Abel-Jacobi theorem
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1 Introduction

The Abel-Jacobi theorem is an important result of algebraic geometry. The theory of divisors and the Riemann bilinear relations are fundamental to the development of this result: if a point $O$ is fixed in a Riemann compact surface $X$ of genus $g$, the Abel-Jacobi map identifies the Picard group $Pic_0(X)$ the quotient of divisors of a group of degree zero on the subgroup of divisors associated to meromorphic functions. The Riemann surface of genus $g \geq 1$ can be embedded in the Jacobian variety $Jac(X)$ via the Abel-Jacobi. In fact we generally have a map:

$$X^{(s)} = X^g/\mathcal{C}_g \to Jac(X)$$

such that $X^{(s)}$ may be provided with an analytical structure. Indeed the two sets $X^{(s)} = X^g/\mathcal{C}_g$, $Jac(X)$ are algebraic varieties and the map

$$X^{(s)} \to Jac(X)$$

is surjective. For reasons of dimension we can verify that is finite fibers. In fact this is a birational map.

2 Riemann bilinear relations

Let $X$ be a compact Riemannian surface. Recalling that,

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g} \text{ and } H^1_{dR}(X, \mathbb{R}) \cong \mathbb{R}^{2g}$$

where $g$ is the genus of $S$. The following map

$$H_1(X, \mathbb{Z}) \times H^1_{dR}(X, \mathbb{R}) \longrightarrow \mathbb{R}
\quad (\gamma, \omega) \quad \longrightarrow \quad \int_\gamma \omega$$

makes these two spaces in duality: for a basis $(\gamma_1, ..., \gamma_{2g})$ in $H_1(X, \mathbb{Z})$ there exist a dual basis

$$(\omega_1, ..., \omega_{2g}) \in H^1_{dR}(X, \mathbb{R})$$

such that for $i, j = 1, ..., 2g$

$$\int_{\gamma_i} \omega_j = \delta_{ij}$$
The intersection product

$$H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \rightarrow \mathbb{Z}$$

$$(\gamma_1, \gamma_2) \rightarrow \gamma_1 \# \gamma_2$$

defines an antisymmetric bilinear form on $H_1(X,\mathbb{Z})$, which has a corresponding symplectic bases

**Proposition 1** For any symplectic basis $(a_1, ..., a_g, b_1, ..., b_g)$ of $H_1(X,\mathbb{Z})$ and for any closed 1-forms $\eta$ and $\eta'$ on the surface $X$ we have

$$\int_X \eta \wedge \eta' = \sum_{k=1}^g \left( \int_{a_i} \eta \int_{b_i} \eta' - \int_{a_i} \eta' \int_{b_i} \eta \right)$$

**Preuve.** Let $(a_1, ..., a_g, b_1, ..., b_g)$ be a symplectic basis of $H_1(X,\mathbb{Z})$ associated with a cutting $S$ into a $4g$-Gones quotes $\Delta$: $A_1B_1A'_1B'_1, ..., A_gB_gA'_gB'_g$, where $A_i$ and $A'_i$ are identified by the map $\varphi_i$ and $B_i, B'_i$ are identified by the map $\psi_i$ as in the following figure. Differential forms can be seen as differential forms on $\Delta$. Since this last is simply connected, so there exist a function $f$ such that $df = \eta$. So for each $x \in A$ and for each $y \in B$ we have:

(1) : \[ \int_{b_i(x)} df = \int_{b_i} \eta = f \circ \varphi_i (x) - f (x) \]

(2) : \[ \int_{a_i(x)} df = \int_{a_i} \eta = f (x) - f \circ \psi_i (x) \]
Stokes formula implies
\[
\int_S \eta \wedge \eta' = \int_\Delta \eta \wedge \eta' \\
= \int_D d(f\eta') \\
= \int_\Delta f\eta' \\
= \sum_{k=1}^g \int_{A_i+B_i-A_i'-B_i'} f\eta'
\]
and it follows from the formulas (1) and (2):
\[
\int_{A_i-A_i'} f\eta' = \int_{A_i} (f - f \circ \varphi_i(x)) \eta' = - \int_{b_i} \eta \int_{a_i} \eta' \\
\int_{B_i-B_i'} f\eta' = \int_{B_i} (f - f \circ \psi_i(x)) \eta' = \int_{a_i} \eta \int_{b_i} \eta'
\]
which proves equality \(\blacksquare\)

