Groups with infinitely many ends acting analytically on the circle

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Abstract

This article is inspired by two milestones in the study of non-minimal group actions on the circle: Duminy's theorem about the number of ends of semi-exceptional leaves, and Ghys' freeness result in real-analytic regularity. Our first result concerns groups of real-analytic diffeomorphisms with infinitely many ends: if the action is non expanding, then the group is virtually free. The second result is a Duminy type theorem for minimal codimension-one foliations: either non-expandable leaves have infinitely many ends, or the holonomy pseudogroup preserves a projective structure.

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1 Introduction

1.1 Foreword and results

The projective linear group $\mathrm{PSL}(2,\mathbb{R})$ is the main source of inspiration for understanding groups of circle diffeomorphisms. Although not as huge as $\mathrm{Diff}_+(\mathbf{S}^1)$ – it is only a three-dimensional Lie group versus an infinite dimensional group –, it is a good model to study several important aspects of subgroups of $\mathrm{Diff}_+(\mathbf{S}^1)$.

To begin, recall that $\mathrm{PSL}(2,\mathbb{R})$ naturally acts on the circle $\mathbf{S}^1$ viewed either as the projective real line $\mathbb{R}\mathbf{P}^1$ or as the boundary of the hyperbolic plane. This action is clearly real analytic, thus we can see $\mathrm{PSL}(2,\mathbb{R})$ as a subgroup of the group $\mathrm{Diff}^\omega_+(\mathbf{S}^1)$ of orientation-preserving real-analytic circle diffeomorphisms.

Several works have already described "non-discrete" (more precisely, *non locally discrete*) subgroups of $\mathrm{Diff}^\omega_+(\mathbf{S}^1)$, if not thoroughly, at least in a very satisfactory way (see Ghys [23], Shcherbakov *et al.* [16], Nakai [33], Loray and Rebelo [30, 37, 38], Eskif and Rebelo [17], etc.). Morally, they resemble *non-discrete* subgroups in $\mathrm{PSL}(2,\mathbb{R})$, in the sense that their dynamics approximate *continuous* dynamics. We will be more precise later.

A decade ago or so, some of the authors, in collaboration with Bertrand Deroin, started a systematic study of *locally discrete* groups of $\mathrm{Diff}^\omega_+(\mathbf{S}^1)$ [8, 10, 11, 18, 19]. They introduced an auxiliary property, named $(\star)$ (and $(\Lambda\star)$, but we do not make a distinction here), under which groups behave roughly like *Fuchsian groups*, *i.e.* discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. Informally speaking, property $(\star)$ requires that the action is *non-uniformly hyperbolic*: points at which hyperbolicity is lost must be *parabolic* fixed points (or more generally the fixed point of some element, with derivative 1). This is indeed the case for non-elementary Fuchsian groups.

Starting from this, a relevant part of the work aims to show that property $(\star)$ is always satisfied. Conjecturally, it should be satisfied even in the lowest regularity setting where one disposes of control of affine distortion, namely C^2. However, the attention should be focused first on real-analytic actions, where arguments are often less technical.

In order to ensure property $(\star)$, one relates the dynamics with the algebraic structure of the group. So far the program proceeds by distinction of the number of *ends* of the group. Extending the previous work [11] on *virtually free groups* (*i.e.* groups containing free subgroups of finite index), our first main result proves that property $(\star)$ holds for groups with *infinitely many ends*:

Theorem A. *Let G be a finitely generated, locally discrete subgroup of* $\mathrm{Diff}^\omega_+(\mathbf{S}^1)$*. If G has infinitely many ends, then it satisfies property $(\star)$, and it is virtually free.*

Our second result goes in the reverse direction: property $(\star)$ determines the structure of the group. As we already mentioned, classical examples of locally discrete groups with property $(\star)$ are Fuchsian groups. Similarly, one can consider *finite central extensions* of Fuchsian groups (*i.e.* discrete subgroups of a k-fold cover $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$). A discrete group $\Gamma\subset\mathrm{PSL}^{(k)}(2,\mathbb{R})$ is *cocompact* if the quotient $\mathrm{PSL}^{(k)}(2,\mathbb{R})/\Gamma$ is compact. Cocompact discrete groups have only one end.

Theorem B. *Let G be a finitely generated, locally discrete subgroup of* $\mathrm{Diff}^\omega_+(\mathbf{S}^1)$ *satisfying property $(\star)$. Then*

- *either G is C^ω-conjugate to a finite central extension of a cocompact Fuchsian group*
- *or it is virtually free.*

An exhaustive description of virtually free, locally discrete groups of $\mathrm{Diff}^\omega_+(\mathbf{S}^1)$ will be the object of a forthcoming work [1].

1.2 Dynamics: Basic definitions and preliminaries

Locally discrete groups of real-analytic circle diffeomorphisms – If a group G acts (continuously) on the circle $\mathbf{S}^1$ and there is no finite orbit, then the group admits a unique *minimal invariant compact set*, which can be the whole circle or a Cantor set. The most interesting dynamics takes place on this minimal set. For example, only minimal sets "survive" under topological semi-conjugacies.

Because of the minimality of the action on the minimal set Λ, the local dynamics around a point $x \in \Lambda$ is essentially the same as the local dynamics around any other point $y \in \Lambda$. Roughly speaking, the dynamics of G on Λ is encoded in the restriction of the action of G to any open interval I intersecting Λ.

Definition 1.1. A group $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ is *locally discrete* (more precisely, C^1 locally discrete) if for any interval $I \subset \mathbf{S}^1$ intersecting a minimal set, the restriction of the identity to I is isolated in the C^1 topology among the set of restrictions to I of the diffeomorphisms in G.

The previous definition makes sense also for subgroups of $\mathrm{Diff}^{1}_{+}(\mathbf{S}^1)$. However, to avoid discussing different classes of regularity, we restrict to $\mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$. The huge difference between C^ω and lower regularity is the following:

Theorem 1.2 (see Proposition 3.7 of [32])**.** *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ be a finitely generated, locally discrete subgroup. Then the stabilizer in G of every point is either trivial or infinite cyclic.*

The next corollary essentially describes locally discrete groups with finite orbits.

Corollary 1.3. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ be a finitely generated, locally discrete subgroup with a finite orbit. Then G is either cyclic or contains an index-2 subgroup which is the direct product of an infinite cyclic group with a finite cyclic group.*

Remark 1.4. Notice that the index-2 subgroup below arises when a rotation conjugates an element with fixed points into its inverse (as it is the case of involution $x \to -1/x$ with respect to the hyperbolic Möbius transformation $x \to \lambda x$, with $\lambda \neq 1$, both viewed as maps of the circle $\mathbf{S}^1 \sim \mathbb{R}\mathbf{P}^1$).

Theorem 1.2 is a consequence of a well-known result due to Hector, and we refer to it as "Hector's lemma" (see [21, Théorème 2.9] and [23, 34]). Generalizing Hector's lemma to lower regularity is a longstanding major problem in codimension-one foliations [12, pp. 448–449]. It is also the major reason why our results hold in this wide generality only for subgroups of $\mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$.

Non locally discrete groups of analytic circle diffeomorphisms – If a subgroup $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ is locally discrete, then it is also discrete (with respect to the C^1 topology). As a matter of fact, there is no deep reason for privileging locally discreteness above discreteness: we believe that the two notions coincide, but this would rather be a consequence of our aimed classification. Indeed, appropriate dynamical tools are known only when working with local (non-)discreteness.

As we mentioned at the beginning, non locally discrete groups have been studied in several works, mainly by Shcherbakov, Nakai, Loray and Rebelo. The fundamental tool, which goes back to [16, 30, 33], is the following result that establishes the existence of *local flows in the local closure* of the group. We state it in the form of [11, Proposition 2.8]:

Proposition 1.5. *Let I be an interval on which nontrivial real-analytic diffeomorphisms $f_k \in \mathrm{Diff}^{\omega}(I, \mathbf{S}^1)$ are defined. Suppose that the sequence f_k converges to the identity in the C^1 topology on I, and let f be another C^ω diffeomorphism having a hyperbolic fixed point on I. Then there exists a (local) C^1 change of coordinates $\phi : I \longrightarrow [-1, 2]$ after which the pseudogroup G generated by the f_k's and f contains in its $C^1([0,1],[-1,2])$-closure a (local) translation sub-pseudogroup:*

$$\overline{\{\phi g \phi^{-1}|_{[0,1]} \mid g \in G\}} \supset \{x \mapsto x + s \mid s \in [-1, 1]\}.$$

Existence of elements with hyperbolic fixed points is often guaranteed, as the classical Sacksteder's theorem claims ([39], see also [9, 15, 35]). We state a more general version (in class C^1) due to Deroin, Kleptsyn and Navas, inspired by a similar result of Ghys in the C^2 context.

