{\mathbb{R}^N \times \mathbb{R}^N}$. As shown in [4, Ch. 2], this is equivalent to saying that the function $\xi \mapsto (I + \gamma A(\xi))^{-1} x$ is measurable from Ξ to $\mathbb{R}^N$ for any $\gamma > 0$ and any $x \in \mathbb{R}^N$. Denoting by $D(\xi)$ the domain of $A(\xi)$, the measurability of $\xi \mapsto \operatorname{gr}(A(\xi))$ implies that the set-valued function $\xi \mapsto \overline{D(\xi)}$ is measurable. Moreover, recalling that $A(\xi, x)$ is the image of a given $x \in \mathbb{R}^N$ under the operator $A(\xi)$, the set-valued function $\xi \mapsto A(\xi, x)$ is measurable [4, Ch. 2]. Denote by $A_0(\xi, x)$ is the element of least norm in $A(\xi, x)$ for any given $x \in D(\xi)$ (namely, $A_0(\xi, x) = \operatorname{proj}_{A(\xi,x)}(0)$). The function $\xi \mapsto A_0(\xi, x)$ is measurable (see again [4, Ch. 2]).

For any $\gamma > 0$, we denote by

$$J_\gamma(\xi, x) = (I + \gamma A(\xi))^{-1}(x)$$

the resolvent of $A(\xi)$. As is well-known, $J_\gamma(\xi, \cdot)$ is a non-expansive function on $\mathbb{R}^N$. Since $J_\gamma(\xi, x)$ is measurable in ξ and continuous in x, Caratheodory's theorem shows that $J_\gamma : \Xi \times \mathbb{R}^N \to \mathbb{R}^N$ is $\mathscr{T} \otimes \mathscr{B}(\mathbb{R}^N)$ measurable. We also introduce the Yosida approximation $A_\gamma(\xi)$ of $A(\xi)$, which is defined for any $\gamma > 0$ as the $\mathscr{T} \otimes \mathscr{B}(\mathbb{R}^N)$ measurable function

$$A_\gamma(\xi, x) = \frac{x - J_\gamma(\xi, x)}{\gamma}.$$

The function $A_\gamma(\xi, \cdot)$ is γ^{-1}-Lipschitz continuous, satisfies $\|A_\gamma(\xi, x)\| \uparrow \|A_0(\xi, x)\|$ and $A_\gamma(\xi, x) \to A_0(\xi, x)$ for any $x \in D(\xi)$ when $\gamma \downarrow 0$. Moreover, the inclusion $A_\gamma(\xi, x) \in A(\xi, J_\gamma(\xi, x))$ holds true for all $x \in \mathbb{R}^N$ [17, 10].

We now introduce the mean operator. The *essential intersection* $\mathcal{D}$ of the domains $D(\xi)$ is [21]

$$\mathcal{D} = \bigcup_{E \in \mathscr{T} : \mu(E) = 0} \; \bigcap_{\xi \in \Xi \setminus E} D(\xi)$$

in other words,

$$x \in \mathcal{D} \;\Leftrightarrow\; \mu(\{\xi \,:\, x \in D(\xi)\}) = 1.$$

Let us assume that $\mathcal{D} \neq \emptyset$ and that this function is integrable for each $x \in \mathcal{D}$. On $\mathcal{D}$, we define $\mathcal{A}$ as the Aumann integral

$$\mathcal{A}(x) = \int_\Xi A(\xi, x) \mu(d\xi).$$

One can immediately see that the operator $\mathcal{A} : \mathcal{D} \to 2^{\mathbb{R}^N}$ so defined is a monotone operator.

2.3 Evolution equations and almost sure APT

Given $\mathsf{A} \in \mathcal{M}$, consider the differential inclusion

$$\begin{cases} \dot{z}(t) \in -\mathsf{A}(z(t)), & \forall t \in \mathbb{R}_+ \text{ a.e.} \\ z(0) = z_0 \end{cases} \tag{3}$$

for a given $z_0 \in \operatorname{dom}(\mathsf{A})$. It is known from [17, 5] that for any $z_0 \in \operatorname{dom}(\mathsf{A})$, there exists a unique absolutely continuous function $z : \mathbb{R}_+ \to \mathbb{R}^N$ satisfying (3) - referred to as the *solution* to (3). Consider the map $\Psi : \operatorname{dom}(\mathsf{A}) \times \mathbb{R}_+ \to \operatorname{dom}(\mathsf{A})$, $(z_0, t) \mapsto z(t)$ where $z(t)$ is the solution to (3) with the initial value z_0. Then for any $t \geq 0$, $\Psi(\cdot, t)$ is a non-expansive map from $\operatorname{dom}(\mathsf{A})$ to $\operatorname{dom}(\mathsf{A})$ that can be extended by continuity to a non-expansive mapping from $\overline{\operatorname{dom}(\mathsf{A})}$ to $\overline{\operatorname{dom}(\mathsf{A})}$ that we still denote as $\Psi(\cdot, t)$ [17, 5]. The function Ψ so defined is a *semiflow* on $\overline{\operatorname{dom}(\mathsf{A})} \times \mathbb{R}_+$, being a continuous function from $\overline{\operatorname{dom}(\mathsf{A}) \times \mathbb{R}_+}$ to $\overline{\operatorname{dom}(\mathsf{A})}$ satisfying $\Psi(\cdot, 0) = I$ and $\Psi(z_0, t+s) = \Psi(\Psi(z_0, s), t)$ for every $z_0 \in \overline{\operatorname{dom}(\mathsf{A})}$, $t, s \geq 0$. The set $\gamma(x) = \{\Psi(x, t) : t \geq 0\}$ is the *orbit* of x. Although orbits of Ψ are not necessarily convergent in general, any solution to (3) converges to a zero of A (assumed to exist) whenever A is *demipositive* (see [18]). By demipositive, we mean that there exists $w \in Z(\mathsf{A})$ such that for every sequence $((u_n, v_n) \in \mathsf{A})$ such that (u_n) converges to u and $\{v_n\}$ is bounded,

$$\langle u_n - w, v_n \rangle \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow \quad u \in Z(\mathsf{A}).$$

We need to introduce some important notions associated to the semiflow Ψ. A comprehensive treatment of the subject can be found in [12, 11]. A set $S \subset \overline{\operatorname{dom}(\mathsf{A})}$ is said *invariant* for the semiflow Ψ if $\Psi(S, t) = S$ for all $t \geq 0$. Given $\varepsilon > 0$ and $T > 0$, an (ε, T)*-pseudo orbit* from a point a to a point b in $\mathbb{R}^N$ is a finite sequence of n partial orbits $(\{\Psi(y_i, s) : s \in [0, t_i]\})_{i=0,\ldots,n-1}$ such that $t_i \geq T$ for $i = 0, \ldots, n-1$ and

$$\begin{aligned} \|y_0 - a\| &< \varepsilon, \\ \|\Psi(y_i, t_i) - y_{i+1}\| &< \varepsilon \quad i = 0, \ldots, n-1, \\ y_n &= b. \end{aligned}$$

Let S be a compact invariant set S for Ψ. If for every $\varepsilon > 0$, $T > 0$ and every $a, b \in S$, there is an (ε, T)-pseudo orbit from a to b, then the set S is said *Internally Chain Transitive* (ICT). We shall say that a $\mathbb{R}^N$-valued random process $v(t)$ on $\mathbb{R}_+$ is an almost sure asymptotic pseudo trajectory [12, 13] for the differential inclusion (3) if

$$\sup_{s\in[0,T]} \|v(t+s) - \Psi(\operatorname{proj}_{\overline{\operatorname{dom}(\mathsf{A})}}(v(t)), s)\| \xrightarrow[t\to\infty]{} 0 \quad \text{a.s.}$$

for any $T > 0$ (in the APT definition of [12, 13], no projection is considered because the flow is defined there on the whole space. Projecting on $\overline{\operatorname{dom}(\mathsf{A})}$ here does not alter the conclusions). Let

$$L(v) = \bigcap_{t\geq 0} \overline{v([t, \infty))}$$

be the *limit set* of the trajectory $v(t)$, *i.e.*, the set of the limits of the convergent subsequences $v(t_k)$ as $t_k \to \infty$. An important result is the following: if $\{v(t)\}_{t\in\mathbb{R}_+}$ is bounded a.s., and if v is an almost sure APT for (3), then with probability one, the compact set $L(v)$ is ICT for the semiflow Ψ [12].

The article [14] establishes a useful property of asymptotic pseudo trajectories pertaining to the asymptotic behavior of their empirical measures. We now consider that $v : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ is a random process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. As is well known, v is said *progressively measurable* if for each $t \geq 0$, the restriction to $\Omega \times [0, t]$ of v is $\mathscr{F}_t \otimes \mathscr{B}([0, t])$-measurable, where $\mathscr{B}([0, t])$ is the Borel field over $[0, t]$. For $t \geq 0$, the *empirical measure* $\nu_t(\omega, \cdot)$ of v is then the random probability measure defined by the identity

$$\int f(x)\, \nu_t(\omega, dx) = \frac{1}{t} \int_0^t f(v(\omega, s))\, ds$$

for any measurable function $f : \mathbb{R}^N \to \mathbb{R}_+$. We also note that a probability measure ν on $\mathbb{R}^N$ is said *invariant* for the semiflow Ψ if

$$\int f(x)\,\nu(dx) = \int f(\Phi(x,t))\,\nu(dx)$$

for any $t \ge 0$ and any measurable function $f : \mathbb{R}^N \to \mathbb{R}_+$.

Now, if v is progressively measurable and if it is an almost sure APT for the semiflow Ψ, then on a probability one set, all the accumulation points of the set $\{\nu_t(\omega,\cdot)\}_{t\ge 0}$ for the weak convergence of probability measures are invariant measures for Ψ [14, Th. 1]. [1]

2.4 Algorithm description and main results

Let $B : \Xi \to \mathcal{M}$ be a mapping such that, similarly to the mapping A introduced in Section 2.2, the function $\xi \mapsto \operatorname{gr}(B(\xi))$ is measurable. We moreover assume throughout the paper that $\operatorname{dom}(B(\xi)) = \mathbb{R}^N$ for almost every $\xi \in \Xi$. We also assume that for every $x \in \mathbb{R}^N$, $B(\cdot,x)$ is integrable and we set $\mathcal{B}(x) = \int B(\xi,x)\mu(d\xi)$. Note that $\operatorname{dom}\mathcal{B} = \mathbb{R}^N$. Let $(u_n)_{n\in\mathbb{N}^*}$ be an iid sequence of random variables from a probability space $(\Omega,\mathscr{F},\mathbb{P})$ to $(\Xi,\mathscr{T})$ having the distribution μ. Starting with some arbitrary $x_0 \in \mathbb{R}^N$, our purpose is to study the behavior of the iterates

$$x_{n+1} = J_{\gamma_{n+1}}(u_{n+1}, x_n - \gamma_{n+1} b(u_{n+1},x_n)), \qquad (n\in\mathbb{N}),$$

where the positive sequence $(\gamma_n)_{n\in\mathbb{N}^*}$ belongs to $\ell^2\setminus\ell^1$ and where b is a measurable map on $(\Xi\times\mathbb{R}^N, \mathscr{T}\otimes\mathscr{B}(\mathbb{R}^N)) \to (\mathbb{R}^N,\mathscr{B}(\mathbb{R}^N))$ such that for every $x\in\mathbb{R}^N$, $b(\,.\,,x)\in\mathcal{S}^1_{B(\,.\,,x)}$ (a possible choice for b is for instance $b(\xi,x) = B_0(\xi,x)$ which is $\mathscr{T}\otimes\mathscr{B}(\mathbb{R}^N)$–measurable as the limit as $\gamma\downarrow 0$ of $B_\gamma(\xi,x)$). Recall the definition of the affine interpolated process

$$x(t) = x_n + \frac{x_{n+1}-x_n}{\gamma_{n+1}}(t-\tau_n)$$

for every $t\in[\tau_n,\tau_{n+1})$ where $\tau_n = \sum_{k\ge n}\gamma_k$. Consider the differential inclusion

$$\left\{\begin{array}{l} \dot z(t) \in -(\mathcal{A}+\mathcal{B})(z(t)), \quad \forall t\in\mathbb{R}_+ \text{ a.e.} \\ z(0) = z_0\,. \end{array}\right. \tag{4}$$

If $\mathcal{A}+\mathcal{B}$ is maximal, then for any $z_0\in\mathcal{D}$, (4) has a unique solution. The case being, let $\Phi : \overline{\mathcal{D}}\times\mathbb{R}_+ \to \overline{\mathcal{D}}$ be the semiflow associated to (4).

