Stochastic Coalescence Multi-Fragmentation Processes

Eduardo Cepeda

STOCHASTIC COALESCENCE MULTI-FRAGMENTATION PROCESSES

EDUARDO CEPEDA

Abstract. We study infinite systems of particles which undergo coalescence and fragmentation, in a manner determined solely by their masses. A pair of particles having masses x and y coalesces at a given rate $K(x,y)$. A particle of mass x fragments into a collection of particles of masses $\theta_1 x, \theta_2 x, \dots$ at rate $F(x)\beta(d\theta)$. We assume that the kernels K and F satisfy Hölder regularity conditions with indices $\lambda \in (0,1]$ and $\alpha \in [0,\infty)$ respectively. We show existence of such infinite particle systems as strong Markov processes taking values in ℓ_λ, the set of ordered sequences $(m_i)_{i\geq 1}$ such that $\sum_{i\geq 1} m_i^\lambda < \infty$. We show that these processes possess the Feller property. This work relies on the use of a Wasserstein-type distance, which has proved to be particularly well-adapted to coalescence phenomena.

Mathematics Subject Classification (2000): 60K35, 60J25.

Keywords: Stochastic coalescence multi-Fragmentation process, Stochastic interacting particle systems.

To appear in *Stochastic Processes And Their Applications*

1. Introduction

A coalescence-fragmentation process is a stochastic process which models the evolution in time of a system of particles undergoing coalescence and fragmentation. The size of a particle increases and decreases due to successive mergers and dislocations. We assume that each particle is fully identified by its mass $x \in (0,\infty)$. We consider the mean-field setting, so that the positions of particles in space, their shapes, and other geometric properties are not considered. Examples of applications of these models arise in the study of polymers, aerosols and astronomy; see the survey papers [8, 1] for more details.

In this paper, we will concern ourselves with the phonomena of coalescence and fragmentation at a macroscopic scale. Consider a (possibly infinite) system of particles. The framework we consider is as follows. The coalescence of particles of masses x and y results in the formation of a new particle of mass $x+y$. We assume that this coalescence occurs at rate $K(x,y)$, where K is some symmetric coagulation kernel. Particles may also fragment: we assume that a particle of mass x splits into a collection of particles of smaller masses $\theta_1 x, \theta_2 x, \dots$ at rate $F(x)\beta(d\theta)$. Here, $F : (0,\infty) \to [0,\infty)$ and β is a positive measure on the set $\Theta := \{\theta = (\theta_i)_{i\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \dots \geq 0\}$. In particular, this means that the distribution of the ratios of the masses of the child particles to the mass of the parent particle is a function of these ratios only, and not of the mass of the parent particle. In this setting, our coalescence-fragmentation processes will be defined through their infinitesimal generators. Note that at a fixed time the state may be composed of an infinite number of particles which have finite total mass.

In previous works by Evans and Pitman [11], Fournier [15] and Fournier and Löcherbach [17], pure stochastic coalescents with an infinite number of particles have been constructed for a large class of kernels K. See also the survey paper by Aldous [1]. On the other hand, the fragmentation model we study was first introduced by Bertoin [3] where the author takes into account an infinite measure β and a mechanism of dislocation with a possibly infinite number of fragments. The properties of the only fragmentation model are studied in Bertoin [3, 4] and in Hass [19, 20]. We refer also to the book [5] where a extensive study of coalescence and fragmentation is carried out.

The present paper combines the two phenomena. We are mainly concerned with a general existence and uniqueness result, and seek to impose as few conditions on K, F and β as possible. Roughly speaking, our assumptions are that the coalescence and fragmentation kernels each satisfy a Hölder regularity condition which makes them bounded near the origin ($(0,0)$ and 0 respectively) but not near ∞. The measure β is allowed to be infinite.

We follow ideas developed in [15, 17] in the context of pure coalescence ($F \equiv 0$). We work on the set ℓ_λ of ordered sequences of non-negative real numbers $(m_i)_{i\geq 1}$ which are such that $\sum_{i\geq 1} m_i^\lambda < \infty$. We endow this space with a Wasserstein-type distance δ_λ: for $m, \tilde{m} \in \ell_\lambda$, let

$$\delta_\lambda(m,\tilde{m}) = \inf_{\pi,\sigma\in \mathrm{Perm}(\mathbb{N})} \sum_{i\geq 1} \left| m^\lambda_{\pi(i)} - \tilde{m}^\lambda_{\sigma(i)} \right|,$$

where $\mathrm{Perm}(\mathbb{N})$ denotes the set of finite permutations of $\mathbb{N}$.

Extending results in [15, 17], we construct a stochastic particle system undergoing coalescence and fragmentation. In Theorem 3.3, we show existence and uniqueness of a stochastic coalescence-fragmentation process as a Markov process in $\mathbb{D}([0,\infty),\ell_\lambda)$ which enjoys the Feller property. We use a convergence method, starting from a finite process, for which the initial number of particles in the system is bounded, and fragmentation occurs at a bounded rate and produces a bounded number of fragments. Existence and uniqueness are obtained for these finite processes in a straightforward manner. We pass to the limit using a Poisson-driven stochastic differential equation (SDE) associated to the model, and coupling techniques.

In the finite case (see Proposition 3.1), we require only that the coalescence and fragmentation kernels be locally bounded (in the sense that for all $a > 0$, $\sup_{(0,a]^2} K(x,y) < \infty$ and $\sup_{(0,a]} F(x) < \infty$) and that there be only a finite number of particles. In order to extend the results to a system composed of an infinite number of particles and with fragmentation into an infinite number of fragments, it is necessary to impose the additional continuity conditions on the kernels.

The first works known to us on coalescence-fragmentation processes focussed on binary fragmentation, where a particle may only split into *two* child particles: denoting $c_t(x)$ the concentration of particles of mass $x \in (0,\infty)$ at time t, in this case the dynamics is given by

$$\begin{aligned} \partial_t c_t(x) &= \frac{1}{2}\int_0^x K(y,x-y)c_t(y)c_t(x-y)\,dy - c_t(x)\int_0^\infty K(x,y)c_t(y)\,dy \\ &\quad + \int_x^\infty F_b(x,y-x)c_t(y)dy - \frac{1}{2}c_t(x)\int_0^x F_b(y,x-y)dy, \end{aligned} \tag{1.1}$$

The binary fragmentation kernel $F_b(\cdot,\cdot)$ is a symmetric function and $F_b(x,y)$ gives the rate at which a particle of mass $x+y$ fragments into particles of masses x and y. In this case, the total fragmentation rate of a particle of mass x is given by $\frac{1}{2}\int_0^x F_b(y,x-y)dy$.

The deterministic setting of our model has been studied in Cepeda [6] where existence and uniqueness of the corresponding equation is proved, using the same notation as for equation (1.1), the equation reads as follows:

$$(1.2) \qquad \partial_t c_t(x) = \frac{1}{2}\int_0^x K(y,x-y)c_t(y)c_t(x-y)\,dy - c_t(x)\int_0^\infty K(x,y)c_t(y)\,dy + \int_\Theta \Big[\sum_{i=1}^\infty \frac{1}{\theta_i}F\left(\tfrac{x}{\theta_i}\right)c_t\left(\tfrac{x}{\theta_i}\right) - F(x)c_t(x)\Big]\beta(d\theta).$$

Note that we can obtain the continuous coagulation binary-fragmentation equation (1.1), for example, by considering β with support in $\{\theta : \theta_1+\theta_2 = 1\}$ and $\beta(d\theta) = h(\theta_1)d\theta_1\delta_{\{\theta_2=1-\theta_1\}}$ and setting $F_b(x,y) = \frac{2}{x+y}F(x+y)h\left(\frac{x}{x+y}\right)$, where $h(\cdot)$ is a continuous function on $[0,1]$ which is symmetric about $1/2$.

In the binary framework, and under the additional assumptions that the kernel K and the total fragmentation rate are bounded, some results on existence, uniqueness and convergence to the solution of the deterministic equation (1.1) may be found in Guiaş [18]. Jeon [12] considered the discrete coagulation-fragmentation equation. He showed that the weak limit points of the stochastic particle system exist and provide a solution. He assumed that $K(x,y) = o(x)o(y)$ and that the total rate of fragmentation of a particle of mass x is $o(x)$.

In Fournier and Giet [13], the authors study the behaviour of small particles in the coagulation-fragmentation equation (1.1) using a probabilistic approach. They assume a linear bound on the coagulation kernel, but allow for the total fragmentation rate to be infinite. Eibeck and Wagner [9] proved tightness of the corresponding stochastic particle systems and characterize the weak limit points as solutions. A continuous coagulation kernel satisfying $K(x,y) = o(x)o(y)$ for $x, y \to \infty$ is required, as is a weakly continuous fragmentation measure for which the total fragmentation rate of a particle of size x is $o(x)$ as $x \to \infty$. We refer also to Eibeck and Wagner [10], where a general model is studied which is used to approach general nonlinear kinetic equations.

Kolokoltsov [21] shows a hydrodynamic limit result for a mass exchange Markov process in the discrete case. In Kolokoltsov [22], existence and uniqueness are proved under different assumptions to ours; there, the author assumes a multiplicative bound on the coagulation rates and a linear growth for the fragmentation rates. For that model, the author also proves convergence to the deterministic equation. An extensive study of the methods used by the author is given in the books [23, 24]. Finally, we also refer to Berestycki [2], who proves a similar result to ours for a class of exchangeable coalescence-fragmentation processes.

We believe that it is possible to obtain a *hydrodynamic limit* result concerning our model: making tend simultaneously the number of particles to infinite and their sizes to 0 may allow to prove convergence of the stochastic coalescence-fragmentation process to the solution to equation (1.2). Considering d_λ an equivalent distance to δ_λ on measures; see [14], and $(\mu_t)_{t\geq 0}$ the solution to the deterministic equation which is a measure, we can proceed in the following way. We fix $n \in \mathbb{N}$, we begin by constructing a system consisting on a finite number of particles, the initial number of particles N_0 is set in such a way that $d_\lambda(\mu_0, \mu_0^n) \leq C/\sqrt{n}$; see [7, Proposition 3.2.] where a way to construct such systems is already provided. Thus, μ_0^n is set as a discretisation of the initial condition μ_0 consisting in N_0 atoms of weight $1/n$, this is $\mu_0^n = \frac{1}{n}\sum_{i=1}^{N_0}\delta_{m_i}$, here δ_{m_i} holds for the Dirac measure on m_i.

Next, we make the system μ_t^n to evolve following the dynamics of a coalescence-fragmentation process where the number of particles at each time $t > 0$ is determined by the successive mergers and fragmentations, so that $\mu_t^n = \frac{1}{n}\sum_{i=1}^{N_t}\delta_{m_i}$. This method requires some finite moments to μ_0, but we believe that is possible to control $d_\lambda(\mu_t, \mu_t^n)$ by roughly $C_t/\sqrt{n}$ where $C_t > 0$, allowing to show convergence and furthermore deduce a rate. Norris [25, 26] gives a first result on convergence for the pure coalescence case $F \equiv 0$; see Cepeda and Fournier [7] for an explicit rate of convergence where the method discribed in this paragraph is applied also to pure coalescence.

The rest of this paper is organized as follows. We introduce the notation and formal definitions in Section 2. The main results may be found in Section 3. Stochastic coalescence-fragmentation processes are studied in Section 4, and in Appendix A we give some useful technical details.

2. Notation and Definitions

Let $\mathcal{S}^\downarrow$ be the set of non-increasing sequences $m = (m_n)_{n\geq 1}$ with values in $[0, +\infty)$. A state m in $\mathcal{S}^\downarrow$ represents the sequence of the ordered masses of the particles in a particle system. Next, for $\lambda \in (0, 1]$, consider

$$\ell_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^\downarrow, \|m\|_\lambda := \sum_{k=1}^{\infty} m_k^\lambda < \infty \right\}. \tag{2.1}$$

Consider also the sets of finite particle systems, completed for convenience with infinite 0-s.

$$\ell_{0+} = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^\downarrow, \inf\{k \geq 1, m_k = 0\} < \infty \right\}.$$

Remark 2.1. *Note that for all $0 < \lambda_1 < \lambda_2$, $\ell_{0+} \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$. Note also that, since $\|m\|_1 \leq \|m\|_\lambda^{\frac{1}{\lambda}}$ the total mass of $m \in \ell_\lambda$ is always finite.*

Hypothesis 2.2. *We present now the general hypotheses.*

Coagulation and Fragmentation Kernels.- *We consider coagulation kernel K, symmetric $K(x, y) = K(y, x)$ for $(x, y) \in [0, \infty)^2$ and bounded on every compact subset in $(0, \infty)^2$. There exists $\lambda \in (0, 1]$ such that for all $a > 0$ there exists a constant $\kappa_a > 0$ such that for all x, y, $\tilde{x}$, $\tilde{y} \in (0, a]$,*

$$|K(x, y) - K(\tilde{x}, \tilde{y})| \leq \kappa_a \left[|x^\lambda - \tilde{x}^\lambda| + |y^\lambda - \tilde{y}^\lambda| \right], \tag{2.2}$$

We consider also a fragmentation kernel $F : (0, \infty) \mapsto [0, \infty)$, bounded on every compact subset in $(0, \infty)$. There exists $\alpha \in [0, \infty)$ such that for all $a > 0$ there exists a constant $\mu_a > 0$ such that for all x, $\tilde{x} \in (0, a]$,

$$|F(x) - F(\tilde{x})| \leq \mu_a \, |x^\alpha - \tilde{x}^\alpha|. \tag{2.3}$$

We define the set of ratios by

$$\Theta = \{\theta = (\theta_k)_{k\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}.$$

The β measure.- *We consider on Θ a measure $\beta(\cdot)$ and assume that it satisfies*

$$\beta\left(\sum_{k\geq 1} \theta_k > 1 \right) = 0, \tag{2.4}$$

$$C_\beta^\lambda := \int_\Theta \left[\sum_{k\geq 2} \theta_k^\lambda + (1 - \theta_1)^\lambda \right] \beta(d\theta) < \infty, \quad \textit{for some } \lambda \in (0, 1]. \tag{2.5}$$

For example, the coagulation kernels listed below, taken from the mathematical and physical literature, satisfy Hypothesis 2.2.

$K(x,y) \equiv 1$ (2.2) holds with $\kappa_a = 0$,

$K(x,y) = (x^\alpha + y^\alpha)^\beta$ with $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$ and $\lambda = \alpha\beta \in (0,1]$,

$K(x,y) = x^\alpha y^\beta + x^\beta y^\alpha$ with $0 \le \alpha \le \beta \le 1$ and $\lambda = \alpha + \beta \in (0,1]$,

$K(x,y) = (xy)^{\alpha/2}(x+y)^{-\beta}$ with $\alpha \in (0,1]$, $\beta \in [0,\infty)$ and $\lambda = \alpha - \beta \in (0,1]$,

$K(x,y) = (x^\alpha + y^\alpha)^\beta |x^\gamma - y^\gamma|$ with $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, $\gamma \in (0,1]$ and $\lambda = \alpha\beta + \gamma \in (0,1]$,

$K(x,y) = (x+y)^\lambda e^{-\beta(x+y)^{-\alpha}}$ with $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, and $\lambda \in (0,1]$.

On the other hand, the following fragmentation kernels satisfy Hypothesis 2.2.