**Remarque 2** If the surface \(X\) is provided with a riemann structure, and if \(\eta, \eta'\) are holomorphic 1-forms, then \(\int_X \eta \wedge \eta' = 0\)

**Proposition 3** Let \(X\) be a compact Riemannian of which is fixed 2\(g\) simple closed curves \((a_1, ..., a_g, b_1, ..., b_g)\), forming a symplectic basis of the space \(H_1(X, \mathbb{Z})\) and let \(\omega_1\) be a holomorphic 1-form on \(X\) and \(\omega_2\) non-singular 1-meromorphic form along all the curves \(a_i b_i\). Given a point \(z_0 \in X - \{a_i b_i\}\) such that, \(u(z) = \int_{z_0}^{z} \omega_1\), then

\[
2i\pi \sum \text{Res}(u, \omega_2) = \sum_{i=1}^g \left( \int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{a_i} \omega_2 \int_{b_i} \omega_1 \right)
\]

**Preuve.** The proposal follows from the Residue formula and equations (1) and (2): \(2i\pi \sum \text{Res}(u, \omega_2) = \int_{\partial \Delta} u \cdot \omega_2 \ \blacksquare\)

Whether now \((a_1, ..., a_g, b_1, ..., b_g)\) is a 2\(g\) simple closed curves on a compact Riemann surface \(X\) which form basis of the space \(H_1(X, \mathbb{Z})\) and \((\omega_1, ..., \omega_g)\) is a fixed basis of the space of 1-holomorphic forms on \(X\).
Definition 4 Let’s call the period matrices \( A, B \in \mathcal{M}_g(\mathbb{C}) \) defined by
\[
A_{ij} = \int_{a_i} \omega_j \\
B_{ij} = \int_{b_i} \omega_j
\]

Thorme 5 (Riemann bilinear relations)
1. The matrix \( A \) is invertible
2. The matrix \( \Omega = A^{-1}B \) is symmetrical and its imaginary part
\[
\text{Im} \Omega = (\text{Im} \Omega_{ij})_{i,j \leq g}
\]
is positive definite

Proof. Whether \( \lambda = (\lambda_1, \ldots, \lambda_g) \in \mathbb{C}^g \) such that \( \sum_{i=1}^g \lambda_i A_{ij} : j = 1, \ldots, g. \)
Consider the holomorphic 1-form
\[
\omega = \sum_{i=1}^g \lambda_i \omega_i
\]
By definition of the matrix \( A \), we have:
\[
\int_{a_i} \omega = 0 = \sum_{i=1}^g \lambda_i A_{ij}
\]
so is
\[
\int_{a_i} \overline{\omega} = 0
\]
Then it follows from the Proposition1,
\[
\int_{a_i} \omega \wedge \overline{\omega} = 0 : \omega = 0
\]
so \( \lambda_i = 0, \ i = 1, \ldots, g. \) For the other one, we easily verify that \( \Omega \) is independent of the basis \( (\omega_1, \ldots, \omega_g) \). Since the matrix \( A \) is invertible, so a base change we can consider \( A = I: A_{ij} = \delta_{ij}. \) Hence \( \Omega_{ij} = B_{ij}, \) and it still follows from the Proposition1:
\[
0 = \int_X \omega_i \wedge \omega_j = \sum_{k=1}^g \left( \int_{a_k} \omega_i \int_{b_k} \omega_j - \int_{a_k} \omega_j \int_{b_k} \omega_i \right)
= \int_{b_i} \omega_j - \int_{b_j} \omega_i
\]
Finally, if \( v = (v_1, \ldots, v_g) \in \mathbb{R}^g - \{0\} \), then we have:

\[
\iota_v \text{Im } \Omega \cdot v = \frac{i}{2} \int_X \eta \wedge \eta > 0, \text{ when } \eta = \sum_{k=1}^g v_k \omega_k
\]

\[\blacksquare\]

3 Lattice of periods

Let \( X \) be a compact Riemannian surface with two \( 2g \) fixed simply closed curves which form a basis of the space \( H_1(X, \mathbb{Z}) \), \((\omega_1, \ldots, \omega_g)\) a basis of the space \( \Omega^1(X) \) of holomorphic 1-forms is fixed. The image of the following map

\[
p : H_1(X, \mathbb{Z}) \to \Omega^1(X)^* \\
\gamma \mapsto p(\gamma)
\]

is a lattice \( \Lambda \) in \( \Omega^1(X)^* \), where \( p(\gamma)(\omega) = \int_\gamma \omega \).