Theorem 1.6. *Let G be a finitely generated group of C^1 circle diffeomorphisms. If G admits no invariant probability measure on $\mathbf{S}^1$, then it contains an element that has a hyperbolic fixed point in the minimal invariant set of G.*

Observe that a group with an invariant measure either is semi-conjugate to a group of rotations or has a finite orbit. Joining Proposition 1.5 and Sacksteder's theorem together, we have that if a finitely generated group $G \subset \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ acts minimally with no invariant measure and is non locally discrete, then it has local vector flows in its local closure.

Non-expandable points – The existence of local flows in the closure of a group of circle diffeomorphisms yields rich dynamics. For instance, the action must be minimal and Lebesgue ergodic. If, besides, there is no invariant probability measure, one deduces from Sacksteder's theorem that the action must be *expanding*, in the following sense:

Definition 1.7. A point $x \in \mathbf{S}^1$ is *non expandable* for the action of a group G of circle diffeomorphisms if for every $g \in G$, the derivative of g at x is not greater than 1. We denote by $\mathrm{NE} = \mathrm{NE}(G)$ the set of non-expandable points of G. The action of a group of circle diffeomorphisms is *expanding* if $\mathrm{NE} = \emptyset$.

Since we have $\mathrm{NE} = \{x \mid g'(x) \le 1 \text{ for every } g \in G\} = \bigcap_{g\in G} \{x \mid g'(x) \le 1\}$, the set of non-expandable points is always closed. Notice that one can define the set of non-expandable points for any group of C^1 circle diffeomorphisms. However, it is important to point out that, *a priori*, the definition does not well behave under C^1 conjugacy: only the property $\mathrm{NE} = \emptyset$ is invariant under C^1 conjugacy. The problem is that the notion of non-expandable points is not a *dynamical* one. The following definition, introduced in [9], forces a conjugacy-invariant condition.

Definition 1.8 (Property ($\star$) – C^ω case)**.** Let $G \subset \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ be a group with no finite orbit, and let Λ be its minimal invariant set. The group G has *property* ($\star$) if for every $x \in \mathrm{NE} \cap \Lambda$ there is $g \in G \setminus \{id\}$ such that x is a fixed point of g.

Property ($\star$) makes sense even for C^1 actions. However it turns to be a useful notion only when working with actions that are at least of class C^2 (because control of distortion is needed). In most issues, there is no relevant difference between C^2 and C^ω actions with property ($\star$). However, the definition of property ($\star$) in class C^2 is slightly more complicated, as one has to take into account that there could be elements that are the identity on some interval.

Definition 1.9 (Property ($\star$) – C^2 case)**.** Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a group with no finite orbit, and let Λ be its minimal invariant set. The group G has *property* ($\star$) if for every $x \in \mathrm{NE} \cap \Lambda$ there are g_+ and g_- in G such that x is an isolated fixed point in Λ from the right (resp. from the left) for $g_+|_\Lambda$ (resp. $g_-|_\Lambda$).

Property ($\star$) entails several strong properties for the dynamics of the group action. For a detailed discussion, the reader may consult [9] or [35, § 3.5]. Here we collect the results that are relevant to our purposes. First of all, if $\mathrm{NE} \neq \emptyset$, then the group is locally discrete. Secondly, the set $\mathrm{NE} \cap \Lambda$ intersects only finitely many orbits (also, in the case where an exceptional minimal set arises, there are only finitely many orbits of connected components of the complement $\mathbf{S}^1 \setminus \Lambda$). This can be seen as a consequence of the work [10] where an *expansion procedure* was introduced, and later improved by Filimonov and Kleptsyn in [18]. In this latter work the authors show that, if $\mathrm{NE} \neq \emptyset$, the dynamics on the minimal set can be encoded by a Markovian dynamics (see Theorem 3.4 below). We will give a more precise account later in § 3.3, as this fact is one fundamental ingredient for the proof of Theorem B, as well as for its generalization to lower regularity, namely Theorem C further on.

Example 1.10. Morally, when property $(\star)$ is satisfied, orbits of non-expandable points should be geometrically interpreted as *cusps*. Actually, this is the case for the action of non-uniform lattices Γ in $\mathrm{PSL}(2,\mathbb{R})$, that is subgroups for which the quotient $\mathbf{H}^2/\Gamma$ is not compact but has finite volume. The most classical examples are $\mathrm{PSL}(2,\mathbb{Z})$ and its finite index free subgroups like

$$\Gamma = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle$$

(the quotient $\mathbf{H}^2/\Gamma$ is a sphere with three cusps). In these cases, the orbit of the set of non-expandable points is made of the rational numbers together with the point at infinity in $\mathbb{R}\mathbf{P}^1 \cong \mathbb{R} \cup \{\infty\}$. In the quotient space $\mathbf{H}^2/\Gamma$, these points coincide with the cusps.

Although property $(\star)$ is always verified by non locally discrete groups, whether it holds or not is definitively a challenging question for locally discrete groups. In some sense, as we plan to clarify in future works, locally discrete groups with property $(\star)$ present a dynamics strongly related to *geometry*: roughly, the dynamics should be described by combining elementary "Fuchsian" pieces.

It is strongly believed that property $(\star)$ holds for any (finitely generated) subgroup of $\mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$. This has already been verified for certain classes of groups: virtually free groups [11] and finitely presented one-ended groups of bounded torsion [19]. Theorem A enlarges this list. We will describe the state of the art on this point later.

1.3 Groups: Basic definitions and preliminaries

Definition 1.11. Let X be a connected topological space. Let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets $K_n \subset X$, such that the union of their interiors covers X. An *end* of X is a decreasing sequence

$$\mathcal{C}_1 \supset \mathcal{C}_2 \supset \ldots \supset \mathcal{C}_n \supset \ldots,$$

where $\mathcal{C}_n$ is a connected component of $X \setminus K_n$. We denote by $e(X)$ the *space of ends* of X; it does not depend on the choice of (K_n).

Note the cardinal of $e(X)$, called the *number of ends* of X, is the least upper bound, possibly infinite, for the number of unbounded connected components of the complementary sets $X \setminus K$, where K runs through the compact subsets of X.

The space of ends carries a natural *topology*: an open set V in X induces an open set in $e(X)$ given by the set of sequences $(\mathcal{C}_n)$ so that $\mathcal{C}_n \subset V$ for all but finitely many n's. Also the topology does not depend on the choice of (K_n). For nice topological spaces (connected and locally connected) the space of ends defines a compactification of X:

Definition 1.12. A sequence of points (x_n) in X goes to an end if it eventually goes outside of every compact set, that is, for every compact set $K \subset X$, there exists n_0 such that $x_n \notin K$ for all $n \geq n_0$.

If G is a group generated by a finite set $\mathcal{G}$, we define the space of ends $e(G)$ of G to be the space of ends of the *Cayley graph* of G relative to $\mathcal{G}$. This is the graph whose vertices are the elements of G, and two elements $g, h \in G$ are joined by an edge if $g^{-1}h \in \mathcal{G}$. The graph metric induces the *length metric* in G given by the following expression:

$$d_{\mathcal{G}}(g,h) = \min\{\ell \mid g^{-1}h = s_1 \cdots s_\ell,\ s_j \in \mathcal{G} \cup \mathcal{G}^{-1}\}.$$

The *length of an element* $g \in G$ is defined as $\|g\| = d_{\mathcal{G}}(id, g)$.

It is a classical fact [3, § 8.30] that the space of ends, and hence the number of ends, of a group does not depend on the choice of the finite generating set (this easily follows from the fact that Cayley graphs associated with different finite generating systems are bilipschitz equivalent). Moreover, the number of ends does not change when passing to finite extensions or finite-index subgroups. Furthermore, finitely generated groups can only have 0, 1, 2 or infinitely many ends. Groups with 0 or 2 ends are not of particular interest: they are respectively finite or virtually infinite cyclic, *i.e.* they contain $\mathbb{Z}$ as a finite index subgroup (we refer to [3, § 8.32] for further details). Although they represent a broader class, groups with infinitely many ends may also be algebraically characterized, according to the celebrated Stallings' theorem. Before stating it, we recall two basic operations on groups.

Definition 1.13. Let G_1 and G_2 be two groups, and denote by $\operatorname{rel} G_i$ the set of relations in G_i. Let Z be a group which embeds in both G_1 and G_2 via morphisms $\phi_i : Z \hookrightarrow G_i$, $i = 1, 2$. The *amalgamated product* $G_1 *_Z G_2$ of G_1 and G_2 over the group Z is defined by the presentation

$$\langle G_1, G_2 \mid \operatorname{rel} G_1, \operatorname{rel} G_2 \text{ and } \phi_1(z) = \phi_2(z) \text{ for every } z \in Z\rangle.$$

Amalgamated products arise, for example, in the classical van Kampen theorem. It is clear that if G_1 and G_2 are finitely generated, then any amalgamated product $G_1 *_Z G_2$ is also finitely generated. Conversely, if Z and $G_1 *_Z G_2$ are finitely generated, then G_1 and G_2 are also finitely generated.