$F(x) \equiv 1$,

$F(x) = x^\alpha$, with $\alpha > 0$.

Remark 2.3. *i) The property (2.4) means that there is no gain of mass due to the dislocation of a particle. Nevertheless, it does not exclude a loss of mass due to the dislocation of the particles.*

ii) Note that under (2.4) we have $\sum_{k\ge1}\theta_k - 1 \le 0$ β-a.e., and since $\theta_k \in [0,1)$ for all $k \ge 1$, $\theta_k \le \theta_k^\lambda$, we have

$$\left\{\begin{array}{l} 1-\theta_1^\lambda \le 1-\theta_1 \le (1-\theta_1)^\lambda,\ \beta - a.e., \\ \\ \sum_{k\ge1}\theta_k^\lambda - 1 = \sum_{k\ge2}\theta_k^\lambda - (1-\theta_1^\lambda) \le \sum_{k\ge2}\theta_k^\lambda,\ \beta - a.e. \end{array}\right. \tag{2.6}$$

implying the following bounds:

$$\left\{\begin{array}{l} \displaystyle\int_\Theta (1-\theta_1)\beta(d\theta) \le C_\beta^\lambda,\ \int_\Theta \left[\sum_{k\ge2}\theta_k^\lambda + (1-\theta_1^\lambda)\right]\beta(d\theta) \le C_\beta^\lambda, \\ \displaystyle\int_\Theta \left(\sum_{k\ge1}\theta_k^\lambda - 1\right)^+ \beta(d\theta) \le C_\beta^\lambda. \end{array}\right. \tag{2.7}$$

We point out that $\int_\Theta \left|\sum_{k\ge1}\theta_k^\lambda - 1\right|\beta(d\theta) \le 2C_\beta^\lambda$ but when the term $\sum_{k\ge1}\theta_k^\lambda - 1$ is negative our calculations can be realized in a simpler manner. We will thus use the positive bound given in the last inequality.

Within the whole paper, we will use the convention that, when dealing with sequences in ℓ_λ,

$$K(x,0) = 0 \quad \text{for all } x \in [0,\infty),$$
$$F(0) = 0.$$

We will always use this convention, even in the case where, e.g., $K \equiv 1$ on $(0,\infty)\times(0,\infty)$ and $F \equiv 1$ on $(0,\infty)$. Actually, 0 is a symbol used to refer to a particle that does not exist. For $\theta \in \Theta$ and $x \in (0,\infty)$ we will write $\theta \cdot x$ to say that the particle of mass x of the system splits into $\theta_1 x, \theta_2 x, \ldots$

Furthermore, we will refer to the property of "local boundedness" of the coagulation and fragmentation kernels in the sense that for all $a > 0$, $\sup_{(0,a]^2} K(x,y) < \infty$ and $\sup_{(0,a]} F(x) < \infty$.

Considering $m \in \ell_\lambda$, the dynamics of the process is as follows. A pair of particles m_i and m_j coalesce with rate given by $K(m_i,m_j)$ and this is described by the map $c_{ij} : \ell_\lambda \to \ell_\lambda$ (see below). A particle m_i fragments following the dislocation configuration $\theta \in \Theta$ with rate given by $F(m_i)\beta(d\theta)$ and this is described by the map $f_{i\theta} : \ell_\lambda \to \ell_\lambda$, with

$$\begin{aligned} c_{ij}(m) &= \text{reorder}(m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots), \\ f_{i\theta}(m) &= \text{reorder}(m_1, \ldots, m_{i-1}, \theta \cdot m_i, m_{i+1}, \ldots), \end{aligned} \tag{2.8}$$

the reordering being in the decreasing order.

Distances on $S^{\downarrow}$

We endow $S^{\downarrow}$ with the pointwise convergence topology, which can be metrized by the distance

$$d(m, \tilde{m}) = \sum_{k \geq 1} 2^{-k} |m_k - \tilde{m}_k|. \tag{2.9}$$

Also, for $\lambda \in (0,1]$ and m, $\tilde{m} \in \ell_\lambda$, we recall that since the masses are decreasingly ordered, from [15, Lemma 3.1.] we have the equality

$$\delta_\lambda(m, \tilde{m}) = \inf_{\pi, \sigma \in Perm(\mathbb{N})} \sum_{i \geq 1} |m^\lambda_{\pi(i)} - \tilde{m}^\lambda_{\sigma(i)}| = \sum_{k \geq 1} |m^\lambda_k - \tilde{m}^\lambda_k|, \tag{2.10}$$

In this paper we will use the second equality.

Infinitesimal generator $\mathcal{L}^\beta_{K,F}$

Considering some coagulation and fragmentation kernels K and F and a measure β. We define the infinitesimal generator $\mathcal{L}^\beta_{K,F}$ for any $\Phi : \ell_\lambda \to \mathbb{R}$ sufficiently regular and for any $m \in \ell_\lambda$ by

$$\mathcal{L}^\beta_{K,F}\Phi(m) = \sum_{1 \leq i < j < \infty} K(m_i, m_j) \left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right] + \sum_{i \geq 1} F(m_i) \int_\Theta \left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right] \beta(d\theta). \tag{2.11}$$

3. Results

We first define the finite coalescence - fragmentation process. In order to properly define this process we need to add two properties to the measure β. Namely, the measure of Θ must be finite and the number of fragments at each fragmentation must be bounded:

$$\begin{cases} \beta(\Theta) < \infty, & \\ \beta(\Theta \setminus \Theta_k) = 0 & \text{for some } k \in \mathbb{N}, \end{cases} \tag{3.1}$$

where

$$\Theta_k = \{\theta = (\theta_n)_{n \geq 1} \in \Theta : \theta_{k+1} = \theta_{k+2} = \cdots = 0\}.$$

Proposition 3.1 (Finite Coalescence - Fragmentation processes)**.** *Consider $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$. Assume that a coagulation kernel K bounded on compact subsets on $[0,\infty)^2$, a fragmentation kernel F bounded on compact subsets of $[0,\infty)$ and a measure β satisfy Hypotheses 2.2. Furthermore, suppose that β satisfies (3.1).*

Then, there exists a unique (in law) strong Markov process $(M(m,t))_{t \geq 0}$ starting at $M(m,0) = m$ and with infinitesimal generator $\mathcal{L}^\beta_{K,F}$.

We point out that in order to prove existence and uniqueness of the Finite Coalescence - Fragmentation process, kernels K and F do not need to satisfy the continuity conditions (2.2) and (2.3), respectively. The proof is based on the existence and uniqueness of its Poissonian representation Proposition 4.3. for which the jump intensity remains bounded on finite time-intervals Lemma 4.5.

We wish to extend this process to the case where the initial condition consists of an infinite number of particles and for more general fragmentation measures β under some additional continuity conditions for the kernels. For this, we will build a particular sequence of finite coalescence - fragmentation processes, the result will be obtained by passing to the limit.

We introduce the following notation that will be useful when working with finite processes. We consider a measure β satisfying Hypotheses 2.2., $n \in \mathbb{N}$ and the set $\Theta(n)$ defined by $\Theta(n) = \left\{\theta \in \Theta : \theta_1 \leq 1 - \frac{1}{n}\right\}$, we consider also the projector

$$\begin{array}{rccl} \psi_n : & \Theta & \to & \Theta_n \\ & \theta & \mapsto & \psi_n(\theta) = (\theta_1, \ldots, \theta_n, 0, \ldots), \end{array} \tag{3.2}$$

and we put

$$\beta_n = \mathbb{1}_{\theta \in \Theta(n)} \beta \circ \psi_n^{-1}. \tag{3.3}$$

The measure β_n can be seen as the restriction of β to the projection of $\Theta(n)$ onto Θ_n. Note that $\Theta(n) \subset \Theta(n+1)$ and that since we have excluded the degenerated cases $\theta_1 = 1$ we have $\bigcup_n \Theta(n) = \Theta$.

Lemma 3.2 (Definition.- The finite process $M^n(m,t)$)**.** *Consider $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$. Assume that the coagulation kernel K, the fragmentation kernel F and the measure β satisfy Hypotheses 2.2. Furthermore, recall β_n as defined by (3.3).*

Then, there exists a unique (in law) strong Markov process $(M^n(m,t))_{t \geq 0}$ starting at m and with infinitesimal generator $\mathcal{L}_{K,F}^{\beta_n}$.

This lemma is straightforward, it suffices to note that β_n satisfies (3.1), that the kernels K and F are locally bounded since they satisfy respectively (2.2) and (2.3) and to use Proposition 3.1. Indeed, recall (2.7), for $n \geq 1$

$$\beta_n(\Theta) = \int_\Theta \mathbb{1}_{\{1 - [\psi_n(\theta)]_1 \geq \frac{1}{n}\}} \beta(d\theta) \leq n \int_\Theta (1 - \theta_1)\, \beta(d\theta) \leq n\, C_\beta^\lambda < \infty.$$

We have chosen an explicit sequence of measures $(\beta_n)_{n \geq 1}$ because it will be easier to manipulate when coupling two coalescence-fragmentation processes. Nevertheless, more generally, taking any sequence of measures β_n satisfying (3.1) and converging towards β in a suitable sense as n tends to infinity should provide the same result.

Our main result concerning stochastic Coalescence-Fragmentation processes is the following.

Theorem 3.3. *Consider $\lambda \in (0,1]$, $\alpha \geq 0$. Assume that the coagulation K and the fragmentation F kernels and a measure β satisfy Hypotheses 2.2. Endow ℓ_λ with the distance δ_λ.*

i) For any $m \in \ell_\lambda$, there exists a (necessarily unique in law) strong Markov process $(M(m,t))_{t \geq 0} \in \mathbb{D}([0,\infty), \ell_\lambda)$ satisfying the following property.

For any sequence $m^n \in \ell_{0+}$ such that $\lim_{n \to \infty} \delta_\lambda(m^n, m) = 0$, the sequence $(M^n(m^n,t))_{t \geq 0}$ defined in Lemma 3.2, converges in law, in $\mathbb{D}([0,\infty), \ell_\lambda)$, to $(M(m,t))_{t \geq 0}$.

ii) The obtained process is Feller in the sense that for all $t \geq 0$, the map $m \mapsto Law(M(m,t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$ (endowed with the distance δ_λ).

iii) Recall the expression (2.9) of the distance d. For all bounded application $\Phi : \ell_\lambda \to \mathbb{R}$ satisfying $|\Phi(m) - \Phi(\tilde{m})| \leq a\, d(m, \tilde{m})$ for some $a > 0$, the process

$$\Phi\left(M(m,t)\right) - \Phi\left(m\right) - \int_0^t \mathcal{L}_{K,F}^\beta \left(M(m,s)\right) ds$$

is a local martingale.

This result extends those of Fournier [15] concerning solely coalescence and Bertoin [4, 3] concerning only fragmentation. We point out that in [4] is not assumed $C_\beta^\lambda < \infty$ but only $\int_\Theta (1-\theta_1)\beta(d\theta) < \infty$. However, we believe that in the presence of coalescence our hypotheses on β are optimal.

Theorem 3.3. will be proved in two steps, the first step consists in proving existence and uniqueness of the Finite Coalescence-Fragmentation process, finite in the sense that it is composed by a finite number of particles for all $t \geq 0$. Next, we will use a sequence of finite processes to build a process as its limit, where the system is composed by an infinite number of particles. The construction of such processes uses a Poissonian representation which is introduced in the next section.

4. A Poisson-driven S.D.E.

We now introduce a representation of the stochastic processes of coagulation - fragmentation in terms of Poisson measures, in order to couple two of these processes with different initial data.

Definition 4.1. *Assume that a coagulation kernel K, a fragmentation kernel F and a measure β satisfy Hypotheses 2.2.*

a) For the coagulation, we consider a Poisson measure $\mathcal{N}(dt, d(i,j), dz)$ on $[0,\infty)\times\{(i,j)\in\mathbb{N}^2, i < j\}\times[0,\infty)$ with intensity measure $dt\,\left[\sum_{k<l}\delta_{(k,l)}(d(i,j))\right]dz$, and denote by $(\mathcal{F}_t)_{t\geq 0}$ the associated canonical filtration.

b) For the fragmentation, we consider $\mathcal{M}(dt, di, d\theta, dz)$ a Poisson measure on $[0,\infty)\times\mathbb{N}\times\Theta\times[0,\infty)$ with intensity measure $dt\,\left(\sum_{k\geq 1}\delta_k(di)\right)\beta(d\theta)\,dz$, and denote by $(\mathcal{G}_t)_{t\geq 0}$ the associated canonical filtration. $\mathcal{M}$ is independent of $\mathcal{N}$.

Finally, we consider $m \in \ell_\lambda$. A càdlàg $(\mathcal{H}_t)_{t\geq 0} = (\sigma(\mathcal{F}_t, \mathcal{G}_t))_{t\geq 0}$-adapted process $(M(m,t))_{t\geq 0}$ is said to be a solution to $SDE(K, F, m, \mathcal{N}, \mathcal{M})$ if it belongs a.s. to $\mathbb{D}([0,\infty), \ell_\lambda)$ and if for all $t \geq 0$, a.s.

$$
\begin{aligned}
M(m,t) \quad = \quad & m + \int_0^t \int_{i<j} \int_0^\infty \left[c_{ij}\left(M(m,s-)\right) - M(m,s-)\right] \mathbb{1}_{\{z \leq K(M_i(m,s-), M_j(m,s-))\}} \\
& \qquad \mathcal{N}(ds, d(i,j), dz) \\
& + \int_0^t \int_i \int_\Theta \int_0^\infty \left[f_{i\theta}\left(M(m,s-)\right) - M(m,s-)\right] \mathbb{1}_{\{z \leq F(M_i(m,s-))\}} \\
& \qquad \mathcal{M}(ds, di, d\theta, dz).
\end{aligned} \tag{4.1}
$$

Remark that due to the independence of the Poisson measures only a coagulation or a fragmentation event occurs at each instant t.

We begin by checking that the integrals in (4.1) always make sense.

Lemma 4.2. *Let $\lambda \in (0,1]$ and $\alpha \geq 0$, the coagulation kernel K be bounded on compact subsets on $[0,\infty)^2$, the fragmentation kernel F be bounded on compact subsets of $[0,\infty)$, and the β and the Poisson measures $\mathcal{N}$ and $\mathcal{M}$ as in Definition 4.1. For any $(\mathcal{H}_t)_{t\geq 0}$-adapted process $(M(t))_{t\geq 0}$ belonging a.s. to $\mathbb{D}([0,\infty), \ell_\lambda)$, a.s.*

$$
\begin{aligned}
I_1 &= \int_0^t \int_{i<j} \int_0^\infty \left[c_{ij}\left(M(s-)\right) - M(s-)\right] \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \mathcal{N}(ds, d(i,j), dz), \\
I_2 &= \int_0^t \int_i \int_\Theta \int_0^\infty \left[f_{i\theta}\left(M(s-)\right) - M(s-)\right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \mathcal{M}(ds, di, d\theta, dz),
\end{aligned}
$$

are well-defined and finite for all $t \geq 0$.