**Definition 6** We call \( \Lambda \) the lattice of periods. The dual basis \((\omega_1, \ldots, \omega_g)\) identifies the space \( \Omega^1(X)^* \) to \( \mathbb{C}^g \). As a lattice in the space \( \mathbb{C}^g \), \( \Lambda = AZ + BZ \)

**Remark 7** Note that the set \( \Lambda \) is a lattice since it comes from the Riemann bilinear relations and the real range of \((A, B)\) is equal \(2g\). The Riemann bilinear relations even show that \( \Lambda \) is a particular lattice.

**Definition 8** A divisor on a Riemannian surface is the data of a finite set the points \((P_i, n_i)\), weighted by nonzero integers. The set of divisors is naturally equipped with a commutative group structure. It is a \( \mathbb{Z} \)-module generated by \( X \). A divisor is called effective if its degree \( \sum_i n_i = 0 \), and the divisor \( D \) is principal if \( D = \text{div}(f) \) is given by the poles and zeros of a meromorphic function \( f \).

**Notation 9** \( D = \sum_{i} n_i P_i, \) \( \text{deg } D = \sum_{i} n_i \)

4 Abel-Jacobi map

Wether \( O \) and \( P \) are two points of a Riemann compact surface \( X \). Two paths \( \gamma \) and \( \gamma' \) link \( O \) to \( P \) in \( X \) differ only by a factor of \( H_1(X, \mathbb{Z}) \). In another word: \( p(\gamma) = p(\gamma') \mod \Lambda \). For any path \( \gamma \) the following map

\[
\nu_O : X \to \mathbb{C}^g / \Lambda \\
P \to (\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g)
\]
is well defined, but depending on the point $O$. Moreover, for each point $P \in X$ we can associate the divisor $P - O$ of degree zero. A divisor $\text{div}(f)$ associated to a meromorphic function $f$ is also of degree zero.

**Definition 10** The set of divisors of degree zero is naturally an Abelian group. We call group of Picard $\text{Pic}_O(X)$ the quotient of divisor group of degree zero by the sub-group of divisors associated to meromorphic functions.

**Proposition 11** The map $u_O$ extends naturally into a group morphism:

$$
\begin{align*}
  u : \text{Pic}_O(X) &\longrightarrow \mathbb{C}^g/\Lambda \\
  \sum_{P} n_P P &\longrightarrow \sum_{P} n_P u_O(P)
\end{align*}
$$

which does not depend on the point $O$.

**Proof.** Let's show first the map $u$ is well defined. Wether

$$
\text{div}(f) = \sum_{P} n_P
$$

where $f$ is a meromorphic function and we set

$$
\omega = \frac{df}{2i\pi f}
$$

We note $F_k(z) = \int_{O}^{z} \omega_k$ for $k = 1, \ldots, g$. So Proposition 3 implies

$$
\sum \text{Res} \left( F_k \frac{df}{f} \right) = \sum_{j=1}^{g} \left( \int_{a_j}^{b_j} \omega \int_{b_j}^{a_j} \omega_k - \int_{b_j}^{a_j} \omega_k \int_{a_j}^{b_j} \omega \right)
$$

The right side is a linear combination in integers of periods $\int_{a_j}^{b_j} \omega_k$, as integer, because the periods of the 1-form $\omega$ are integers (Residue formula). The left side is equal to

$$
\sum_{P} n_P F_k(P)
$$

Finally the $k^{th}$ coordinate of the image $u_O(P)$ equals $F_k(P)$. Whether we change the point $O$ in another one $O' \in X$ in another one, then

$$
\left( u_O - u_{O'} \right) \left( \sum_{P} n_P P \right) = - \sum_{P} n_P \left( \int_{O}^{O'} \omega_1, \ldots, \int_{O}^{O'} \omega_g \right)
$$

But the sum of the right hand is zero, because the degree $\sum_{P} n_P = 0$ ☐
Definition 12 The map \( u \) defined as above is called the Abel-Jacobi map.

Theorem 13 (Abel) The Abel-Jacobi map is injective.