Before stating our main result, we need a preliminary remark. A point $x_\star$ is an element of $\mathcal{Z} = Z(\mathcal{A}+\mathcal{B})$ if and only if there exists $\varphi\in\mathcal{S}^1_{A(\cdot,x_\star)}$ and $\psi\in\mathcal{S}^1_{B(\cdot,x_\star)}$ such that $\int\varphi d\mu + \int\psi d\mu = 0$. We will refer to such couple (φ,ψ) as a *representation* of the zero $x^\star$. In Theorem 2.1 below, we shall moreover assume that there exists such a zero $x_\star$ for which the above functions φ and ψ can be chosen in $\mathcal{L}^{2p}(\Xi,\mathscr{T},\mu;\mathbb{R}^N)$ where $p\ge 1$ is some integer possibly strictly larger than one. We thus introduce the set of $2p$-integrable representations

$$\mathcal{R}_{2p}(x_\star) = \left\{(\varphi,\psi)\in\mathcal{S}^{2p}_{A(\cdot,x_\star)}\times\mathcal{S}^{2p}_{B(\cdot,x_\star)} : \int\varphi d\mu + \int\psi d\mu = 0\right\}.$$

We denote by $\Pi(\xi,.)$ the projection operator onto $\overline{D(\xi)}$ and by $d(\xi,\cdot)$ (resp. $\boldsymbol{d}(\cdot)$) the distance function to $D(\xi)$ (resp. to $\mathcal{D}$).

[1] The result is stated in [14] when v is a so-called *weak APT*. It turns out that any almost sure APT is a weak APT by Levy's conditional form of Borel-Cantelli's lemma.

Theorem 2.1. *Assume the following facts:*

1. *The monotone operator $\mathcal{A}$ is maximal,*
2. *There exists an integer $p \geq 1$ and an element $x_\star \in \mathcal{Z}$ such that $\mathcal{R}_{2p}(x_\star) \neq \emptyset$.*
3. *For any compact set K of $\mathbb{R}^N$, there exists $\varepsilon \in (0,1]$ such that*
$$\sup_{x \in K \cap \mathcal{D}} \int \|A_0(\xi, x)\|^{1+\varepsilon} \mu(d\xi) < \infty,$$
moreover, there exists $y_0 \in \mathcal{D}$ such that
$$\int \|A_0(\xi, y_0)\|^{1+1/\varepsilon} \mu(d\xi) < \infty,$$
4. *There exists $C > 0$ such that for all $x \in \mathbb{R}^N$,*
$$\int d(\xi, x)^2 \mu(d\xi) \geq C \boldsymbol{d}(x)^2$$
and furthermore, $\gamma_{n+1}/\gamma_n \to 1$,
5. *There exists $C > 0$ such that for any $x \in \mathbb{R}^N$ and any $\gamma > 0$,*
$$\frac{1}{\gamma^4} \int \|J_\gamma(\xi, x) - \Pi(\xi, x)\|^4 \mu(d\xi) \leq C(1 + \|x\|^{2p})$$
where the integer p is the one specified in 2.
6. *There exists $M : \Xi \to \mathbb{R}_+$ such that M^{2p} is μ-integrable and for all $x \in \mathbb{R}^N$,*
$$\|b(\xi, x)\| \leq M(\xi)(1 + \|x\|) \,.$$
Moreover, $\int \|b(\xi, x)\|^4 \mu(d\xi) \leq C(1 + \|x\|^{2p})$.

Then the monotone operator $\mathcal{A}+\mathcal{B}$ is maximal. Moreover, with probability one, the continuous time process $x(t)$ is bounded and is an APT of the differential inclusion (4).

Sufficient conditions for the maximality of $\mathcal{A}$ are provided below in Sections 2.5 and 3.1. Setting $\varepsilon = 1$, Assumption 3 can be replaced by the stronger condition that for any compact set K of $\mathbb{R}^N$,
$$\sup_{x \in K \cap \mathcal{D}} \int \|A_0(\xi, x)\|^2 \mu(d\xi) < \infty.$$
In the particular case where μ is a finite sum of Dirac measures, Assumption 4 reduces to the linear regularity condition introduced by [9]. Let us finally discuss Assumption 5. As $\gamma \to 0$, it is a known that for every (ξ, x), $J_\gamma(\xi, x)$ tends to $\Pi(\xi, x)$. Assumption 5 moreover provides a control on the convergence rate. The fourth order moment of $\|J_\gamma(\xi, x) - \Pi(\xi, x)\|$ is assumed to vanish at rate γ^4 with a multiplicative constant of order $\|x\|^{2p}$. The integer p can potentially be chosen as large as needed, provided that one is able to find a zero $x_\star$ satisfying Assumption 2.

The results of Theorem 2.1 can first be used to study the convergence of the sequence $(\bar{x}_n)$ of empirical means:

Corollary 2.1. *Let the assumptions in the statement of Theorem 2.1 hold true. Assume that for any $x_\star \in \mathcal{Z}$, the set $\mathcal{R}_2(x_\star)$ is nonempty. Then for any initial value x_0, there exists a random variable U supported by $\mathcal{Z}$ such that the sequence $(\bar{x}_n)$ of empirical means converges almost surely to U as $n \to \infty$.*

We now consider the issue of the convergence of the sequence $\{x_n\}$ to a point of $\mathcal{Z}$. Note that the conditions of Theorem 2.1 are generally unsufficient to ensure that x_n converges. A counterexample is obtained by setting $N = 2$ and taking $\mathcal{A}$ as a $\pi/2$-rotation matrix, $\mathcal{B} = 0$. However, the statement will be proved valid when $\mathcal{A} + \mathcal{B}$ is moreover assumed demipositive. We start by listing some known verifiable conditions ensuring that the maximal monotone operator $\mathcal{A} + \mathcal{B}$ is demipositive:

1. $\mathcal{A} + \mathcal{B} = \partial G$ where $G \in \Gamma_0$ has a minimum.
2. $\mathcal{A} + \mathcal{B} = I - T$ where T is a non-expansive mapping having a fixed point.
3. The interior of $\mathcal{Z}$ is nonempty.
4. $\mathcal{Z} \neq \emptyset$ and $\mathcal{A} + \mathcal{B}$ is 3-monotone, *i.e.*, for every triple $(x_i, y_i) \in \mathcal{A} + \mathcal{B}$ for $i = 1, 2, 3$, it holds that $\sum_{i=1}^{3} \langle y_i, x_i - x_{i-1} \rangle \geq 0$ by setting $x_0 = x_3$.
5. $\mathcal{A} + \mathcal{B}$ is strongly monotone, *i.e.*, there exists $\alpha > 0$ such that $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \alpha \|x_1 - x_2\|^2$ for all (x_1, y_1) and (x_2, y_2) in $\mathcal{A} + \mathcal{B}$.
6. $\mathcal{Z} \neq \emptyset$ and $\mathcal{A} + \mathcal{B}$ is cocoercive, *i.e.*, there exists $\alpha > 0$ such that $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \alpha \|y_1 - y_2\|^2$ for all (x_1, y_1) and (x_2, y_2) in $\mathcal{A} + \mathcal{B}$.

The above conditions can be found in [30]. Specifically, conditions 1–3 can be found in [18] while Condition 4 can be found in [29]. Conditions 5 and 6 can be easily verified to lead to the demipositivity of $\mathcal{A} + \mathcal{B}$. Condition 1 is further discussed in Section 3.1 below. Condition 2 is satisfied if $\mathcal{Z} \neq \emptyset$ and if for any ξ, the operator $I - (\mathcal{A} + \mathcal{B})(\xi)$ is a non-expansive mapping. Condition 4 is satisfied if $\mathcal{Z} \neq \emptyset$ and if all the operators $A(\xi) + B(\xi)$ are 3-monotone. The last two conditions are most often easily verifiable.
We now have:

Corollary 2.2. *Let the assumptions in the statement of Theorem 2.1 hold true. Assume in addition that the operator $\mathcal{A} + \mathcal{B}$ is demipositive and that for any $x_\star \in \mathcal{Z}$, the set $\mathcal{R}_2(x_\star)$ is nonempty. Then for any initial value x_0, there exists a random variable U supported by $\mathcal{Z}$ such that $x_n \to U$ almost surely as $n \to \infty$.*

We now address the important problem of the maximality of $\mathcal{A}$.

2.5 Maximality of $\mathcal{A}$

By extending a well-known result on the maximality of the sum of two maximal monotone operators, it is obvious that $\mathcal{A}$ is maximal is the case where μ is a finite sum of Dirac measures and where the interior of $\mathcal{D}$ is non empty [17, 10]. For more general measures μ, we have the following result.

Proposition 2.1. *Assume the following facts:*

1. *The interior of* $\mathcal{D}$ *is non empty, and there exists a closed ball in* $\mathcal{D}$ *such that* $\|A_0(\xi, x)\| \leq M(\xi)$ *for any* x *in this ball, and such that* $M(\xi)$ *is* μ*-integrable,*

2. *For any compact set* K *of* $\mathbb{R}^N$*, there exists* $\varepsilon > 0$ *such that*

$$\sup_{x \in K \cap \mathcal{D}} \int \|A_0(\xi, x)\|^{1+\varepsilon} \mu(d\xi) < \infty,$$

moreover, there exists $y_0 \in \mathcal{D}$ *such that*

$$\int \|A_0(\xi, y_0)\|^{1+1/\varepsilon} \mu(d\xi) < \infty,$$

3. *There exists* $C > 0$ *such that for any* $x \in \mathbb{R}^N$*,*

$$\int d(\xi, x)\mu(d\xi) \geq C\boldsymbol{d}(x),$$

4. $\int \|J_\gamma(\xi, x) - \Pi(\xi, x)\|\mu(d\xi) \leq \gamma C(x)$ *where* $C(x)$ *is bounded on compact sets of* $\mathbb{R}^N$*,*

Then the monotone operator $\mathcal{A}$ *is maximal.*

3 Application to convex optimization

We start this section by briefly reproducing some known results related to the case where $A(\xi)$ is the subdifferential of a proper closed convex function $g(\xi, \cdot)$.

3.1 Known facts about the Aumann integral of subdifferentials

A function $g : \Xi \times \mathbb{R}^N \to (-\infty, \infty]$ is called a *normal integrand* [32] if the set-valued mapping $\xi \mapsto \operatorname{epi} g(\xi, \cdot)$ is closed-valued and measurable. Let us assume in addition that $g(\xi, \cdot)$ is convex and proper for every ξ.