Definition 1.14. Let H be a group and Z another group that embeds in two distinct ways into H via morphisms $\phi_i : Z \hookrightarrow H$, $i = 1, 2$. The *HNN extension* $H *_Z$ of H over Z is defined by the presentation

$$\langle H, \sigma \mid \operatorname{rel} H,\ \text{and } \phi_1(z) = \sigma\phi_2(z)\sigma^{-1} \text{ for every } z \in Z\rangle.$$

The generator σ is usually called the *stable letter* of the extension.

The most basic examples are the Baumslag-Solitar groups $\mathrm{BS}(m,n) = \langle t, \sigma \mid t^n = \sigma t^m \sigma^{-1}\rangle$, which correspond to HNN extensions of the type $\mathbb{Z}*_{\mathbb{Z}}$ (here the embeddings $\phi_i : \mathbb{Z} \hookrightarrow \mathbb{Z}$ are the multiplications by m and n, respectively).

From an algebraic point of view, an HNN extension $H *_Z$ is isomorphic to the semi-direct product of $\mathbb{Z}$ (generated by σ) and a bi-infinite chain of amalgamated products of copies of H. As before, if H is finitely generated, then any HNN extension $H *_Z$ is also finitely generated. Conversely, if Z and $H *_Z$ are finitely generated, then H is also finitely generated. We refer the reader to [2, 40] for more details.

Theorem 1.15 (Stallings)**.** *Let G be a finitely generated group with infinitely many ends. Then G is either an amalgamated product $G_1 *_Z G_2$ over a finite group Z (different from G_1 and G_2) or an HNN extension $H *_Z$ over a finite group Z (different from H).*

Given a finitely generated group G with infinitely many ends, we shall call *Stallings' decomposition* any possible decomposition of G as an amalgamated product or as an HNN extension over a finite group. Typical examples of groups with infinitely many ends are non-abelian (virtually) free groups.

In the second part of this work we study the geometry of orbits. To this extent, we recall the notion of *Schreier graph*, which is nothing but the generalization of Cayley graphs to group actions.

Definition 1.16. Let G be a finitely generated group acting on a space, let $\mathcal{G}$ be a finite generating system and X an orbit for the action. The *Schreier graph* of the orbit X, denoted by $\mathrm{Sch}(X, \mathcal{G})$, is the graph whose vertices are the elements of X, and two vertices $x, y \in X$ are joined by an edge if there exists $s \in \mathcal{G}$ such that $s(x) = y$. The graph metric on X is induced by the length metric on G:

$$d_{\mathcal{G}}^X(x, y) = \min \left\{ d_{\mathcal{G}}(id, g) \mid g(x) = y \right\}.$$

As for Cayley graphs, the number of ends of Schreier graphs does not depend on the choice of the finite generating set. Remark, however, that a Schreier graph might not have the same number of ends as the Cayley graph: for example, Thompson's group T is one-ended, it acts on the circle by C^∞ diffeomorphisms [20], and there are Schreier graphs associated with this action that have infinitely many ends (as Duminy's theorem below guarantees).

Finally, we introduce a graph structure for the *grupoid of germs* G_{x_0} defined at a point x_0. Fix a generating system $\mathcal{G}$ for G. Recall that two diffeomorphisms f and g define the same *germ* at a point x_0 if there exists a neighbourhood U of x_0 such that the restrictions of f and g to U coincide. In the following, we identify a germ with any diffeomorphism representing it. The germs usually do not define a group, but rather a grupoid. For our purposes, it is enough to consider G_{x_0} simply as a metric space as follows: G_{x_0} is formed by all the germs defined at x_0 and equipped with the distance

$$d_{\mathcal{G},x_0}(g, h) = \min \left\{ \ell \in \mathbb{N} \,\middle|\, g^{-1}h|_U = s_1 \cdots s_\ell|_U,\ s_j \in \mathcal{G} \cup \mathcal{G}^{-1},\ \text{for some neighbourhood } U \ni x_0 \right\}.$$

Remark 1.17. The grupoid of germs G_{x_0} is a covering of the Schreier graph of $X = G \cdot x_0$, defined by the natural map $g \in G_{x_0} \mapsto g(x_0) \in X$. In foliation theory, this is called the *holonomy covering* of X.

1.4 Background and perspective

Let us return to the discussion of our results, putting them in a neater context.

A previous result – Virtually free groups are the typical examples of groups with infinitely many ends. In [11] Deroin, Kleptsyn and Navas succeeded in showing that virtually free groups have property $(\star)$:

Theorem 1.18 (Deroin, Kleptsyn, Navas)**.** *Let $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a virtually free group acting minimally on the circle. Then G has property $(\star)$.*

Hence, Theorem A extends the main result of [11]. In fact, the proof of Theorem A relies on an interplay between the proof of Theorem 1.18 and Stallings' theorem, following ideas of Hector and Ghys [21] that we sketch in § 2.1.

Duminy's and Ghys' theorems – Our second result, Theorem B, also shows that minimal actions with non-expandable points are very close to actions with an exceptional minimal set. The orbit of a non-expandable point plays the role of the gaps associated with an exceptional minimal set. In this analogy, the non-expandable point is identified with a maximal gap which cannot be expanded.

Example 1.19. If we think on classical Fuchsian groups, actions with an exceptional minimal set (usually called Fuchsian groups *of the second kind*) are semi-conjugate to minimal actions (Fuchsian groups *of the first kind*). Geometrically, the semi-conjugacy is realized by contracting all boundary components of a hyperbolic surface $\mathbf{H}^2/\Gamma_0$ of infinite volume to cusps, so to obtain a new hyperbolic surface $\mathbf{H}^2/\Gamma$ of finite volume. Here, the groups Γ_0 and Γ are isomorphic (and free). The deformation also goes in the reverse way: given a non compact surface of finite volume, we can deform it by making cusps become circular boundary components.

In this perspective, our results are natural analogues of the more classical Duminy's and Ghys' theorems [21, 34]:

Theorem 1.20 (Duminy). *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group acting on $\mathbf{S}^1$ with an exceptional minimal set Λ. Consider a connected component (a gap) J_0 of $\mathbf{S}^1 \setminus \Lambda$. Then the Schreier graph of the orbit $X = G \cdot J_0$ has infinitely many ends.*

In the particular case where $G \subset \mathrm{Diff}^\omega_+(\mathbf{S}^1)$, this implies that the group G itself has infinitely many ends.

Theorem 1.21 (Ghys). *Let $G \subset \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ be a finitely generated group acting with an exceptional minimal set. Then G is virtually free.*

The latter theorem is more than an analogy here, as it is fundamental in the proof of Theorem A.

In its full generality (namely, codimension-one foliations that are transversally of class C^2), the proof of Duminy's Theorem is a gemstone. Here in this work we present a proof for groups of real-analytic diffeomorphisms, which apparently was already known to Hector. This is relatively simple because in C^ω regularity we can use Hector's lemma, but it is enlightening enough in view of the proof of Theorem B.

Duminy's Theorem and property $(\star)$ – In lower regularity, the statement of Theorem B cannot hold, as one sees from the example of Thompson's group T. However, there is an intermediate result, on which Theorem B relies, that still holds for C^r minimal non-expandable actions ($r \geq 3$):

Theorem C. *Let $G \subset \mathrm{Diff}^r_+(\mathbf{S}^1)$ be finitely generated group of C^r diffeomorphisms, $r \geq 3$, such that the action of G is minimal, satisfies property $(\star)$ and has a non-expandable point $x_0 \in \mathbf{S}^1$. Then the Schreier graph of the orbit of x_0 has infinitely many ends.*

The best plausible extension of the theorem above would be the following:

Conjecture 1.22. *Under the hypotheses of Theorem C, also the grupoid of germs G_{x_0} has infinitely many ends.*

In the statement of the conjecture, one could take for G_{x_0} the groupoid of germs defined on a right or left neighbourhood of the orbit of x_0. A local C^r diffeomorphism representing a germ in G_{x_0} is defined on a right (or left) neighbourhood of a point in the orbit of x_0. In the following, we keep the convention of considering G_{x_0} as the groupoid of *right* germs.

Despite our many efforts, we have not been able to prove Conjecture 1.22 in all its generality. However, with the perspective of proving Theorem B, we have the following:

Theorem D. *Let $G \subset \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ be a finitely generated group of C^ω diffeomorphisms, such that the action of G is minimal, has property $(\star)$ and a non-expandable point $x_0 \in \mathbf{S}^1$. Suppose that the Schreier graph $\mathrm{Sch}(X, \mathcal{G})$ of the orbit X of the non-expandable point $x_0 \in \mathrm{NE}$ has infinitely many ends. Then the groupoid of one-sided germs G_{x_0} has also infinitely many ends.*

Remark 1.23. In the statement of Theorem D, one may replace the grupoid G_{x_0} by the group G, as in real-analytic regularity there is no difference. However, we prefer to talk of grupoids because our proof is formulated in this language, hoping one day to get rid of the strong regularity assumption.