Proof. Whether \( D = \sum n_P P \) is a divisor of degree zero such that \( u(D) = 0 \), we will find a meromorphic function \( f \) such that \( D = \text{div}(f) \). Indeed we will construct a 1-form

\[
\omega = \frac{df}{2i\pi f}
\]

Let \( \omega \) be a 1-meromorphic form on the surface \( S \) with simple poles in the points \( P \) of divisor \( D \) with residues \( n_P \). Hence once again by Proposition 1:

\[
u(D) = \sum_P n_P u_0(P) = \sum \text{Res}(u_0\omega)
= \sum_{j=1}^g \left( \int_{a_j} \omega \int_{b_j} \omega_k - \int_{a_j} \omega_k \int_{b_j} \omega \right)_{k=1,...,g}
\]

We will modify \( \omega \) so that all its periods will become integers.

Lemma 14 Whether \( x_1, .., x_g, y_1, .., y_g \) are complex numbers, then there exists a holomorphic 1-form \( \eta \) such that

\[\int_{a_i} \eta = x_i \quad \text{and} \quad \int_{b_i} \eta = y_i\]

if and only if

\[
\sum_{k=1}^g \left( y_k \int_{a_k} \omega_i - x_k \int_{b_k} \omega_i \right) = 0 \quad i = 1, ..., g
\]

Proof. As the matrix \( A \) is invertible, then the vectors

\[
\left( \int_{a_1} \omega, ..., \int_{a_g} \omega \right) \quad i = 1, ..., g
\]

are linearly independent. Now the following linear map is surjective

\[
\Phi : \mathbb{C}^{2g} \rightarrow \mathbb{C}^g \quad (x_1, ..., x_g, y_1, ..., y_g) \mapsto \left( \sum_{k=1}^g \left( y_k \int_{a_k} \omega_k - x_k \int_{b_k} \omega_i \right) \right)_{i=1,...,g}
\]
So \( \text{dim ker } \Phi = g \). But if \( \eta \) is a holomorphic 1-form, \( \eta \wedge \omega_i = 0 : i = 1, ..., g \), and then Proposition 1 implies

\[
\left( \int_{a_1} \eta, ..., \int_{a_g} \eta, \int_{b_1} \eta, ..., \int_{b_g} \eta \right) \in \ker \Phi
\]

The lemma follows from that the dimension of the space of the holomorphic 1-forms is equal to the geneus \( g \). Since \( u(D) = 0 \) in the quotient \( \mathbb{C}^g / \Lambda \), then there exists integers \( (A_1, A_g, B_1, ..., B_g) \) such that

\[
\sum_{k=1}^{g} \left( \left( \int_{a_k} \omega - B_k \right) \int_{a_k} \omega_i - \left( \int_{a_k} \omega - A_k \right) \int_{a_k} \omega_i \right) \quad i = 1, ..., g
\]

So by the lemma above, there exists a holomorphic 1-form \( \eta \) such that all the periods of the 1-form \( \eta - \omega \) are integers. Hence we can consider that \( \omega \) has integer periods. A primitive of the form between \( O \) and \( z \) gives the meromorphic function

\[
f(z) = \exp \left( 2i\pi \int_{O}^{z} \omega \right)
\]

which is well defined, satisfying \( \text{div}(f) = D \).

**Thorme 15 (Jacobi)** The Abel-Jacobi map is injective

**Preuve.** The map \( u \) is a group morphism. So it suffices to show that the image of the map \( u \) contains a neighborhood of the point \( O \). This will follow from the inverse function theorem: ■

**Lemme 16** There exists \( g \) distincts points \( P_1, ..., P_g \in X \) such that any holomorphic 1-form which vanishes in each \( P_k \) is identically zero

**Preuve.** For any point \( P \in X \) the sub-space

\[
H_P = \{ \omega \in \Omega^1 (X)^* : \omega(P) = 0 \}
\]

is of codimension \( \leq 1 \) in \( \Omega^1 (X) \). But the intersection

\[
\bigcap_{P \in S} H_P
\]

is trivial and \( \dim \Omega^1 (X) = g \). Then there exists points \( P_1, ..., P_g \in S \) such that

\[
H_{P_1} \cap ... \cap H_{P_2} \cap H_{P_g} = 0
\]
Let $P_1, \ldots, P_g \in X$ be fixed points as in the lemma with simply connected disjoint local coordinates $(U_i, z_i)$ around these points and $z_i(P_i) = 0$ $i \leq g$. In fact each 1-form $\omega_i$ is written as:

$$\omega_i = \varphi_{ij} dz_j \text{ on } U_j$$

The matrix $(\varphi_{ij})_{1 \leq i,j \leq g}$ is invertible by lemma above.