Consider the case where $A(\xi) = \partial g(\xi, \cdot)$. The mean operator $\mathcal{A}$ is given by[2]

$$\mathcal{A}(x) = \int \partial g(\xi, x)\mu(d\xi) \,. \tag{5}$$

Under some conditions that will be discussed below, the integral and subdifferential signs can be exchanged in (5). In this case,

$$\mathcal{A}(x) = \partial G(x) \tag{6}$$

where G is the integral functional given by $G(x) = \int g(\xi, x) \, \mu(d\xi)$, the integral being defined here as the sum

$$\int_{\{\xi \,:\, g(\xi,x) \in \mathbb{R}_+\}} g(\xi, x) \, \mu(d\xi) + \int_{\{\xi \,:\, g(\xi,x) \in (-\infty,0)\}} g(\xi, x) \, \mu(d\xi) + I(x)$$

[2]By [4, 34], it holds that the mapping $A : \Xi \to \mathcal{M}$ defined as $A(\xi) = \partial g(\xi, \cdot)$ is measurable in the sense of Section 2.2.

where

$$I(x) = \begin{cases} +\infty & \text{if } \mu(\{\xi : g(\xi, x) = \infty\}) > 0, \\ 0 & \text{otherwise} \end{cases}$$

and where the convention $(+\infty) + (-\infty) = +\infty$ is used. The function G is a lower semi continuous convex function if $G(x) > -\infty$ for all x [38]. Assuming in addition that G is proper, the identity (6) ensures that:

- $\mathcal{A}$ is a maximal monotone demipositive operator,
- the zeros of $\mathcal{A}$ are the minimizers of G.

Sufficient conditions for obtaining (6) can be found in [33]. Namely, denoting as $\operatorname{dom} g(\xi, \cdot)$ the domain of this function, assume that $G(x) < \infty$ whenever $x \in \operatorname{dom} g(\xi, \cdot)$ μ-almost everywhere. Suppose moreover that G is continuous at some point and that the set-valued function $\xi \mapsto \overline{\operatorname{dom} g(\xi, \cdot)}$ is constant almost everywhere. Assume finally that the right hand side of (6) is a closed set. Then (6) holds true.

3.2 A constrained optimization problem

In this paragraph, we consider the case of Example 2 described in the introduction of this paper. Let $(\mathsf{X}, \mathscr{X}, \nu)$ be a probability space. Let the functions $f : \mathsf{X} \times \mathbb{R}^N \to (-\infty, \infty)$ and $g : \mathsf{X} \times \mathbb{R}^N \to (-\infty, \infty)$ be normal convex integrand. Here we assume that g is everywhere finite to simplify the presentation, however we note that the results can be extended to the case where g is allowed to take the value $+\infty$. Recall the optimization problem

$$\min_{x \in C} F(x) + G(x), \quad C = \bigcap_{i=1}^{m} C_i \tag{7}$$

where $F(x) = \int f(\eta, x)\nu(d\eta)$, $G(x) = \int g(\eta, x)\nu(d\eta)$ and $C_1, \ldots, C_m$ are closed convex sets. Consider a measurable function $\tilde{\nabla} f : \mathsf{X} \times \mathbb{R}^N \to \mathbb{R}$ such that for every $\eta \in \mathsf{X}$ and $x \in \mathbb{R}^N$, $\tilde{\nabla} f(\eta, x)$ is a subgradient of $f(\eta, \,.\,)$ at x. Let $(v_n)_n$ be an iid sequence on X with probability distribution ν. Finally, let (I_n) be an iid sequence on $\{0, 1, \ldots, m\}$ with distribution $\alpha_i = \mathbb{P}(I_1 = i)$ for every i and satisfying $\alpha_i > 0$ for every i. We consider the iterates

$$x_{n+1} = \begin{cases} \operatorname{prox}_{\alpha_0^{-1}\gamma_{n+1} g(v_{n+1}, \cdot)}(x_n - \gamma_{n+1}\tilde{\nabla} f(v_{n+1}, x_n)) & \text{if } I_{n+1} = 0, \\ \operatorname{proj}_{C_{I_{n+1}}}(x_n - \gamma_{n+1}\tilde{\nabla} f(v_{n+1}, x_n)) & \text{otherwise.} \end{cases} \tag{8}$$

We denote by $\partial g_0(\eta, x)$ the element of least norm in the subdifferential of $g(\eta, \,.\,)$ at point x. If H is a subset of $\mathbb{R}^N$, we use the notation $|H| = \sup\{\|v\| : v \in H\}$.

Corollary 3.1. *We assume the following. Let $p \geq 1$ be an integer.*

1. *For every $x \in \mathbb{R}^N$, $\int |f(\eta, x)|\nu(d\eta) + \int |g(\eta, x)|\nu(d\eta) < \infty$.*
2. *For any solution $x_\star$ to Problem (7), there exists a measurable function $M_\star : \mathsf{X} \to \mathbb{R}$ such that $\int M_\star(\eta)^2 \nu(d\eta) < \infty$ and for all $\eta \in \mathsf{X}$, $|\partial f(\eta, x_\star)| + |\partial g(\eta, x_\star)| \leq M_\star(\eta)$. Moreover, there exists a solution $x_\star$ for which $\int M_\star(\eta)^{2p} \nu(d\eta) < \infty$.*

3. *For any compact set K of $\mathbb{R}^N$, there exists $\varepsilon \in (0,1]$ such that*

$$\sup_{x\in K} \mathbb{E}\|\partial g_0(\Theta, x)\|^{1+\varepsilon} < \infty\,.$$

   Moreover, there exists $y_0 \in C$ such that $\mathbb{E}\|\partial g_0(\Theta, y_0)\|^{1+1/\varepsilon} < \infty$.

4. *The closed convex sets $C_1, \ldots, C_m$ are linearly regular* i.e.,

$$\exists \kappa > 0, \forall x \in \mathbb{R}^N, \max_{i=1,\ldots,m} d(x, C_i) \geq \kappa\, d(x, \bigcap_{i=1}^m C_i)$$

   and $\gamma_n/\gamma_{n+1} \to 1$. Moreover, C has a non-empty interior.

5. *There exists $M : \mathsf{X} \to \mathbb{R}$ such that $\int M(\eta)^{2p}\nu(d\eta) < \infty$ and for all $(\eta, x) \in \mathsf{X} \times \mathbb{R}^N$, $\|\tilde{\nabla} f(\eta, x)\| \leq M(\eta)(1 + \|x\|)$.*

6. *There exists $c > 0$ such that for all $x \in \mathbb{R}^N$, $\int \|\tilde{\nabla} f(\eta, x)\|^4 \nu(d\eta) \leq c(1 + \|x\|^{2p})$.*

Then, the sequence (x_n) given by (8) converges almost surely to a solution to Problem (7).

4 Proofs

We start with the proof of Proposition 2.1 because it contains many elements of the proof of the main theorem.

4.1 Proof of Proposition 2.1

We recall that for any $\xi \in \Xi$ and any $\gamma > 0$, the Yosida approximation $A_\gamma(\xi)$ is a single-valued γ^{-1}-Lipschitz monotone operator defined on $\mathbb{R}^N$. As a consequence, the operator $\mathcal{A}^\gamma : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\mathcal{A}^\gamma(x) = \int A_\gamma(\xi, x)\mu(d\xi)$$

is a single-valued continuous monotone operator defined on $\mathbb{R}^N$. As such, $\mathcal{A}^\gamma$ is maximal [17, Prop. 2.4]. Thus, given any $y \in \mathbb{R}^N$, there exists $x_\gamma \in \mathbb{R}^N$ such that $y = x_\gamma + \mathcal{A}^\gamma(x_\gamma)$. We shall find a sequence $\gamma_n \to 0$ such that $x_{\gamma_n} \to x_\star \in \mathcal{D}$ with $y - x_\star \in \mathcal{A}x_\star$. The maximality of $\mathcal{A}$ then follows by Minty's theorem [17].
Denote respectively by z_0 and ρ the center and the radius of the ball alluded to in Assumption 1, and set

$$u(\xi) = z_0 + \rho \frac{A_\gamma(\xi, x_\gamma)}{\|A_\gamma(\xi, x_\gamma)\|} \in \mathcal{D}$$

where the convention $0/0 = 0$ is used. By the monotonicity of $A_\gamma(\xi)$,

$$0 \leq \int \langle x_\gamma - u(\xi), A_\gamma(\xi, x_\gamma) - A_\gamma(\xi, u(\xi))\rangle\, \mu(d\xi).$$

Writing $C = \int M(\xi)\mu(d\xi) < \infty$ (see Assumption 1), we obtain

$$\begin{aligned}
\int \langle x_\gamma, A_\gamma(\xi, x_\gamma)\rangle \, \mu(d\xi) &= \langle x_\gamma, y\rangle - \|x_\gamma\|^2, \\
\int \langle -u(\xi), A_\gamma(\xi, x_\gamma)\rangle \, \mu(d\xi) &= \langle z_0, x_\gamma - y\rangle - \rho \int \|A_\gamma(\xi, x_\gamma)\| \, \mu(d\xi), \\
\int |\langle x_\gamma, A_\gamma(\xi, u(\xi))\rangle| \, \mu(d\xi) &\le \|x_\gamma\| \int \|A_0(\xi, u(\xi)\| \, \mu(d\xi) \le C\|x_\gamma\|, \\
\int |\langle u(\xi), A_\gamma(\xi, u(\xi))\rangle| \, \mu(d\xi) &\le C(\|z_0\| + \rho).
\end{aligned}$$

Therefore,

$$\rho \int \|A_\gamma(\xi, x_\gamma)\| \, \mu(d\xi) + \|x_\gamma\|^2 \le \|x_\gamma\|(\|y\| + \|z_0\| + C) + C(\|z_0\| + \rho) + \|z_0\| \, \|y\| \,.$$

This shows that the families $\{\|x_\gamma\|\}$ and $\{\int \|A_\gamma(\xi, x_\gamma)\| \, \mu(d\xi)\}$ are both bounded. Writing $A_\gamma(\xi, x_\gamma) = \gamma^{-1}(\Pi(\xi, x_\gamma) - J_\gamma(\xi, x_\gamma)) + \gamma^{-1}(x_\gamma - \Pi(\xi, x_\gamma))$ and using Assumption 4, we obtain that the set $\{\gamma^{-1} \int \|x_\gamma - \Pi(\xi, x_\gamma)\|\mu(d\xi)\}$ is bounded. By Assumption 3, $\{\boldsymbol{d}(x_\gamma)/\gamma\}$ is bounded. Given x_γ, let us choose $\tilde{x}_\gamma \in \mathcal{D}$ such that $\|x_\gamma - \tilde{x}_\gamma\| \le 2\boldsymbol{d}(x_\gamma)$. By the boundedness of $\{\|x_\gamma\|\}$, there exists a compact set $K \subset \mathbb{R}^N$ such that $\tilde{x}_\gamma \in K$. Associating to K a positive number ε as in Assumption 2, we obtain

$$\begin{aligned}
\int \|A_\gamma(\xi, x_\gamma)\|^{1+\varepsilon} \, \mu(d\xi) &\le 2^\varepsilon \int \Big(\|A_\gamma(\xi, \tilde{x}_\gamma)\|^{1+\varepsilon} + \|A_\gamma(\xi, x_\gamma) - A_\gamma(\xi, \tilde{x}_\gamma)\|^{1+\varepsilon} \Big) \mu(d\xi) \\
&\le 2^\varepsilon \int \|A_0(\xi, \tilde{x}_\gamma)\|^{1+\varepsilon} \, \mu(d\xi) + 2^{1+2\varepsilon} \Big| \frac{\boldsymbol{d}(x_\gamma)}{\gamma} \Big|^{1+\varepsilon}
\end{aligned}$$

which is bounded by a constant independent of γ thanks to Assumption 2. Thus, the family of $\Xi \to \mathbb{R}^N$ functions $\{A_\gamma(\xi, x_\gamma)\}$ is bounded in the Banach space $\mathcal{L}^{1+\varepsilon}(\Xi, \mathscr{T}, \mu; \mathbb{R}^N)$.