Remark 1.24. It is important to stress that the assumption for C^3 regularity is unavoidable for our proof of Theorem C. Indeed, we are able to offer a proof only using control on the *projective distortion* of the elements of the group, which classically uses the Schwarzian derivative and hence requires three derivatives. However, we hope that Theorem C can be generalized to actions of class C^2.

Motivations – It is perhaps worthwhile to extend this discussion to the description of the dynamical properties of actions on the circle. First, notice that the notion of ergodicity can be naturally extended to transformations with *quasi-invariant* measures (as for example the Lebesgue measure for any C^1 action) as saying that any G-invariant subset of the circle has either full or zero Lebesgue measure. Now, going back to the 80s, it was observed by Shub and Sullivan [41] that expanding actions of groups $G \subset \mathrm{Diff}^{1+\alpha}_+(\mathbf{S}^1)$ have nice *ergodic* properties: if the action is minimal then it is also ergodic with respect to the Lebesgue measure, whereas if the action has an exceptional minimal set Λ, then the Lebesgue measure of Λ is zero and the complementary set $\mathbf{S}^1 \setminus \Lambda$ splits into finitely many distinct orbits of intervals (or *gaps*). An analogous result was known for $\mathbb{Z}$ actions by C^2 circle diffeomorphisms: in case of minimality (which, according to Denjoy's theorem, is equivalent to that nontrivial elements have irrational rotation number [7]), the action is Lebesgue ergodic (this was independently proven by Katok [28] and Herman [26]).

One of the motivations for studying local flows for non locally discrete groups (see for instance [37]) was to extend the method of Katok and Herman to more general actions. Indeed, the group generated by a minimal circle diffeomorphism f is the most natural example of a non locally discrete group: if (q_n) is the sequence of denominators of the rational approximations of the rotation number of f, then the sequence f^{q_n} tends to the identity in the C^1 topology (see [26, Ch. VII] and also [36]).

One of the key ingredients behind these results is the technique of control of the affine distortion of the action (highly exploited throughout this paper as well). In the 80s, this suggested the conjecture that the picture above should hold as soon as control of distortion can be sought.

Conjecture 1.25 (Ghys, Sullivan)**.** *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group whose action on the circle is minimal. Then the action is also Lebesgue ergodic.*

Conjecture 1.26 (Ghys, Sullivan; Hector)**.** *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group whose action on the circle has an exceptional minimal set Λ. Then the Lebesgue measure of Λ is zero, and the complementary set $\mathbf{S}^1 \setminus \Lambda$ splits into finitely many orbits of intervals.*

Property $(\star)$ was first introduced in [10] as a property under which these conjectures can be established by somewhat standard techniques. Roughly, as we already mentioned, from the set $\mathrm{NE} \cap \Lambda$ of non-expandable points it is possible to define an expansion procedure. More precisely, as done in [18], one defines Markov partition of the minimal set, with a non-uniformly expanding map encoding the dynamics of G. This allows to extend the technique of Shub and Sullivan and prove the Conjectures 1.25 and 1.26 for groups with property $(\star)$.

State of the art – We hope that at this point the reader has got a flavour of why it is very important to verify that locally discrete groups have property $(\star)$. After the results discussed in this work, in the real-analytic framework, we are still left with one class of groups.

"Missing Piece" Conjecture. *Let $G \subset \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ be a finitely generated, one-ended group. Assume that G is neither finitely presented nor has a sequence of torsion elements of unbounded order. Then G cannot be locally discrete.*

For a brief summary, see also Table 1. This simplified conjecture needs further comments. Our impression is that if any counter-example existed, it should be very pathological. The feeling is that a locally discrete group of $\mathrm{Diff}_+^\omega(\mathbf{S}^1)$ should be *Gromov-hyperbolic*. Finitely generated Gromov-hyperbolic groups are always finitely presented and have bounded torsion (see [3, Ch. III.Γ]). Even if we are still not able to prove Gromov-hyperbolicity for general locally discrete groups, this has been done in one particular case:

Theorem 1.27 (Deroin)**.** *Let $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a locally discrete, finitely generated group whose action on the circle is minimal and expanding. Then G is Gromov-hyperbolic.*

Unfortunately, Deroin has not published the proof of this result yet. His announced result is actually stronger, and suggests that locally discrete groups of $\mathrm{Diff}_+^\omega(\mathbf{S}^1)$ carry some Fuchsian geometry:

Theorem 1.28 (Deroin)**.** *Let $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a locally discrete, finitely generated group whose action on the circle is minimal and expanding. Then G is analytically conjugate to a finite central extension of a cocompact Fuchsian group.*

The interested reader may consult the survey [8] for getting an idea of the landscape growing around the study of locally discrete groups.

infinitely many ends	**two ends**	**one end and** $\mathrm{NE} = \emptyset$	**one end and** $\mathrm{NE} \neq \emptyset$
virtually free	virtually $\mathbb{Z}$	finite central extension of a cocompact Fuchsian group	conjectured to be impossible

Table 1: Classification of locally discrete subgroups of $\mathrm{Diff}_+^\omega(\mathbf{S}^1)$.

2 Theorem A: Property ($\star$) for groups with infinitely many ends

2.1 Stallings' theorem and virtually free groups

An idea which can be traced back to Hector (and Ghys) [21] is that we can use the knowledge of the action of G to restrict the possible Stallings' decompositions of a group G acting by real-analytic diffeomorphisms of the circle and admitting an exceptional minimal set. As a first illustrative example, let us sketch an argument by Hector under the additional assumption of no torsion [21, Proposition 4.1].

Theorem 2.1 (Hector)**.** *Let $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a finitely generated, torsion-free group acting with an exceptional minimal set. Then G is free.*

Proof. Duminy's theorem (Theorem 1.20) implies that G has infinitely many ends, so there is a Stallings' decomposition. Since the group is torsion free, the Stallings' decomposition must be a free product $G = G_1 * G_2$ of finitely generated groups G_1 and G_2. Now, neither factor acts minimally (otherwise G does). If one of the factors acts with an exceptional minimal set, then we can expand the free product $G_1 * G_2$ until the moment we get $G = H_1 * \ldots * H_n$ with every H_i acting with some periodic orbit. Indeed, this procedure has to stop in a finite number of steps, for the rank (the least number of generators) of the factors is less than the rank of the group (this follows from a classical formula of Grushko; see [31]). Now we use that the action is by real-analytic diffeomorphisms. Namely, Corollary 1.3 implies that the groups H_i's must be either cyclic or direct products of an infinite cyclic group with a finite group. Since the group G is torsion-free, the only possibility is that every H_i is infinite cyclic. Thus, G is free, as claimed. □

In a similar manner, we can sketch the proof of Ghys' Theorem 1.21 under the assumption that the group G acting on the circle with an exceptional minimal set verifies a certain hypothesis, called *Dunwoody's accessibility.* Finitely generated groups with 0 or 1 ends are accessible (by definition) and, in general, accessible groups are all those groups that can be obtained as amalgamated products or HNN extensions of accessible groups over finite groups. Dunwoody proved that finitely presented groups are accessible [13], but there are finitely generated groups that are not accessible [14].

Theorem 2.2 (Ghys)**.** *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ be a finitely generated, accessible group acting with an exceptional minimal set. Then G is virtually free.*

Proof. Starting with a Stallings' decomposition of G, say $G = G_1 *_Z G_2$ or $H*_Z$, we argue as before that the groups G_1 and G_2 or H cannot act minimally. If the action of one of these groups has a finite orbit, then the group is virtually cyclic (Corollary 1.3). Otherwise, it acts with an exceptional minimal set and Duminy's Theorem 1.20 applies, so we can take a Stallings' decomposition and keep repeating this argument. Accessibility guarantees that this process stops after a finite number of steps, so the group G is obtained by a (finite) combination of amalgamated products and HNN extensions over finite groups, with virtually cyclic groups as basic pieces. Finally, these groups are *virtually free*, as one deduces from the following classical theorem [27]:

Theorem 2.3 (Karrass, Pietrowski, Solitar)**.** *Let G_1, G_2 and H denote finitely generated, virtually free groups and Z a finite group. Then the amalgamated product $G_1 *_Z G_2$ and the HNN extension $H*_Z$ are also virtually free.*

□

The rest of this section is dedicated to the proof of Theorem A.

2.2 Proof of Theorem A: Preliminaries

Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ be a locally discrete, finitely generated subgroup with infinitely many ends acting minimally on the circle. Assume that G admits non-expandable points. By Stallings' theorem, we know that either $G = G_1 *_Z G_2$ or $G = H*_Z$, with Z a finite group. For the proof of Theorem A, we analyse the factors appearing in Stallings' decompositions, as we have just illustrated.

