Stability and analytic expansions of local solutions of systems of quadratic BSDEs with applications to a price impact model

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Abstract

We obtain stability estimates and derive analytic expansions for local solutions of multidimensional quadratic backward stochastic differential equations. We apply these results to a financial model where the prices of risky assets are quoted by a representative dealer in such a way that it is optimal to meet an exogenous demand. We show that the prices are stable under the demand process and derive their analytic expansions for small risk aversion coefficients of the dealer.

Keywords: multidimensional quadratic BSDEs, stability of quadratic BSDEs, asymptotic behavior of quadratic BSDEs, liquidity, price impact.

AMS Subject Classification (2010): 60H10, 91B24, 91G80.

JEL Classification: D53, G12, C62.

1 Introduction

One-dimensional backward stochastic differential equations (BSDEs) with quadratic growth are well studied. Existence, uniqueness, and stability of their solutions for bounded terminal conditions have been established in the pioneering paper Kobylanski [17]. Alternative proofs have been proposed in Tevzadze [20] and Briand and Elie [1]. Generalizations to the unbounded case have been obtained in Briand and Hu [2, 3] among others.

The situation with systems of quadratic BSDEs is more cumbersome. Unless there is a special structure (as, for instance, in Darling [6], Tang [19], El-Karoui and Hamadène [8], Cheridito and Nam [4], and Hu and Tang [14]), they may fail to have a solution even with bounded terminal conditions. A counterexample can be found in Frei and dos Reis [10]. On a positive side, existence and local uniqueness of solutions have been obtained for sufficiently small terminal conditions, first, in Tevzadze [20] for the $\mathscr{L}_\infty$-norm and then in Frei [9] and Kramkov and Pulido [18] for the bounded mean oscillation (BMO) norm.

In this paper, we establish stability properties and derive analytic expansions of such local solutions. Our main results are stated in Theorems 2.1 and 3.2. In Theorem 2.1, we get stability estimates in $\mathscr{S}_p$ and BMO spaces with respect to the driver and the terminal condition. In Theorem 3.2 we obtain analytic expansions in BMO with respect to the terminal condition.

The coefficients of these power series can be calculated recursively up to an arbitrary order.

This work is motivated by our study in [18] of a price impact model from the market microstructure theory, which builds upon earlier works by Grossman and Miller [13], Garleanu, Pedersen, and Poteshman [11], and German [12]. In this model, a representative dealer provides liquidity for risky stocks and quotes prices in such a way that it is optimal to meet an exogenous demand for stocks. As we have proved in [18], the resulting stock prices can be characterized in terms of solutions to a system of quadratic BSDEs parametrized by the demand process.

It has been shown in [12] that under simple demands the stock prices exist, are unique and can be constructed explicitly by backward induction and martingale representation. For general (nonsimple) demands the situation is more involved. In this case, the existence and uniqueness of prices can be obtained only if the product of certain model parameters is sufficiently small as in (4.6) below. A natural question to ask is whether under such constraint the output stock prices are stable under demands and, in particular, whether they can be well approximated by the prices originated from simple demands. A positive answer is given in Theorem 4.3 and relies on the general stability estimates from Theorem 2.1.

As the dealer's risk aversion coefficient a approaches zero, the price impact effect vanishes and we arrive at a classical model of Mathematical Finance. In Theorem 4.5 we derive an analytic expansion of prices for sufficiently small values of a, thus getting a natural scale of price impact corrections. The leading term of these corrections has been obtained in [12] for simple demands.

Notations

For a matrix $A = (A^{ij})$ we denote its transpose by A^* and define its norm as

$$|A| \triangleq \sqrt{\operatorname{trace} AA^*} = \sqrt{\sum_{i,j} |A^{ij}|^2}.$$

We will work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the standard conditions of right-continuity and completeness; the initial σ-algebra $\mathscr{F}_0$ is trivial, $\mathscr{F} = \mathscr{F}_T$, and the maturity T is finite. The expectation is denoted as $\mathbb{E}[\cdot]$ and the conditional expectation with respect to $\mathscr{F}_t$ as $\mathbb{E}_t[\cdot]$.

For an n-dimensional integrable random variable ξ set

$$\begin{aligned}
\|\xi\|_{\mathscr{L}_p} &\triangleq (\mathbb{E}[|\xi|^p])^{1/p}, \quad p \geq 1, \\
\|\xi\|_{\mathscr{L}_\infty} &\triangleq \inf\{c \geq 0: \ |\xi(\omega)| \leq c, \quad \mathbb{P}[d\omega] - a.s.\}.
\end{aligned}$$

We shall use the following spaces of stochastic processes:

$\mathrm{BMO}(\mathbf{R}^n)$ is the Banach space of continuous n-dimensional martingales M with $M_0 = 0$ and the norm

$$\|M\|_{\mathrm{BMO}} \triangleq \sup_\tau \|\{\mathbb{E}_\tau[\langle M\rangle_T - \langle M\rangle_\tau]\}^{1/2}\|_{\mathscr{L}_\infty},$$

where the supremum is taken with respect to all stopping times τ and $\langle M\rangle$ is the quadratic variation of M.

$\mathscr{S}_{\mathrm{BMO}}(\mathbf{R}^n)$ is the Banach space of continuous n-dimensional semimartingales $X = X_0 + M + A$, where M is a continuous martingale and A is a process of finite variation, with the norm

$$\|X\|_{\mathscr{S}_{\mathrm{BMO}}} \triangleq |X_0| + \|M\|_{\mathrm{BMO}} + \sup_\tau \|\mathbb{E}_\tau[\int_\tau^T |dA|]\|_{\mathscr{L}_\infty}.$$

Here the supremum is taken over all stopping times τ and $\int |dA|$ is the total variation of A.

$\mathscr{S}_p(\mathbf{R}^n)$ for $p \geq 1$ is the Banach space of continuous n-dimensional semimartingales $X = X_0 + M + A$, where M is a continuous martingale and A is a process of finite variation, with the norm

$$\|X\|_{\mathscr{S}_p} \triangleq |X_0| + \|\langle M\rangle_T^{1/2}\|_{\mathscr{L}_p} + \|\int_0^T |dA|\|_{\mathscr{L}_p}.$$

$\mathscr{H}_0(\mathbf{R}^{n\times d})$ is the vector space of predictable processes ζ with values in $n \times d$-matrices such that $\int_0^T |\zeta_s|^2\, ds < \infty$. This is precisely the space of $n \times d$-dimensional integrands ζ for a d-dimensional Brownian motion B. We shall identify α and β in $\mathscr{H}_0(\mathbf{R}^{n\times d})$ if $\int_0^T |\alpha_s - \beta_s|^2\, ds = 0$ or, equivalently, if the stochastic integrals $\alpha \cdot B$ and $\beta \cdot B$ coincide.