Proof. The processes in the integral being càdlàg and adapted, it suffices to check the compensators are a.s. finite. We have to show that a.s., for all $k \geq 1$, all $t \geq 0$,

$$\begin{aligned} C_k(t) &= \int_0^t ds \sum_{i<j} K(M_i(s), M_j(s)) |[c_{ij}(M(s))]_k - M_k(s)| \\ &\quad + \int_0^t ds \int_\Theta \beta(d\theta) \sum_{i\geq 1} F(M_i(s)) |[f_{i\theta}(M(s))]_k - M_k(s)| < \infty. \end{aligned}$$

Note first that for all $s \in [0,t]$, $\sup_i M_i(s) \leq \sup_{[0,t]} \|M(s)\|_1 \leq \sup_{[0,t]} \|M(s)\|_\lambda^{1/\lambda} =: a_t < \infty$ a.s. since M belongs a.s. to $\mathbb{D}([0,\infty), \ell_\lambda)$. Next, let

$$\overline{K}_t = \sup_{(x,y)\in[0,a_t]^2} K(x,y) \qquad \text{and} \qquad \overline{F}_t = \sup_{x\in[0,a_t]} F(x), \tag{4.2}$$

which are $a.s.$ finite since K and F are bounded on every compact in $[0,\infty)^2$ and $[0,\infty)$, respectively. Then using (A.15) and (A.17) with (2.6) and (2.7), we write:

$$\begin{aligned} \sum_{k\geq 1} 2^{-k} C_k(t) &= \int_0^t ds \sum_{i<j} K(M_i(s), M_j(s))\, d\left(c_{ij}(M(s)), M(s)\right) \\ &\quad + \int_0^t ds \int_\Theta \beta(d\theta) \sum_{i\geq 1} F(M_i(s))\, d\left(f_{i\theta}(M(s)), M(s)\right) \\ &\leq \overline{K}_t \int_0^t ds \sum_{i<j} \frac{3}{2} 2^{-i} M_j(s) + C_\beta^\lambda \overline{F}_t \int_0^t ds \sum_{i\geq 1} 2^{-i} M_i(s) \\ &\leq \left(\frac{3}{2}\overline{K}_t + C_\beta^\lambda \overline{F}_t\right) \int_0^t \|M(s)\|_1 ds \\ &\leq t\left(\frac{3}{2}\overline{K}_t + C_\beta^\lambda \overline{F}_t\right) \sup_{[0,t]} \|M(s)\|_\lambda^{1/\lambda} \;<\; \infty. \end{aligned}$$

□

4.1. **Existence and uniqueness for *SDE*: finite case.** The aim of this paragraph is to prove Proposition 3.1. This proposition is a consequence of Proposition 4.3. bellow. We will first prove existence and uniqueness of the Finite Coalescence - Fragmentation processes satisfiying (SDE) and then some fundamental inequalities.

Proposition 4.3. *Let $m \in \ell_{0+}$. Consider a coagulation kernel K bounded on compact subsets of $[0,\infty)^2$, a fragmentation kernel F bounded on compact subsets of $[0,\infty)$ and a measure β and the Poisson measures $\mathcal{N}$ and $\mathcal{M}$ as in Definition 4.1, suppose furthermore that β satisfies (3.1).*

Then there exists a unique process $(M(m,t))_{t\geq 0}$ which solves $SDE(K,F,m,\mathcal{N},\mathcal{M})$. This process is a finite Coalescence-Fragmentation process in the sense of Proposition 3.1.

We recall that in order to prove Proposition 4.3. the kernels K and F do not need to satisfy the continuity conditions (2.2) and (2.3), we need only to assume local boundness to prove that the jump intensity is bounded on finite time-intervals. The continuity conditions on both kernels are needed, in general, when considering an infinite number of particles in the system and in particular, to control the distance δ_λ between two solutions to *SDE* Proposition 4.4. *ii)* below.

4.1.1. *A Gronwall type inequality.* We will also check a fundamental inequality, which shows that the distance between two coagulation-fragmentation processes introduced in Proposition 4.3. cannot increase excessively while their moments of order λ remain finite. For this, we need to consider the additional continuity conditions (2.2) and (2.3).

Proposition 4.4. *Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m, \tilde{m} \in \ell_{0+}$. Consider K, F, β and the Poisson measures $\mathcal{N}$ and $\mathcal{M}$ as in Definition 4.1, we furthermore suppose that β satisfies (3.1). Consider the unique solutions $M(m,t)$ and $M(\tilde{m},t)$ to $SDE(K,F,m,\mathcal{N},\mathcal{M})$ and $SDE(K,F,\tilde{m},\mathcal{N},\mathcal{M})$ constructed in Proposition 4.3. and recall C^λ_β (2.5).*

i) The map $t \mapsto \|M(m,t)\|_1$ is a.s. non-increasing. Futhermore, for all $t \geq 0$

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|M(m,s)\|_\lambda\right] \leq \|m\|_\lambda \, e^{\overline{F}_m C^\lambda_\beta \; t},$$

where $\overline{F}_m = \sup_{[0,\|m\|_1]} F(x)$.

ii) We define, for all $x > 0$, the stopping time $\tau(m,x) = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x\}$. Then for all $t \geq 0$ and all $x > 0$,

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau(m,x)\wedge\tau(\tilde{m},x)]} \delta_\lambda\left(M(m,s), M(\tilde{m},s)\right)\right] \leq \delta_\lambda\left(m,\tilde{m}\right) e^{C(x+1)\,t}.$$

where C is a positive constant depending on K, F, C^λ_β, $\|m\|_1$ and $\|\tilde{m}\|_1$.

This proposition will be useful to construct a process in the sense of Definition 4.1. as the limit of a sequence of approximations. It will provide some important uniform bounds not depending on the approximations but only on the initial conditions and C^λ_β.

4.1.2. *Proofs.* In this section we provide proofs to propositions 4.3., 3.1. and 4.4.

Proof of Proposition 4.3. This proposition will be proved considering that in such a system the number of particles remains finite. We will conclude using the fact that the total rate of jumps of the system is bounded by the number of particles.

Lemma 4.5. *Let $m \in \ell_{0+}$, consider a coagulation kernel K bounded on compact subsets on $[0,\infty)^2$, a fragmentation kernel F bounded on compact subsets of $[0,\infty)$, and β and the Poisson measures $\mathcal{N}$ and $\mathcal{M}$ as in Definition 4.1. and assume that β satisfies (3.1). Assume that there exists $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,\mathcal{N},\mathcal{M})$.*

i) The number of particles in the system remains a.s. bounded on finite time-intervals,

$$\sup_{s\in[0,t]} N_s < \infty, \;\; a.s. \text{ for all } \;\; t \geq 0,$$

where $N_t = card\{M_i(m,t) : M_i(m,t) > 0\} = \sum_{i\geq 1} \mathbb{1}_{\{M_i(m,t)>0\}}$.

ii) The coalescence and fragmentation jump rates of the process $(M(m,t))_{t\geq 0}$ are a.s. bounded on finite time-intervals, this is

$$\sup_{s\in[0,t]} \left(\rho_c(s) + \rho_f(s)\right) < \infty, \;\; a.s. \text{ for all } \;\; t \geq 0,$$

where $\rho_c(t) := \sum_{i<j} K(M_i(m,t), M_j(m,t))$ and $\rho_f(t) := \beta(\Theta)\sum_{i\geq 1} F(M_i(m,t))$.

Proof. First, denoting $\overline{K}_m := \sup_{[0,\|m\|_1]^2} K(x,y)$ and $\overline{F}_m := \sup_{[0,\|m\|_1]} F(x)$, note that we have $\rho_c(0) \leq \overline{K}_m N_0^2$ and $\rho_f(0) \leq \beta(\Theta)\overline{F}_m N_0$, which shows that the initial total jump intensity of the system is finite and that the first jump time is strictly positive $T_1 > 0$. We can thus prove by recurrence that there exists a sequence $0 < T_1 < \ldots < T_j < \ldots < T_\infty$ of jumping times with $T_\infty = \lim_{j\to\infty} T_j$. We now prove that $T_\infty = \infty$.

Let $L^f(t) := card\{j \geq 1 : T_j \leq t \text{ and } T_j \text{ is a jump of } M\}$ be the number of fragmentations in the system until the instant $t \geq 0$. Recall that the measure β satisfies (3.1), since k is the maximum number of fragments, it is easy to see that

$$N_t \leq N_0 + (k-1)L^f(t) < \infty \ a.s., \ \text{for all}\, t < T_\infty.$$

Applying now (2.11) with $\Psi(m) = \sum_{n\geq 1} m_n$ and since that $\Psi(c_{ij}(m)) - \Psi(m) = 0$ and $\Psi(f_{i\theta}(m)) - \Psi(m) = m_i \left(\sum_{i=1}^k \theta_i - 1\right) \leq 0$, $\beta - a.e.$, we obtain

$$\sup_{s\in[0,t]} \|M(m,s)\|_1 \leq \|m\|_1, a.s., \ \text{for all } t < T_\infty,$$

which implies, $a.s.$ for all $t < T_\infty$,

$$\begin{cases} \rho_c(t) & \leq & \overline{K}_m N_{t-}^2, \\ \rho_f(t) & \leq & \beta(\Theta)\overline{F}_m N_{t-}. \end{cases} \tag{4.3}$$

Next, define $\Phi(m) = \sum_{n\geq 1} \mathbb{1}_{\{m_n > 0\}}$, recall (2.11) and use $\Phi(c_{ij}(m)) - \Phi(m) \leq 0$, to obtain

$$\begin{aligned} \mathcal{L}_{K,F}^{\beta}\Phi(m) &\leq \sum_{i\geq 1} \int_\Theta F(m_i) \left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right] \beta(d\theta) \\ &\leq \overline{F}_m \sum_{i\geq 1} \int_\Theta \left[\sum_{n\geq 1} \mathbb{1}_{\{\theta_n m_i > 0\}} - \mathbb{1}_{\{m_i > 0\}}\right] \beta(d\theta) \\ &\leq (k-1)\,\overline{F}_m\,\beta(\Theta)\Phi(m), \end{aligned}$$

we used $\theta_j m_i = 0$ for all $j \geq k+1$.

Hence, we have for all $t \geq 0$,

$$\begin{aligned} \mathbb{E}\left[\sup_{s\in[0,t\wedge T_\infty)} N_s\right] &\leq N_0 + (k-1)\,\overline{F}_m\,\beta(\Theta)\mathbb{E}\left[\int_0^{t\wedge T_\infty} N_{s-} ds\right] \\ &\leq N_0 + (k-1)\,\overline{F}_m\,\beta(\Theta) \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge T_\infty)} N_u\right] du. \end{aligned}$$

We use the Gronwall Lemma to obtain

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge T_\infty)} N_s\right] \leq N_0\, e^{(k-1)\,\overline{F}_m\,\beta(\Theta)t},$$

for all $t \geq 0$. We thus deduce,

$$\sup_{s\in[0,t\wedge T_\infty)} N_s < \infty, \ a.s., \tag{4.4}$$

for all $t \geq 0$.

Suppose now that $T_\infty < \infty$, then from (4.4) we deduce that $\sup_{t\in[0,T_\infty)} N_t < \infty,\ a.s.$. which means that, using (4.3), $\sup_{t\in[0,T_\infty)}(\rho_c(t)+\rho_f(t)) < \infty,\ a.s.$ This is in contradiction with $T_\infty < \infty$ since the total jump intensity necessarily explodes to infinity on T_∞ when $T_\infty < \infty$.

We deduce that,

$$\mathbb{E}\left[\sup_{s\in[0,t]} N_s\right] \quad \leq \quad N_0\, e^{(k-1)\,\overline{F}_m\,\beta(\Theta)t},$$

for all $t \geq 0$, and *i)* readily follows. Finally, *ii)* follows easily from *i)* and (4.3).

This ends the proof of Lemma 4.5. □

From Lemma 4.5. we deduce that the total rate of jumps of the system is uniformly bounded. Thus, pathwise existence and uniqueness holds for $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,\mathcal{N},\mathcal{M})$.

This ends the proof of Proposition 4.3. □

Proof of Proposition 3.1. Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider K, F, β and the Poisson measures $\mathcal{N}$ and $\mathcal{M}$ as in Proposition 3.1.

Consider the process $(M(m,t))_{t\geq 0}$, the unique solution to $SDE(K,F,m,\mathcal{N},\mathcal{M})$ built in Proposition 4.3. The system $(M(m,t))_{t\geq 0}$ is a strong Markov process in continuous time with infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$ and Proposition 3.1. follows. □

Proof of Proposition 4.4. Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider $(M(m,t))_{t\geq 0}$ the solution to $SDE(K,F,m,\mathcal{N},\mathcal{M})$ constructed in Proposition 4.3. We begin studying the behavior of the moments of this solution.

First, we will see that under our assumptions the total mass $\|\cdot\|_1$ does $a.s.$ not increase in time. This property is fundamental in this approach since we will use the bound $\sup_{[0,\|M(m,0)\|_1]} F(x)$, which is finite whenever $\|M(m,0)\|_\lambda$ is. This will allows us to bound lower moments of $M(m,t)$ for $t \geq 0$.

Next, we will prove that the λ-moment remains finite in time. Finally, we will show that the distance δ_λ between two solutions to (4.1) is bounded in time while their λ-moments remain finite.

We point out that in these paragraphs we will use more general estimates for $m \in \ell_\lambda$ and β satisfying Hypotheses 2.2. and not necessarily (3.1). This will provide uniform bound when dealing with finite processes.

Moments Estimates.- The aim of this paragraph is to prove *Proposition 4.4. i)*.

The solution to $SDE(K,F,m,\mathcal{N},\mathcal{M})$ will be written $M(t) := M(m,t)$ for simplicity. From Lemma 4.5. *i)*, we know that the number of particles in the system is $a.s.$ finite and thus the following sums are obviously well-defined.

First, from (4.1) we have for $k \geq 1$,

$$
\begin{aligned}
M_k(t) &= M_k(0) + \int_0^t \int_{i<j} \int_0^\infty [[c_{ij}(M(s-))]_k - M_k(s-)] \mathbb{1}_{\{z \le K(M_i(s-), M_j(s-))\}} \\
&\qquad\qquad \mathcal{N}(ds, d(i,j), dz) \\
&\quad + \int_0^t \int_i \int_\Theta \int_0^\infty [[f_{i\theta}(M(s-))]_k - M(s-)_k] \mathbb{1}_{\{z \le F(M_i(s-))\}} \\
&\qquad\qquad \mathcal{M}(ds, di, d\theta, dz),
\end{aligned} \tag{4.5}
$$

and summing on k, we deduce

$$
\begin{aligned}
\|M(t)\|_1 &= \|m\|_1 + \int_0^t \int_{i<j} \int_0^\infty [\|c_{ij}(M(s-))\|_1 - \|M(s-)\|_1] \mathbb{1}_{\{z \le K(M_i(s-), M_j(s-))\}} \\
&\qquad\qquad \mathcal{N}(ds, d(i,j), dz) \\
&\quad + \int_0^t \int_i \int_0^\infty [\|f_{i\theta}(M(s-))\|_1 - \|M(s-)\|_1] \mathbb{1}_{\{z \le F(M_i(s-))\}} \\
&\qquad\qquad \mathcal{M}(ds, di, d\theta, dz).
\end{aligned} \tag{4.6}
$$

Note that, clearly $\|c_{ij}(m)\|_1 = \|m\|_1$ and $\|f_{i\theta}(m)\|_1 = \|m\|_1 + m_i \left(\sum_{k\ge 1} \theta_k - 1\right) \le \|m\|_1$ for all $m \in \ell_\lambda$, since $\sum_{k\ge1} \theta_k \le 1$ β-a.e. Then,

$$
\sup_{[0,t]} \|M(s)\|_1 \le \|m\|_1, \; a.s. \;\; \forall t \ge 0.
$$

This implies for all $s \in [0,t]$, $\sup_i M_i(s) \le \sup_{[0,t]} \|M(s)\|_1 \le \|m\|_1$ $a.s.$ We set

$$
\overline{K}_m = \sup_{(x,y)\in[0,\|m\|_1]^2} K(x,y) \qquad \text{and} \qquad \overline{F}_m = \sup_{x\in[0,\|m\|_1]} F(x) \tag{4.7}
$$

which are finite since K and F are bounded on every compact in $[0,\infty)^2$ and $[0,\infty)$ respectively.