2.3 First (possible) case: No Stallings' factor acts minimally

If such a factor has a finite orbit, then it is virtually cyclic by Corollary 1.3. Otherwise, it acts with an exceptional minimal set, and Ghys' Theorem 1.21 implies that it is virtually free. Therefore, G is either an amalgamated product of virtually free groups over a finite group or an HNN extension of a virtually free group over a finite group. By the already mentioned theorem of Karrass, Pietrowski and Solitar (Theorem 2.3), the group G itself is virtually free. We deduce that the group satisfies ($\star$) by Theorem 1.18.

2.4 Second (impossible) case: At least one factor acts minimally

Under this assumption, we will prove that G is non locally discrete borrowing one of the main arguments from [11]. To do this, remark that it is enough to study the case where $G = G_1 *_Z G_2$ is an amalgamated product, since any HNN extension $H*_Z$ contains copies of $H *_Z H$ as subgroups. Indeed, if we denote by σ the *stable letter* (that is, the element conjugating the two embedded copies of Z) in $H*_Z$, then H and $\sigma H \sigma^{-1}$ generate a subgroup isomorphic to $H *_Z H$.

Thus, from now on, we suppose that G is an amalgamated product $G_1 *_Z G_2$ over a finite group Z, and we assume that G_1 acts minimally. In particular G_1 is infinite, while G_2 can possibly be finite. For simplicity, we let $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$ be a finite system of generators for G, with $\mathcal{G}_i$ generating G_i and symmetric. We consider the length metric on the group G associated with this generating system, and for every $n \in \mathbb{N}$ we let $B(n)$ be the ball of radius n centred at the identity. Given a finite set $E \subset G$, let $\rho(E)$ denote the *outer radius* of E, that is, the minimal $n \in \mathbb{N}$ such that $E \subset B(n)$.

Let us illustrate the main lines of the proof before getting involved in technicalities. This will be also the opportunity to introduce some notation.

We fix a non-expandable point $x_0 \in \mathrm{NE}$, and for any finite set $E \subset G$, we let x_E denote the closest point on the right of x_0 among the points in the image set $E \cdot x_0$ distinct from x_0 (such a point exists for any E which is not contained in the stabilizer of x_0). This point corresponds to some $g_E \in E$, that is, $x_E = g_E(x_0)$. Besides, g_E is uniquely defined modulo right multiplication with an element in $\mathrm{Stab}_G(x_0)$. The length of the interval $J_E = [x_0, x_E]$ will be denoted by $\ell(E)$.

In order to take account of the number of elements fixing x_0, and hence of possible overlaps of the intervals $g(J_E)$, for $g \in E$, we define

$$c_E = \max_{h \in E} \#\left(E \cap h\,\mathrm{Stab}_G(x_0)\right).$$

Recall that, under our assumption of real-analytic regularity, the stabilizer of x_0 is either trivial or infinite cyclic (Theorem 1.2).

As in [11,19], the proof is carried on in three different stages, which will be exposed separately in the next paragraphs (with an intermezzo concerning some group theoretical aspects).

Step 1. – The first and most important step (Proposition 2.6) is to describe a *sufficient condition* guaranteeing that for a prescribed sequence of finite subsets $E(n) \subset G$, setting $F(n) = E(n)^{-1}E(n)$, the elements $g_{F(n)}$ "locally converge" (in the C^1 topology) to the identity. In concrete terms, letting

$$S_E = \sum_{g \in E} g'(x_0),$$

we will show that, in order to ensure the desired convergence, it is enough that

$$\rho(E(n))\frac{c_{E(n)}}{S_{E(n)}} = o(1) \quad \text{as } n \text{ goes to infinity.} \tag{2.1}$$

Notice, however, that this criterion does not provide a contradiction to the hypothesis of local discreteness of G, since we are only able to show that $g_{F(n)}$ is closer and closer to *id when restricted* to (a complex extension of) an interval containing $J_{F(n)}$, which is unfortunately shrinking to x_0.

Remark 2.4. In the following, we will see that will deal both with C^0 and C^1 local convergence. In fact, as the elements are real-analytic, the classical Cauchy estimates imply that the two notions are equivalent. The point is that for proving that the sequence of elements $g_{F(n)}$ converges C^0 to the identity, we first prove that the derivatives converge to 1 and deduce from the control of the affine distortion that the elements converge C^0.

Step 2. – We then show that it is very easy to find examples of sequences $(E(n))_{n\in\mathbb{N}}$ which satisfy the criterion above, even in a very strong way. For this, we use three key facts:

1. G_1 acts minimally, hence taking a sufficiently large integer $n \in \mathbb{N}$, the sum $\sum_{g \in B_1(n)} g'(x)$ can be made as large as we want (Proposition 2.27). Here, $B_1(n)$ is the ball of radius n in G_1 with respect to the generating set $\mathcal{G}_1$.

2. Using the tree-like structure and the normal form in amalgamated products, we move from a G_1-slice in G to another. Doing this, we increase the lower bound for $S_{E(n)}$ in an exponential way (Proposition 2.30). As a consequence, there exists $a > 1$, such that

$$S_{E(n)} \geq a^{\rho(E(n))}.$$

3. At the same time, studying how the stabilizer $\mathrm{Stab}_G(x_0)$ sits inside G, we prove that $c_{E(n)}$ has at most linear growth in terms of $\rho(E(n))$ (Proposition 2.31). This estimate turns to be fine enough, since $S_{E(n)}$ grows exponentially.

Step 3. – The key idea here relies on a result of Ghys [23, Proposition 2.7] (that can be traced back to Gromov [6, § 7.11.E1]) about groups of analytic local diffeomorphisms defined on the complex neighbourhood $U_r^{\mathbb{C}}(x_0)$ of radius $r > 0$ of $x_0 \in \mathbb{C}$:

Proposition 2.5. *For any $r > 0$ there exists $\varepsilon_0 > 0$ with the following property: Assume that the complex analytic local diffeomorphisms f_1, $f_2 : U_r^{\mathbb{C}}(x_0) \to \mathbb{C}$ are ε_0-close (in the C^0 topology) to the identity, and let the sequence f_k be defined by the recurrence relation*

$$f_{k+2} = [f_k, f_{k+1}], \quad k = 1, 2, 3, \ldots$$

Then all the maps f_k are defined on the disc $U_{r/2}^{\mathbb{C}}(x_0)$ of radius $1/2$, and f_k converges to the identity in the C^1 topology on $U_{r/2}^{\mathbb{C}}(x_0)$.

The main point of this proposition is that if the sequence of iterated commutators $(f_k)_{k\in\mathbb{N}}$ is not eventually trivial, then f_1 and f_2 generate a group which is non locally discrete.

From the previous steps, it is not difficult to find elements f_1, f_2 of the form $g_{E(m)}$ which are very close to the identity on some neighbourhood of x_0, but we must exhibit explicit f_1 and f_2 for which we are able to show that the sequence of iterated commutators f_k is not eventually the identity. This is certainly the case if f_1 and f_2 generate a free group: we prove in Proposition 2.33 that it is possible to find such two elements. In order to do this, we introduce a good geometric setting (namely the action on the Bass-Serre tree) and find a ping-pong configuration.

2.5 Step 1: Getting close to the identity

Here we review the argument given in [11, § 3.2] and [19, § 2.5], which explains how to find elements which are close to the identity in a neighbourhood of a non-expandable point. The result is stated in a general form, because of the algebraic issues that we have to overcome in § 2.8. The main result of this section is a variation of [11, Lemma 3.15]:

Proposition 2.6. *Let $(E(n))_{n\in\mathbb{N}}$ be a sequence of subsets of G containing the identity. If*

$$\rho(E(n))\frac{c_{E(n)}}{S_{E(n)}} = o(1) \quad \text{as } n \text{ goes to infinity,}$$

then the sequence $g_{F(n)}$ for $F(n) = E(n)^{-1}E(n)$ converges to the identity in the C^1 topology on a complex disc of radius $o(1/\rho(E(n)))$ around x_0. More precisely, considering $r_n = o\left(1/\rho(E(n))\right)$ such that

$$\frac{c_{E(n)}}{S_{E(n)}} = o\left(r_n\right) \quad \text{as } n \text{ goes to infinity,}$$

the (affinely) rescaled sequence

$$\widetilde{g}_{F(n)}(t) = \frac{g_{F(n)}(x_0 + r_n\, t) - x_0}{r_n}$$

converges to the identity in $C^0(U_1^{\mathbb{C}}(0))$*.*

We avoid the (somehow technical) details of the proof and prefer to explain the relevant ideas, which mostly rely on the classical technique of *control of affine distortion* (see [11, Lemma 3.7]). To remind this, recall that if $J \subset \mathbf{S}^1$ is an interval, the *distortion coefficient* of a diffeomorphism $g : J \to g(J)$ on J is defined as

$$\varkappa(g; J) = \sup_{x,y \in J} \left| \log \frac{g'(x)}{g'(y)} \right| .$$

This measures how far is g to be an affine map. Besides, this is well behaved under composition and inversion:

$$\varkappa(gh; J) \le \varkappa(g; h(J)) + \varkappa(h; J), \quad \varkappa(g; J) = \varkappa(g^{-1}; g(J)).$$

If we fix a finite generating system $\mathcal{G}$ of the group G and set $C_{\mathcal{G}} = \max_{g \in \mathcal{G} \cup \mathcal{G}^{-1}} \sup_{\mathbf{S}^1} |g''/g'|$, then

$$\varkappa(g; J) \le C_{\mathcal{G}}\, |J| \quad \text{for every } g \in \mathcal{G}.$$

This implies that if $g = g_n \cdots g_1$ belongs to the ball of radius n in G, $g_i \in \mathcal{G}$, then

$$\varkappa(g_n \cdots g_1; J) \le C_{\mathcal{G}} \sum_{i=0}^{n-1} |g_i \cdots g_1(J)|, \tag{2.2}$$

where $g_i \cdots g_1 = id$ for $i = 0$.