Consider now the following map

$$F : U_1 \times \ldots \times U_g \rightarrow C^g$$

$$z = (z_1, \ldots, z_g) \rightarrow (F_1(z), \ldots, F_g(z))$$

such that

$$F_i(z) = \sum_{j=1}^g \int_{P_j}^{z_j} \omega_i : i = 1, \ldots, g$$

The integral

$$\int_{P_j}^{z_j} \omega_i$$

is well defined since each $U_i$ is simply connected. Hence the map $F$ is differentiable in complexe coordonates $z_1, \ldots, z_g$ and the expression of the jacobian matrix is

$$\left( \frac{\partial F_i}{\partial x_j} \right)_{1 \leq i,j \leq g} (P) = (\varphi_{ij}(P))_{1 \leq i,j \leq g}$$

This matrix is invertible in the point $P = (P_1, \ldots, P_g)$. So by the local inverse theorem we have a neighborhood of $F(P) = 0$:

$$W = F(U_1 \times \ldots \times U_g) \subset C^g$$

Finally if $\xi \in W$ then there exists points $Q_1, \ldots, Q_g \in C^g$ such that

$$\left( \sum_{j=1}^g \int_{P_j}^{Q_j} \omega_1, \ldots, \sum_{j=1}^g \int_{P_j}^{Q_j} \omega_g \right) = \xi$$

In another wordrs

$$u \left( \sum_{j=1}^g (Q_j - P_j) \right) = \xi$$

Summarizing the theorem of Abel-Jacobi:
Thorme 17 (Abel-Jacobi) The Abel-Jacobi map \( u : Pic(X) \rightarrow Jac(X) = \mathbb{C}^g / \Lambda \) is bijective.

Furthermore whether a point \( O \in X \) is fixed, we have the following map

\[
u_O : X \rightarrow Jac(X)
\]

\[
P \rightarrow u(P - O)
\]

When \( g = 1 \) this map is an isomorphism. In general it is still:

Proposition 18 If the genus \( g \geq 1 \), the map \( u_O : X \rightarrow Jac(X) \) is an embedding.

Preuve. Since \( S \) is compact, it suffices to show that \( u_O \) is an injective immersion map. Let’s prove firstable \( u_O \) is injective. Suppose by contradiction that \( u_O(P) = u_O(P') \). So the map \( u \) concels on the divisor of degree zero, \( P - P' \). This last is the divisor of a meromorphic function \( f \). This one has a single pole and a single zero; so it is a map:

\[
X \rightarrow \mathbb{CP}^1
\]

of degree one. Thus is absurd since \( g \geq 1 \). Let’s prove that \( u_O \) is an immersion map. As in the proof the Abel-Jacobi theorem:

\[
d_p u_O(\xi) = (\omega_1(P)(\xi), \ldots, \omega_g(P)(\xi))
\]

The proposition follows again from the local inverse theorem and the next lemma.

Lemme 19 The holomorphic 1-forms \( (\omega_1, \ldots, \omega_g) \) have no common zero.

Preuve. Once again by contradiction: if a point \( P \) is a common zero. According to Riemann-Roch theorem: the dimension of the space of holomorphic functions having more then one simple pole in \( P \) equals:

\[
\deg u_O - g + 1 + \dim \{ \omega \in \Omega^1(X) : \omega(P) = 0 \} = 1 - g + 1 + \dim \{ \omega \in \Omega^1(X) : \omega(P) = 0 \} = 2
\]

Then there exists a function \( f \in X \), which has a unique simple pole in \( P \). So it is a map \( f : X \rightarrow \mathbb{CP}^1 \) of one degree, when even an absurdity since \( g \geq 1 \).
**Remarque 20** Once a point \( O \in X \) is fixed we have more generally a map

\[
X^{(g)} = X^g / \mathfrak{S}_g \longrightarrow \text{Jac}(X)
\]

\[
(P_1, \ldots, P_g) \longrightarrow u \left( \sum_{j=1}^{g} (P_j - O) \right)
\]

and \( X^{(g)} \) can be provided with an analytical structure. We showed that the map \( X^{(g)} \rightarrow \text{Jac}(X) \) is surjective. For reasons of dimensions we can verify that is finite fibers. We can show:

- \( X^{(g)} \) and \( \text{Jac}(X) \) are algebraic variety
- The map \( X^{(g)} \rightarrow \text{Jac}(X) \) is birational
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