In the same way, from (4.1) for $\lambda \in (0,1)$ we have for $k \ge 1$,

$$
\begin{aligned}
[M_k(t)]^\lambda &= [M_k(0)]^\lambda + \int_0^t \int_{i<j} \int_0^\infty \left[[c_{ij}(M(s-))]_k^\lambda - [M_k(s-)]^\lambda\right] \mathbb{1}_{\{z \le K(M_i(s-), M_j(s-))\}} \\
&\qquad\qquad \mathcal{N}(ds, d(i,j), dz) \\
&\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left[[f_{i\theta}(M(s-))]_k^\lambda - [M(s-)]_k^\lambda\right] \mathbb{1}_{\{z \le F(M_i(s-))\}} \\
&\qquad\qquad \mathcal{M}(ds, di, d\theta, dz),
\end{aligned}
$$

and summing on k, we deduce

$$
\begin{aligned}
\|M(t)\|_\lambda &= \|m\|_\lambda + \int_0^t \int_{i<j} \int_0^\infty [\|c_{ij}(M(s-))\|_\lambda - \|M(s-)\|_\lambda] \mathbb{1}_{\{z \le K(M_i(s-), M_j(s-))\}} \\
&\qquad\qquad \mathcal{N}(ds, d(i,j), dz) \\
&\quad + \int_0^t \int_i \int_0^\infty [\|f_{i\theta}(M(s-))\|_\lambda - \|M(s-)\|_\lambda] \mathbb{1}_{\{z \le F(M_i(s-))\}} \\
&\qquad\qquad \mathcal{M}(ds, di, d\theta, dz).
\end{aligned} \tag{4.8}
$$

We take the expectation, use (A.4) and (A.5) with (2.7) and (4.7), to obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\|M(s)\|_\lambda\right] &\le \|m\|_\lambda + C_\beta^\lambda \int_0^t \mathbb{E}\left[\sum_{i\ge1} F\left(M_i(s)\right) M_i^\lambda(s)\right] ds\\ &\le \|m\|_\lambda + \overline{F}_m C_\beta^\lambda \int_0^t \mathbb{E}\left[\|M(s)\|_\lambda\right] ds.\end{aligned}$$

We conclude using the Gronwall Lemma.

Bound for δ_λ.- The aim of this paragraph is to prove *Proposition 4.4. ii)*. For this, we consider for $m, \tilde{m} \in \ell_\lambda$ some solutions to $SDE(K, F, m, \mathcal{N}, \mathcal{M})$ and $SDE(K, F, \tilde{m}, \mathcal{N}, \mathcal{M})$ which will be written $M(t) := M(m,t)$ and $\tilde{M}(t) := M(\tilde{m}, t)$ for simplicity. Since M and $\tilde{M}$ solve (4.1) with the same Poisson measures $\mathcal{N}$ and $\mathcal{M}$, and since the numbers of particles in the systems are *a.s.* finite, we have

$$\delta_\lambda(M(t), \tilde{M}(t)) = \delta_\lambda(m, \tilde{m}) + A_t^c + B_t^c + C_t^c + A_t^f + B_t^f + C_t^f, \tag{4.9}$$

where

$$\begin{aligned}
A_t^c &= \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda\left(c_{ij}(M(s-)), c_{ij}(\tilde{M}(s-))\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\}\\
&\qquad \mathbb{1}_{\left\{z\le K(M_i(s-),M_j(s-))\wedge K\left(\tilde{M}_i(s-),\tilde{M}_j(s-)\right)\right\}} \mathcal{N}(ds, d(i,j), dz),\\
B_t^c &= \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda\left(c_{ij}(M(s-)), \tilde{M}(s-)\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\}\\
&\qquad \mathbb{1}_{\left\{K\left(\tilde{M}_i(s-),\tilde{M}_j(s-)\right)\le z\le K(M_i(s-),M_j(s-))\right\}} \mathcal{N}(ds, d(i,j), dz),\\
C_t^c &= \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda\left(M(s-), c_{ij}(\tilde{M}(s-))\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\}\\
&\qquad \mathbb{1}_{\left\{K(M_i(s-),M_j(s-))\le z\le K\left(\tilde{M}_i(s-),\tilde{M}_j(s-)\right)\right\}} \mathcal{N}(ds, d(i,j), dz),\\
A_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\theta}(M(s-)), f_{i\theta}(\tilde{M}(s-))\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\}\\
&\qquad \mathbb{1}_{\left\{z\le F(M_i(s-))\wedge F\left(\tilde{M}_i(s-)\right)\right\}} \mathcal{M}(ds, di, d\theta, dz),\\
B_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\theta}(M(s-)), \tilde{M}(s-)\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\}\\
&\qquad \mathbb{1}_{\left\{F\left(\tilde{M}_i(s-)\right)\le z\le F(M_i(s-))\right\}} \mathcal{M}(ds, di, d\theta, dz),\\
C_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(M(s-), f_{i\theta}(\tilde{M}(s-))\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\}\\
&\qquad \mathbb{1}_{\left\{F(M_i(s-))\le z\le F\left(\tilde{M}_i(s-)\right)\right\}} \mathcal{M}(ds, di, d\theta, dz).
\end{aligned}$$

Note also that

$$\left|\delta_\lambda\left(c_{ij}(M(s-)),\tilde{M}(s-)\right)-\delta_\lambda\left(M(s-),\tilde{M}(s-)\right)\right| \leq \delta_\lambda\left(c_{ij}(M(s-)),M(s-)\right) \tag{4.10}$$

$$\left|\delta_\lambda\left(f_{i\theta}(M(s-)),\tilde{M}(s-)\right)-\delta_\lambda\left(M(s-),\tilde{M}(s-)\right)\right| \leq \delta_\lambda\left(f_{i\theta}(M(s-)),M(s-)\right) \tag{4.11}$$

We now search for an upper bound to the expression in (4.9). We define, for all $x>0$, the stopping time $\tau(m,x):=\inf\{t\geq 0;\|M(m,t)\|_\lambda\geq x\}$. We set $\tau_x=\tau(m,x)\wedge\tau(\tilde{m},x)$.

Furthermore, since for all $s\in[0,t]$, $\sup_i M_i(s)\leq\sup_{[0,t]}\|M(s)\|_1\leq\|m\|_1:=a_m\,a.s$, equivalently for $\tilde{M}$, we put $a_{\tilde{m}}=\|\tilde{m}\|_1$. For $a:=a_m\vee a_{\tilde{m}}$ we set κ_a and μ_a the constants for which the kernels K and F satisfy (2.2) and (2.3). Finally, we set $\overline{F}_m$ as in (4.7).

Term A_t^c: using (A.8) we deduce that this term is non-positive, we bound it by 0.

Term B_t^c: we take the expectation, use (4.10), (A.6) and (2.2), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}B_s^c\right] &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i<j}2M_j^\lambda(s)\left|K\left(M_i(s),M_j(s)\right)-K\left(\tilde{M}_i(s),\tilde{M}_j(s)\right)\right|ds\right]\\
&\leq 2\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i<j}M_j^\lambda(s)\Big(\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|+\left|M_j^\lambda(s)-\tilde{M}_j^\lambda(s)\right|\Big)ds\right]\\
&\leq 2\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\sum_{j\geq i+1}M_j^\lambda(s)ds\right]\\
&\quad+2\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{j\geq 2}\left|M_j^\lambda(s)-\tilde{M}_j^\lambda(s)\right|\sum_{i=1}^{j-1}M_i^\lambda(s)ds\right]\\
&\leq 4\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\|M(s)\|_\lambda\,\delta_\lambda\left(M(s),\tilde{M}(s)\right)ds\right]\\
&\leq 4\kappa_a\,x\int_0^t\mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\left(M(u),\tilde{M}(u)\right)\right]ds, \qquad (4.12)
\end{aligned}$$

we used that for $m\in\ell_\lambda$, $\sum_{i=1}^{j-1}m_j^\lambda\leq\sum_{i=1}^{j-1}m_i^\lambda\leq\|m\|_\lambda$.

Term C_t^c: it is treated exactly as B_t^c.

Term A_t^f: We take the expectation, and use (A.9) together with (2.7), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}A_s^f\right] &\leq C_\beta^\lambda\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left(F(M_i(s))\wedge F(\tilde{M}_i(s))\right)\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\right]ds\\
&\leq \overline{F}_m\,C_\beta^\lambda\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\right]ds\\
&\leq \overline{F}_m\,C_\beta^\lambda\int_0^t\mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\left(M(u),\tilde{M}(u)\right)\right]ds. \qquad (4.13)
\end{aligned}$$

Term B_t^f: we take the expectation and use (2.3) (recall $a := a_m \vee a_{\tilde{m}}$), (4.11), (A.7) together with (2.7), (A.3) and finally Proposition 4.4. *ii)*, to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} B_s^f\right] &\leq C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|F(M_i(s)) - F(\tilde{M}_i(s))\right| M_i^\lambda(s)\right] ds \\
&\leq \mu_a\, C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|M_i(s)^\alpha - \tilde{M}_i(s)^\alpha\right|\left(M_i^\lambda(s) + \tilde{M}_i^\lambda(s)\right)\right] ds \\
&\leq \mu_a\, C_\beta^\lambda\, C\, \mathbb{E}\left[\int_0^{t\wedge\tau_x}\left(\|M(s)\|_1^\alpha + \|\tilde{M}(s)\|_1^\alpha\right)\times\sum_{i\geq 1}\left|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)\right|\right] ds \\
&\leq 2\mu_a\, C_\beta^\lambda\, C\left(\|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha\right)\times\int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\left(M(u),\tilde{M}(u)\right)\right] ds. \qquad (4.14)
\end{aligned}$$

Term C_t^f: it is treated exactly as B_t^f.

Conclusion.- we take the expectation on (4.9) and gather (4.12), (4.13) and (4.14) to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}\delta_\lambda\left(M(s),\tilde{M}(s)\right)\right] &\leq \delta_\lambda(m,\tilde{m}) \\
&\quad + \left[8\kappa_a\, x + 4\mu_a\, C_\beta^\lambda\, C\left(\|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha\right) + \overline{F}_m C_\beta^\lambda\right] \\
&\qquad \times \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\left(M(u),\tilde{M}(u)\right)\right] ds. \qquad (4.15)
\end{aligned}$$

We conclude using the Gronwall Lemma:

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}\delta_\lambda\left(M(s),\tilde{M}(s)\right)\right] &\leq \delta_\lambda(m,\tilde{m}) \times e^{C\,(x\vee 1\vee\|m\|_1^\alpha\vee\|\tilde{m}\|_1^\alpha)\,t} \\
&\leq \delta_\lambda(m,\tilde{m})\; e^{C\,(x+1)\,t}.
\end{aligned}$$

Where C is a positive constant depending on λ, α, κ_a, μ_a, K, F, C_β^λ, $\|m\|_1$ and $\|\tilde{m}\|_1$.

This ends the proof of Proposition 4.4. □

4.2. **Existence for *SDE*: general case.** We may now prove existence for (SDE). For this, we will build a sequence of coupled finite Coalescence-Fragmentation processes which will be proved to be a Cauchy sequence in $\mathbb{D}([0,\infty),\ell_\lambda)$.

Theorem 4.6. *Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_\lambda$. Consider the coagulation kernel K, the fragmentation kernel F, the measure β and the Poisson measures $\mathcal{N}$ and $\mathcal{M}$ as in Definition 4.1.*

Then, there exists a solution $(M(m,t))_{t\geq 0}$ to $SDE(K,F,m,\mathcal{N},\mathcal{M})$.

We point out that we do not provide a pathwise uniqueness result for such processes. This is because, under our assumptions, we cannot take advantage of Proposition 4.4. for this process since the expressions in (4.6), (4.8) and (4.9) are possibly not true in general.

Nevertheless, when adding the hypothesis $\lim_{x+y\to 0} K(x,y) = 0$ to the coagulation kernel we can prove that these expressions hold by considering finite sums and passing to the limit. We believe that this is due to a possible injection of *dust* (particles of mass 0) into the system which could produce an increase in the total mass of the system; see [16].

In order to prove this theorem, we first need the following lemma.

Lemma 4.7. *Let $\lambda \in (0,1]$ and $\alpha \geq 0$ be fixed. Assume that the coagulation kernel K, the fragmentation kernel F and a measure β satisfy Hypotheses 2.2. Consider for all $k \geq 1$ the measure β_k defined by (3.3). Finally, consider also a subset $\mathcal{A}$ of ℓ_{0+} such that $\sup_{m\in\mathcal{A}} \|m\|_\lambda < \infty$ and $\lim_{i\to\infty} \sup_{m\in\mathcal{A}} \sum_{k\geq i} m_k^\lambda = 0$.*

For each $m \in \mathcal{A}$ and each $k \geq 1$, let $(M^k(m,t))_{t\geq 0}$ be the unique solution to $SDE(K,F,m,\mathcal{N},\mathcal{M}_k)$ constructed in Lemma 3.2., define $\tau_k(m,x) = \inf\{t \geq 0 : \|M^k(m,t)\|_\lambda \geq x\}$. Then for each $t \geq 0$ we have $\lim_{x\to\infty} \gamma(t,x) = 0$, where

$$\gamma(t,x) := \sup_{m\in\mathcal{A}} \sup_{k\geq 1} P\left[\sup_{s\in[0,t]} \|M^k(m,s)\|_\lambda \geq x\right].$$

Remark that this convergence does not depend on β_k since is based on a bound not depending in the number of fragments but only on C^λ_β.

4.2.1. *Proofs.*

Proof of Lemma 4.7. It suffices to remark that from Proposition 4.4. *i)*, we have

$$\begin{aligned}
\sup_{m\in\mathcal{A}} \sup_{k\geq 1} P\left[\sup_{[0,t]} \|M^k(m,s)\|_\lambda \geq x\right] &\leq \frac{1}{x} \sup_{m\in\mathcal{A}} \sup_{k\geq 1} \mathbb{E}\left[\sup_{[0,t]} \|M^k(m,s)\|_\lambda\right] \\
&\leq \frac{1}{x} \sup_{m\in\mathcal{A}} \|m\|_\lambda e^{\overline{F}_m C^\lambda_\beta t}.
\end{aligned}$$

We make x tend to infinity and the lemma follows. □

Proof of Theorem 4.6. First, recall ψ_n defined by (3.2) and the measure $\beta_n = \mathbb{1}_{\theta\in\Theta(n)} \beta \circ \psi_n^{-1}$. Consider the Poisson measure $\mathcal{M}(dt,di,d\theta,dz)$ associated to the fragmentation, as in Definition 4.1.