The inequality (2.2) suggests that the control of the affine distortion of g on some small interval J can be controlled by the *intermediate compositions* $g_i \cdots g_1$. This is better explained in the following way: Let

$$S = \sum_{i=0}^{n-1} (g_i \cdots g_1)'(x_0) \tag{2.3}$$

denote the sum of the intermediate derivatives at some *single* point $x_0 \in \mathbf{S}^1$. Then the affine distortion of g can be controlled in a (complex) neighbourhood of radius $\sim 1/S$ about x_0. More precisely, we have the following statement (which goes back to A. Schwartz and, later, to Sullivan):

Proposition 2.7. *For a point* $x_0 \in \mathbf{S}^1$ *and* $g \in B(n)$*, let* S *be as in* (2.3) *and* $c = \log 2/4C_{\mathcal{G}}$*. For every* $r \le c/S$*, we have the following bound on the affine distortion of* g*:*

$$\varkappa(g; U_r^{\mathbb{C}}(x_0)) \le 4C_{\mathcal{G}} S r.$$

The key observation in our framework (and originally of [11,19]) is that at non-expandable points $x_0 \in \mathrm{NE}$, we obviously have $S \le n$ for $g \in B(n)$. Therefore, for a very large n, in a neighbourhood of size $r \ll 1/n$ about x_0, the maps in $B(n)$ are almost affine. In particular, the element $g_{F(n)}$ (resp. $\widetilde{g}_{F(n)}$) is almost affine on a neighbourhood of radius $r_n = o\,(1/\rho(E(n)))$ (resp. 1) about x_0 (resp. 0).

To see that the derivative of $g_{F(n)}$ (and $\widetilde{g}_{F(n)}$) is close to 1, we consider the inverse map $g_{F(n)}^{-1}$, which satisfies

$$(g_{F(n)}^{-1})'(x_0) \le 1 \quad \text{and} \quad (g_{F(n)}^{-1})'(x_{F(n)}) = \frac{1}{g_{F(n)}'(x_0)} \ge 1.$$

The point $x_{F(n)}$ is at distance $\ell_{F(n)}$ from x_0. If $\ell_{F(n)} = o(r_n)$, then the control on the affine distortion guarantees that the derivative of $g_{F(n)}^{-1}$, and hence of $g_{F(n)}$, is close to 1 on the neighbourhood of radius r_n. Indeed, for every $z \in U_r(x_0)$ one has

$$\log(g_{F(n)}^{-1})'(z) = \log \frac{(g_{F(n)}^{-1})'(z)}{(g_{F(n)}^{-1})'(x_0)} + \log(g_{F(n)}^{-1})'(x_0) \leq \sup_{x,y \in U_r(x_0)} \log \frac{(g_{F(n)}^{-1})'(x)}{(g_{F(n)}^{-1})'(y)}$$

and

$$\log(g_{F(n)}^{-1})'(z) = \log \frac{(g_{F(n)}^{-1})'(z)}{(g_{F(n)}^{-1})'(x_{F(n)})} + \log(g_{F(n)}^{-1})'(x_{F(n)}) \geq \inf_{x,y \in U_r(x_0)} \log \frac{(g_{F(n)}^{-1})'(x)}{(g_{F(n)}^{-1})'(y)}.$$

Thus $\sup_{U_r(x_0)} |\log(g_{F(n)}^{-1})'| \leq \varkappa(g_{F(n)}^{-1}, U_r^{\mathbf{C}}(x_0))$.

The asymptotic condition $\ell_{F(n)} = o(r_n)$ assures that also the map $\widetilde{g}_{F(n)}$ is almost the identity, since $\widetilde{g}_{F(n)}(0) = \ell_{F(n)}/r_n$. Therefore, we get the desired conclusion from the following key estimate:

Lemma 2.8. *Let $E \subset G$ be a finite subset of G containing the identity and define $F = E^{-1}E$. Then the length ℓ_F verifies*

$$\ell_F \leq C\,\frac{c_E}{S_E},$$

where the constant $C > 0$ does not depend on E.

Sketch of the proof. We observe that any two intervals $g(J_F)$ and $h(J_F)$, for $g, h \in E$, are either disjoint or have the same leftmost points, with equality if and only if $g \in h\,\mathrm{Stab}_G(x_0)$. Indeed, suppose that the left endpoint of $h(J_F)$ belongs to $g(J_F)$. Then $h^{-1}g(x_0)$ is closer than x_F to x_0 on the right, and since $h^{-1}g \in E^{-1}E = F$, we must have $h^{-1}g(x_0) = x_0$, that is, $g \in h\,\mathrm{Stab}_G(x_0)$.

Therefore, the union of the intervals $g(J_F)$, for $g \in E$, covers the circle $\mathbf{S}^1$ at most c_E times. With the (quite subtle) argument in [11, Lemma 3.15] relying on the control of the affine distortion, we find

$$\ell_F \leq C\,\frac{c_E}{S_E},$$

as desired. □

2.6 Intermezzo: Basic Bass-Serre theory for amalgamated products and actions on trees

In this part we recall some elementary facts about groups acting on trees. These are well-known results, but we give details to develop the geometrical intuition behind the combinatorial work needed for the rest of the proof of Theorem A.

Normal forms − Every element in an amalgamated product can be written in a *normal form* (see [31, 40]).

Lemma 2.9. *Fix transversal sets of cosets $T_1 \subset G_1$ and $T_2 \subset G_2$ for ${}_Z\backslash^{G_1}$ and ${}_Z\backslash^{G_2}$ respectively, both containing the identity. Then every element $g \in G$ has a unique factorization as $g = \gamma\, t_n \cdots t_1$, with $\gamma \in Z$ and $t_j \in T_{i_j} \setminus \{id\}$, with none of two consecutive i_j's equal.*

We sketch a geometrical proof of this lemma using Bass-Serre theory [40]. Every amalgamated product acts isometrically on a simplicial tree without edge-inversion, namely the *Bass-Serre tree*, that we denote it by X. Bass-Serre theory holds more generally, but for an amalgamated product

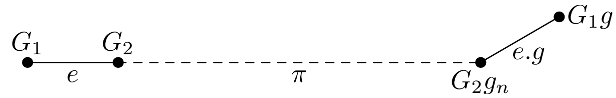

Figure 2.1: The geodesic path in the Bass-Serre which determines the normal form. Here is an example where we have $t_1 \in G_2$ and $t_n \in G_2$.

$G = G_1 *_Z G_2$, the tree and the action on it have a simple description: the vertices are the cosets $\{G_i g \mid g \in G, i = 1, 2\}$, and the edges are $\{(G_1 g, G_2 g) \mid g \in G\}$. The group G acts by *right* multiplication: $G_i g.\varphi = G_i g\varphi$. The edge $e = (G_1, G_2)$ is a *fundamental domain* for the action of G on X: each factor group G_i coincides with the stabilizer of the vertex G_i, and $Z = G_1 \cap G_2$ is the stabilizer of the edge e.

Remark that if $(G_1 g, G_2 g)$ and $(G_1 g', G_2 g')$ represent the same edge, then we have $G_i g = G_i g'$ for $i = 1, 2$. We deduce that $g' g^{-1}$ belongs to the intersection $G_1 \cap G_2 = Z$. So $g' = \gamma g$ for some $\gamma \in Z$.

Proof of Lemma 2.9. If an element $g \in G$ belongs to a factor group G_i, then there is a unique $t \in T_i$ and $\gamma \in Z$ such that $g = \gamma t$.