$\mathscr{H}_p(\mathbf{R}^{n\times d})$ for $p \geq 1$ consists of $\zeta \in \mathscr{H}_0(\mathbf{R}^{n\times d})$ such that $\zeta \cdot B \in \mathscr{S}_p(\mathbf{R}^n)$ for a d-dimensional Brownian motion B. It is a Banach space under the norm:

$$\|\zeta\|_{\mathscr{H}_p} \triangleq \|\zeta \cdot B\|_{\mathscr{S}_p} = \left\{ \mathbb{E}\left[\left(\int_0^T |\zeta_s|^2 \, ds \right)^{p/2} \right] \right\}^{1/p}.$$

$\mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{n\times d})$ consists of $\zeta \in \mathscr{H}_0(\mathbf{R}^{n\times d})$ such that $\zeta \cdot B \in \mathrm{BMO}(\mathbf{R}^n)$ for a d-dimensional Brownian motion B. It is a Banach space under the norm:

$$\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} \triangleq \|\zeta \cdot B\|_{\mathrm{BMO}} = \sup_{\tau} \| \left\{ \mathbb{E}_\tau \left[\int_\tau^T |\zeta_s|^2 \, ds \right] \right\}^{1/2} \|_{\mathscr{L}_\infty}.$$

$\mathscr{H}_\infty(\mathbf{R}^n)$ is the Banach space of bounded n-dimensional predictable processes γ with the norm:

$$\|\gamma\|_{\mathscr{H}_\infty} \triangleq \inf \{ c \geq 0 : \ |\gamma_t(\omega)| \leq c, \quad dt \times \mathbb{P}[d\omega] - a.s. \}.$$

For an n-dimensional integrable random variable ξ with $\mathbb{E}[\xi] = 0$ set

$$\|\xi\|_{\mathscr{L}_{\mathrm{BMO}}} \triangleq \|(\mathbb{E}_t[\xi])_{t\in[0,T]}\|_{\mathrm{BMO}}.$$

2 Stability estimates

Hereafter, we shall assume that

(A1) There exists a d-dimensional Brownian motion B such that every local martingale M is a stochastic integral with respect to B:

$$M = M_0 + \zeta \cdot B.$$

Of course, this assumption holds if the filtration is generated by B.

Consider the n-dimensional BSDE:

$$Y_t = \Xi + \int_t^T f(s, \zeta_s) \, ds - \int_t^T \zeta \, dB, \quad t \in [0, T]. \tag{2.1}$$

Here Y is an n-dimensional semimartingale, ζ is a predictable process with values in the space of $n \times d$ matrices, and the terminal condition Ξ and the driver $f = f(t, z)$ satisfy the following assumptions:

(A2) Ξ is an integrable random variable with values in $\mathbf{R}^n$ such that the martingale

$$L_t \triangleq \mathbb{E}_t[\Xi] - \mathbb{E}[\Xi], \quad t \in [0,T],$$

belongs to BMO.

(A3) $t \mapsto f(t,z)$ is a predictable process with values in $\mathbf{R}^n$,

$$f(t,0) = 0, \tag{2.2}$$

and there is a constant $\Theta > 0$ such that

$$|f(t,u) - f(t,v)| \leq \Theta(|u-v|)(|u|+|v|), \tag{2.3}$$

for all $t \in [0,T]$ and $u, v \in \mathbf{R}^{n\times d}$.

Note that $f = f(t,z)$ has a quadratic growth in z. We shall discuss (A3) in Remark 2.3 at the end of this section.

Recall that there is a constant $\kappa = \kappa(n)$ such that, for every martingale $M \in \mathrm{BMO}(\mathbf{R}^n)$,

$$\frac{1}{\kappa}\|M\|_{\mathrm{BMO}} \leq \|M\|_{\mathrm{BMO}_1} \triangleq \sup_{\tau}\|\mathbb{E}_\tau[|M_T - M_\tau|]\|_{\mathscr{L}_\infty} \leq \|M\|_{\mathrm{BMO}}, \tag{2.4}$$

see [16], Corollary 2.1. Theorem A.1 in [18] shows that under (A1), (A2), and (A3), if

$$\|L\|_{\mathrm{BMO}} < \frac{1}{8\kappa\Theta}, \tag{2.5}$$

then there exists a unique solution (Y,ζ) to (2.1) such that

$$\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} \leq \frac{1}{4\kappa\Theta}.$$

The analogous local existence and uniqueness result was first shown in [20, Proposition 1] for small bounded terminal conditions and then in [9, Proposition 2.1] in the current BMO setting, but with different constants.

Theorem 2.1 below provides stability estimates for such *local* solution (Y,ζ) with respect to the terminal condition and the driver. Along with (2.1), we consider a similar n-dimensional BSDE

$$Y'_t = \Xi' + \int_t^T f'(s,\zeta'_s)\,ds - \int_t^T \zeta'\,dB, \quad t \in [0,T], \tag{2.6}$$

whose terminal condition Ξ' and the driver $f' = f'(t,z)$ satisfy the same conditions (A2) and (A3) as Ξ and f. We denote

$$L'_t \triangleq \mathbb{E}_t[\Xi'] - \mathbb{E}[\Xi'], \quad t \in [0,T],$$

and assume that there exists a nonnegative process $\delta = (\delta_t)$ such that

$$|f(t,z) - f'(t,z)| \leq \delta_t \, |z|^2 \, . \tag{2.7}$$

Theorem 2.1. *Assume that the BSDEs* (2.1) *and* (2.6) *satisfy (A1), (A2), (A3), and* (2.7) *and let* (Y,ζ) *and* (Y',ζ') *be their respective solutions. For* $p>1$ *there are positive constants* $c = c(n,p)$ *and* $C = C(n,p)$ *(depending only on* n *and* p*) such that if*

$$\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}} \leq \frac{c}{\Theta}, \tag{2.8}$$

then

$$\|\zeta' - \zeta\|_{\mathscr{H}_p} \leq C\left(\|L'_T - L_T\|_{\mathscr{L}_p} + \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{2p}}\right), \tag{2.9}$$

$$\|Y' - Y\|_{\mathscr{S}_p} \leq C\left(\|\Xi' - \Xi\|_{\mathscr{L}_p} + \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{2p}}\right). \tag{2.10}$$

Moreover, there exist constants $\widetilde{c} = \widetilde{c}(n)$ *and* $\widetilde{C} = \widetilde{C}(n)$ *(depending only on* n*) such that if*

$$\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}} \leq \frac{\widetilde{c}}{\Theta},$$

then

$$\|\zeta' - \zeta\|_{\mathscr{H}_{\mathrm{BMO}}} \leq \widetilde{C}\left(\|L' - L\|_{\mathrm{BMO}} + \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{\mathrm{BMO}}}\right), \tag{2.11}$$

$$\|Y' - Y\|_{\mathscr{S}_{\mathrm{BMO}}} \leq \widetilde{C}\left(|\mathbb{E}[\Xi' - \Xi]| + \|L' - L\|_{\mathrm{BMO}} + \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{\mathrm{BMO}}}\right). \tag{2.12}$$

Proof. To shorten the notations set $\Delta\zeta \triangleq \zeta' - \zeta$, etc. Define the martingale M and the process A of bounded variation as

$$M \triangleq \Delta\zeta \cdot B,$$
$$A_t \triangleq \int_0^t (f'(s,\zeta'_s) - f(s,\zeta_s))ds, \quad t \in [0,T].$$

We deduce that

$$M_T = \Delta L_T + A_T - \mathbb{E}[A_T],$$

which readily implies that

$$\|M_T\|_{\mathscr{L}_p} \leq \|\Delta L_T\|_{\mathscr{L}_p} + 2\|A_T\|_{\mathscr{L}_p}.$$

From Doob's and Burkholder-Davis-Gundy's (BDG) inequalities we deduce the existence of a constant $C_1 = C_1(n,p)$ such that

$$\|\Delta\zeta\|_{\mathscr{H}_p} \leq C_1\|M_T\|_{\mathscr{L}_p}.$$

To estimate $\|A_T\|_{\mathscr{L}_p}$ we use the potential Z associated with the variation of A:

$$\begin{aligned} Z_t &\triangleq \mathbb{E}_t\left[\int_t^T |dA|\right] = \mathbb{E}_t\left[\int_t^T |f'(s,\zeta'_s) - f(s,\zeta_s)|\,ds\right] \\ &\leq \mathbb{E}_t\left[\int_t^T |f'(s,\zeta'_s) - f'(s,\zeta_s)|\,ds\right] + \mathbb{E}_t\left[\int_t^T |f'(s,\zeta_s) - f(s,\zeta_s)|\,ds\right]. \end{aligned}$$

Denoting $Z^* \triangleq \sup_{t\in[0,T]}|Z_t|$ we have, by the Garsia-Neveu inequality,

$$\|A_T\|_{\mathscr{L}_p} \leq \|\int_0^T |dA|\|_{\mathscr{L}_p} \leq p\|Z^*\|_{\mathscr{L}_p}.$$