We set $\mathcal{M}_n = \mathbb{1}_{\Theta(n)} \mathcal{M} \circ \psi_n^{-1}$. This means that writing $\mathcal{M}$ as $\mathcal{M} = \sum_{k\geq 1} \delta_{(T_k,i_k,\theta_k,z_k)}$, we have $\mathcal{M}_n = \sum_{k\geq 1} \delta_{(T_k,i_k,\psi_n(\theta_k),z_k)} \mathbb{1}_{\theta\in\Theta(n)}$. Defined in this way, $\mathcal{M}_n$ is a Poisson measure on $[0,\infty) \times \mathbb{N} \times \Theta \times [0,\infty)$ with intensity measure $dt \left(\sum_{k\geq 1} \delta_k(di)\right) \beta_n(d\theta)\, dz$. In this paragraph $\delta_{(\cdot)}$ holds for the Dirac measure on $(\cdot)$.

We define $m^n \in \ell_{0+}$ by $m^n = (m_1, m_2, \cdots, m_n, 0, \cdots)$ and denote $M^n(t) := M(m^n,t)$ the unique solution to $SDE(K,F,m^n,\mathcal{N},\mathcal{M}_n)$ obtained in Proposition 4.3. Note that $M^n(t)$ satisfies the following equation

(4.16)
$$\begin{aligned}
M^n(t) &= m^n + \int_0^t \int_{i<j} \int_0^\infty \left[c_{ij}\left(M^n(s-)\right) - M^n(s-)\right] \mathbb{1}_{\{z\leq K(M^n_i(s-),M^n_j(s-))\}} \\
&\qquad \mathcal{N}(ds,d(i,j),dz) \\
&+ \int_0^t \int_i \int_\Theta \int_0^\infty \left[f_{i\psi_n(\theta)}\left(M^n(s-)\right) - M^n(s-)\right] \mathbb{1}_{\{z\leq F(M^n_i(s-))\}} \mathbb{1}_{\{\theta\in\Theta(n)\}} \\
&\qquad \mathcal{M}(ds,di,d\theta,dz).
\end{aligned}$$

This setting allows us to couple the processes since they are driven by the same Poisson measures.

Convergence $M_t^n \to M_t$**.**– Consider $p, q \in \mathbb{N}$ with $1 \le p < q$, from (4.16) we obtain

$$\delta_\lambda\left(M^p(t), M^q(t)\right) \le \delta_\lambda(m^p, m^q) + A_c^{p,q}(t) + B_c^{p,q}(t) + C_c^{p,q}(t) + A_f^{p,q}(t) + B_f^{p,q}(t) + C_f^{p,q}(t) + D_f^{p,q}(t). \tag{4.17}$$

We obtain this equality, exactly as in (4.9), by replacing M by M^p and $\tilde{M}$ by M^q. The terms concerning the coalescence are the same. The terms concerning the fragmentation are, equivalently:

$$A_f^{p,q}(t) = \int_0^t \int_i \int_\Theta \int_0^\infty \Big\{\delta_\lambda\left(f_{i\psi_p(\theta)}(M^p(s-)), f_{i\psi_p(\theta)}(M^q(s-))\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right)\Big\} \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{z \le F\left(M_i^p(s-)\right) \wedge F\left(M_i^q(s-)\right)\right\}} \mathcal{M}(ds, di, d\theta, dz),$$

$$B_f^{p,q}(t) = \int_0^t \int_i \int_\Theta \int_0^\infty \left\{\delta_\lambda\left(f_{i\psi_p(\theta)}(M^p(s-)), M^q(s-)\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right)\right\} \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{F\left(M_i^q(s-)\right) \le z \le F\left(M_i^p(s-)\right)\right\}} \mathcal{M}(ds, di, d\theta, dz),$$

$$C_f^{p,q}(t) = \int_0^t \int_i \int_\Theta \int_0^\infty \left\{\delta_\lambda\left(f_{i\psi_p(\theta)}(M^q(s-)), M^p(s-)\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right)\right\} \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{F\left(M_i^p(s-)\right) \le z \le F\left(M_i^q(s-)\right)\right\}} \mathcal{M}(ds, di, d\theta, dz),$$

Finally, the term $D_f^{p,q}(t)$ is the term that collects the errors.

$$D_f^{p,q}(t) = \int_0^t \int_i \int_\Theta \int_0^\infty \delta_\lambda\left(f_{i\psi_p(\theta)}(M^q(s-)), f_{i\psi_q(\theta)}(M^q(s-))\right) \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{z \le F\left(M_i^q(s-)\right)\right\}} \mathcal{M}(ds, di, d\theta, dz) + \int_0^t \int_i \int_\Theta \int_0^\infty \left\{\delta_\lambda\left(f_{i\psi_q(\theta)}(M^q(s-)), M^p(s-)\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right)\right\} \mathbb{1}_{\left\{z \le F\left(M_i^q(s-)\right)\right\}} \mathbb{1}_{\{\theta \in \Theta(q) \setminus \Theta(p)\}} \mathcal{M}(ds, di, d\theta, dz).$$

The first term of $D_f^{p,q}(t)$ results from the utilization of the triangle inequality that gives $A_f^{p,q}(t)$ and $C_f^{p,q}(t)$. The second term is issued from fragmentation of M^q when θ belongs to $\Theta(q) \setminus \Theta(p)$. This induces a fictitious jump to M^p which does not undergo fragmentation.

We proceed to bound each term. We define, for all $x > 0$ and $n \ge 1$, the stopping time $\tau_n^x = \inf\{t \ge 0 : \|M^n(t)\|_\lambda \ge x\}$.

From Proposition 4.4. we have for all $s \in [0, t]$,

$$\sup_{n \ge 1} \sup_{i \ge 1} M_i^n(s) \le \sup_{n \ge 1} \sup_{i \ge 1} \sup_{[0,t]} \|M^n(s)\|_1 \le \|m\|_1 := a_m \, a.s.$$

We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (2.2) and (2.3). Finally, we set $\overline{F}_m = \sup_{[0,a_m]} F(x)$.

The terms concerning coalescence are upper bounded on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ with $t \ge 0$, exactly as in (4.9).

Term $A_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (A.9) together with (2.7). We thus obtain exactly the same bound as for A_t^f.

Term $B_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (4.11), (A.7) with (2.7) and (2.3). We thus obtain exactly the same bound as for B_t^f.

Term $C_f^{p,q}(t)$: it is treated exactly as $B_f^{p,q}(t)$.

Term $D_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. For the first term we use (A.10). For the second term we use (4.11) and (A.7) together with (2.7). Finally, we use Proposition 4.4. *i)*. and the notation $C(\theta) := \sum_{k\geq 2} \theta_k^\lambda + (1-\theta_1^\lambda)$, to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]} D_f^{p,q}(t)\right] \\
&\quad\leq \mathbb{E}\left[\int_0^{t\wedge\tau_p^x\wedge\tau_q^x} \sum_{i\geq 1} F\left(M_i^q(s)\right) \int_\Theta \mathbb{1}_{\{\theta\in\Theta(p)\}} \sum_{k=p+1}^{q} \theta_k^\lambda [M_i^q(s)]^\lambda \beta(d\theta) ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^{t\wedge\tau_p^x\wedge\tau_q^x} \sum_{i\geq 1} F\left(M_i^q(s)\right)[M_i^q(s)]^\lambda ds \int_\Theta C(\theta)\mathbb{1}_{\{\theta\in\Theta(q)\setminus\Theta(p)\}}\beta(d\theta)\right] \\
&\quad\leq \overline{F}_m \int_\Theta \sum_{k>p} \theta_k^\lambda \beta(d\theta) \int_0^t \mathbb{E}\left[\sup_{u\in[0,t]} \|M^q(u)\|_\lambda\right] ds \\
&\qquad + \overline{F}_m \int_\Theta C(\theta)\mathbb{1}_{\{\theta\in\Theta\setminus\Theta(p)\}}\beta(d\theta) \int_0^t \mathbb{E}\left[\sup_{u\in[0,t]} \|M^q(u)\|_\lambda\right] ds \\
&\quad\leq \overline{F}_m\, t\, \|m\|_\lambda\, e^{\overline{F}_m\, C_\beta^\lambda\, t}(A(p)+B(p)),
\end{aligned}$$

where $A(p) := \int_\Theta \sum_{k>p} \theta_k^\lambda \beta(d\theta)$ and $B(p) := \int_\Theta C(\theta)\mathbb{1}_{\{\theta\in\Theta\setminus\Theta(p)\}}\beta(d\theta)$. Note that by (2.5) and since $\Theta\setminus\Theta(p)$ tends to the empty set, $A(p)$ and $B(p)$ tend to 0 as p tends to infinity.

Thus, gathering the terms as for the bound (4.15), we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]} \delta_\lambda\left(M^p(s), M^q(s)\right)\right] \\
&\qquad\leq \delta_\lambda\left(m^p, m^q\right) + D_1 t[A(p)+B(p)] \\
&\qquad\quad + \left(8\kappa_1\, x + CC_\beta^\lambda \|m\|_1^\alpha\right) \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_q^x\wedge\tau_p^x]} \delta_\lambda\left(M^p(u), M^q(u)\right)\right] ds,
\end{aligned} \tag{4.18}$$

where $D_1 = \overline{F}_m\, \|m\|_\lambda\, e^{\overline{F}_m\, C_\beta^\lambda\, t}$. The Gronwall Lemma allows us to obtain

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]} \delta_\lambda\left(M^p(s), M^q(s)\right)\right] \leq \{\delta_\lambda\left(m^p, m^q\right) + D_1[A(p)+B(p)]t\} \times e^{D_2\, x\, t}, \tag{4.19}$$

where D_2 is a positive constants depending on λ, α, κ_{a_m}, μ_{a_m}, K, F, C_β^λ and $\|m\|_1$.

Since $\lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$, we deduce from Lemma 4.7. that for all $t \geq 0$,

$$\lim_{x\to\infty} \overline{\gamma}(t,x) = 0 \text{ where } \overline{\gamma}(t,x) := \sup_{n\geq 1} P[\tau(m^n, x) \leq t]. \tag{4.20}$$

This means that the stopping times τ_n^x tend to infinity as $x \to \infty$, uniformly in n.

Next, from (4.19), (4.20) and since $(m^n)_{n\geq 1}$ is a Cauchy sequence for δ_λ and $(A(n))_{n\geq 1}$ and $(B(n))_{n\geq 1}$ converge to 0, we deduce that for all $\varepsilon > 0$, $T > 0$ we may find $n_\varepsilon > 0$ such that for $p, q \geq n_\varepsilon$ we have

$$P\left[\sup_{[0,T]} \delta_\lambda \left(M^p(t), M^q(t)\right) \geq \varepsilon\right] \leq \varepsilon. \tag{4.21}$$

Indeed, for all $x > 0$,

$$\begin{aligned} P\left[\sup_{[0,T]} \delta_\lambda \left(M^p(t), M^q(t)\right) \geq \varepsilon\right] &\leq P[\tau_p^x \leq T] + P[\tau_q^x \leq T] + \frac{1}{\varepsilon}\mathbb{E}\left[\sup_{[0,T\wedge\tau_p^x\wedge\tau_q^x]} \delta_\lambda \left(M^p(t), M^q(t)\right)\right] \\ &\leq 2\overline{\gamma}(T,x) + \frac{1}{\varepsilon}[\delta_\lambda \left(m^p, m^q\right) + D_1 T(A(p) + B(p))] \times e^{D_2\, x\, T}. \end{aligned}$$

Choosing x large enough so that $\overline{\gamma}(T,x) \leq \varepsilon/8$ and n_ε large enough to have both $A(p)$ and $B(p) \leq (\varepsilon^2/4D_1 T)e^{-D_2 x T}$ and in a such a way that for all $p, q \geq n_\varepsilon$, $\delta_\lambda\left(m^p, m^q\right) \leq (\varepsilon^2/4)e^{-D_2 x T}$, we conclude that (4.21) holds.

We deduce from (4.21) that the sequence of processes $(M_t^n)_{t\geq 0}$ is Cauchy in probability in $\mathbb{D}([0,\infty), \ell_\lambda)$, endowed with the uniform norm in time on compact intervals. We are thus able to find a subsequence (not relabelled) and a $(\mathcal{H}_t)$-adapted process $(M(t))_{t\geq 0}$ belonging *a.s.* to $\mathbb{D}([0,\infty), \ell_\lambda)$ such that for all $T > 0$,

$$\lim_{n\to\infty} \sup_{[0,T]} \delta_\lambda \left(M^n(t), M(t)\right) = 0. \; a.s. \tag{4.22}$$

Setting now $\tau^x := \inf\{t \geq 0 : \|M(t)\|_\lambda \geq x\}$, due to Lebesgue Theorem,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{[0,T\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda \left(M^n(t), M(t)\right)\right] = 0. \tag{4.23}$$

We have to show now that the limit process $(M(t))_{t\geq 0}$ defined by (4.22) solves the equation $SDE(K, F, m, \mathcal{N}, \mathcal{M})$ defined in (4.1).

We want to pass to the limit in (4.16), it suffices to show that $\lim_{n\to\infty} \Delta_n(t) = 0$, where

$$\begin{aligned} \Delta_n(t) = \mathbb{E}\Bigg[&\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_{i<j} \int_0^\infty \sum_{k\geq 1} 2^{-k} \Big| \left([c_{ij}\left(M(s-)\right)]_k - M_k(s-)] \right) \mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}} \\ &- \left([c_{ij}\left(M^n(s-)\right)]_k - M_k^n(s-)\right) \mathbb{1}_{\{z\leq K(M_i^n(s-),M_j^n(s-))\}} \Big| \mathcal{N}(ds, d(i,j), dz) \\ &+ \int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_i \int_\Theta \int_0^\infty \sum_{k\geq 1} 2^{-k} \Big| \left([f_{i\theta}\left(M(s-)\right)]_k - [M(s-)]_k\right) \mathbb{1}_{\{z\leq F(M_i(s-))\}} \\ &- \left([f_{i\psi_n(\theta)}\left(M^n(s-)\right)]_k - M_k^n(s-)\right) \mathbb{1}_{\{z\leq F(M_i^n(s-))\}} \mathbb{1}_{\{\theta\in\Theta(n)\}} \Big| \mathcal{M}(ds, di, d\theta, dz) \Bigg]. \end{aligned}$$

Indeed, due to (4.22), for all $x > 0$ and for n large enough, *a.s.* $\tau_n^x \geq \tau^{x/2}$. Thus M will solve $SDE(K, F, M(0), \mathcal{N}, \mathcal{M})$ on the time interval $[0, \tau^{x/2})$ for all $x > 0$, and thus on $[0,\infty)$ since *a.s.* $\lim_{x\to\infty} \tau^x = \infty$, because $M \in \mathbb{D}([0,\infty), \ell_\lambda)$.