If an element g is not in a factor group, then the fundamental domain e and its image $e.g$ do not intersect. Therefore, since X is a tree, there is a unique geodesic path π connecting e to $e.g$ (see Figure 2.1). The path is of the form

$$\pi = (G_{i_1} = G_{i_1} g_1, G_{i_2} g_2, G_{i_3} g_3, \dots, G_{i_n} g_n = G_{i_n} g),$$

with the g_k's verifying $G_{i_k} g_k = G_{i_k} g_{k-1}$ for every $k = 2, \dots n$, and none of two consecutive i_j's equal. From the remark above, the g_k's are uniquely defined modulo Z. However, if the transversal sets T_1 and T_2 are given, then we can write every g_k in the form

$$\begin{aligned} g_1 &= t_1, \\ g_2 &= t_2 t_1, \\ &\dots \\ g_n &= t_n \cdots t_1, \text{ with every } t_j \in T_{i_j} \setminus \{id\}, \end{aligned}$$

which is unique. Since $G_{i_n} g_n = G_{i_n} g$, $G_{i_{n+1}} g_{n+1} = G_{i_{n+1}} g$ and $G_{i_{n+1}} g_{n+1} = G_{i_{n+1}} g_n$, the product $g g_n^{-1} = \gamma$ belongs to $Z = G_{i_n} \cap G_{i_{n+1}}$. $\square$

Remark 2.10. Consider an element $g \in G = G_1 *_Z G_2$, written in normal form as $g = \gamma\, t_n \cdots t_1$. Observe that if g is written differently as $g = s_k \cdots s_1$ with every $s_j \in G_{i_j} \setminus Z$ and none of two consecutive i_j's equal, then $k = n$, and for every $j = 1, \dots, n$, the factor t_j belongs to G_{i_j}. Moreover every quotient $t_j^{-1} s_j$ belongs to Z.

Indeed, the length n is exactly the length of the geodesic path in the Bass-Serre tree π from the edge $e = (G_1, G_2)$ to the edge $e.g = (G_1 g, G_2 g)$, and the indices i_j's correspond to the vertices visited by the path.

We also have that the inverse g^{-1} can be written in a normal form of length n, since the geodesic path from e to $e.g^{-1}$ is the translation $\pi.g^{-1}$ (with opposite orientation), of the path π from e to $e.g$.

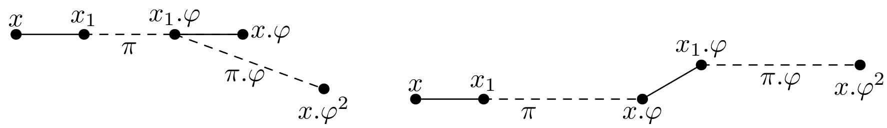

Figure 2.2: Existence of the translation axis for a hyperbolic element.

Tree isometries – Let us study in more detail the action of G on its Bass-Serre tree. The reader may consult [5] for a general description of actions on (real) trees.

As in the previous proof, we denote by X the Bass-Serre tree of G and by d the graph metric on the tree X. We keep the convention of right action.

Definition 2.11. Given an isometry φ of the tree (X, d), we denote by $\ell(\varphi)$ its *translation length*:

$$\ell(\varphi) = \min\{d(x, x.\varphi) \mid x \in X\}. \tag{2.4}$$

If $\ell(\varphi) = 0$ then φ is *elliptic*, otherwise, φ is *hyperbolic*.

Observe that the minimum in (2.4) is attained because the distance d on X takes discrete values. In particular, we have:

Lemma 2.12. *Let $G = G_1 *_Z G_2$ be an amalgamated product and let X be its Bass-Serre tree. Take an element $\varphi \in G$. The following statements are equivalent:*

1. *the element φ belongs to a conjugate factor group (i.e. a subgroup of G of the form gG_ig^{-1});*
2. *φ fixes a point in X;*
3. *φ is elliptic, that is, $\ell(\varphi) = 0$.*

Any tree isometry φ has a natural *invariant set* $X(\varphi)$, which is a convex subset of X. This is the union of the minimal invariant sets. More explicitly, for an elliptic element, $X(\varphi)$ is defined as the set of fixed points of φ. Observe that φ fixes more than one point if and only if φ belongs to some conjugate of the edge group Z.

For hyperbolic elements, the invariant set is described as follows:

Lemma 2.13. *If $\varphi \in G$ is hyperbolic, the invariant set $X(\varphi)$ is a* translation axis, *i.e. an invariant bi-infinite geodesic line in X, on which φ acts as a translation of displacement $\ell(\varphi)$.*

Moreover, for any vertex $x \in X$, one has

$$d(x, x.\varphi) = \ell(\varphi) + 2d(x, X(\varphi)). \tag{2.5}$$

Proof. Consider a point $x \in X$ that minimizes the translation length: $d(x, x.\varphi) = \ell(\varphi)$. We denote by $\pi = (x = x_0, x_1, \ldots, x_{\ell(\varphi)} = x.\varphi)$ the geodesic path from x to $x.\varphi$ in X. We claim that the segments π and $\pi.\varphi$ only intersect at $x.\varphi$. Indeed, since X is a tree, the intersection $\pi \cap \pi.\varphi$ is connected, and if it were not a point, then the points x_1 and $x_1.\varphi$ would be at distance $\ell(\varphi) - 2$, contradicting the minimality (see Figure 2.2 on the left). Therefore, the union

$$X(\varphi) = \bigcup_{n \in \mathbb{Z}} \pi.\varphi^n \tag{2.6}$$

is a bi-infinite geodesic in X, on which φ acts as a translation by $\ell(\varphi)$ (see Figure 2.2 on the right).

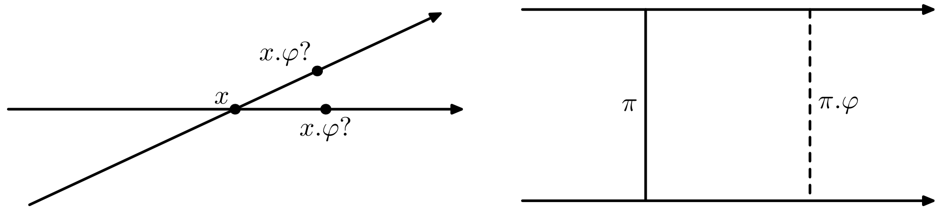

Figure 2.3: Uniqueness of the translation axis for a hyperbolic element.

Let us prove that the translation axis is unique. First suppose that two translation axes intersect. Then the intersection is a (possibly infinite) interval that is φ-invariant. Since φ acts as a translation on the intersection, the only possibility is that both axes actually coincide. Secondly, if the intersection is empty, then the two axes must be parallel. More precisely, the set of shortest geodesic paths connecting the two axes is invariant under the isometry φ. However, such a geodesic path is unique in a tree, hence it must be fixed by φ (see Figure 2.3). This contradicts $\ell(\varphi) > 0$.

Finally, let us prove (2.5). Let γ be the geodesic segment from x to $X(\varphi)$, with endpoint $y \in X(\varphi)$. Then $\gamma.\varphi$ is the geodesic segment from $x.\varphi$ to $X(\varphi)$, with endpoint $y.\varphi$. Then we have

$$d(x, x.\varphi) = d(x, y) + d(y, y.\varphi) + d(y.\varphi, x.\varphi),$$

from which one easily deduces (2.5). □

Remark 2.14. The relation (2.5) holds even for an elliptic isometry φ, in which case we simply have

$$d(x, x.\varphi) = 2d(x, X(\varphi)).$$

More precisely, if γ is the geodesic segment from x to $X(\varphi)$, with endpoint $y \in X(\varphi)$, then $\gamma.\varphi$ is the geodesic segment from $x.\varphi$ to $X(\varphi)$, with endpoint $y.\varphi = y$.

The following result gives a geometric condition for detecting the position of the invariant set of a tree isometry:

Proposition 2.15. *Let G be a group acting isometrically on a tree X. Let $x \in X$ be a fixed vertex of the tree and $\varphi \in G$ be any element. Let π^-, π^+ denote respectively the oriented geodesic paths in X connecting x to $x.\varphi^-$, $x.\varphi^+$, starting from x. Suppose that π^- and π^+ share the first edge e. Then the invariant set $X(\varphi)$ is contained in the subtree of X which is the connected component of $X \setminus \{x\}$ containing e.*

Remark 2.16. As it will appear clear from the proof (*cf.* also Remark 2.14), if the element φ is elliptic, then it is enough to look at the geodesic path from x to $x.\varphi$: if π^+ starts with the edge e, so does π^-.

Proof. Suppose first that the element $\varphi \in G$ is elliptic. The statement is clearly empty if $x = x.\varphi$, so we suppose that we are not in this situation. Let π^+ be the geodesic segment connecting x to $x.\varphi$. As described in Remark 2.14, the invariant set $X(\varphi)$ intersects the path π^+ exactly at its middle point.