Let us take a sequence (γ_n, x_{γ_n}) converging to $(0, x_\star)$. Let us extract from the sequence of indices (n) a subsequence (still denoted as (n)) such that $(A_{\gamma_n}(\xi, x_{\gamma_n}))_n$ converges weakly in $\mathcal{L}^{1+\varepsilon}$ towards a function $f(\xi)$. By Mazur's theorem, there exists a function $J : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of weights $(\{\alpha_{k,n}, k = n \ldots, J(n) \, : \, \alpha_{k,n} \ge 0, \sum_{k=n}^{J(n)} \alpha_{k,n} = 1\})_n$ such that the sequence of functions $(g_n(\xi) = \sum_{k=n}^{J(n)} \alpha_{k,n} A_{\gamma_k}(\xi, x_{\gamma_k}))$ converges strongly to f in $\mathcal{L}^{1+\varepsilon}$. Taking a further subsequence, we obtain the μ-almost everywhere convergence of (g_n) to f.

Observe that $x_\star \in \overline{\mathcal{D}}$ since $\boldsymbol{d}(x_{\gamma_n}) \to 0$. Choose a sequence (z_n) in $\mathcal{D}$ that converges to $x_\star$, and for each n, let $T_n = \{\xi \in \Xi \, : \, z_n \in D(\xi)\}$. Then on the probability one set $T = \cap_n T_n$, it holds that $x_\star \in \overline{D(\xi)}$. On the intersection of T and the set where $g_n \to f$, set $\eta_n(\xi) = J_{\gamma_n}(\xi, x_{\gamma_n}) - x_\star$, and write

$$\|\eta_n(\xi)\| \le \|J_{\gamma_n}(\xi, x_{\gamma_n}) - J_{\gamma_n}(\xi, x_\star)\| + \|J_{\gamma_n}(\xi, x_\star) - x_\star\|.$$

Since $J_{\gamma_n}(\xi, \cdot)$ is non-expansive and since $x_\star \in \overline{D(\xi)}$, it holds that $\eta_n(\xi) \to_n 0$. Considering Assumption 2 we also have

$$\begin{aligned}
\|\eta_n(\xi)\| &\le \|x_\star\| + \|J_{\gamma_n}(\xi, x_{\gamma_n}) - J_{\gamma_n}(\xi, y_0)\| + \|J_{\gamma_n}(\xi, y_0) - y_0\| + \|y_0\| \\
&\le \|x_\star\| + \sup_\gamma \|x_\gamma\| + 2\|y_0\| + \|A_0(\xi, y_0)\|
\end{aligned}$$

as soon as $\gamma_n \le 1$. By Assumption 2 and the dominated convergence theorem, we obtain that $\eta_n \to 0$ in $\mathcal{L}^{1+1/\varepsilon}$. With this at hand,

$$\int |\langle \eta_n(\xi), A_{\gamma_n}(\xi, x_{\gamma_n})\rangle| \mu(d\xi) \\ \le \left(\int \|\eta_n(\xi)\|^{1+1/\varepsilon} \mu(d\xi) \right)^{\varepsilon/(1+\varepsilon)} \left(\int \|A_{\gamma_n}(\xi, x_{\gamma_n})\|^{1+\varepsilon} \mu(d\xi) \right)^{1/(1+\varepsilon)}$$

and we obtain that the left hand side converges to zero. Consequently, the random variable

$$e_n = \sum_{k=n}^{J(n)} \alpha_{k,n} \langle J_{\gamma_k}(\xi, x_{\gamma_k}) - x_\star, A_{\gamma_k}(\xi, x_{\gamma_k})\rangle$$

converges to zero in probability, hence in the μ-almost sure sense along a subsequence. Fix ξ in the new probability one set so defined, choose $(u, v) \in A(\xi)$ arbitrarily, and write

$$X_n = \sum_{k=n}^{J(n)} \langle u - J_{\gamma_k}(\xi, x_{\gamma_k}), \alpha_{k,n} v - \alpha_{k,n} A_{\gamma_k}(\xi, x_{\gamma_k})\rangle.$$

It holds by the monotonicity of $A(\xi)$ that $X_n \ge 0$. Writing

$$X_n = \langle u - x_\star, v - g_n(\xi)\rangle + e_n - \sum_{k=n}^{J(n)} \alpha_{k,n} \langle \eta_k, v\rangle$$

and taking $n \to \infty$, we obtain that $\langle u - x_\star, v - f(\xi)\rangle \ge 0$. By the maximality of $A(\xi)$, it holds that $(x_\star, f(\xi)) \in A(\xi)$.

To conclude, we have

$$y = \sum_{k=n}^{J(n)} \alpha_{k,n} x_{\gamma_k} + \int g_n(\xi)\, \mu(d\xi),$$

$\sum_{k=n}^{J(n)} \alpha_{k,n} x_{\gamma_k} \to_n x_\star \in \mathcal{D}$, and $g_n \xrightarrow{\mathcal{L}^1(\mu)} f \in \mathcal{S}^1_{A(\cdot, x_\star)}$. Making $n \to \infty$, we obtain

$$y - x_\star = \int f(\xi)\, \mu(d\xi) \in \mathcal{A}(x_\star)$$

which is the desired result.

4.2 Proof of Theorem 2.1

Noting that $\operatorname{dom} \mathcal{B} = \mathbb{R}^N$ and using Assumption 6 of Theorem 2.1, one can check that the assumptions of Proposition 2.1 are satisfied for B. It results that $\mathcal{B}$ is maximal. Because $\mathcal{B}$ has moreover a full domain and $\mathcal{A}$ is maximal, $\mathcal{A} + \mathcal{B}$ is maximal by [10, Corollary 24.4]. Thus, the first assertion of Theorem 2.1 is shown, and moreover, the differential inclusion (4) admits a unique solution, and the associated semiflow Φ is well defined.

Defining $Y_\gamma(\xi, x) = A_\gamma(\xi, x - \gamma b(\xi, x))$, the iterates x_n be rewritten as

$$\begin{aligned} x_{n+1} &= x_n - \gamma_{n+1} b(u_{n+1}, x_n) - \gamma_{n+1} Y_{\gamma_{n+1}}(u_{n+1}, x_n) \\ &= x_n - \gamma_{n+1} h_{\gamma_{n+1}}(x_n) + \gamma_{n+1} \eta_{n+1}, \end{aligned}$$

where we define

$$h_\gamma(x) = \int (Y_\gamma(\xi, x) + b(\xi, x))\mu(d\xi)$$

and

$$\eta_{n+1} = -Y_{\gamma_{n+1}}(u_{n+1}, x_n) + \mathbb{E}_n(Y_{\gamma_{n+1}}(u_{n+1}, x_n)) - b(u_{n+1}, x_n) + \mathbb{E}_n(b(u_{n+1}, x_n))$$

where $\mathbb{E}_n$ denotes the expectation conditionally to the sub σ-field $\sigma(u_1, \ldots, u_n)$ of $\mathscr{F}$ (we also write $\mathbb{E}_0 = \mathbb{E}$). Consider the martingale

$$M_n = \sum_{k=1}^{n} \gamma_k \eta_k$$

and let $M(t)$ be the affine interpolated process defined for any $n \in \mathbb{N}$ and any $t \in [\tau_n, \tau_{n+1})$ as

$$M(t) = M_n + \eta_{n+1}(t - \tau_n) = M_n + \frac{M_{n+1} - M_n}{\gamma_{n+1}}(t - \tau_n).$$

For any $t \geq 0$, let

$$r(t) = \max\{k \geq 0 \ : \ \tau_k \leq t\}.$$

Then for any $t \geq 0$, we obtain

$$\begin{aligned} x(\tau_n + t) - x(\tau_n) &= -\int_0^t h_{\gamma_{r(\tau_n+s)+1}}(x_{r(\tau_n+s)})\, ds + M(\tau_n + t) - M(\tau_n) \\ &= H(\tau_n + t) - H(\tau_n) + M(\tau_n + t) - M(\tau_n) \end{aligned} \tag{9}$$

where $H(t) = \int_0^t h_{\gamma_{r(s)+1}}(x_{r(s)})\, ds$. The idea of the proof is to establish that on a $\mathbb{P}$-probability one set, the sequence of continuous time processes $(x(\tau_n + \cdot))_{n\in\mathbb{N}}$ is equicontinuous and bounded. The accumulation points for the uniform convergence on a compact interval $[0, T]$ (who are guaranteed to exist by the Arzelà-Ascoli theorem) will be shown moreover to have the form

$$z(t) - z(0) = -\lim_{n\to\infty} \int_0^t ds \int_\Xi \mu(d\xi)\, (Y_{\gamma_{r(\tau_n+s)+1}}(\xi, x_{r(\tau_n+s)}) + b(\xi, x_{r(\tau_n+s)})) \tag{10}$$

where the limit is taken over a subsequence. We then show that the sequence of $\Xi \times [0, T] \to \mathbb{R}^{2N}$ functions $((\xi, s) \mapsto Y_{\gamma_{r(\tau_n+s)+1}}(\xi, x_{r(\tau_n+s)}), b(\xi, x_{r(\tau_n+s)}))_n$ is bounded in the Banach space $\mathcal{L}^{1+\varepsilon}(\Xi \times [0, T], \mu \otimes \lambda)$ where λ is the Lebesgue measure on $[0, T]$. Analyzing the accumulation points and following an approach similar to the one used in the proof of Proposition 2.1, we prove that the limit in the right hand side of (10) coincides with

$$z(t) - z(0) = -\lim_{n\to\infty} \int_0^t ds \left(\int_\Xi f^{(a)}(\xi, s)\mu(d\xi) + \int_\Xi f^{(b)}(\xi, s)\mu(d\xi) \right)$$

where for almost every $s \in [0, T]$, $f^{(a)}(\cdot, s)$ and $f^{(b)}(\cdot, s)$ are integrable selections of $A(\cdot, s)$ and $B(\cdot, s)$ respectively. This shows that z satisfies the differential inclusion (4). Hence, almost surely, the accumulation points of the sequence of processes $(x(\tau_n + \cdot))_{n\in\mathbb{N}}$ are solutions to (4). Recalling that the latter defines a semiflow $\Phi : \overline{\mathcal{D}} \times \mathbb{R}_+ \to \overline{\mathcal{D}}$, it follows that the process $x(t)$

is a.s. an APT of (4).

Throughout the proof, C denotes a positive constant that can change from line to line but that remains independent of n. We denote by c, c_1, etc. random variables on $\Omega \to \mathbb{R}_+$ that do not depend on n. For a fixed event $\omega \in \Omega$, these will act as constants.