Note that

$$\begin{aligned}
&\Big| \left([c_{ij}\left(M(s)\right)]_k - M_k(s)]\right) \mathbb{1}_{\{z \le K(M_i(s),M_j(s))\}} - \left([c_{ij}\left(M^n(s)\right)]_k - M^n_k(s)\right) \mathbb{1}_{\{z \le K(M^n_i(s),M^n_j(s))\}} \Big| \\
&\quad \le \Big| \left([c_{ij}\left(M(s)\right)]_k - M_k(s)]\right) - \left([c_{ij}\left(M^n(s)\right)]_k - M^n_k(s)\right) \Big| \mathbb{1}_{\{z \le K(M_i(s),M_j(s))\}} \\
&\qquad + \Big| [c_{ij}\left(M^n(s)\right)]_k - M^n_k(s) \Big| \Big| \mathbb{1}_{\{z \le K(M_i(s),M_j(s))\}} - \mathbb{1}_{\{z \le K(M^n_i(s),M^n_j(s))\}} \Big|
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) \mathbb{1}_{\{z \le F(M_i(s))\}} - \left([f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M^n_k(s)\right) \mathbb{1}_{\{z \le F(M^n_i(s))\}} \mathbb{1}_{\{\theta \in \Theta(n)\}} \Big| \\
&\quad \le \Big| \left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) - \left([f_{i\theta}\left(M^n(s)\right)]_k - M^n_k(s)\right) \Big| \mathbb{1}_{\{z \le F(M_i(s))\}} \\
&\qquad + \Big| \left([f_{i\theta}\left(M^n(s)\right)]_k - [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k\right) \Big| \mathbb{1}_{\{z \le F(M_i(s))\}} \\
&\qquad + \Big| [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M^n_k(s) \Big| \Big| \mathbb{1}_{\{z \le F(M_i(s))\}} - \mathbb{1}_{\{z \le F(M^n_i(s))\}} \Big| \\
&\qquad + \Big| [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M^n_k(s) \Big| \mathbb{1}_{\{z \le F(M^n_i(s))\}} \mathbb{1}_{\{\theta \in \Theta(n)^c\}},
\end{aligned}$$

where $\Theta(n)^c = \Theta \setminus \Theta(n)$. We thus obtain the following bound

$$\Delta_n(t) \le A^c_n(t) + B^c_n(t) + A^f_n(t) + B^f_n(t) + C^f_n(t) + D^f_n(t).$$

First, $A^c_n(t) = \sum_{i<j} A^{ij}_n(t)$ with

$$\begin{aligned}
A^{ij}_n(t) \quad = \quad \mathbb{E}\Bigg[\int_0^{t \wedge \tau^x_n \wedge \tau^x} & K\left(M_i(s), M_j(s)\right) \sum_{k \ge 1} 2^{-k} \\
& \left| \left([c_{ij}\left(M(s)\right)]_k - M_k(s)]\right) - \left([c_{ij}\left(M^n(s)\right)]_k - M^n_k(s)\right) \right| ds \Bigg],
\end{aligned}$$

and using

$$\begin{aligned}
&\Big| \mathbb{1}_{\{z \le K(M_i(s),M_j(s))\}} - \mathbb{1}_{\{z \le K(M^n_i(s),M^n_j(s))\}} \Big| \\
&\qquad = \mathbb{1}_{\{K(M_i(s),M_j(s)) \wedge K(M^n_i(s),M^n_j(s)) \le z \le K(M_i(s),M_j(s)) \vee K(M^n_i(s),M^n_j(s))\}},
\end{aligned}$$

$$\begin{aligned}
B^c_n(t) \quad = \quad \mathbb{E}\Bigg[\int_0^{t \wedge \tau^x_n \wedge \tau^x} & \sum_{i<j} \Big| K\left(M_i(s), M_j(s)\right) - K\left(M^n_i(s), M^n_j(s)\right) \Big| \\
& \sum_{k \ge 1} 2^{-k} \left| [c_{ij}\left(M^n(s)\right)]_k - M^n_k(s) \right| ds \Bigg].
\end{aligned}$$

For the fragmentation terms we have

$$\begin{aligned}
A^f_n(t) \quad = \quad \mathbb{E}\Bigg[\int_0^{t \wedge \tau^x_n \wedge \tau^x} & \int_\Theta \sum_{i \ge 1} F\left(M_i(s)\right) \\
& \sum_{k \ge 1} 2^{-k} \left| \left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) - \left([f_{i\theta}\left(M^n(s)\right)]_k - M^n_k(s)\right) \right| \beta(d\theta) ds \Bigg],
\end{aligned}$$

$$B^f_n(t) \quad = \quad \mathbb{E}\Bigg[\int_0^{t \wedge \tau^x_n \wedge \tau^x} \int_\Theta \sum_{i \ge 1} F\left(M_i(s)\right) \sum_{k \ge 1} 2^{-k} \Big| \left([f_{i\theta}\left(M^n(s)\right)]_k - [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k\right) \Big| \beta(d\theta) ds \Bigg],$$

using

$$\left|\mathbb{1}_{\{z\le F(M_i(s))\}}-\mathbb{1}_{\{z\le F(M_i^n(s))\}}\right|=\mathbb{1}_{\{F(M_i(s))\wedge F(M_i^n(s))\le z\le F(M_i(s))\vee F(M_i^n(s))\}},$$

$$\begin{aligned} C_n^f(t) &= \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)\}}\sum_{i\ge1}\left|F\left(M_i(s)\right)-F\left(M_i^n(s)\right)\right| \\ &\qquad \sum_{k\ge1}2^{-k}\left|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k-M_k^n(s)\right|\beta(d\theta)ds\Bigg], \end{aligned}$$

and finally,

$$\begin{aligned} D_n^f(t) &= \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)^c\}}\sum_{i\ge1}F\left(M_i^n(s)\right) \\ &\qquad \sum_{k\ge1}2^{-k}\left|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k-M_k^n(s)\right|\beta(d\theta)ds\Bigg], \end{aligned}$$

We will show that each term converges to 0 as n tends to infinity.

Note first that from (4.22) we have, *a.s.* $\sup_{[0,t]}\|M(s)\|_1\le\limsup_{n\to\infty}\sup_{[0,t]}\|M^n(s)\|_1$ and *a.s.* $\sup_{[0,t]}\|M(s)\|_\lambda\le\limsup_{n\to\infty}\sup_{[0,t]}\|M^n(s)\|_\lambda$ and from Proposition 4.4 *i)*, we get $\sup_{n\ge1}\sup_{[0,t]}\|M^n(s)\|_1\le\|m\|_1$, implying for all $t\ge0$

$$\sup_{s\in[0,t]}\|M(s)\|_1\le\|m\|_1:=a_m<\infty,\ a.s., \tag{4.24}$$

equivalently for M^n, we have $a_{m^n}=\|m^n\|_1\le\|m\|_1$. We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (2.2) and (2.3). Finally, we set $\overline{K}_m=\sup_{[0,a_m]^2}K(x,y)$ and $\overline{F}_m=\sup_{[0,a_m]}F(x)$.

We prove that $A_n^c(t)$ tends to 0 using the Lebesgue dominated convergence Theorem. It suffices to show that:

a) for each $1\le i<j$, $A_n^{ij}(t)$ tends to 0 as n tends to infinity,
b) $\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{i+j\ge k}A_n^{ij}(t)=0$.

Now, for $A_n^{ij}(t)$ using (A.16), (A.14), (4.24) and Proposition 4.4. *i)*, we have

$$\begin{aligned} A_n^{ij}(t) &\le \overline{K}_m\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} d\left(c_{ij}\left(M(s)\right),c_{ij}\left(M^n(s)\right)\right)+d\left(M(s),M^n(s)\right)\,ds\right] \\ &\le \overline{K}_m\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\left(2^i+2^j+1\right)d\left(M(s),M^n(s)\right)\,ds\right] \\ &\le C\,\overline{K}_m\left(2^i+2^j+1\right)\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\left(\|M(s)\|_1^{1-\lambda}\vee\|M^n(s)\|_1^{1-\lambda}\right) \\ &\qquad\qquad \times\delta_\lambda\left(M(s),M^n(s)\right)\,ds\Bigg] \\ &\le C\,\overline{K}_m\left(2^i+2^j+1\right)t\|m\|_1^{1-\lambda}\mathbb{E}\left[\sup_{[0,t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s),M^n(s)\right)\right]. \end{aligned}$$

which tends to 0 as $n \to \infty$ due to (4.23). On the other hand, using (A.15) we have

$$\begin{aligned} A_n^{ij}(t) &\leq \overline{K}_m \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} d\left(c_{ij}\left(M(s)\right), M(s)\right) + d\left(c_{ij}\left(M^n(s)\right), M^n(s)\right)\, ds\right] \\ &\leq \frac{3\overline{K}_m}{2} 2^{-i} \int_0^t \mathbb{E}\left[M_j(s) + M_j^n(s)\right] ds. \end{aligned}$$

Since $\sum_{i\geq 1} 2^{-i} = 1$ and $\sum_{j\geq 1} \int_0^t \mathbb{E}[M_j(s)]ds \leq \|m\|_1 t$, b) reduces to

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j\geq k} \int_0^t \mathbb{E}[M_j^n(s)]ds = 0.$$

For each $k \geq 1$, since $M^n(s)$ and $M(s)$ belong to ℓ_1 for all $s \geq 0$ $a.s$ and since the map $m \mapsto \sum_{j=1}^{k-1} m_j$ is continuous for the pointwise convergence topology,

$$\begin{aligned} \limsup_{n\to\infty} \int_0^t \mathbb{E}\left[\sum_{j\geq k} M_j^n(s)\right] &= \int_0^t ds \left\{\lim_{n\to\infty} \|M^n(s)\|_1 - \lim_{n\to\infty} \mathbb{E}\left[\sum_{j=1}^{k-1} M_j^n(s)\right]\right\} ds \\ &= \int_0^t \left\{\|M(s)\|_1 - \mathbb{E}\left[\sum_{j=1}^{k-1} M_j(s)\right]\right\} ds \\ &= \int_0^t \mathbb{E}\left[\sum_{j=k}^{\infty} M_j(s)\right] ds. \end{aligned}$$

We easily conclude using that $a.s.$ $\|M(s)\|_1 < \|m\|_1$ for all $s \geq 0$.

Using (2.2), (A.15) and Proposition 4.4. *i)*, we obtain

$$\begin{aligned} B_n^c(t) &\leq \kappa_{a_m} \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \sum_{i<j} \left[\left|M_i^n(s)^\lambda - M_i(s)^\lambda\right| + \left|M_j^n(s)^\lambda - M_j(s)^\lambda\right|\, ds\right] \\ &\qquad\qquad \times d\left(c_{ij}\left(M^n(s)\right), M^n(s)\right)\Bigg] \\ &\leq \frac{3}{2}\kappa_{a_m} \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \sum_{i<j} \left[\,\left|M_i^n(s)^\lambda - M_i(s)^\lambda\right| + \left|M_j^n(s)^\lambda - M_j(s)^\lambda\right|\,\right] 2^{-i} M_j^n(s)\, ds\right] \\ &\leq 3t\, \kappa_{a_m} \|m\|_1 \mathbb{E}\left[\sup_{[0,[t\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda\left(M(s), M^n(s)\right)\right], \end{aligned}$$

which tends to 0 as $n \to \infty$ due to (4.23).

We use (A.18) and (A.14) both with (4.24) and Proposition 4.4. *i)* and (A.17) to obtain

$$\begin{aligned} A_n^f(t) &\leq \overline{F}_m \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \sum_{i\geq 1}\int_\Theta \Bigg[\Bigg(d\left(f_{i\theta}\left(M(s)\right), f_{i\theta}\left(M^n(s)\right)\right) + d\left(M(s), M^n(s)\right) \Bigg) \\ &\qquad \wedge \Bigg(d\left(f_{i\theta}\left(M(s)\right), M(s)\right) + d\left(f_{i\theta}\left(M^n(s)\right), M^n(s)\right) \Bigg) \Bigg] \beta(d\theta) ds \Bigg] \\ &\leq \overline{F}_m \mathbb{E}\Bigg\{ \int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \sum_{i\geq 1} \Bigg[\Bigg(2C\|m\|_\lambda^{1-\lambda} \delta_\lambda\left(M(s), M^n(s)\right) \Bigg) \wedge \\ &\qquad \Bigg(2^{-i}(1-\theta_1)\left(M_i(s) + M_i^n(s)\right) \Bigg) \Bigg] \beta(d\theta) ds \Bigg\}. \end{aligned}$$

We split the integral on Θ and the sum on i into two parts. Consider $\Theta_\varepsilon = \{\theta \in \Theta : \theta_1 \leq 1-\varepsilon\}$ and $N \in \mathbb{N}$. Using (4.24) and Proposition 4.4. *i)* and relabelling the constant C, we deduce

$$\begin{aligned} &\int_\Theta \sum_{i\geq 1} \Bigg[\Bigg(C\|m\|_\lambda^{1-\lambda} \delta_\lambda\left(M(s), M^n(s)\right) \Bigg) \wedge \Bigg(2^{-i}(1-\theta_1)\left(M_i(s) + M_i^n(s)\right) \Bigg) \Bigg] \beta(d\theta) \\ &\qquad \leq C\|m\|_\lambda^{1-\lambda} \int_{\Theta_\varepsilon} \sum_{i=1}^N \delta_\lambda\left(M(s), M^n(s)\right) \beta(d\theta) + \int_{\Theta_\varepsilon^c} (1-\theta_1)\beta(d\theta) \sum_{i\geq 1} \left(M_i(s) + M_i^n(s)\right) \\ &\qquad\quad + \int_\Theta \sum_{i>N} 2^{-i}(1-\theta_1)\left(M_i(s) + M_i^n(s)\right)\beta(d\theta) \\ &\qquad \leq C\|m\|_1^{1-\lambda} N\beta(\Theta_\varepsilon)\delta_\lambda\left(M(s), M^n(s)\right) + 2\|m\|_1 \int_{\Theta_\varepsilon^c} (1-\theta_1)\beta(d\theta) \\ &\qquad\quad + 2\|m\|_1 \int_\Theta (1-\theta_1)\beta(d\theta) \sum_{i>N} 2^{-i}. \end{aligned}$$

Note that $\beta(\Theta_\varepsilon) = \int_\Theta \mathbb{1}_{\{1-\theta_1\geq\varepsilon\}}\, \beta(d\theta) \leq \frac{1}{\varepsilon}\int_\Theta (1-\theta_1)\, \beta(d\theta) \leq \frac{1}{\varepsilon}\, C_\beta^\lambda < \infty$. Thus, we get

$$\begin{aligned} A_n^f(t) &\leq \frac{t}{\varepsilon} C_\beta^\lambda N \overline{F}_m C\|m\|_1^{1-\lambda} \mathbb{E}\left[\sup_{[0,[t\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda\left(M(s), M^n(s)\right) \right] \\ &\quad + 2t\overline{F}_m\|m\|_1 \int_{\Theta_\varepsilon^c} (1-\theta_1)\beta(d\theta) + 4t\overline{F}_m\|m\|_1 C_\beta^\lambda\, 2^{-N}. \end{aligned}$$

Thus, due to (4.23) we have for all $\varepsilon > 0$ and $N \geq 1$,

$$\limsup_{n\to\infty} A_n^f(t) \leq 2t\overline{F}_m\|m\|_1 \int_{\Theta_\varepsilon^c} (1-\theta_1)\beta(d\theta) + 4t\overline{F}_m\|m\|_1 C_\beta^\lambda\, 2^{-N}.$$

Since Θ_ε^c tends to the empty set as $\varepsilon \to 0$ we conclude using (2.7) with (2.5) and making $\varepsilon \to 0$ and $N \to \infty$.