Take $1 < p' < p$ and denote $q' \triangleq p'/(p'-1)$. From (A3) and the Cauchy-Schwarz and Hölder inequalities we obtain

$$\begin{aligned} U_t &\triangleq \mathbb{E}_t\left[\int_t^T |f'(s,\zeta'_s) - f'(s,\zeta_s)|\,ds\right] \leq \Theta\mathbb{E}_t\left[\int_t^T (|\zeta_s| + |\zeta'_s|)\,|\Delta\zeta_s|\,ds\right] \\ &\leq \Theta\mathbb{E}_t\left[\left(\int_t^T (|\zeta_s| + |\zeta'_s|)^2 ds\right)^{1/2}\left(\int_t^T |\Delta\zeta_s|^2\,ds\right)^{1/2}\right] \\ &\leq \Theta\left(\mathbb{E}_t\left[\left(\int_t^T (|\zeta_s| + |\zeta'_s|)^2 ds\right)^{q'/2}\right]\right)^{1/q'}\left(\mathbb{E}_t\left[\left(\int_t^T |\Delta\zeta_s|^2\,ds\right)^{p'/2}\right]\right)^{1/p'}. \end{aligned}$$

From Doob's inequality, the BDG inequalities, and the equivalence of BMO_p-norms, see [16, Corollary 2.1, p. 28], we deduce the existence of a constant $C_2 = C_2(n,q')$ such that

$$\left(\mathbb{E}_t\left[\left(\int_t^T (|\zeta_s| + |\zeta'_s|)^2 ds\right)^{q'/2}\right]\right)^{1/q'} \leq C_2(\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}}).$$

Using the obvious estimate

$$\mathbb{E}_t\left[\left(\int_t^T |\Delta\zeta_s|^2\,ds\right)^{p'/2}\right] \le \mathbb{E}_t\left[\left(\int_0^T |\Delta\zeta_s|^2\,ds\right)^{p'/2}\right] \triangleq N_t,$$

and denoting $r \triangleq p/p' > 1$, we deduce from Doob's inequality the existence of a constant $C_3 = C_3(r)$ such that

$$\|N^*\|_{\mathscr{L}_r} \le C_3\|N_T\|_{\mathscr{L}_r} = C_3\|\Delta\zeta\|^{p'}_{\mathscr{H}_p}.$$

Combining the above estimates we obtain

$$\|U^*\|_{\mathscr{L}_p} \le C_4\Theta(\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}})\|\Delta\zeta\|_{\mathscr{H}_p},$$

where the constant $C_4 = C_2C_3^{1/p'}$ depends only on n and p.

From (2.7) we deduce

$$\begin{aligned}V_t &\triangleq \mathbb{E}_t\left[\int_t^T |f'(s,\zeta_s) - f(s,\zeta_s)|\,ds\right] \le \mathbb{E}_t\left[\int_t^T \delta_s\,|\zeta_s|^2\,ds\right]\\ &\le \mathbb{E}_t\left[\int_0^T \delta_s\,|\zeta_s|^2\,ds\right].\end{aligned}$$

Another application of Doob's inequality yields a constant $C_5 = C_5(p) > 1$ such that

$$\|V^*\|_{\mathscr{L}_p} \le C_5\|\int_0^T \delta_s\,|\zeta_s|^2\,ds\|_{\mathscr{L}_p}.$$

Defining the constants $c = c(n,p)$ and $C = C(n,p)$ as

$$c = \frac{1}{4pC_1C_4}, \quad C = 4pC_1C_5,$$

and assuming (2.8), we obtain

$$\begin{aligned}\|\Delta\zeta\|_{\mathscr{H}_p} &\le C_1p(\|\Delta L_T\|_{\mathscr{L}_p} + 2\|Z^*\|_{\mathscr{L}_p})\\ &\le C_1p(\|\Delta L_T\|_{\mathscr{L}_p} + 2\|U^*\|_{\mathscr{L}_p} + 2\|V^*\|_{\mathscr{L}_p})\\ &\le C_1p\Big(\|\Delta L_T\|_{\mathscr{L}_p} + 2C_4\Theta(\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}})\|\Delta\zeta\|_{\mathscr{H}_p}\\ &\qquad + 2C_5\|\int_0^T \delta_s\,|\zeta_s|^2\,ds\|_{\mathscr{L}_p}\Big)\\ &\le \frac{1}{2}\|\Delta\zeta\|_{\mathscr{H}_p} + \frac{C}{2}\Big(\|\Delta L_T\|_{\mathscr{L}_p} + \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{2p}}\Big),\end{aligned}$$

which implies (2.9).

The estimate (2.10) follows from (2.9) and estimates above for $\|\int_0^T |dA|\|_{\mathscr{L}_p}$ with appropriate $C = C(n,p)$ as soon as we write

$$\Delta Y_0 = \mathbb{E}[\Delta\Xi + A_T],$$
$$\Delta Y = \Delta Y_0 + M - A,$$

with M and A defined at the beginning of the proof.

Estimates (2.11) and (2.12) are more straightforward. Using the same processes M, A, U, and V we have that

$$M_t = \Delta L_t + \mathbb{E}_t[A_T] - \mathbb{E}[A_T]$$

and for a stopping time τ

$$\mathbb{E}_\tau[|A_T - \mathbb{E}_\tau[A_T]|] \leq 2\mathbb{E}_\tau\left[\int_\tau^T |f'(s,\zeta'_s) - f(s,\zeta_s)|\, ds\right] \leq 2(U_\tau + V_\tau).$$

By the Cauchy-Schwarz inequality we deduce from (A3) that

$$U_\tau = \mathbb{E}_\tau\left[\int_\tau^T |f'(s,\zeta'_s) - f'(s,\zeta_s)|\, ds\right] \leq \Theta\mathbb{E}_\tau\left[\int_\tau^T (|\zeta_s| + |\zeta'_s|)\,|\Delta\zeta_s|\, ds\right]$$
$$\leq \Theta(\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}})\|\Delta\zeta\|_{\mathscr{H}_{\mathrm{BMO}}}$$

and from (2.7) that

$$V_\tau = \mathbb{E}_\tau\left[\int_\tau^T |f'(s,\zeta_s) - f(s,\zeta_s)|\, ds\right] \leq \mathbb{E}_\tau\left[\int_\tau^T \delta_s\,|\zeta_s|^2\, ds\right]$$
$$\leq \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{\mathrm{BMO}}}.$$

Accounting for (2.4) we obtain that

$$\frac{1}{2\kappa}\|A_T - \mathbb{E}[A_T]\|_{\mathscr{L}_{\mathrm{BMO}}} \leq \Theta(\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\zeta'\|_{\mathscr{H}_{\mathrm{BMO}}})\|\Delta\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\sqrt{\delta}\zeta\|^2_{\mathscr{H}_{\mathrm{BMO}}}$$

and since

$$\|\Delta\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} = \|M\|_{\mathrm{BMO}} \leq \|\Delta L\|_{\mathrm{BMO}} + \|A_T - \mathbb{E}[A_T]\|_{\mathscr{L}_{\mathrm{BMO}}},$$

estimate (2.11) readily follows if we choose

$$\widetilde{c} = \frac{1}{4\kappa}, \quad \widetilde{C} = 4\kappa.$$

Finally, we deduce (2.12) (with appropriate $\widetilde{C} = \widetilde{C}(n)$) from the estimates for $\|M\|_{\mathrm{BMO}}$ and $\|A_T - \mathbb{E}[A_T]\|_{\mathscr{L}_{\mathrm{BMO}}}$ as soon as we write

$$\Delta Y = \mathbb{E}[\Delta\Xi] + M - (A - \mathbb{E}[A_T]).$$

□

Remark 2.2. Assume that BSDEs (2.1) and (2.6) satisfy (A1), (A2), (A3), and (2.7) and let (Y, ζ) and (Y', ζ') be their respective solutions. If we set

$$\begin{aligned} A_t &\triangleq \int_0^t (f'(s, \zeta_s) - f(s, \zeta_s))ds, \\ \gamma_t &\triangleq \frac{1}{|\zeta_t' - \zeta_t|}(f'(s, \zeta_s') - f'(s, \zeta_s))1_{\{|\zeta_t'-\zeta_t|>0\}}, \end{aligned}$$

then we can write

$$Y_t' - Y_t = \Xi' - \Xi + A_T - A_t + \int_t^T \gamma_s \, |\zeta_s' - \zeta_s| \, ds - \int_t^T (\zeta' - \zeta) \, dB.$$

This linearization argument allows us to deduce (2.10) from the stability estimate for Lipschitz BSDEs given in Theorem 2.2 of [7] if we observe that the local bound (2.8) implies the conditions of item (ii) of this theorem. We thank a referee for pointing out this connection.