Proposition 4.1. *Let Assumptions 2 and 6 of Theorem 2.1 hold true. Then*

1. *The sequence (x_n) is bounded almost surely and in $\mathcal{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^N)$,*
2. $\mathbb{E}[\sum_n \gamma_n^2 \int \|Y_{\gamma_n}(\xi, x_n)\|^2 \mu(d\xi)] < \infty$,
3. *The sequence $(\|x_n - x_\star\|)_n$ converges almost surely.*

Proof. By expanding $\|x_{n+1} - x_\star\|^2 = \|x_n - x_\star\|^2 + 2\langle x_{n+1} - x_n, x_n - x_\star\rangle + \|x_{n+1} - x_n\|^2$, we obtain

$$\begin{aligned}\|x_{n+1} - x_\star\|^2 = \|x_n - x_\star\|^2 - 2\gamma_{n+1}\langle Y_{\gamma_{n+1}}(u_{n+1}, x_n), x_n - x_\star\rangle - 2\gamma_{n+1}\langle b(u_{n+1}, x_n), x_n - x_\star\rangle \\ + \gamma_{n+1}^2 \|b(u_{n+1}, x_n) + Y_{\gamma_{n+1}}(u_{n+1}, x_n)\|^2\end{aligned}$$

Thanks to Assumption 2, we can choose $\varphi \in \mathcal{S}^2_{A(\,\cdot\,, x_\star)}$ and $\psi \in \mathcal{S}^1_{B(\,\cdot\,, x_\star)}$ such that $0 = \int (\varphi + \psi) d\mu$. Writing $u = u_{n+1}$, $\gamma = \gamma_{n+1}$, $Y_\gamma = Y_{\gamma_{n+1}}(u_{n+1}, x_n)$, $J_\gamma = J_{\gamma_{n+1}}(u_{n+1}, x_n - \gamma_{n+1} b(u_{n+1}, x_n))$, and $b = b(u_{n+1}, x_n)$ for conciseness, and recalling that $Y_\gamma = (x - \gamma b - J_\gamma)/\gamma$, we write

$$\begin{aligned}\langle Y_\gamma, x_n - x_\star\rangle &= \langle Y_\gamma - \varphi(u), J_\gamma - x_\star\rangle + \gamma\langle Y_\gamma - \varphi(u), Y_\gamma\rangle + \gamma\langle Y_\gamma - \varphi(u), b\rangle + \langle \varphi(u), x_n - x_\star\rangle \\ &\geq \gamma\|Y_\gamma\|^2 - \gamma\langle \varphi(u), Y_\gamma\rangle + \gamma\langle Y_\gamma - \varphi(u), b\rangle + \langle \varphi(u), x_n - x_\star\rangle\end{aligned}$$

since $Y_\gamma \in A(u, J_\gamma)$ and $A(\xi)$ is monotone. Also, $\langle b, x_n - x_\star\rangle \geq \langle \psi(u), x_n - x_\star\rangle$ by the monotonicity of $B(\xi)$. By expanding $\gamma^2\|b + Y_\gamma\|^2$, we obtain altogether

$$\begin{aligned}\|x_{n+1} - x_\star\|^2 &\leq \|x_n - x_\star\|^2 - \gamma^2\|Y_\gamma\|^2 + 2\gamma^2\langle \varphi(u), Y_\gamma\rangle + 2\gamma^2\langle \varphi(u), b\rangle + \gamma^2\|b\|^2 \\ &\quad - 2\gamma\langle \varphi(u) + \psi(u), x_n - x_\star\rangle \\ &\leq \|x_n - x_\star\|^2 - \gamma^2(1 - \beta^{-1})\|Y_\gamma\|^2 + \gamma^2(1 + \beta^{-1})\|b\|^2 + 2\gamma^2\beta\|\varphi(u)\|^2 \\ &\quad - 2\gamma\langle \varphi(u) + \psi(u), x_n - x_\star\rangle \end{aligned} \tag{11}$$

where we used the inequality $|\langle a, b\rangle| \leq (\beta/2)\|a\|^2 + \|b\|^2/(2\beta)$ where $\beta > 0$ is arbitrary. By Assumption 6, $\mathbb{E}_n\|b\|^2 \leq C(1 + \|x_n\|^2) \leq 2C(1 + \|x_\star\|^2 + \|x_n - x_\star\|^2)$ for some (other) constant C. Moreover $\mathbb{E}_n\langle \varphi(u) + \psi(u), x_n - x_\star\rangle = 0$. Thus,

$$\mathbb{E}_n\|x_{n+1} - x_\star\|^2 \leq (1 + C\gamma_{n+1}^2)\|x_n - x_\star\|^2 - \gamma_{n+1}^2(1 - \beta^{-1}) \int \|Y_{\gamma_{n+1}}(\xi, x_n)\|^2 \mu(d\xi) + C\gamma_{n+1}^2.$$

Choose $\beta > 1$. Using the Robbins-Siegmund Lemma [31] along with $(\gamma_n) \in \ell^2$, the conclusion follows. $\square$

The following lemma provides a moment control over the iterates x_n.

Lemma 4.1. *Let Assumptions 2 and 6 of Theorem 2.1 hold true. Then* $\sup_n \mathbb{E}\|x_n\|^{2p} < \infty$.

Proof. We shall establish the result by recurrence over p. Proposition 4.1 shows that it holds for $p = 1$. Assume that it holds for $p-1$. Using Assumption 2, choose $\varphi \in \mathcal{S}^{2p}_{A(\,.\,,x_\star)}$ and $\psi \in \mathcal{S}^{2p}_{B(\,.\,,x_\star)}$ such that $0 = \int(\varphi + \psi)d\mu$. Inequality (11) shows that for some constant $C > 0$,

$$\|x_{n+1} - x_\star\|^2 \le \|x_n - x_\star\|^2 - 2\gamma_{n+1}\langle \varphi(u_{n+1}) + \psi(u_{n+1}), x_n - x_\star\rangle \\ + C\gamma_{n+1}^2(\|\varphi(u_{n+1})\|^2 + \|b(u_{n+1}, x_n)\|^2)\,.$$

Raising to the power p and taking the expectation at both sides, we obtain

$$\mathbb{E}\|x_{n+1} - x_\star\|^{2p} \le \sum_{k_1+k_2+k_3=p} \binom{p}{k_1, k_2, k_3} C^{k_2}(-2)^{k_3}\gamma_{n+1}^{2k_2+k_3} T_n^{(k_1,k_2,k_3)} \tag{12}$$

where we set for every $\vec{k} = (k_1, k_2, k_3)$,

$$T_n^{\vec{k}} = \mathbb{E}\Big[\|x_n - x_\star\|^{2k_1}(\|\varphi(u_{n+1})\|^2 + \|b(u_{n+1}, x_n)\|^2)^{k_2}\langle \varphi(u_{n+1}) + \psi(u_{n+1}), x_n - x_\star\rangle^{k_3}\Big]\,.$$

We make the following observations.

- By choosing $k_2 = k_3 = 0$, we notice that $\mathbb{E}\|x_{n+1} - x_\star\|^{2p}$ is no greater than $\mathbb{E}\|x_n - x_\star\|^{2p}$ plus some additional terms involving only smaller powers of $\|x_n - x_\star\|$,
- The term corresponding to $(k_1, k_2, k_3) = (p-1, 0, 1)$ is zero since u_{n+1} and $\sigma(u_1, \ldots, u_n)$ are independent and $\mathbb{E}_n\langle \varphi(u_{n+1}) + \psi(u_{n+1}), x_n - x_\star\rangle = 0$. This implies that any term in the sum except $\mathbb{E}\|x_n - x_\star\|^{2p}$ is multiplied by γ_{n+1} raised to a power greater than 2,
- Consider the case $(k_1, k_2, k_3) \neq (p-1, 0, 1)$ and $(k_1, k_2, k_3) \neq (p, 0, 0)$. Using Jensen's inequality and the inequality $x^k y^\ell \le x^{k+\ell} + y^{k+\ell}$ for nonnegative x, y, k and ℓ, we get

  $$\begin{aligned}|T_n^{\vec{k}}| &\le \mathbb{E}\Big[\|x_n - x_\star\|^{2k_1+k_3}(\|\varphi(u_{n+1})\|^2 + \|b(u_{n+1}, x_n)\|^2)^{k_2}\|\varphi(u_{n+1}) + \psi(u_{n+1})\|^{k_3}\Big] \\ &\le C\mathbb{E}\Big[\|x_n - x_\star\|^{2k_1+k_3}(\|\varphi(u_{n+1})\|^{2k_2} + \|b(u_{n+1}, x_n)\|^{2k_2})(\|\varphi(u_{n+1})\|^{k_3} + \|\psi(u_{n+1})\|^{k_3})\Big] \\ &\le C\mathbb{E}\Big[\|x_n - x_\star\|^{2k_1+k_3}\|b(u_{n+1}, x_n)\|^{2k_2+k_3}\Big] \\ &\quad + C\mathbb{E}\Big[\|x_n - x_\star\|^{2k_1+k_3}\Big]\mathbb{E}\Big[\|\varphi(u_{n+1})\|^{2k_2+k_3} + \|\psi(u_{n+1})\|^{2k_2+k_3}\Big].\end{aligned}$$

  By conditioning on $\sigma(u_1, \ldots, u_n)$ and by using Assumption 6, we get

  $$\begin{aligned}\mathbb{E}\Big[\|x_n - x_\star\|^{2k_1+k_3}\|b(u_{n+1}, x_n)\|^{2k_2+k_3}\Big] &\le C\mathbb{E}\Big[\|x_n - x_\star\|^{2k_1+k_3}(1 + \|x_n\|^{2k_2+k_3})\Big] \\ &\le C(\mathbb{E}\|x_n - x_\star\|^{2p} + 1).\end{aligned}$$

  Noting that $2k_1 + k_3 \le 2(p-1)$, we get that $\mathbb{E}\|x_n - x_\star\|^{2k_1+k_3} < C$ by the induction hypothesis. Since $2k_2 + k_3 \le 2p$ and since φ and ψ are $2p$-integrable selections, it follows that $|T_n^{\vec{k}}| \le C(1 + \mathbb{E}\|x_n - x_\star\|^{2p})$. Note also the in the considered case, one has $2k_2 + k_3 \ge 2$ which implies that all terms $T_n^{\vec{k}}$ are multiplied by γ_{n+1}^2.

In conclusion, we obtain that $\mathbb{E}\|x_{n+1} - x_\star\|^{2p} \le \mathbb{E}(1 + C\gamma_{n+1}^2)\|x_n - x_\star\|^{2p} + C\gamma_{n+1}^2$ for some constant $C > 0$. Starting from $n = 0$ and iterating, we obtain that $\sup_n \mathbb{E}\|x_n - x_\star\|^{2p} < \infty$. □

We now need to provide a control over the distances to $\mathcal{D}$ of the iterates x_n. We start with an easy technical result.

Lemma 4.2. *For any $\varepsilon > 0$, there exist $C(\varepsilon) > 0$ and $C'(\varepsilon) > 0$ such that for any vectors $x, y \in \mathbb{R}^N$,*

$$\|x+y\|^2 \le (1+\varepsilon)\|x\|^2 + C(\varepsilon)\|y\|^2 \quad \text{and} \quad \|x+y\|^4 \le (1+\varepsilon)\|x\|^4 + C'(\varepsilon)\|y\|^4.$$

Proof. Observe that $|\langle x, y\rangle| \le (\beta/2)\|x\|^2 + \|y\|^2/(2\beta)$ is true for any $\beta > 0$. Taking $\beta = \varepsilon$, we obtain the first inequality. We also have

$$\begin{aligned}
\|x+y\|^4 &\le ((1+\beta)\|x\|^2 + (1+1/\beta)\|y\|^2)^2 \\
&= (1+\beta)^2\|x\|^4 + (1+1/\beta)^2\|y\|^4 + 2(1+\beta)(1+1/\beta)\|x\|^2\|y\|^2 \\
&\le (1+\beta)^3\|x\|^4 + (1+1/\beta)^3\|y\|^4.
\end{aligned}$$

By choosing β small enough, we obtain the second inequality. □

Proposition 4.2. *Let Assumptions 2, 4, 5, and 6 of Theorem 2.1 hold true. Then $\boldsymbol{d}(x_n)$ tends a.s. to zero. Moreover, for every ω in a probability one set, there exists $c(\omega) > 0$ and a positive sequence $(c_m(\omega))_{m\in\mathbb{N}}$ converging to zero such that for every integers n, m such that $n \ge m$,*

$$\sum_{k=m}^{n} \frac{\boldsymbol{d}(x_k)^2}{\gamma_k} \le c_m(\omega) + c(\omega)\sum_{k=m}^{n} \gamma_k \,.$$

Proof. We start by writing $x_{n+1} = \Pi(u_{n+1}, x_n) + \gamma_{n+1}\delta_{n+1}$ where

$$\delta_{n+1} = \frac{J_{\gamma_{n+1}}(u_{n+1}, x_n - \gamma_{n+1} b(u_{n+1}, x_n)) - \Pi(u_{n+1}, x_n)}{\gamma_{n+1}}.$$

Upon noting that $J_\gamma(\xi, \,.\,)$ is non expansive for every ξ,

$$\|\delta_{n+1}\| \le \|b(u_{n+1}, x_n)\| + \frac{\|J_{\gamma_{n+1}}(u_{n+1}, x_n) - \Pi(u_{n+1}, x_n)\|}{\gamma_{n+1}}.$$