Next, use (A.19) and Proposition 4.4. *i)* to obtain

$$B_n^f(t) \leq t\overline{F}_t\|m\|_1 \int_\Theta \sum_{k>n} \theta_k \beta(d\theta).$$

which tends to 0 as $n \to \infty$ due to (2.4).

Using (2.3), (A.17) with (2.6) and (2.7), (A.3), (A.14), (4.24) and Proposition 4.4. *i)*, we obtain

$$\begin{aligned} C_n^f(t) &\leq 2\mu_{a_m}\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_{\Theta(n)}\sum_{i\geq 1}\left|[M_i(s)]^\alpha - [M_i^n(s)]^\alpha\right| 2^{-i}(1-\theta_1)M_i(s)\beta(d\theta)ds\right] \\ &\leq 2\mu_{a_m}\, C_\beta^\lambda\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i\geq 1}2^{-i}\left|M_i(s)-M_i^n(s)\right|\left([M_i^n(s)]^\alpha + [M_i(s)]^\alpha\right)\, ds\right] \\ &\leq 2\mu_{a_m}\, C\, C_\beta^\lambda t\|m\|_1^{1-\lambda+\alpha}\mathbb{E}\left[\sup_{[0,t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s),M^n(s)\right)\right], \end{aligned}$$

which tends to 0 as $n\to\infty$ due to (4.23).

Finally, we use (A.17) with (2.6) and (2.7) and Proposition 4.4. *i)*, to obtain

$$D_n^f(t) \leq 2t\,\overline{F}_t\|m\|_1\int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)^c\}}(1-\theta_1)\beta(d\theta),$$

which tends to 0 as n tends to infinity since $\int_\Theta(1-\theta_1)\,\beta(d\theta)\leq C_\beta^\lambda$ and $\Theta(n)^c$ tends to the empty set.

This ends the proof of Theorem 4.6. □

4.3. **Conclusion.** It remains to conclude the proof of Theorem 3.3.

We start with some boundedness of the operator $\mathcal{L}_{K,F}^\beta$.

Lemma 4.8. *Let $\lambda\in(0,1]$, $\alpha\geq 0$, the coagulation kernel K, fragmentation kernel F and the measure β satisfying Hypotheses 2.2. Let $\Phi:\ell_\lambda\to\mathbb{R}$ satisfy, for all $m,\tilde{m}\in\ell_\lambda$, $|\Phi(m)|\leq a$ and $|\Phi(m)-\Phi(\tilde{m})|\leq ad(m,\tilde{m})$. Recall (2.11). Then $m\mapsto\mathcal{L}_{K,F}^\beta\Phi(m)$ is bounded on $\{m\in\ell_\lambda,\|m\|_\lambda\leq c\}$ for each $c>0$.*

Proof. This Lemma is a straightforward consequence of the hypotheses on the kernels and Lemma A.3. Let $c>0$ be fixed, and set $A:=c^{1/\lambda}$. Notice that if $\|m\|_\lambda\leq c$, then for all $k\geq 1$ $m_k\leq A$.

Setting $\sup_{[0,A]^2}K(x,y)=\overline{K}$ and $\sup_{[0,A]}F(x)=\overline{F}$. We use (A.15) and (A.17) with (2.6) and (2.7), and deduce that for all $m\in\ell_\lambda$ such that $\|m\|_\lambda\leq c$,

$$\begin{aligned} |\mathcal{L}_{K,F}^\beta\Phi(m)| &\leq \overline{K}\sum_{1\leq i<j<\infty}\left|\Phi\left(c_{ij}(m)\right)-\Phi(m)\right| + \overline{F}\sum_{i\geq 1}\int_\Theta\left|\Phi\left(f_{i\theta}(m)\right)-\Phi(m)\right|\beta(d\theta) \\ &\leq a\overline{K}\sum_{1\leq i<j<\infty}d(c_{ij}(m),m) + a\overline{F}\int_\Theta\sum_{i\geq 1}d(f_{i\theta}(m),m)\beta(d\theta) \\ &\leq \frac{3}{2}a\overline{K}\,\|m\|_1 + 2a\,\overline{F}\,C_\beta^\lambda\|m\|_1 \;\leq\; \left(\frac{3}{2}\overline{K}+2\overline{F}\,C_\beta^\lambda\right)ac^{1/\lambda}. \end{aligned}$$

□

Finally, it remains to conclude the proof of Theorem 3.3.

Proof of Theorem 3.3. We consider the Poisson measures N and M as in Definition 4.1., and we fix $m\in\ell_\lambda$. We consider $M(t):=M(m,t)$ a solution to $SDE(K,F,m,\mathcal{N},\mathcal{M})$ built in Section 4.2. M is a strong Markov Process, since it solves a time-homogeneous Poisson-driven $S.D.E.$ We now check the points *i)* and *ii)*.

Consider any sequence $m^n \in \ell_{0+}$ such that $\lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$ and $M^n(t) := M(m^n, t)$ the unique solution to $SDE(K, F, m^n, \mathcal{N}, \mathcal{M}_n)$ obtained in Proposition 4.3. Denote by $\tau^x = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x\}$ and by τ_n^x the stopping time concerning M^n. We will prove that for all $T \geq 0$ and $\varepsilon > 0$

$$\lim_{n\to\infty} P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^n(t)\right) > \varepsilon\right] = 0. \tag{4.25}$$

For this, consider the sequence $m^{(n)} \in \ell_{0+}$ defined by $m^{(n)} = (m_1, \cdots, m_n, 0, \dots)$ and $M^{(n)}(t) := M(m^{(n)}, t)$ the solution to $SDE(K, F, m^{(n)}, \mathcal{N}, \mathcal{M}_n)$ obtained in Proposition 4.3. and denote by $\tau_{(n)}^x$ the stopping time concerning $M^{(n)}$.

First, note that since $\lim_{n\to\infty} \delta_\lambda(m^{(n)}, m) = \lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$, we deduce that $\sup_{n\geq 1} \|m^{(n)}\|_\lambda < \infty$ and from Lemma 4.7. that for all $t \geq 0$,

$$\lim_{x\to\infty} \gamma_1(t,x) = 0 \text{ where } \gamma_1(t,x) := \sup_{n\geq 1} P[\tau_{(n)}^x \leq t], \tag{4.26}$$

$$\lim_{x\to\infty} \gamma_2(t,x) = 0 \text{ where } \gamma_2(t,x) := \sup_{n\geq 1} P[\tau_n^x \leq t]. \tag{4.27}$$

Thus, using Proposition 4.4. *ii)* we get for all $x > 0$

$$\begin{aligned}
&P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^n(t)\right) > \varepsilon\right] \\
&\quad \leq P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^{(n)}(t)\right) > \frac{\varepsilon}{2}\right] + P\left[\sup_{[0,T]} \delta_\lambda\left(M^{(n)}(t), M^n(t)\right) > \frac{\varepsilon}{2}\right] \\
&\quad \leq P[\tau^x \leq T] + \gamma_1(T,x) + \frac{2}{\varepsilon}\mathbb{E}\left[\sup_{[0,T\wedge\tau_{(n)}^x\wedge\tau^x]} \delta_\lambda\left(M(t), M^{(n)}(t)\right)\right] \\
&\qquad + \gamma_1(T,x) + \gamma_2(T,x) + \frac{2}{\varepsilon}e^{C(x+1)T}\delta_\lambda\left(m^{(n)}, m^n\right).
\end{aligned}$$

We first make n tend to infinity and use (4.23), then x to infinity and use (4.26) and (4.27). We thus conclude that (4.25) holds.

We may prove point *ii)* using a similar computation that for *i)*. The proof is easier since we do not need to use a triangle inequality.

Finally, consider $(M(m,t))_{t\geq 0}$ solution to $SDE(K, F, m, \mathcal{N}, \mathcal{M})$ and the sequence of stopping times $(\tau^{x_n})_{n\geq 1}$ where $\tau^{x_n} = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x_n\}$, with $x_n = n$. Since $M \in \mathbb{D}([0,\infty), \ell_\lambda)$, we have that $(\tau^{x_n})_{n\geq 1}$ is non-decreasing and $\tau^{x_n} \underset{n\to\infty}{\longrightarrow} \infty$ and from Lemma 4.8. we deduce that $(\mathcal{L}_{K,F}^\beta \Phi(M(m,s)))_{s\in[0,\tau^{x_n})}$ is uniformly bounded.

We thus apply Itô's Formula to $\Phi(M(m,t))$ on the interval $[0, t\wedge\tau^{x_n})$ to obtain

$$\begin{aligned}\Phi(M(m,t\wedge\tau^{x_n}))-\Phi(m) \ =& \\ \int_0^{t\wedge\tau^{x_n}}\int_{i<j}\int_0^\infty \left[\Phi\left(c_{ij}\left(M(m,s-)\right)\right)-\Phi\left(M(m,s-)\right)\right]\mathbb{1}_{\{z\le K(M_i(m,s-),M_j(m,s-))\}}& \\ \tilde{\mathcal{N}}(dt,d(i,j),dz)& \\ +\int_0^{t\wedge\tau^{x_n}}\int_i\int_\Theta\int_0^\infty \left[\Phi\left(f_{i\theta}\left(M(m,s-)\right)\right)-\Phi\left(M(m,s-)\right)\right]\mathbb{1}_{\{z\le F(M_i(m,s-))\}}& \\ \tilde{\mathcal{M}}(dt,di,d\theta,dz)& \\ +\int_0^{t\wedge\tau^{x_n}}\mathcal{L}^\beta_{K,F}\left(M(m,s)\right)ds,&\end{aligned}$$

where $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{M}}$ are two compensated Poisson measures and point *iii)* follows.

This ends the proof of Theorem 3.3.

I would like to express my deepest thanks to my Ph.D. advisor Prof. Nicolas Fournier for his insightful comments and advices during the preparation of this work. I would like also to thank Bénédicte Haas and James R. Norris for the lecture and their remarks. Finally, I sincerely thank the anonymous referee for pointing out many problems in the first version of this paper and for helping improving it.

Appendix A. Estimates concerning c_{ij}, $f_{i\theta}$, d and δ_λ

Here we put all the auxiliary computations needed in Sections 4.1.2 and 4.2.

Lemma A.1. *Fix $\lambda\in(0,1]$. Consider any pair of finite permutations σ, $\tilde{\sigma}$ of $\mathbb{N}$. Then for all m and $\tilde{m}\in\ell_\lambda$,*

$$d(m,\tilde{m}) \ \le\ \sum_{k\ge1}2^{-k}|m_k-\tilde{m}_{\tilde{\sigma}(k)}|, \tag{A.1}$$

$$\delta_\lambda(m,\tilde{m}) \ \le\ \sum_{k\ge1}|m^\lambda_{\sigma(k)}-\tilde{m}^\lambda_{\tilde{\sigma}(k)}|. \tag{A.2}$$

This lemma is a consequence of [15, Lemma 3.1].

We also have the following inequality: for all $\alpha,\beta>0$, there exists a positive constant $C=C_{\alpha,\beta}$ such that for all $x,y\ge0$,

$$(x^\alpha+y^\alpha)|x^\beta-y^\beta|\le 2|x^{\alpha+\beta}-y^{\alpha+\beta}|\le C(x^\alpha+y^\alpha)|x^\beta-y^\beta|. \tag{A.3}$$

We now give the inequalities concerning the action of c_{ij} and $f_{i\theta}$ on δ_λ and $\|\cdot\|_\lambda$.

Lemma A.2. *Let $\lambda\in(0,1]$ and $\theta\in\Theta$. Then for all m and $\tilde{m}\in\ell_\lambda$, all $1\le i<j<\infty$,*

$$\|c_{ij}(m)\|_\lambda = \|m\|_\lambda + (m_i+m_j)^\lambda - m_i^\lambda - m_j^\lambda \le \|m\|_\lambda, \tag{A.4}$$

$$\|f_{i\theta}(m)\|_\lambda = \|m\|_\lambda + m_i^\lambda \left(\sum_{k\geq 1}\theta_k^\lambda - 1\right), \tag{A.5}$$

$$\delta_\lambda(c_{ij}(m),m) \le 2m_j^\lambda, \tag{A.6}$$

$$\delta_\lambda(f_{i\theta}(m),m) \le m_i^\lambda\left[\sum_{k\geq 2}\theta_k^\lambda + \left(1-\theta_1^\lambda\right)\right], \tag{A.7}$$

$$\delta_\lambda(c_{ij}(m),c_{ij}(\tilde m)) \le \delta_\lambda(m,\tilde m), \tag{A.8}$$

$$\delta_\lambda(f_{i\theta}(m),f_{i\theta}(\tilde m)) \le \delta_\lambda(m,\tilde m) + |m_i^\lambda - \tilde m_i^\lambda|\left(\sum_{k\geq 1}\theta_k^\lambda - 1\right). \tag{A.9}$$

On the other hand, recall (3.2), we have, for $u, v \in \mathbb{N}$ with $1 \le u < v$,

$$\delta_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) \le \sum_{k=u+1}^{v} \theta_k^\lambda m_i^\lambda. \tag{A.10}$$

Note that in the case $\sum_{k\geq 1}\theta_k^\lambda - 1 < 0$, we have that $\|\cdot\|_\lambda$ and δ_λ are respectively, decreasing and contracting under the action of fragmentation and the calculations in precedent sections would be simpler.

Proof. First (A.4) and (A.5) are evident. Next, (A.6) and (A.8) are proved in [17, Lemma A.2].

To prove (A.7) let $\theta = (\theta_1, \cdots) \in \Theta$, $i \ge 1$ and $p \ge 2$ and set $l := l(m) = \min\{k \ge 1 : m_k \le \theta_p\, m_i\}$, we consider the largest particle of the original system (before dislocation of m_i) that is smaller than the p-th fragment of m_i, this is m_l. Consider now σ, the finite permutation of $\mathbb{N}$ that achieves:

$$\begin{aligned}(f_k)_{k\geq 1} &:= \Big([f_{i\theta}(m)]_{\sigma(k)}\Big)_{k\geq 1} \\ &= (m_1,\cdots,m_{i-1},\theta_1 m_i, m_{i+1},\cdots,m_{l-1},m_l,\theta_2 m_i,\theta_3 m_i,\cdots,\theta_p m_i,[f_{i\theta}(m)]_{l+1},\cdots).\end{aligned} \tag{A.11}$$

It suffices to compute the δ_λ-distance of the sequences $(f_k)_k$ and $(m_k)_k$:

$$\begin{matrix} m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots \\ m_1 & \cdots & m_{i-1} & m_i & m_{i+1} & \cdots & m_{l-1} & m_l & m_{l+1} & m_{l+2} & \cdots & m_{l+p-1} & m_{l+p} & \cdots \end{matrix} \tag{A.12}$$

Thus, using (A.2), we have

$$\begin{aligned}
\delta_\lambda(f_{i\theta}(m),m) &\le \sum_{k\ge 1}\left|f_k^\lambda - m_k^\lambda\right| = \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\ge l+p}\right)\left|f_k^\lambda - m_k^\lambda\right| \\
&\le (1-\theta_1^\lambda)m_i^\lambda + \sum_{k=l+1}^{l+p-1}\left|\theta_{k-l+1}^\lambda m_i^\lambda - m_k^\lambda\right| + \sum_{k\ge l+p}\left|f_k^\lambda - m_k^\lambda\right| \\
&\le (1-\theta_1^\lambda)m_i^\lambda + \left(\sum_{k=2}^{p}\theta_k^\lambda m_i^\lambda + \sum_{k=l+1}^{l+p-1} m_k^\lambda\right) + \sum_{k\ge l+p}\left(f_k^\lambda + m_k^\lambda\right) \\
&= (1-\theta_1^\lambda)m_i^\lambda + m_i^\lambda\sum_{k=2}^{\infty}\theta_k^\lambda + 2\sum_{k>l} m_k^\lambda.
\end{aligned}$$

For the last equality it suffices to remark that $\sum_{k\ge l} f_k^\lambda$ contains all the remaining fragments of m_i^λ and all the particles m_k^λ with $k>l$.