Remark 2.3. Assumption (A3) on the driver $f = f(t, z; \omega)$ allows us to obtain existence of solutions to (2.1) under the condition (2.5) which does *not* involve the maturity T. Such T-independence property is not valid without (2.2) (this fact is easy to see) or if (2.3) is replaced with a weaker condition:

$$|f(t, u) - f(t, v)| \leq \Theta(|u - v|)(1 + |u| + |v|). \tag{2.13}$$

We construct below a related counterexample using the same idea as in [10].

We take a one-dimensional Brownian motion B and define the stopping time τ and the bounded martingale M as

$$\begin{aligned} \tau &\triangleq \min\left\{ t \in (0, 1) : \left| \int_0^t \frac{1}{1-s} dB_s \right| = \frac{\pi}{2} \right\}, \\ M_t &\triangleq \int_0^{t\wedge\tau} \frac{1}{1-s} dB_s, \quad t \geq 0. \end{aligned}$$

We deduce that $\tau \in (0,1)$ and $|M_\tau| = \pi/2$. Standard arguments (see [16, Lemma 1.3]) show that

$$\mathbb{E}\left[\exp\left(\frac{a^2}{2}\langle M\rangle_\tau\right)\right] = \begin{cases} 1/\cos(a\pi/2), & 0 \le a < 1, \\ \infty, & a \ge 1. \end{cases} \tag{2.14}$$

We consider the three-dimensional quadratic BSDE, parametrized by $a > 0$ and $T > 1$:

$$\begin{aligned} Y_t^1 &= \frac{a}{\sqrt{T}} B_T - \int_t^T \zeta^1 dB, \\ Y_t^2 &= \int_t^T M_\tau 1_{\{s\ge 1\}} \zeta_s^1 ds - \int_t^T \zeta^2 dB, \\ Y_t^3 &= \frac{1}{2}\int_t^T \left((\zeta_s^2)^2 + (\zeta_s^3)^2\right) ds - \int_t^T \zeta^3 dB. \end{aligned}$$

We notice that the driver in the second equation depends on ζ^1 linearly, that (2.13) holds with $\Theta = \frac{\pi}{2}$ and that

$$\|\Xi\|_{\mathscr{L}_{\mathrm{BMO}}} = \|\big(\frac{a}{\sqrt{T}} B_T, 0, 0\big)\|_{\mathscr{L}_{\mathrm{BMO}}} = a. \tag{2.15}$$

If $\int \zeta^1 dB$ and $\int \zeta^2 dB$ are true martingales on $[0,T]$, then the first two equations yield that

$$\begin{aligned} \zeta^1 &= \frac{a}{\sqrt{T}}, \\ \int_0^T \zeta^2 dB &= \frac{a(T-1)}{\sqrt{T}} M_\tau. \end{aligned}$$

The third equation implies that

$$\begin{aligned} \exp\left(Y_0^3 + \int_0^T \zeta^3 dB - \frac{1}{2}\int_0^T (\zeta_s^3)^2 ds\right) &= \exp\left(\frac{1}{2}\int_0^T (\zeta_s^2)^2 ds\right) \\ &= \exp\left(\frac{a^2(T-1)^2}{2T}\langle M\rangle_\tau\right). \end{aligned}$$

In view of (2.14) and (2.15), the existence of ζ^3 is then equivalent to

$$\frac{a(T-1)}{\sqrt{T}} = \frac{\|\Xi\|_{\mathscr{L}_{\mathrm{BMO}}}(T-1)}{\sqrt{T}} < 1.$$

Thus, the solvability of our BSDE depends on *both* $\|\Xi\|_{\mathscr{L}_{\mathrm{BMO}}}$ and T.

3 Analytic expansion for purely quadratic BSDE

Consider an n-dimensional BSDE

$$Y_t = a\Xi + \int_t^T f(s, \zeta_s)\, ds - \int_t^T \zeta\, dB, \quad t \in [0, T], \tag{3.1}$$

where the terminal condition depends on a parameter $a \in \mathbf{R}$. If Ξ and f satisfy (A2) and (A3) and $|a| < \rho$, where

$$\rho \triangleq \frac{1}{8\kappa\Theta\|L\|_{\mathrm{BMO}}}, \tag{3.2}$$

then, by [18, Theorem A.1], there is only one solution $(Y(a), \zeta(a))$ such that

$$\|\zeta(a)\|_{\mathscr{H}_{\mathrm{BMO}}} \leq \frac{1}{4\kappa\Theta}, \tag{3.3}$$

and for this solution we have an estimate:

$$\|\zeta(a)\|_{\mathscr{H}_{\mathrm{BMO}}} \leq 2|a|\|L\|_{\mathrm{BMO}}.$$

In particular, $\zeta(a)$ converges to 0 in $\mathscr{H}_{\mathrm{BMO}}$ as a approaches 0.

In Theorem 3.2 below we obtain an analytic expansion for $\zeta(a)$ in the neighborhood of $a = 0$ under the additional assumption that the driver $f = f(t, z)$ is purely quadratic in z:

$$f(t, z) = \sum_{ijkl} z^{ij} z^{kl} \alpha_t^{ijkl}$$

for some $\mathbf{R}^n$-valued predictable bounded processes (α^{ijkl}) or, equivalently,

(A4) $f(t, z) = \widetilde{f}(t, z, z)$, where for all $u, v \in \mathbf{R}^{n\times d}$ the map $t \mapsto \widetilde{f}(t, u, v)$ is an $\mathbf{R}^n$-valued predictable process and for every $t \in [0, T]$ the map $(u, v) \mapsto \widetilde{f}(t, u, v)$ is symmetric, bilinear on $\mathbf{R}^{n\times d} \times \mathbf{R}^{n\times d}$, and bounded by a constant $\Theta > 0$. In other words,

$$\widetilde{f}(t, \lambda u, v + w) = \widetilde{f}(t, \lambda(v + w), u) = \lambda(\widetilde{f}(t, u, v) + \widetilde{f}(t, u, w)),$$
$$\left|\widetilde{f}(t, u, v)\right| \leq \Theta\, |u|\, |v|\,,$$

for all $t \in [0, T]$, $\lambda \in \mathbf{R}$, and $u, v, w \in \mathbf{R}^{n\times d}$.

Notice that (A4) implies (A3):

$$
\begin{aligned}
|f(t,u)-f(t,v)| &= \left|\widetilde{f}(t,u,u)-\widetilde{f}(t,v,v)\right| = \left|\widetilde{f}(t,u-v,u+v)\right| \\
&\le \Theta(|u-v|)(|u+v|) \le \Theta(|u-v|)(|u|+|v|).
\end{aligned}
$$

Condition (A4) naturally arises in financial applications dealing with exponential utilities, see e.g., [10], [18], and [15].