Using Assumptions 5 and 6, we have

$$\begin{aligned}
\mathbb{E}_n\|\delta_{n+1}\|^4 &= 4\int \|b(\xi, x_n)\|^4 \mu(d\xi) + 4\gamma_{n+1}^{-4}\int \|J_{\gamma_{n+1}}(\xi, x_n) - \Pi(\xi, x_n)\|^4 \mu(d\xi) \\
&\le C(1 + \|x_n\|^{2p}).
\end{aligned}$$

Therefore, by Proposition 4.1-1., there exists a nonnegative $c_1(\omega)$ which is a.s. finite and satisfies $\mathbb{E}_n\|\delta_{n+1}\|^4 \le c_1(\omega)$ almost surely. By Lemma 4.1, it also holds that $\sup_n \mathbb{E}\|\delta_n\|^4 < \infty$. Consider an arbitrary point $u \in \overline{\mathcal{D}}$. For any $\varepsilon > 0$, we have by Lemma 4.2

$$\|x_{n+1} - u\|^2 \le (1+\varepsilon)\|\Pi(u_{n+1}, x_n) - u\|^2 + \gamma_{n+1}^2 C\|\delta_{n+1}\|^2.$$

Since $\Pi(u_{n+1}, \cdot)$ is firmly non expansive as being the projector onto a closed convex set, we have

$$\|\Pi(u_{n+1}, x_n) - u\|^2 \le \|x_n - u\|^2 - \|\Pi(u_{n+1}, x_n) - x_n\|^2.$$

Taking $u = \mathbf{\Pi}(x_n)$, we obtain

$$\boldsymbol{d}(x_{n+1})^2 \leq \|x_{n+1} - \mathbf{\Pi}(x_n)\|^2 \leq (1+\varepsilon)(\boldsymbol{d}(x_n)^2 - d(u_{n+1}, x_n)^2) + C\gamma_{n+1}^2 \|\delta_{n+1}\|^2$$

Taking the conditional expectation $\mathbb{E}_n$ at both sides of this inequality, using Assumption 4 and choosing ε small enough, we obtain the inequality $\mathbb{E}_n \boldsymbol{d}^2(x_{n+1}) \leq \rho \boldsymbol{d}^2(x_n) + \gamma_{n+1}^2 C \mathbb{E}_n \|\delta_{n+1}\|^2$ where $\rho \in [0,1)$. It implies that $\boldsymbol{d}^2(x_n)$ tends to zero by the Robbins-Siegmund Theorem [31]. Setting $\Delta_n = \boldsymbol{d}(x_n)^2/\gamma_n$ and using the fact that $\gamma_n/\gamma_{n+1} \to 1$, we obtain that

$$\mathbb{E}_n \Delta_{n+1} \leq \rho \Delta_n + \gamma_{n+1} C \mathbb{E}_n \|\delta_{n+1}\|^2$$

for n larger than some n_0.
By Lemma 4.2 and the firm nonexpansiveness of $\Pi(u_{n+1}, \cdot)$, we also have

$$\begin{aligned} \|x_{n+1} - u\|^4 &\leq (1+\varepsilon)\|\Pi(u_{n+1}, x_n) - u\|^4 + \gamma_{n+1}^4 C \|\delta_{n+1}\|^4 \\ &\leq (1+\varepsilon)(\|x_n - u\|^2 - \|\Pi(u_{n+1}, x_n) - x_n\|^2)^2 + \gamma_{n+1}^4 C \|\delta_{n+1}\|^4. \end{aligned} \tag{13}$$

We also set $u = \mathbf{\Pi}(x_n)$ and apply the operator $\mathbb{E}_n$ at both sides of this inequality. By Assumption 4, we have

$$\begin{aligned} \int (\boldsymbol{d}(x)^2 - d(\xi, x)^2)^2 \mu(d\xi) &= \boldsymbol{d}(x)^4 + \int d(\xi, x)^4 \mu(d\xi) - 2\boldsymbol{d}(x)^2 \int d(\xi, x)^2 \mu(d\xi) \\ &\leq \boldsymbol{d}(x)^4 - \boldsymbol{d}(x)^2 \int d(\xi, x)^2 \mu(d\xi) \leq (1-C)\boldsymbol{d}(x)^4 \end{aligned}$$

since $d(\xi, x) \leq \boldsymbol{d}(x)$. Integrating (13), we obtain $\mathbb{E}_n \boldsymbol{d}^4(x_{n+1}) \leq \rho \boldsymbol{d}^4(x_n) + \gamma_{n+1}^4 C \mathbb{E}_n \|\delta_{n+1}\|^4$ where $\rho \in [0,1)$, hence $\mathbb{E}_n \Delta_{n+1}^2 \leq \rho \Delta_n^2 + \gamma_n^2 C \mathbb{E}_n \|\delta_{n+1}\|^4$ for n larger than some n_0. Taking the expectation at each side, iterating, and using the boundedness of $(\mathbb{E}\|\delta_n\|^4)$, we obtain that $\mathbb{E}\Delta_n^2 \leq C(\rho^n + \sum_{k=1}^n \gamma_k^2 \rho^{n-k})$. Therefore,

$$\sum_{n=0}^{\infty} \mathbb{E}\Delta_n^2 \leq C\Big(1 + \sum_{n=0}^{\infty} \gamma_n^2\Big) < \infty.$$

Consequently, $\Delta_n \to 0$ almost surely. Moreover, the martingale

$$Y_n = \sum_{k=1}^{n} (\Delta_k - \mathbb{E}_{k-1}\Delta_k)$$

converges almost surely and in $\mathcal{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R})$. Given any two integers $0 < m < n$, let

$$D_m^n = \sum_{k=m+1}^{n} \Delta_k.$$

We can write

$$\begin{aligned} D_m^n &= \sum_{k=m+1}^{n} \mathbb{E}_{k-1}\Delta_k + Y_n - Y_m \\ &\leq \rho \sum_{k=m}^{n-1} (\Delta_k + C\gamma_{k+1}\mathbb{E}_k \|\delta_{k+1}\|^2) + Y_n - Y_m \\ &\leq \rho \Delta_m + \rho D_m^n + \rho C \sqrt{c_1(\omega)} \sum_{k=m+1}^{n} \gamma_k + Y_n - Y_m. \end{aligned}$$

To conclude, we have

$$D_m^n \leq \frac{\rho}{1-\rho}\Delta_m + \frac{Y_n - Y_m}{1-\rho} + \frac{\rho C\sqrt{c_1(\omega)}}{1-\rho}\sum_{k=m+1}^{n}\gamma_k\,.$$

Since $\Delta_m \to 0$ and since $(Y_n(\omega))_{n\in\mathbb{N}}$ is almost surely a Cauchy sequence, we obtain the desired result. □

Lemma 4.3. *Let Assumptions 3 and 6 hold true. For any compact set K, there exists a constant $C > 0$ and $\varepsilon \in (0,1]$ such that for all $x \in K$ and all $\gamma > 0$,*

$$\|h_\gamma(x)\| \leq C + 2\frac{\boldsymbol{d}(x)}{\gamma}$$

and moreover

$$\int (\|Y_\gamma(\xi,x)\|^2 + \|b(\xi,x)\|^2)^{\frac{1+\varepsilon}{2}}\mu(d\xi) \leq C\left[1 + \left(\frac{\boldsymbol{d}(x)}{\gamma}\right)^{1+\varepsilon}\right].$$

Proof. Set $x \in K$ and introduce some $\tilde{x} \in \mathcal{D}$ such that $\|x - \tilde{x}\| \leq 2\boldsymbol{d}(x)$. Using that $A_\gamma(\xi,\,.\,)$ is $\frac{1}{\gamma}$-Lipschitz continuous,

$$\begin{aligned}\|Y_\gamma(\xi,x)\| &\leq \|A_\gamma(\xi,\tilde{x})\| + \frac{1}{\gamma}\|x - \gamma b(\xi,x) - \tilde{x}\| \\ &\leq \|A_0(\xi,\tilde{x})\| + \|b(\xi,x)\| + 2\frac{\boldsymbol{d}(x)}{\gamma}\,.\end{aligned}$$

Therefore,

$$\|h_\gamma(x)\| \leq \int \|A_0(\xi,\tilde{x})\|\mu(d\xi) + 2\int \|b(\xi,x)\|\mu(d\xi) + 2\frac{\boldsymbol{d}(x)}{\gamma}\,.$$

The first two terms are independent of γ and, by Assumptions 3 and 6, are bounded functions of x on the compact K. This proves the first statement of the Lemma. Let $\varepsilon = \varepsilon(K)$ be the exponent defined in Assumption 3. There exists a constant C such that

$$\begin{aligned}(\|Y_\gamma(\xi,x)\|^2 + \|b(\xi,x)\|^2)^{\frac{1+\varepsilon}{2}} &\leq C(\|Y_\gamma(\xi,x)\|^{1+\varepsilon} + \|b(\xi,x)\|^{1+\varepsilon}) \\ &\leq C\left((\|A_0(\xi,\tilde{x})\| + \|b(\xi,x)\| + 2\frac{\boldsymbol{d}(x)}{\gamma})^{1+\varepsilon} + \|b(\xi,x)\|^{1+\varepsilon}\right) \\ &\leq C'\left(2^\varepsilon\|A_0(\xi,\tilde{x})\|^{1+\varepsilon} + 2^{1+2\varepsilon}\|b(\xi,x)\|^{1+\varepsilon} + 2^{1+3\varepsilon}\left(\frac{\boldsymbol{d}(x)}{\gamma}\right)^{1+\varepsilon}\right).\end{aligned}$$

By Assumption 6 and since $\int \|b(\xi,x)\|^{1+\varepsilon}\mu(d\xi) \leq 1 + \int \|b(\xi,x)\|^2\mu(d\xi)$, there exists some (other) constant C such that

$$\int (\|Y_\gamma(\xi,x)\|^2 + \|b(\xi,x)\|^2)^{\frac{1+\varepsilon}{2}}\mu(d\xi) \leq C\left(\int \|A_0(\xi,\tilde{x})\|^{1+\varepsilon}\mu(d\xi) + 1 + \|x\|^2 + \left(\frac{\boldsymbol{d}(x)}{\gamma}\right)^{1+\varepsilon}\right).$$

The proof is concluded using Assumption 3. □

End of the proof of Theorem 2.1

Recall Equation (9). Given an arbitrary real number $T > 0$, we shall study the asymptotic behavior of the family of functions $\{x(\tau_n + \cdot)\}_{n\in\mathbb{N}}$ on the compact interval $[0, T]$.