Note that if $m\in\ell_{0+}$ the last sum consists of a finite number of terms and it suffices to take p large enough (implying l large) to cancel this term. On the other hand, if $m\in\ell_\lambda\setminus\ell_{0+}$ then the last sum is the tail of a convergent serie and since $l\to\infty$ whenever $p\to\infty$, we conclude by making p tend to infinity and (A.7) follows.

To prove (A.9) consider $\tilde m$, $l := l(m)\vee l(\tilde m)$ and the permutations σ and $\tilde\sigma$ associated to this l, exactly as in (A.11). Let f and $\tilde f$ be the corresponding objects concerning m and $\tilde m$:

$$\text{(A.13)}\quad \begin{array}{cccccccccccccc}
m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots \\
\tilde m_1 & \cdots & \tilde m_{i-1} & \theta_1 \tilde m_i & \tilde m_{i+1} & \cdots & \tilde m_{l-1} & \tilde m_l & \theta_2 \tilde m_i & \theta_3 \tilde m_i & \cdots & \theta_p \tilde m_i & \tilde f_{l+p} & \cdots
\end{array}$$

Using again (A.2) for $(f_k)_k$ and $(\tilde f_k)_k$, we have

$$\begin{aligned}
&\delta_\lambda(f_{i\theta}(m), f_{i\theta}(\tilde m)) \\
&\quad\le \sum_{k\ge 1}\left|f_k^\lambda - \tilde f_k^\lambda\right| = \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\ge l+p}\right)\left|f_k^\lambda - \tilde f_k^\lambda\right| \\
&\quad= \sum_{k=1}^{l}\left|m_k^\lambda - \tilde m_k^\lambda\right| - \left|m_i^\lambda - \tilde m_i^\lambda\right| + \sum_{k=1}^{p}\theta_k^\lambda\left|m_i^\lambda - \tilde m_i^\lambda\right| + \sum_{k\ge l+p}\left(f_k^\lambda + \tilde f_k^\lambda\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{l}\left|m_k^\lambda - \tilde m_k^\lambda\right| - \left|m_i^\lambda - \tilde m_i^\lambda\right| + \sum_{k=1}^{p}\theta_k^\lambda\left|m_i^\lambda - \tilde m_i^\lambda\right| + \sum_{k>p}\theta_k^\lambda\left(m_i^\lambda + \tilde m_i^\lambda\right) + \sum_{k>l}\left(m_k^\lambda + \tilde m_k^\lambda\right) \\
&= \sum_{k=1}^{l}\left|m_k^\lambda - \tilde m_k^\lambda\right| + |m_i^\lambda - \tilde m_i^\lambda|\left(\sum_{k=1}^{p}\theta_k^\lambda - 1\right) + \left(m_i^\lambda + \tilde m_i^\lambda\right)\sum_{k>p}\theta_k^\lambda + \sum_{k>l}\left(m_k^\lambda + \tilde m_k^\lambda\right).
\end{aligned}$$

Notice that the last two sums are the tails of convergent series, note also that $l\to\infty$ whenever $p\to\infty$. We thus conclude making p tend to infinity.

Finally, to prove (A.10) we consider the permutation σ as in (A.11) with $p = v$ and $l := l(m)$. Recall (A.13), we have

$$\begin{aligned}\delta_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) &= \delta_\lambda(f_{i\psi_v(\psi_u(\theta))}(m), f_{i\psi_v(\theta)}(m))\\ &\leq \sum_{k\geq 1}\left|[f_{i\psi_v(\psi_u(\theta))}(m)]^\lambda_{\sigma(k)} - [f_{i\psi_v(\theta)}(m)]^\lambda_{\sigma(k)}\right|\\ &\leq \sum_{k=u+1}^{v}\theta_k^\lambda m_i^\lambda + 2\sum_{k>l} m_k^\lambda.\end{aligned}$$

We used that $[\psi_v(\psi_u(\theta))]_k = 0$ for $k = u+1, \cdots, v$. Since $m \in \ell_\lambda$, we conclude making l tend to infinity.

□

Lemma A.3. *Consider m, $\tilde{m} \in S^\downarrow$ and $1 \leq i < j < \infty$. Recall the definition of d (2.9), δ_λ (2.10), $c_{ij}(m)$ and $f_{i\theta}(m)$ (2.8) and $\psi_n(\theta)$ (3.2). For $\lambda \in (0,1)$ and for all $m, \tilde{m} \in \ell_\lambda$ there exists a positive constant C depending on λ such that*

$$d(m,\tilde{m}) \leq \delta_1(m,\tilde{m}) \leq C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda})\,\delta_\lambda(m,\tilde{m}). \tag{A.14}$$

Next,

$$d(c_{ij}(m), m) \leq \frac{3}{2}2^{-i}m_j, \qquad \sum_{1\leq k<l<\infty} d(c_{kl}(m), m) \leq \frac{3}{2}\|m\|_1, \tag{A.15}$$

$$d(c_{ij}(m), c_{ij}(\tilde{m})) \leq (2^i + 2^j)d(m,\tilde{m}). \tag{A.16}$$

$$d(f_{i\theta}(m), m) \leq 2(1-\theta_1)2^{-i}m_i, \tag{A.17}$$

$$d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \leq C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda})\,\delta_\lambda(m,\tilde{m}), \tag{A.18}$$

$$d(f_{i\theta}(m), f_{i\psi_n(\theta)}(m)) \leq m_i\sum_{k>n}\theta_k. \tag{A.19}$$

Proof. The first inequality in (A.14) follows readily from the definition of d and the second one comes from (A.3), with $\alpha = 1-\lambda$ and $\beta = \lambda$. The inequalities (A.15) and (A.16) involving d are proved in [15, Corollary 3.2.].

We prove (A.17) exactly as (A.7). Consider p, l and the permutation σ defined by (A.11), from (A.1) and since $i \leq l+1 \leq l+p$, we obtain

$$\begin{aligned}d(f_{i\theta}(m), m) &\leq \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\geq l+p}\right) 2^{-k}\,|f_k - m_k|\\ &\leq (1-\theta_1)2^{-i}m_i + \sum_{k=l+1}^{l+p-1} 2^{-k}\,|\theta_{k-l+1}m_i - m_k| + \sum_{k\geq l+p} 2^{-k}\,|f_k - m_k|\\ &\leq (1-\theta_1)2^{-i}m_i + \left(\sum_{k=2}^{p} 2^{-i}\theta_k m_i + \sum_{k=l+1}^{l+p-1} m_k\right) + \sum_{k\geq l+p} 2^{-i}\,(f_k + m_k)\\ &\leq (1-\theta_1)2^{-i}m_i + 2^{-i}m_i\sum_{k=2}^{\infty}\theta_k + 2\sum_{k>l} m_k.\end{aligned}$$

Since $m \in \ell_1$, we conclude using (2.4) and making l tend to infinity.

Next, we prove (A.18) as (A.9) using δ_1. Consider p, l and the permutations σ and $\tilde{\sigma}$ defined by (A.11). Recall (A.13), using (A.14) then (A.2) (applied to δ_1) and since, $i \le l+1 \le l+p$ we obtain

$$\begin{aligned}
&d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \\
&\quad \le \delta_1(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \ \le \ \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\ge l+p}\right) \left|f_k - \tilde{f}_k\right| \\
&\quad \le \sum_{k=1}^{l} |m_k - \tilde{m}_k| + (\theta_1 - 1)\,|m_i - \tilde{m}_i| + \sum_{k=l+1}^{l+p-1} \theta_{k-l+1}\,|m_i - \tilde{m}_i| + \sum_{k\ge l+p} \left(f_k + \tilde{f}_k\right) \\
&\quad \le \sum_{k=1}^{l} |m_k - \tilde{m}_k| + |m_i - \tilde{m}_i| \left(\sum_{k=1}^{p} \theta_k - 1\right) + (m_i + \tilde{m}_i) \sum_{k>p} \theta_k + \sum_{k>l} (m_k + \tilde{m}_k) \\
&\quad \le \sum_{k=1}^{l} |m_k - \tilde{m}_k| + (m_i + \tilde{m}_i) \sum_{k>p} \theta_k + \sum_{k>l} (m_k + \tilde{m}_k)\,.
\end{aligned}$$

We used that for $k \ge l+p$, f_k contains all the remaining fragments of m_i and the particles m_j with $j > l$ and (2.4). Since $m, \tilde{m} \in \ell_1$ we conclude making p tend to infinity and using (A.14).

Finally, for inequality (A.19), let $i \ge 1$, $p \ge 1$ and $l := l_p(m) = \min\{k \ge 1 : m_k \le (\theta_n/p)m_i\}$ and consider σ, the finite permutation of $\mathbb{N}$ that achieves:

$$\begin{aligned}
(f_k)_{k\ge 1} \quad &:= \quad \left([f_{i\theta}(m)]_{\sigma(k)}\right)_{k\ge 1} \\
&= \quad (m_1, \cdots, m_{i-1}, \theta_1 m_i, \cdots, \theta_n m_i, m_{i+1}, \cdots, m_{l-1}, m_l, [f_{i\theta}(m)]_{l+n}, \cdots). \qquad \text{(A.20)}
\end{aligned}$$

Thus, from (A.14) and (A.2), and since $i \le l+1 \le l+n+1$, we deduce

$$\begin{aligned}
d\left((f_{i\theta}(m), f_{i\psi_n(\theta)}(m)\right) \quad &\le \quad \delta_1\left((f_{i\theta}(m), f_{i\psi_n(\theta)}(m)\right) \ = \ \sum_{k\ge 1} \left|[f_{i\theta}(m)]_k - [f_{i\psi_n(\theta)}(m)]_k\right| \\
&\le \quad \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+n-1} + \sum_{k\ge l+n}\right) \left|[f_{i\theta}(m)]_{\sigma(k)} - [f_{i\psi_n(\theta)}(m)]_{\sigma(k)}\right| \\
&\le \quad \sum_{k>n} \theta_k m_i + 2\sum_{k>l} m_k.
\end{aligned}$$

The last sum being the tail of a convergent series we conclude making $l \to \infty$.

This concludes the proof of Lemma A.3. □

References

[1] D.J. Aldous, *Deterministic and Stochastic Models for Coalescence (Aggregation, Coagulation): A Review of the Mean-Field Theory of Probabilists,* Bernoulli, 5 (1999), 3–48.

[2] J. Berestycki. Exchangeable fragmentation-coalescence processes and their equilibrium measures. *Electron. J. Probab.*, 9(25):770–824, 2004.

[3] J. Bertoin. Homogeneous fragmentation processes. *Prob. Theory Relat. Fields*, 121:301–318, 2001.

[4] J. Bertoin. Self-similar fragmentations. *Ann. I. H. Poincaré*, 38:319–340, 2002.

[5] J. Bertoin. *Random Fragmentation and Coagulation Processes.* Cambridge Series on Statistical and Probability Mathematics. 2006.

[6] E. Cepeda. Well-posedness for a coagulation multiple-fragmentation equation. *Differential Integral Equations*, 127(1/2): 105–136, 2013.
[7] E. Cepeda and N. Fournier. Smoluchowski's equation: rate of convergence of the Marcus-Lushnikov process. *Stochastic Process. Appl.*, 121(6):1411–1444, 2011.
[8] R. L. Drake. A general mathematical survey of the coagulation equation, *Topics in current aerosol research (Part 2)*, 3, 201–376, 1972.
[9] A. Eibeck and W. Wagner. Approximative solution of the coagulation-fragmentation equation by stochastic particle systems, *Stochastic Anal. Appl.*, 18, 921–948, 2000.
[10] A. Eibeck and W. Wagner. Stochastic interacting particle systems and nonlinear kinetic equations, *Ann. Appl. Probab.*, 13(3):845–889, 2003.
[11] S. Evans and J. Pitman. Construction of Markovian coalescents, *Ann. Inst. Henri Poincaré*, 13:339–383, 1998.
[12] I. Jeon. Existence of Gelling Solutions for Coagulation-Fragmentation Equations, *Commun. Math. Phys.*, 194, 541–567, 2003.
[13] N. Fournier and J. S. Giet. On small particles in Coagulation-Fragmentation equations, *J. Stat. Phys.*, 111, (5/6), 1299–1329, 2003.
[14] N. Fournier. A distance for coagulation. *Markov Poricess. Related Fields*, 12(4):399–406, 2006.
[15] N. Fournier. On some stochastic coalescents. *Proba. Theory Related Fields*, 136(4):509–523, 2006.
[16] N. Fournier. Standard stochastic coalescence with sum kernels. *Electron. Comm. Probab.*, 11:141–148, 2006.
[17] N. Fournier and E. Löcherbach. Stochastic coalescence with homogeneous-like interaction rates. *Stoch. Proc. Appl.*, 119:45–73, 2009.
[18] F. Guiaş. A Monte Carlo approach to the Smoluchowski equations. *Monte Carlo Methods Appl.*, 3(4):313–326, 1997.
[19] B. Haas. Loss of mass in deterministic and random fragmentations. *Stochastic Process. Appl.*, 106(2):245–277, 2003.
[20] B. Haas. Asymptotic behavior of solutions of the fragmentation equation with shattering: An approach via self-similar Markov processes. *Ann. Appl. Probab.*, 20(2):382–429, 2010.
[21] V. Kolokoltsov. Hydrodynamic limit of coagulation-fragmentation type models of k-nary interacting particles. *J. Statist. Phys.*, 15(5-6):1621–1653, 2004.
[22] V. Kolokoltsov. Kinetic equations for the pure jump models of k-nary interacting particle systems. *Markov Processes Relat. Fields*, 12(1):95–138, 2006.
[23] V. Kolokoltsov. *Nonlinear Markov processes and kinetic equations*. Cambridge Tracts in Mathematics. Cambridge University Press, 2010.
[24] V. Kolokoltsov. *Markov processes, semigroups and generators*. de Gruyter Studies in Mathematics. Walter de Gruyter & Co, 2011.
[25] J. R. Norris. Smoluchowski's coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9(1):78–109, 1999.
[26] J. R. Norris. Cluster coagulation. *Communications in Mathematical Physics*, 209(2):407–435, 2000.

Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050. Université Paris-Est. 61, avenue du Général de Gaulle, 94010 Créteil Cédex
E-mail address: eduardo.cepeda@m4x.org