To state Theorem 3.2 we need the following technical result.

Lemma 3.1. *Assume (A1) and (A4). For $\mu, \nu \in \mathscr{H}_{\mathrm{BMO}}$ there is a unique $\zeta \in \mathscr{H}_{\mathrm{BMO}}$ such that*

$$
(\zeta \cdot B)_t = \mathbb{E}_t\left[\int_0^T \widetilde{f}(s,\mu_s,\nu_s)\,ds\right] - \mathbb{E}\left[\int_0^T \widetilde{f}(s,\mu_s,\nu_s)\,ds\right]. \tag{3.4}
$$

Moreover,

$$
\|\zeta\|_{\mathscr{H}_{\mathrm{BMO}}} \le 2\kappa\Theta\|\mu\|_{\mathscr{H}_{\mathrm{BMO}}}\|\nu\|_{\mathscr{H}_{\mathrm{BMO}}},
$$

where the positive constants κ and Θ are defined in (2.4) *and (A4).*

Proof. Define the martingale

$$
M_t \triangleq \mathbb{E}_t\left[\int_0^T \widetilde{f}(s,\mu_s,\nu_s)\,ds\right] - \mathbb{E}\left[\int_0^T \widetilde{f}(s,\mu_s,\nu_s)\,ds\right].
$$

For a stopping time τ we deduce from (A4) and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\mathbb{E}_\tau[|M_T-M_\tau|] &= \mathbb{E}_\tau\left[\left|\int_\tau^T \widetilde{f}(s,\mu_s,\nu_s)ds - \mathbb{E}_\tau[\int_\tau^T \widetilde{f}(s,\mu_s,\nu_s)ds]\right|\right] \\
&\le 2\mathbb{E}_\tau\left[\int_\tau^T \left|\widetilde{f}(s,\mu_s,\nu_s)\right| ds\right] \le 2\Theta\mathbb{E}_\tau\left[\int_\tau^T |\mu_s|\,|\nu_s|\,ds\right] \\
&\le 2\Theta\left(\mathbb{E}_\tau\left[\int_\tau^T |\mu_s|^2\,ds\right]\right)^{1/2}\left(\mathbb{E}_\tau\left[\int_\tau^T |\nu_s|^2\,ds\right]\right)^{1/2}.
\end{aligned}
$$

Hence by (2.4),

$$
\|M\|_{\mathrm{BMO}} \le \kappa\sup_\tau\|\mathbb{E}_\tau[|M_T-M_\tau|]\|_{\mathscr{L}_\infty} \le 2\kappa\Theta\|\mu\|_{\mathscr{H}_{\mathrm{BMO}}}\|\nu\|_{\mathscr{H}_{\mathrm{BMO}}},
$$

and the result follows because, in view of (A1), M admits an integral representation $\zeta \cdot B$ for some unique $\zeta \in \mathscr{H}_{\mathrm{BMO}}$. □

Lemma 3.1 allows us to define the map

$$\widetilde{F}: \mathscr{H}_{\mathrm{BMO}} \times \mathscr{H}_{\mathrm{BMO}} \to \mathscr{H}_{\mathrm{BMO}}$$

such that $\zeta = \widetilde{F}(\mu, \nu)$ is given by (3.4). This map is bilinear (since $\widetilde{f}(t, \cdot, \cdot)$ is bilinear) and is bounded by $2\kappa\Theta$.

Recall the constant ρ from (3.2) and the BMO martingale L from (A2).

Theorem 3.2. *Assume (A1), (A2), and (A4). Then for $|a| < \rho$ there is only one solution $(Y(a), \zeta(a))$ to (3.1) satisfying (3.3). It is given by the power series*

$$Y(a) = \sum_{k=1}^{\infty} Y^{(k)} a^k \quad \text{and} \quad \zeta(a) = \sum_{k=1}^{\infty} \zeta^{(k)} a^k$$

convergent for $|a| < \rho$ in $\mathscr{S}_{\mathrm{BMO}}$ and $\mathscr{H}_{\mathrm{BMO}}$, respectively, with the coefficients

$$Y_t^{(1)} = \mathbb{E}_t[\Xi], \quad t \in [0, T], \tag{3.5}$$

$$\zeta^{(1)} \cdot B = L, \tag{3.6}$$

and, for $k \geq 2$,

$$\zeta^{(k)} = \sum_{l+m=k} \widetilde{F}(\zeta^{(l)}, \zeta^{(m)}), \tag{3.7}$$

$$Y_t^{(k)} = \sum_{l+m=k} \mathbb{E}_t\Big[\int_t^T \widetilde{f}(s, \zeta_s^{(l)}, \zeta_s^{(m)}) ds\Big], \quad t \in [0, T], \tag{3.8}$$

where we sum with respect to all pairs of positive integers (l, m) which add to k.

Remark 3.3. Based on (3.7)–(3.8) and the definition of the bilinear map $\widetilde{F}$, the power series expansion of $(Y(a), \zeta(a))$ is constructed using a martingale representation approach. We refer the reader to [5] and the references therein for numerical algorithms to approximate martingale representation terms of functionals of Brownian motion.

The proof of Theorem 3.2 relies on some lemmas.

Lemma 3.4. *Assume (A1), (A2), and (A4). Let $(\zeta^{(k)})_{k \geq 1}$ be given by (3.6)–(3.7). Then $(\zeta^{(k)})_{k \geq 1} \subset \mathscr{H}_{\mathrm{BMO}}$ and*

$$\sum_{k=1}^{\infty} \|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^k \leq \frac{1}{4\kappa\Theta}. \tag{3.9}$$

Proof. The claim that each $\zeta^{(k)}$ belongs to $\mathscr{H}_{\mathrm{BMO}}$ follows from its construction and Lemma 3.1. For $n \geq 1$ define the partial sums

$$s_n \triangleq \sum_{k=1}^{n} \|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^k.$$

Using the boundedness of the bilinear map $\widetilde{F}$ by $2\kappa\Theta$ we obtain

$$\begin{aligned} s_n - s_1 &= \sum_{k=2}^{n} \|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^k \leq \sum_{k=2}^{n} \left(\sum_{l+m=k} \|\widetilde{F}(\zeta^{(l)}, \zeta^{(m)})\|_{\mathscr{H}_{\mathrm{BMO}}} \right) \rho^k \\ &\leq 2\kappa\Theta \sum_{k=2}^{n} \sum_{l+m=k} (\|\zeta^{(l)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^l)(\|\zeta^{(m)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^m) \\ &\leq 2\kappa\Theta \sum_{l,m=1}^{n-1} (\|\zeta^{(l)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^l)(\|\zeta^{(m)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^m) \\ &= 2\kappa\Theta \left(\sum_{k=1}^{n-1} \|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^k \right)^2 = 2\kappa\Theta (s_{n-1})^2. \end{aligned}$$

To verify (3.9) we use an induction argument. For $n = 1$ we have

$$s_1 = \|\zeta^{(1)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho = \rho \|L\|_{\mathrm{BMO}} = \frac{1}{8\kappa\Theta}.$$

If now $s_{n-1} \leq 1/(4\kappa\Theta)$, then

$$s_n \leq s_1 + 2\kappa\Theta(s_{n-1})^2 \leq \frac{1}{8\kappa\Theta} + 2\kappa\Theta \left(\frac{1}{4\kappa\Theta} \right)^2 = \frac{1}{4\kappa\Theta}$$

and (3.9) follows. □

Lemma 3.5. *Assume (A1) and (A4). For $\mu, \nu \in \mathscr{H}_{\mathrm{BMO}}$ the process*

$$X_t \triangleq \mathbb{E}_t \left[\int_t^T \widetilde{f}(s, \mu_s, \nu_s)\, ds \right], \quad t \in [0, T],$$

belongs to $\mathscr{S}_{\mathrm{BMO}}$ and

$$\|X\|_{\mathscr{S}_{\mathrm{BMO}}} \leq 2(1+\kappa)\Theta \|\mu\|_{\mathscr{H}_{\mathrm{BMO}}} \|\nu\|_{\mathscr{H}_{\mathrm{BMO}}}.$$