Given $\delta > 0$, we have $\|H(t+\delta) - H(t)\| \le \int_t^{t+\delta} \|h_{\gamma_{r(s)+1}}(x_{r(s)})\| ds$. By Proposition 4.1-1, the sequence (x_n) is bounded a.s. Thus, by Lemma 4.3, there exists a constant $c_1 = c_1(\omega)$ such that for almost every ω,

$$
\begin{aligned}
\|H(t+\delta) - H(t)\| &\le c_1\delta + 2\int_t^{t+\delta} \frac{\boldsymbol{d}(x_{r(s)})}{\gamma_{r(s)+1}} ds \\
&\le c_1\delta + \int_t^{t+\delta} \left(1 + \frac{\boldsymbol{d}(x_{r(s)})^2}{\gamma_{r(s)+1}^2}\right) ds \\
&= (c_1+1)\delta + \int_t^{t+\delta} \frac{\boldsymbol{d}(x_{r(s)})^2}{\gamma_{r(s)+1}^2} ds \\
&\le (c_1 + c_2 + 1)\delta + e(t)
\end{aligned}
$$

for some $e(t) \to_{t\to\infty} 0$, where the last inequality is due to Proposition 4.2. We also observe from Proposition 4.1 and Assumption 6 that M_n is a $\mathcal{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^N)$ martingale and that

$$
\mathbb{E}\|M_n\|^2 \le \mathbb{E}\Big[2\sum_{k=1}^{\infty} \gamma_k^2 \int \|Y_{\gamma_k}(\xi, x_k)\|^2 \mu(d\xi) + 2\sum_{k=1}^{\infty} \gamma_k^2 \int \|b(\xi, x_k)\|^2 \mu(d\xi)\Big]
$$

and the right hand side is finite by Assumption 6 and Proposition 4.1. Hence, M_n converges almost surely. It results that on a probability one set, the family of continuous time processes $(M(\tau_n + \cdot) - M(\tau_n))_{n\in\mathbb{N}}$ converges to zero uniformly on $\mathbb{R}_+$. The consequence of these observations is that on a probability one set, the family of processes $\{z_n(\,.\,)\}_{n\in\mathbb{N}}$ where $z_n(t) = x(\tau_n + t)$ is equicontinuous. Specifically, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
\limsup_n \sup_{0\le t,s\le T, |t-s|\le\delta} \|z_n(t) - z_n(s)\| \le \varepsilon.
$$

This family is moreover bounded by Proposition 4.1-1. By the Arzelà-Ascoli theorem, it admits an accumulation point for the uniform convergence on $[0, T]$, for an arbitrary $T > 0$. From any sequence of integers we can extract a subsequence (which we still denote as (z_n) with slight abuse) and a continuous function $z(\cdot)$ on $[0, T]$ such that (z_n) converges to z uniformly on $[0, T]$. Hence, for $t \in [0, T]$,

$$
\begin{aligned}
z(t) - z(0) &= -\lim_{n\to\infty} \int_0^t h_{\gamma_{r(\tau_n+s)+1}}(x_{r(\tau_n+s)})\, ds \\
&= -\lim_{n\to\infty} \int_0^t ds \int_\Xi \mu(d\xi)\, (g_n^{(a)}(\xi, s) + g_n^{(b)}(\xi, s))
\end{aligned}
$$

where we set

$$
\begin{aligned}
g_n^{(a)}(\xi, t) &= Y_{\gamma_{r(\tau_n+s)+1}}(\xi, x_{r(\tau_n+s)}) \\
g_n^{(b)}(\xi, t) &= b(\xi, x_{r(\tau_n+s)})\,.
\end{aligned}
$$

Define the mapping $g_n = (g_n^{(a)}, g_n^{(b)})$ on $\Xi \times [0,T] \to \mathbb{R}^{2N}$. Recalling that the sequence $(\tilde{x}_n)$ belongs to a compact set, say K, let $\varepsilon \in (0,1]$ be the exponent defined in Lemma 4.3. By the same Lemma,

$$\begin{aligned}\int_0^T ds \int_\Xi \mu(d\xi)\, \|g_n(\xi,s)\|^{1+\varepsilon} &\le c\left[T + \int_0^T \left(\frac{\boldsymbol{d}(x_{r(\tau_n+s)})}{\gamma_{r(\tau_n+s)+1}}\right)^{1+\varepsilon} ds\right] \\ &\le c\left[T + T^{\frac{1-\varepsilon}{2}} \left(\int_0^T \frac{\boldsymbol{d}(x_{r(\tau_n+s)})^2}{\gamma^2_{r(\tau_n+s)+1}} ds\right)^{\frac{1+\varepsilon}{2}}\right] \\ &\le c_1\end{aligned}$$

for some constants c and c_1. Therefore, the sequence of functions (g_n) is bounded in $\mathcal{L}^{1+\varepsilon}(\Xi \times [0,T], \mathscr{T} \otimes \mathscr{B}([0,T]), \mu \otimes \lambda; \mathbb{R}^{2N})$ where λ is the Lebesgue measure on $[0,T]$. The statement extends to the sequence of functions $G_n(\xi,t) = (g_n(\xi,t), \|g_n^{(a)}(\xi,t)\|, \|g_n^{(b)}(\xi,t)\|)$ which is uniformly bounded in $\mathcal{L}^{1+\varepsilon}(\Xi \times [0,T], \mathscr{T} \otimes \mathscr{B}([0,T]), \mu \otimes \lambda; \mathbb{R}^{2N+2})$. We can extract from this sequence a subsequence that converges weakly in this Banach space to a function $F : \Xi \times [0,T] \to \mathbb{R}^{2N+2}$. We decompose F as $F(\xi,t) = (f(\xi,t), \kappa(\xi,t), \upsilon(\xi,t))$ where κ, υ are real-valued and $f(\xi,t) = (f^{(a)}(\xi,t), f^{(b)}(\xi,t))$ where $f^{(a)}, f^{(b)} : \Xi \times [0,T] \to \mathbb{R}^N$. Using the weak convergence $(g_n^{(a)}, g_n^{(b)}) \rightharpoonup (f^{(a)}, f^{(b)})$, we obtain

$$z(t) - z(0) = -\int_0^t ds \left(\int_\Xi f^{(a)}(\xi,s)\mu(d\xi) + \int_\Xi f^{(b)}(\xi,s)\mu(d\xi)\right).$$

It remains to prove that for almost every $t \in [0,T]$, $f^{(a)}(\,.\,,t) \in A(\,.\,,z(t))$ and $f^{(b)}(\,.\,,t) \in B(\,.\,,z(t))$ μ-almost everywhere, along with $z(0) \in \overline{\mathcal{D}}$. This shows indeed that $z(t) = \Phi(z(0),t)$ for every $t \in [0,T]$, and the fact that $x(t)$ is a.s. an APT of the differential inclusion (4) follows.

By Mazur's theorem, there exists a function $J : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of weights $(\{\alpha_{k,n}, k = n \ldots, J(n) \,:\, \alpha_{k,n} \ge 0, \sum_{k=n}^{J(n)} \alpha_{k,n} = 1\})_n$ such that the sequence of functions defined by

$$\bar{G}_n(\xi,s) = \sum_{k=n}^{J(n)} \alpha_{k,n}\, G_k(\xi,s)$$

converges strongly to F. We define in the same way $\bar{g}_n(\xi,s) = \sum_k \alpha_{k,n}\, g_k(\xi,s)$ and similarly for $\bar{g}_n^{(a)}, \bar{g}_n^{(b)}$. Extracting a further subsequence, we obtain the $\mu \otimes \lambda$-almost everywhere convergence of $\bar{G}_n$ to F. By Fubini's theorem, for almost every $t \in [0,T]$, there exists a μ-negligible set such that for every ξ outside this set, $\bar{G}_n(\xi,t) \to F(\xi,t)$. From now on to the end of this proof, we fix such a $t \in [0,T]$.

As $\boldsymbol{d}(x_n) \to 0$, it is clear that $z(t) \in \overline{\mathcal{D}}$ (this holds in particular when $t = 0$, hence $z(0) \in \overline{\mathcal{D}}$). Following the same arguments as in the proof of Proposition 2.1, it holds that $z(t) \in \overline{D(\xi)}$ for all ξ outside a μ-negligible set.

Define $\eta_n(\xi) = J_{\gamma_{m+1}}(\xi, x_m - \gamma_{m+1} b(\xi,x_m)) - z(t) + \gamma_{m+1} b(\xi,x_m)$ with $m = r(\tau_n + t)$. Using the same approach as in the proof of Proposition 2.1, it can be shown that, as $n \to \infty$, $\eta_n(\,.\,)$ tends to zero almost surely along a subsequence. We now consider an arbitrary ξ outside a μ-negligible set, such that $\eta_n(\xi) \to 0$ and $z(t) \in \overline{D}(\xi)$.

Let (u,v) be an arbitrary element of $A(\xi)$. By the monotonicity of $A(\xi)$,

$$\langle v - Y_\gamma(\xi,x), u - J_\gamma(\xi, x - \gamma b(\xi,x))\rangle \ge 0\,, \qquad (\forall x \in \mathbb{R}^N, \gamma > 0)$$

we obtain

$$\begin{aligned}\langle v - \bar{g}_n^{(a)}(\xi,t), u - z(t)\rangle &= \sum_{k=n}^{J(n)} \alpha_{k,n} \langle v - g_k^{(a)}(\xi,t), u - z(t)\rangle \\ &\geq \sum_{k=n}^{J(n)} \alpha_{k,n} \langle v - g_k^{(a)}(\xi,t), \eta_k(\xi,t) - \gamma_{r(\tau_k+t)+1} b(\xi, x_{r(\tau_k+t)})\rangle \\ &\geq -\left(\|v\| + \sum_{k=n}^{J(n)} \alpha_{k,n} \|g_k^{(a)}(\xi,t)\| \right) \sup_{k\geq n} \left(\|\eta_k(\xi,t)\| + \gamma_{r(\tau_k+t)+1} \|b(\xi, x_{r(\tau_k+t)})\| \right).\end{aligned}$$

The term enclosed in the first parenthesis of the above right hand side converges to $\|v\|+\kappa(\xi,t)$ while the supremum converges to zero using Assumption 6. As $\bar{g}_n^{(a)}(\xi,t) \to f^{(a)}(\xi,t)$, it follows that

$$\langle v - f^{(a)}(\xi,t), u - z(t)\rangle \geq 0$$

and by maximality of $A(\xi)$, it holds that $f^{(a)}(\xi,t) \in A(\xi, z(t))$. The proof that $f^{(b)}(\xi,t) \in B(\xi, z(t))$ follows the same lines.

4.3 Proof of Corollary 2.1

The proof is based on the study of the family of empirical measures of a process close to $x(t)$. Using [14], we show that any accumulation point of this family is an invariant measure for the flow Φ. The corollary is then obtained by showing that the mean of such an invariant measure belongs to $\mathcal{Z}$.

Let $\boldsymbol{x}_n = \boldsymbol{\Pi}(x_n)$ be the projection of x_n on $\overline{\mathcal{D}}$, and write

$$\bar{\boldsymbol{x}}_n = \frac{\sum_{k=1}^n \gamma_k \boldsymbol{x}_k}{\sum_{k=1}^n \gamma_k}.$$

Let $\boldsymbol{x}(\omega,t)$ be the $\Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ process obtained from the piecewise constant interpolation of the sequence $(\boldsymbol{x}_n)$, namely $\boldsymbol{x}(\omega,t) = \boldsymbol{x}_n$ for $t \in [\tau_n, \tau_{n+1})$. On $(\Omega, \mathscr{F}, \mathbb{P})$, let $(\mathscr{F}_t)$ be the filtration generated by the process obtained from the similar piecewise constant interpolation of (u_n). With this filtration, it is clear that $\boldsymbol{x}$ is progressively measurable. It is moreover obvious that $\boldsymbol{x}(\omega,\cdot)$ is an APT for (4) for almost all ω. Let $\{\nu_t(\omega,\cdot)\}_{t\geq 0}$ be the family of empirical measures of $\boldsymbol{x}(\omega,\cdot)$. Observe from Theorem 2.1 that for almost all ω, there is a compact set $K(\omega)$ such that the support $\mathrm{supp}(\nu_t(\omega,\cdot))$ is included in $K(\omega)$ for all $t \geq 0$, which shows that the family $\{\nu_t(\omega,\cdot)\}_{t\geq 0}$ is tight. Hence this family has accumulation points. Let ν be the weak limit of (ν_{t_n}) along some sequence (t_n) of times. By [14, Th. 1], ν is invariant for the flow Φ. Clearly, $\mathrm{supp}(\nu)$ is a compact subset of $\overline{\mathcal{D}}$. Moreover, for any $x \in \mathrm{supp}(\nu)$ and any $t \geq 0$, $\Phi(x,t) \in \mathrm{supp}(\nu)$. Indeed, suppose for the sake of contradiction that there exists $t_0 > 0$ such that $\Phi(x,t_0) \notin \mathrm{supp}(\nu)$. Then $\Phi(B(x,\varepsilon) \cap \overline{\mathcal{D}}, t_0) \subset \mathrm{supp}(\nu)^{\mathrm{c}}$ for some $\varepsilon > 0$ by the continuity of Φ and the closedness of $\mathrm{supp}(\nu)$, where $B(x,\varepsilon)$ is the closed ball with center x and radius ε. Since $\nu(\Phi(B(x,\varepsilon)\cap\overline{\mathcal{D}}, 0)) > 0$, we obtain a contradiction. We also know from [8] (see also [30, Th. 5.3]) that there exists $\varphi : \overline{\mathcal{D}} \to \mathcal{Z}$ such that

$$\forall x \in \overline{\mathcal{D}}, \quad \frac{1}{t}\int_0^t \Phi(x,s)\, ds \xrightarrow[t\to\infty]{} \varphi(x).$$