Proof. The canonical decomposition of the semimartingale X has the form

$$X = X_0 + M - A,$$

where

$$\begin{aligned} A_t &= \int_0^t \widetilde{f}(s, \mu_s, \nu_s)\, ds, \\ X_0 &= \mathbb{E}[A_T], \\ M_t &= \mathbb{E}_t[A_T] - \mathbb{E}[A_T]. \end{aligned}$$

By Lemma 3.1 we have

$$\|M\|_{\mathrm{BMO}} \le 2\kappa\Theta \|\mu\|_{\mathscr{H}_{\mathrm{BMO}}} \|\nu\|_{\mathscr{H}_{\mathrm{BMO}}}.$$

As in the proof of Lemma 3.1 we deduce that for any stopping time τ

$$\mathbb{E}_\tau[\int_\tau^T |dA|] = \mathbb{E}_\tau[\int_\tau^T \left|\widetilde{f}(s, \mu_s, \nu_s)\right| ds] \le \Theta \|\mu\|_{\mathscr{H}_{\mathrm{BMO}}} \|\nu\|_{\mathscr{H}_{\mathrm{BMO}}}$$

and the result readily follows. □

Proof of Theorem 3.2. Take $a \in \mathbf{R}$ such that $|a| < \rho$. Recall that (A4) implies (A3). Theorem A.1 in [18] then implies the existence and uniqueness of the solution $\zeta(a)$ satisfying (3.3). To show that

$$\zeta(a) = \beta \triangleq \sum_{k=1}^{\infty} \zeta^{(k)} a^k, \tag{3.10}$$

we need to verify that β is a fixed point of the map $F : \mathscr{H}_{\mathrm{BMO}} \to \mathscr{H}_{\mathrm{BMO}}$ given by

$$F(\zeta) \triangleq a\zeta^{(1)} + \widetilde{F}(\zeta, \zeta).$$

For $n \ge 1$ define the partial sums:

$$\beta_n \triangleq \sum_{k=1}^{n} \zeta^{(k)} a^k.$$

In view of Lemma 3.4, the processes β and β_n belong to $\mathscr{H}_{\mathrm{BMO}}$ and

$$\|\beta - \beta_n\|_{\mathscr{H}_{\mathrm{BMO}}} \le \sum_{k=n+1}^{\infty} \|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}} \rho^k \to 0, \quad n \to \infty.$$

The bilinearity of $\widetilde{F}$ and Lemma 3.1 then yield

$$\begin{aligned}
\|F(\beta) - F(\beta_n)\|_{\mathscr{H}_{\mathrm{BMO}}} &= \|\widetilde{F}(\beta,\beta) - \widetilde{F}(\beta_n,\beta_n)\|_{\mathscr{H}_{\mathrm{BMO}}} \\
&= \|\widetilde{F}(\beta-\beta_n, \beta+\beta_n)\|_{\mathscr{H}_{\mathrm{BMO}}} \\
&\leq 2\kappa\Theta\|\beta-\beta_n\|_{\mathscr{H}_{\mathrm{BMO}}}(\|\beta\|_{\mathscr{H}_{\mathrm{BMO}}} + \|\beta_n\|_{\mathscr{H}_{\mathrm{BMO}}}) \to 0, \quad n \to \infty,
\end{aligned}$$

and to conclude the proof of (3.10) we only have to show that

$$\|F(\beta_n) - \beta_n\|_{\mathscr{H}_{\mathrm{BMO}}} \to 0, \quad n \to \infty. \tag{3.11}$$

From the bilinearity of $\widetilde{F}$ and the construction of $(\zeta^{(k)})$ we deduce that

$$\begin{aligned}
F(\beta_n) - \beta_n &= \widetilde{F}(\sum_{k=1}^{n}\zeta^{(k)}a^k, \sum_{k=1}^{n}\zeta^{(k)}a^k) - \sum_{k=2}^{n}\zeta^{(k)}a^k \\
&= \sum_{l,m=1}^{n}\widetilde{F}(\zeta^{(l)},\zeta^{(m)})a^{l+m} - \sum_{k=2}^{n}\left(\sum_{l+m=k}\widetilde{F}(\zeta^{(l)},\zeta^{(m)})\right)a^k \\
&= \sum_{\substack{1\leq l,m\leq n \\ l+m>n}}\widetilde{F}(\zeta^{(l)},\zeta^{(m)})a^{l+m}.
\end{aligned}$$

Using the boundedness of the bilinear map $\widetilde{F}$ by $2\kappa\Theta$ we obtain

$$\begin{aligned}
\|F(\beta_n) - \beta_n\|_{\mathscr{H}_{\mathrm{BMO}}} &\leq 2\kappa\Theta\sum_{l+m>n}\|\zeta^{(l)}\|_{\mathscr{H}_{\mathrm{BMO}}}\|\zeta^{(m)}\|_{\mathscr{H}_{\mathrm{BMO}}}\rho^{l+m} \\
&\leq 2\kappa\Theta\left(\left(\sum_{k=1}^{\infty}\|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}}\rho^k\right)^2 - \left(\sum_{k=1}^{[n/2]}\|\zeta^{(k)}\|_{\mathscr{H}_{\mathrm{BMO}}}\rho^k\right)^2\right)
\end{aligned}$$

and (3.11) follows from Lemma 3.4.

The power series representation for $Y(a)$ and formulas (3.5) and (3.8) for its coefficients readily follow from the expansion for $\zeta(a)$ as soon as we write $Y(a)$ as

$$Y_t(a) = a\mathbb{E}_t[\Xi] + \mathbb{E}_t[\int_t^T \widetilde{f}(s,\zeta_s(a),\zeta_s(a))ds], \quad t \in [0,T],$$

and use the bilinearity of $\widetilde{f}$ and Lemma 3.5. □

4 Applications to a price impact model

We consider the financial model of price impact studied in Garleanu, Pedersen, and Poteshman [11], German [12], and Kramkov and Pulido [18]. There is a representative dealer whose preferences regarding terminal wealth are modeled by the exponential utility

$$U(x) = -\frac{1}{a}e^{-ax}, \quad x \in \mathbf{R}.$$

The risk aversion coefficient $a > 0$ defines the strength of the price impact effect. In particular, as $a \downarrow 0$ we are getting the classical impact-free model of mathematical finance; see section 4.2.

The financial market consists of a bank account and n stocks. The bank account pays zero interest rate . The stocks pay dividends $\Psi = (\Psi^i)_{i=1,\dots,n}$ at maturity T; each Ψ^i is a random variable. While the terminal stocks' prices S_T are always given by Ψ, their intermediate values S_t on $[0, T)$ are affected by an exogenous demand process γ through the following equilibrium mechanism.

Definition 4.1. A predictable process γ with values in $\mathbf{R}^n$ is called a *demand*. The demand γ is *viable* if there is an n-dimensional semimartingale of *stock prices* S with terminal value $S_T = \Psi$ such that the *pricing* probability measure $\mathbb{Q}$ is well-defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{U'(\int_0^T \gamma dS)}{\mathbb{E}[U'(\int_0^T \gamma dS)]} = \frac{e^{-a\int_0^T \gamma dS}}{\mathbb{E}[e^{-a\int_0^T \gamma dS}]}$$

and S and the stochastic integral $\gamma \cdot S$ are uniformly integrable martingales under $\mathbb{Q}$.

Lemma 2.2 in [18] clarifies the economic meaning of Definition 4.1. It shows that a demand γ is viable if and only if it defines the optimal number of stocks for the dealer trading at stock prices $S = S(\gamma)$.

Under (A1), for a viable demand γ accompanied by stocks' prices S and the pricing measure $\mathbb{Q}$ there are unique processes $\alpha \in \mathscr{H}_0(\mathbf{R}^d)$ and $\sigma \in \mathscr{H}_0(\mathbf{R}^{n\times d})$, called, respectively, the *market price of risk* and the *volatility*, such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \alpha\, dB - \frac{1}{2}\int_0^T |\alpha_t|^2 dt},$$

$$S_t = S_0 + \int_0^t \sigma_s \alpha_s ds + \int_0^t \sigma dB.$$

Theorem 3.1 in [18] characterizes S, α, and σ in terms of solutions to a system of quadratic BSDEs. More precisely, it states that a demand γ is viable and is accompanied by the stock prices S if and only if there are a one-dimensional semimartingale R and predictable processes $\eta \in \mathscr{H}_0(\mathbf{R}^d)$, and $\theta \in \mathscr{H}_0(\mathbf{R}^{n\times d})$, such that, for every $t \in [0,T]$,

$$aR_t = \frac{1}{2}\int_t^T (|\theta_s^* \gamma_s|^2 - |\eta_s|^2) ds - \int_t^T \eta dB, \tag{4.1}$$

$$aS_t = a\Psi - \int_t^T \theta_s(\eta_s + \theta_s^*\gamma_s) ds - \int_t^T \theta dB, \tag{4.2}$$

and such that the stochastic exponential $Z \triangleq \mathscr{E}(-(\eta + \theta^*\gamma)\cdot B)$ and the processes ZS and $Z(\gamma \cdot S)$ are uniformly integrable martingales.

In this case, Z is the density process of the pricing measure $\mathbb{Q}$, and the market price of risk α and the volatility σ are given by

$$\alpha = \eta + \theta^*\gamma, \tag{4.3}$$

$$\sigma = \theta/a. \tag{4.4}$$

The value of the auxiliary process R at time t can be written as

$$R_t \triangleq U^{-1}\left(\mathbb{E}_t\left[U\left(\int_t^T \gamma dS\right)\right]\right) = -\frac{1}{a}\log\left(\mathbb{E}_t\left[e^{-a\int_t^T \gamma dS}\right]\right),$$

and thus represents the dealer's *certainty equivalent value* of the *remaining gain* $\int_t^T \gamma dS$.

Remark 4.2. From Definition 4.1 we deduce that the dependence of stocks' prices $S = S(\gamma, a, \Psi)$ on the viable demand γ, on the risk-aversion coefficient a, and on the dividend Ψ has the following homogeneity properties: for $b > 0$,

$$S(b\gamma, a, \Psi) = S(\gamma, ba, \Psi) = \frac{1}{b}S(\gamma, a, b\Psi).$$

This yields similar properties of the market prices of risk $\alpha = \alpha(\gamma, a, \Psi)$ and of the volatilities $\sigma = \sigma(\gamma, a, \Psi)$:

$$\begin{aligned} \alpha(b\gamma, a, \Psi) &= \alpha(\gamma, ba, \Psi) = \alpha(\gamma, a, b\Psi), \\ \sigma(b\gamma, a, \Psi) &= \sigma(\gamma, ba, \Psi) = \frac{1}{b}\sigma(\gamma, a, b\Psi). \end{aligned} \tag{4.5}$$

4.1 Stability with respect to demand

A demand γ is called *simple* if it has the form:

$$\gamma = \sum_{i=0}^{m-1} \theta_i 1_{(\tau_i, \tau_{i+1}]},$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_m = T$ are stopping times and θ_i is a $\mathscr{F}_{\tau_i}$-measurable random variable with values in $\mathbf{R}^n$, $i = 0, \ldots, m-1$. Theorem 1 in [12] shows that every bounded simple demand γ is viable provided that the dividends $\Psi = (\Psi^i)$ have all exponential moments. Moreover, in this case, the price process $S = S(\gamma)$ is unique and is constructed explicitly by backward induction.

For general (nonsimple) demands the situation is more involved. As [18, Proposition 4.3] shows, even for bounded dividends Ψ and demands γ, either existence or uniqueness of prices $S = S(\gamma)$ may fail. On a positive side, by [18, Theorem 4.1], there is a constant $c = c(n) > 0$ (dependent only on the number of stocks n) such that if

$$a \|\gamma\|_{\mathscr{H}_\infty} \|\Psi - \mathbb{E}[\Psi]\|_{\mathscr{L}_{\mathrm{BMO}}} \leq c, \tag{4.6}$$

then the prices $S = S(\gamma)$ exist and are unique.

The following theorem shows that under (4.6) the prices $S = S(\gamma)$ are stable under small changes in the demand γ. In particular, they can be well-approximated by the prices originated from simple demands.

Theorem 4.3. *Assume (A1) and let $p > 1$. There is a constant $c = c(n, p) > 0$ such that if $(\gamma^m)_{m \geq 1}$ and γ are elements of $\mathscr{H}_\infty(\mathbf{R}^n)$ such that*

$$a \|\gamma^m\|_{\mathscr{H}_\infty} \|\Psi - \mathbb{E}[\Psi]\|_{\mathscr{L}_{\mathrm{BMO}}} \leq c, \quad m \geq 1, \tag{4.7}$$

and

$$\mathbb{E}\left[\int_0^T |\gamma_t^m - \gamma_t| \, dt\right] \to 0, \quad m \to \infty,$$

then $(\gamma^m)_{m\geq 1}$ *and* γ *are viable demands and the corresponding stock prices* $(S^m)_{m\geq 1}$ *and* S*, volatilities* $(\sigma^m)_{m\geq 1}$ *and* σ*, and the market prices of risk* $(\alpha^m)_{m\geq 1}$ *and* α *converge as*

$$\|S^m - S\|_{\mathscr{S}_p} + \|\sigma^m - \sigma\|_{\mathscr{H}_p} + \|\alpha^m - \alpha\|_{\mathscr{H}_p} \to 0, \quad m \to \infty. \tag{4.8}$$

Proof. Observe that the self-similarity relations (4.5) for the market prices of risk and volatilities allow us to assume that

$$a = 1 \geq \|\gamma^m\|_{\mathscr{H}_\infty}, \quad m \geq 1.$$

Clearly, γ satisfies (4.6) with the same constant c as in (4.7). By [18, Theorem 4.1], we can choose $c = c(n)$ so that the demands (γ^m) and γ are viable and are accompanied by unique stock prices. Using Theorem 2.1 and the BSDE characterizations (4.1)–(4.4) we can also choose $c = c(n, p)$ so that

$$\|S^m - S\|_{\mathscr{S}_p} + \|\sigma^m - \sigma\|_{\mathscr{H}_p} + \|\alpha^m - \alpha\|_{\mathscr{H}_p} \leq C \| \int_0^T |\gamma^m_t - \gamma_t| \left(|\alpha_t|^2 + |\sigma_t|^2\right) dt \|_{\mathscr{L}_p}$$

for some $C = C(n, p)$. This yields (4.8) by the dominated convergence theorem. □

Remark 4.4. Using Theorem 2.1 we can prove an analogous convergence result of prices (S^m) and their local characteristics (α^m, σ^m) in BMO provided that $\|\gamma^m - \gamma\|_{\mathscr{H}_\infty} \to 0$. It is important to highlight, however, that simple demands are not dense in the space of bounded demands with respect to the norm $\|\cdot\|_{\mathscr{H}_\infty}$.

4.2 Asymptotic expansion for small risk-aversion

As the risk aversion coefficient a approaches zero, the price impact effect vanishes and we obtain a classical model of mathematical finance. Theorem 4.5 below provides analytic expansions of volatilities and market prices of risk in the neighborhood of $a = 0$. The terms in these expansions are computed recursively, by martingale representation and, thus, are quite explicit.