By the dominated convergence and Fubini's theorems, we now have

$$\int \varphi(x)\,\nu(dx) = \int \nu(dx) \lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, \Phi(x,s) = \lim_{t\to\infty} \frac{1}{t}\int_0^t ds \int \nu(dx)\,\Phi(x,s) = \int x\,\nu(dx)$$

which shows that $\int x\,\nu(dx) \in \mathcal{Z}$ by the convexity of this set. Since $\int x\,d\nu_{t_n} \to \int x\,d\nu$ as $n\to\infty$, we conclude that all the accumulation points of $(\bar{\boldsymbol{x}}_n)$ belong to $\mathcal{Z}$. On the other hand, since $\mathcal{R}_{2p}(x_\star) \neq \emptyset$ for each $x_\star \in \mathcal{Z}$, a straightforward inspection of the proof of Prop. 4.1-3. shows that $\|x_n - x_\star\|$ converges almost surely for each $x_\star \in \mathcal{Z}$. From these two facts, we obtain by [28] or [30, Lm 4.2] that $(\bar{\boldsymbol{x}}_n)$ converges a.s. to a point of $\mathcal{Z}$. Since $x_n - \boldsymbol{x}_n \to 0$ a.s., the convergence of $(\bar{x}_n)$ to the same point follows.

4.4 Proof of Corollary 2.2

We start with a preliminary lemma.

Lemma 4.4. *Let $\mathsf{A} \in \mathcal{M}$ be demipositive. Assume that the set $\mathrm{zer}(\mathsf{A})$ of zeros of A is non-empty. Let $\Psi : \overline{\mathrm{dom}(\mathsf{A})} \times \mathbb{R}_+ \to \overline{\mathrm{dom}(\mathsf{A})}$ be the semiflow associated to the differential inclusion $\dot{z}(t) \in -\mathsf{A}(z(t))$. Then, any ICT set of Ψ is included in $\mathrm{zer}(\mathsf{A})$.*

Proof. Let K be an ICT set and let U be an arbitrary bounded open set of $\mathbb{R}^N$ such that $K \cap U \neq \emptyset$. Define $G_t = \bigcup_{s\geq t} \Psi(U,s)$ for all $t \geq 0$. For any $x_* \in \mathrm{zer}(\mathsf{A})$ and any $x \in U$, $\|\Psi(x,t)\| \leq \|\Psi(x,t) - \Psi(x_*,t)\| + \|x_*\| \leq \|x - x_*\| + \|x_*\|$. Therefore, G_0 is a bounded set. By [13, Proposition 3.10], the set $G = \bigcap_{t\geq 0} \overline{G_t}$ is an attractor for Ψ with a fundamental neighborhood U. As $K \cap U \neq \emptyset$, it follows that $K \subset G$ by [11, Corollary 5.4]. We finally check that $G \subset \mathrm{zer}(\mathsf{A})$. Let $y \in G$ that is, $y = \lim_{k\to\infty} \Psi(x_k, t_k)$ for some sequence (x_k, t_k) such that $x_k \in U$ and $t_k \to \infty$. By compactness of $\overline{U}$, the sequence x_k can be chosen such that $x_k \to \bar{x}$ for some $\bar{x} \in \overline{U}$. Therefore, $y = \lim_{k\to\infty} \Psi(\bar{x}, t_k)$ which by demipositivity of A implies $y \in \mathrm{zer}(\mathsf{A})$ [18, 30]. □

By theorem 2.1 and the discussion of Section 2.3, $L(x)$ is an ICT set. Using Lemma 4.4 and the standing hypotheses, $L(x) \subset \mathcal{Z}$. On the other hand, since $\mathcal{R}_2(x_\star) \neq \emptyset$ for all $x_* \in \mathcal{Z}$, a straightforward inspection of the proof of Proposition 4.1-3. shows that $\|x_n - x_*\|$ converges almost surely for any of those x_*. By Opial's lemma [30, Lm 4.1], we obtain the almost sure convergence of (x_n) to a point of $\mathcal{Z}$.

4.5 Proof of Corollary 3.1

Define on $\{0,1,\ldots,m\}$ the probability distribution $\zeta = \sum_{i=0}^m \alpha_i \delta_i$. On the space $\mathsf{X} \times \{0,\ldots,m\}$ equipped with the probability $\mu = \nu \otimes \zeta$, let $\xi = (\eta, i)$, and define the random operators A and B by (2). The Aumann integral $\mathcal{B}(x) = \int \partial f(\eta,x) d\pi(\eta)$ coincides with $\partial F(x)$ by [33] (see also the discussion in Section 3.1). Similarly,

$$\mathcal{A}(x) = \partial(G(x) + \iota_C)(x)\,.$$

The operator $\mathcal{A}$ is thus maximal. It holds that $\mathcal{A} + \mathcal{B} = \partial(F + G + \iota_C)$ which is maximal, demipositive and whose zeroes coincide with the minimizers of $F + G$ over C. The end of the proof consists in checking the assumptions of Corollary 2.2. It follows the same line as [16] and is left to the reader.

Acknowledgements

This work was partially funded by the Orange - Telecom ParisTech think tank phi-TAB and by the program ASTRID of the French Agence Nationale de la Recherche (project ODISSEE).

References

[1] F. Alvarez and J. Peypouquet. Asymptotic almost-equivalence of abstract evolution systems. *arXiv preprint arXiv:0904.2157*, 2009.

[2] F. Alvarez and J. Peypouquet. Asymptotic almost-equivalence of Lipschitz evolution systems in Banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 73(9):3018–3033, 2010.

[3] Y. F. Atchade, G. Fort, and E. Moulines. On stochastic proximal gradient algorithms. *ArXiv e-prints, 1402.2365*, February 2014.

[4] H. Attouch. Familles d'opérateurs maximaux monotones et mesurabilité. *Annali di Matematica Pura ed Applicata*, 120(1):35–111, 1979.

[5] J.-P. Aubin and A. Cellina. *Differential inclusions*, volume 264 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[6] J.-P. Aubin and H. Frankowska. *Set-valued analysis*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition.

[7] R. J. Aumann. Integrals of set-valued functions. *J. Math. Anal. Appl.*, 12:1–12, 1965.

[8] J. B. Baillon and H. Brézis. Une remarque sur le comportement asymptotique des semigroupes non linéaires. *Houston J. Math.*, 2(1):5–7, 1976.

[9] H. H. Bauschke and J. M. Borwein. On projection algorithms for solving convex feasibility problems. *SIAM review*, 38(3):367–426, 1996.

[10] H. H. Bauschke and P. L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.

[11] M. Benaïm. Dynamics of stochastic approximation algorithms. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 1–68. Springer, Berlin, 1999.

[12] M. Benaïm and M. W. Hirsch. Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equations*, 8(1):141–176, 1996.

[13] M. Benaïm, J. Hofbauer, and S. Sorin. Stochastic approximations and differential inclusions. *SIAM J. Control Optim.*, 44(1):328–348 (electronic), 2005.

[14] M. Benaïm and S. J. Schreiber. Ergodic properties of weak asymptotic pseudotrajectories for semiflows. *J. Dynam. Differential Equations*, 12(3):579–598, 2000.

[15] D. P. Bertsekas. Incremental proximal methods for large scale convex optimization. *Math. Program.*, 129(2, Ser. B):163–195, 2011.

[16] P. Bianchi. Ergodic convergence of a stochastic proximal point algorithm. *ArXiv e-prints, 1504.05400*, April 2015.

[17] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland mathematics studies. Elsevier Science, Burlington, MA, 1973.

[18] R. E. Bruck, Jr. Asymptotic convergence of nonlinear contraction semigroups in Hilbert space. *J. Funct. Anal.*, 18:15–26, 1975.

[19] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions*, volume 580. Springer, 1977.

[20] F. Hiai and H. Umegaki. Integrals, conditional expectations, and martingales of multivalued functions. *Journal of Multivariate Analysis*, 7(1):149 – 182, 1977.

[21] J.-B. Hiriart-Urruty. *Contributions à la programmation mathématique: cas déterministe et stochastique.* Université de Clermont-Ferrand II, Clermont-Ferrand, 1977. Thèse présentée à l'Université de Clermont-Ferrand II pour obtenir le grade de Docteur ès Sciences Mathématiques, Série E, No. 247.

[22] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[23] H. J. Kushner and G. G. Yin. *Stochastic approximation and recursive algorithms and applications*, volume 35 of *Applications of Mathematics (New York).* Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

[24] S. Lee and A. Nedic. Distributed random projection algorithm for convex optimization. *Selected Topics in Signal Processing, IEEE Journal of*, 7(2):221–229, April 2013.

[25] P.-L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.*, 16(6):964–979, 1979.

[26] B. Mercier. *Lectures on topics in finite element solution of elliptic problems*, volume 63 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics.* Tata Institute of Fundamental Research, Bombay, 1979. With notes by G. Vijayasundaram.

[27] A. Nedić. Random algorithms for convex minimization problems. *Math. Program.*, 129(2, Ser. B):225–253, 2011.

[28] G. B. Passty. Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. Appl.*, 72(2):383–390, 1979.

[29] A. Pazy. On the asymptotic behavior of semigroups of nonlinear contractions in Hilbert space. *J. Functional Analysis*, 27(3):292–307, 1978.

[30] J. Peypouquet and S. Sorin. Evolution equations for maximal monotone operators: asymptotic analysis in continuous and discrete time. *J. Convex Anal.*, 17(3-4):1113–1163, 2010.

[31] H. Robbins and D. Siegmund. A convergence theorem for non negative almost supermartingales and some applications. In *Optimizing Methods in Statistics*, pages 233–257. Academic Press, New York, 1971.

[32] R. T. Rockafellar. Measurable dependence of convex sets and functions on parameters. *J. Math. Anal. Appl.*, 28:4–25, 1969.

[33] R. T. Rockafellar and R. J.-B. Wets. On the interchange of subdifferentiation and conditional expectations for convex functionals. *Stochastics*, 7(3):173–182, 1982.

[34] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.

[35] L. Rosasco, S. Villa, and B. C. Vũ. Convergence of stochastic proximal gradient algorithm. *arXiv preprint arXiv:1403.5074*, 2014.

[36] L. Rosasco, S. Villa, and B. C. Vũ. A stochastic inertial forward-backward splitting algorithm for multivariate monotone inclusions. *arXiv preprint arXiv:1507.00848*, 2015.

[37] P. Toulis, D. Tran, and E. M. Airoldi. Stability and optimality in stochastic gradient descent. *ArXiv e-prints, 1505.02417*, May 2015.

[38] D. W. Walkup and R. J.-B. Wets. Stochastic programs with recourse. II: On the continuity of the objective. *SIAM J. Appl. Math.*, 17:98–103, 1969.

[39] M. Wang and D. P Bertsekas. Incremental constraint projection-proximal methods for nonsmooth convex optimization. Technical report, Massachusetts Institute of Technology, 2013.

[40] M. Wang and D. P. Bertsekas. Incremental constraint projection methods for variational inequalities. *Math. Program.*, 150(2, Ser. A):321–363, 2015.