We write $z \in \mathbf{R}^{(n+1)\times m}$ as $z = (z_1, z_2)$ with $z_1 \in \mathbf{R}^m$ and $z_2 \in \mathbf{R}^{n\times m}$, the decomposition of $(n+1) \times m$-dimensional matrix on its first and subsequent rows; hereafter $m = 1$ or d, the dimension of underlying Brownian motion from (A1).

For a vector $w \in \mathbf{R}^n$ consider the bilinear map:

$$g(\cdot; w) = (g_1, g_2)(\cdot; w): \ \mathbf{R}^{(n+1)\times d} \times \mathbf{R}^{(n+1)\times d} \to \mathbf{R}^{n+1}$$

defined for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ from $\mathbf{R}^{(n+1)\times d}$ as

$$\begin{aligned} g_1(u, v; w) &\triangleq \frac{1}{2}(\langle u_2^* w, v_2^* w\rangle - \langle u_1, v_1\rangle) \\ g_2(u, v; w) &\triangleq -\frac{1}{2}(u_2 v_1 + v_2 u_1 + (u_2 v_2^* + v_2 u_2^*) w), \end{aligned}$$

where $\langle x, y\rangle$ denotes the scalar product of $x, y \in \mathbf{R}^m$.

Take $\gamma \in \mathscr{H}_\infty(\mathbf{R}^n)$. Lemma 3.1 shows that for μ and ν in $\mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d})$ there is a unique $\zeta \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d})$ such that

$$\int_0^t \zeta dB = \mathbb{E}_t[\int_0^T g(\mu_s, \nu_s; \gamma_s) ds] - \mathbb{E}[\int_0^T g(\mu_s, \nu_s; \gamma_s) ds]. \tag{4.9}$$

Hence, we can define a bilinear map

$$G(\cdot, \cdot; \gamma): \ \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d}) \times \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d}) \to \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d})$$

with $\zeta = G(\mu, \nu; \gamma)$ given by (4.9).

Denote also by $S(0)$ and $\sigma(0)$ the unperturbed stocks' prices and volatilities corresponding to the case $\gamma = 0$:

$$S_t(0) = \mathbb{E}_t[\Psi] = S_0(0) + \int_0^t \sigma(0) dB, \quad t \in [0, T].$$

Theorem 4.5. *Assume (A1) and that*

$$0 < \|\Psi - \mathbb{E}[\Psi]\|_{\mathscr{L}_{\mathrm{BMO}}} < \infty.$$

There is a constant $c = c(n) > 0$ such that if $\gamma \in \mathscr{H}_\infty(\mathbf{R}^n)$, $\gamma \neq 0$, and the risk-aversion satisfies

$$0 < a < \rho \triangleq \frac{c}{\|\gamma\|_{\mathscr{H}_\infty} \|\Psi - \mathbb{E}[\Psi]\|_{\mathscr{L}_{\mathrm{BMO}}}}, \tag{4.10}$$

then γ is a viable demand. The price $S(\gamma; a)$ is unique and admits the power series expansion in $\mathscr{S}_{\mathrm{BMO}}$:

$$S(\gamma; a) = S(0) + \sum_{k=1}^\infty S^{(k)} a^k, \quad a < \rho.$$

The market price of risk $\alpha(\gamma; a)$ and the volatility $\sigma(\gamma; a)$ have the power series expansions in $\mathscr{H}_{\mathrm{BMO}}$:

$$\alpha(\gamma; a) = \sum_{k=1}^{\infty} (\zeta_1^{(k)} + (\zeta_2^{(k)})^* \gamma) a^k, \quad a < \rho,$$

$$\sigma(\gamma; a) = \sum_{k=0}^{\infty} \zeta_2^{(k+1)} a^k, \quad a < \rho.$$

Here the coefficients $\zeta^{(k)} = (\zeta_1^{(k)}, \zeta_2^{(k)})$, $k \geq 1$, in $\mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d})$ are given recursively as

$$\zeta^{(1)} = (\zeta_1^{(1)}, \zeta_2^{(1)}) = (0, \sigma(0)), \tag{4.11}$$

$$\zeta^{(k)} = \sum_{l+m=k} G(\zeta^{(l)}, \zeta^{(m)}; \gamma), \quad k \geq 2, \tag{4.12}$$

and the coefficients $(S^{(k)})_{k\geq 1} \subset \mathscr{S}_{\mathrm{BMO}}(\mathbf{R}^n)$ are given by

$$S_t^{(k)} = - \sum_{l+m=k+1} \mathbb{E}_t[\int_t^T \zeta_{2,s}^{(l)} (\zeta_{1,s}^{(m)} + (\zeta_{2,s}^{(m)})^* \gamma_s) ds], \quad t \in [0, T].$$

Remark 4.6. The leading price impact coefficient in the expansion for stock prices is given by

$$S_t^{(1)} = -\mathbb{E}_t \left[\int_t^T \sigma_s(0) \sigma_s(0)^* \gamma_s \, ds \right], \quad t \in [0, T].$$

This result had been obtained earlier in [12, Theorem 2] for a simple demand.

Proof of Theorem 4.5. Theorem 4.1 in [18] yields the existence of a constant $c = c(n)$ such that, under (4.10), γ is a viable demand accompanied by the unique price process $S = S(\gamma, a)$. Observe that the dependence of the coefficients $\zeta^{(k)}$ on γ and $\sigma(0)$ has the homogeneity properties:

$$\zeta_1^{(k)}(b\gamma, \sigma(0)) = \zeta_1^{(k)}(\gamma, b\sigma(0)) = b^k \zeta_1^{(k)}(\gamma, \sigma(0)),$$
$$b\zeta_2^{(k)}(b\gamma, \sigma(0)) = \zeta_2^{(k)}(\gamma, b\sigma(0)) = b^k \zeta_2^{(k)}(\gamma, \sigma(0)), \quad b > 0,$$

which can be verified by induction. In view of these properties and the self-similarity relations (4.5) for the market prices of risk and volatilities, we can assume, without a loss in generality, that

$$\|\gamma\|_{\mathscr{H}_\infty} = \|\sigma(0)\|_{\mathscr{H}_{\mathrm{BMO}}} = 1.$$

In this case, the stochastic bilinear map $g(\cdot, \cdot; \gamma)$ is bounded by a constant $\Theta = \Theta(n)$ such that

$$|g(u, v; w)| \leq \Theta\, |u|\, |v| \text{ for every } w \in \mathbf{R}^n \text{ with } |w| \leq 1.$$

Taking now the constant c also smaller than $1/(8\Theta\kappa)$, where $\kappa = \kappa(n)$ is defined in (2.4), we deduce from Theorem 3.2 the existence and uniqueness of $\eta = \eta(a) \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^d)$ and $\theta = \theta(a) \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{n \times d})$ solving (4.1)–(4.2) and such that

$$\sqrt{\|\eta\|^2_{\mathscr{H}_{\mathrm{BMO}}} + \|\theta\|^2_{\mathscr{H}_{\mathrm{BMO}}}} \leq \frac{1}{4\kappa\Theta}.$$

Moreover, the maps $a \mapsto \eta(a)$ and $a \mapsto \theta(a)$ of $(-\rho, \rho)$ to $\mathscr{H}_{\mathrm{BMO}}$ are analytic, and their power series are given by

$$\eta(a) = \sum_{k=1}^{\infty} \zeta_1^{(k)} a^k,$$
$$\theta(a) = \sum_{k=1}^{\infty} \zeta_2^{(k)} a^k,$$

with the coefficients $\zeta^{(k)} = (\zeta_1^{(k)}, \zeta_2^{(k)})$, $k \geq 1$, in $\mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{(n+1)\times d})$ determined by (4.11)–(4.12). The power series expansion for S follow from Theorem 3.2, while the expansions for α and σ follow from the linear invertibility relations (4.3)–(4.4) between (α, σ) and (η, θ). □

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