Suppose now that φ is hyperbolic. Observe that our hypothesis guarantees that x does not belong to the hyperbolic axis $X(\varphi) = X(\phi^{-1})$, because otherwise the paths π^- and π^+ would intersect only at x. Then, as we saw in Lemma 2.13, the paths $\pi^\pm$ from x to $x.\varphi^{\pm 1}$ decompose into three nontrivial pieces: first, the path reaches the translation axis $X(\varphi)$, then it crosses the axis along a segment of length $\ell(\varphi)$, and finally it goes out of $X(\varphi)$ to reach the image $x.\varphi$. By uniqueness of geodesics in a tree, the first pieces for π^- and π^+ must coincide. This gives the result. □

Distorted elements – First, we recall the following:

Definition 2.17. An element φ of a finitely generated group G is undistorted (in G) if the length of the element φ^n grows linearly in n. (Notice that this definition is invariant under quasi-isometries and in particular it does not depend on the finite generating system chosen for defining the length metric on G.)

Lemma 2.18. *Let $G = G_1 *_Z G_2$ be an amalgamated product and let $\varphi \in G$ be a distorted element in G. Then φ is conjugate to an element into one of the two factors (and it is actually distorted in the conjugate factor with respect to the restricted metric).*

Proof. Because of Lemma 2.12, it is enough to prove that if the element φ is hyperbolic, then it is undistorted. Consider a point x on the axis $X(\varphi)$. Since φ acts by translation by $\ell(\varphi)$ on $X(\varphi)$, one has $d(x, x.\varphi^n) = |n|\ell(\varphi)$. If φ was distorted, then $d(x, x.\varphi^n)$ would have sublinear growth (apply the triangular inequality), but we have just proven that it grows linearly. □

Lemma 2.19. *Let G be a finitely generated virtually free group. Then every element of infinite order is undistorted in G.*

Proof. The statement can be seen as a consequence of Lemma 2.18 above and Theorem 2.3. There is however a simpler, more classical, proof. Indeed, up to quasi-isometry, it is enough to prove the result for a group G which is *free*. For free groups there are many ways to see this, here we choose to give an argument relying on actions on trees. Indeed, one of the first byproducts of the Bass-Serre theory is that a group is free if and only if it has a free action on a tree. So consider such a free action: every element acts as a hyperbolic isometry, so it is undistorted. □

Ping-pong and free groups – Let us first give a statement about commutators in a free group:

Lemma 2.20. *In the rank-two free group F_2, consider two free generators a and b. Define the sequence of iterated commutators*

$$\begin{cases} w_0 = a, \\ w_1 = b, \\ w_{k+2} = [w_k, w_{k+1}]. \end{cases}$$

Let H be the free subgroup generated by w_2 and w_3. Given an element $h \in H$, the following property holds: when writing h as a reduced word in the generating system $\{a^{\pm 1}, b^{\pm 1}\}$, then the expression does not contain $a^{\pm 2}$ and $b^{\pm 3}$ as subwords.

The following nice proof has been explained to us by Jarek Kędra on MathOverflow.

Proof. Every element in the commutator subgroup $[F_2, F_2]$ can be represented by an oriented closed path on the square grid $\mathbb{Z}^2$, starting at the origin: the letters a, b are represented by edges going to the right and up, respectively. Since the subgroup generated by w_2, w_3 is contained in $[F_2, F_2]$, we can use this interpretation for any element in H.

In this interpretation, the element w_2 is represented by a simple square loop, while w_3 is represented by a loop describing a "figure eight", namely two vertically adjacent squares (see Figure 2.4).

Thus every element in the group H describes a closed loop that is contained in the figure eight, simply because when concatenating $w_2^{\pm 1}, w_3^{\pm 1}$, the support of the loops cannot escape. In particular the reduced form for an element $h \in H$ cannot contain powers of $a^{\pm}$ exceeding 1, otherwise the support of the loop it represent would escape the figure eight from one of its vertical sides. Similarly we deduce that there is no power of $b^{\pm 1}$ exceeding 2. □

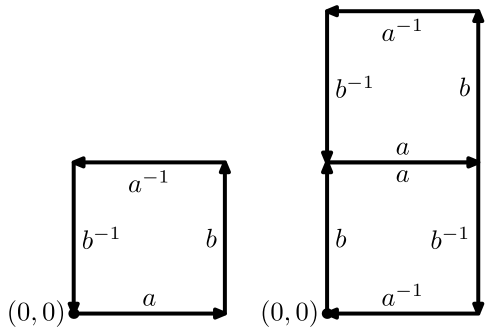

Figure 2.4: The paths representing the commutators w_2 (left) and w_3 (right).

Definition 2.21. Let G be a group acting of isometries of a tree X. Let $\beta \in \mathbb{N}$ be a positive integer. We say that G is β-bounded if for any isometry $\varphi \in G$ fixing an edge of X, then $\varphi^n = id$ for some $|n| \leq \beta$. In other words, β is a uniform upper bound on the order of isometries of G fixing edges.

Lemma 2.22. *Let G be a group of isometries of a tree X, which is β-bounded. If $\varphi \in G$ is such that there exists a positive integer $p \in \mathbb{N}$ such that φ^p fixes an edge, then φ has order at most βp.*

Proof. It follows directly from Definition 2.21 above. □

Lemma 2.23. *Let G be a group of isometries of a tree X, which is β-bounded. Consider an isometry $\varphi \in G$ whose order is at least 5β (possibly infinite).*

Consider a connected component C of the complement $X \setminus X(\varphi)$ of the invariant set of φ. Then for every power $p \in \{\pm 1, \dots, \pm 4\}$, the image $\varphi^p(C)$ has empty intersection with C.

Proof. Since C is a connected component of the complement of the invariant set $X(\varphi)$, there is a unique edge e connecting $X(f)$ to C.

Suppose there is $p > 0$ such that $\varphi^p \neq id$ and the intersection $\varphi^p(C) \cap C$ is not empty. The power φ^p must fix the edge e, so φ^p fixes one edge. As G is β-bounded, Lemma 2.22 implies that we have $\varphi^{p\beta} = id$. Thus, by hypothesis, we must have $p\beta \geq 5\beta$. This implies $p \geq 5$.

When $p < 0$, considering φ^{-1} we find similarly $p \leq -5$. This ends the proof. □

Now we can proceed to the main result of this paragraph, which is a variation on the classical ping-pong lemma:

Proposition 2.24 (Ping-pong)**.** *Let G be a group acting by isometries on a tree X, which is β-bounded. Let $\varphi, \psi \in G$ be two tree isometries such that:*

1. *their invariant sets are disjoint,*
2. *their order is at least 5β (possibly infinite).*

Then $h = [\varphi, \psi]$ and $k = [\psi, [\varphi, \psi]]$ generate a free subgroup of G.

Proof. Let φ and ψ be two isometries with disjoint invariant sets $X(\varphi)$ and $X(\psi)$. Denote by π the geodesic path in X connecting these two sets. Let a and b be the vertices on π that lie on $X(\varphi)$ and $X(\psi)$ respectively. We consider the following two subtrees of X:

1. A is the maximal subtree of X that contains a but not the rest of π;

2. B is the maximal subtree of X that contains b but not the rest of π.

Consider an element g in the group generated by h and k. Up to cyclical rewriting (that is, up to pass to a conjugate by an element in $\langle\varphi,\psi\rangle$)[1] the element g can be written it in the form

$$g = a_1 b_1 \cdots a_n b_n, \quad a_i \in \langle\varphi\rangle, b_i \in \langle\psi\rangle, \tag{2.7}$$

which is (formally) reduced in the free group $F(\varphi,\psi)$. Moreover Lemma 2.20 implies that

$$a_i \in \{\varphi^{\pm 1}, \cdots \varphi^{\pm 4}\}, b_i \in \{\psi^{\pm 1}, \cdots, \psi^{\pm 4}\} \quad \text{for every } i = 1, \ldots, n:$$

indeed the lemma says initially that powers are bounded by 2, however after a cyclical rewriting powers may increase up to 4. Thus, applying Lemma 2.23, we observe the following ping-pong dynamics:

$$B.a_i \subset A \quad \text{and} \quad A.b_i \subset B \quad \text{for every } i = 1, \ldots, n.$$

Therefore, if we apply g to B, we must have

$$g.B \subset A.$$

As A and B are disjoint, this implies that g is not the identity in G. □

Next, we detect the translation axis of certain hyperbolic elements.

Lemma 2.25. *Let $G = G_1 *_Z G_2$. Consider an element $\varphi \in G$ of the form*

$$\varphi = \sigma_n t_n \sigma_{n-1} t_{n-1} \cdots \sigma_1 t_1, \quad \textit{with } t_i \in G_1 \setminus Z, \sigma_i \in G_2 \setminus Z \textit{ for every } i = 1, \ldots n. \tag{2.8}$$

Set $e = (G_1, G_2)$. Then φ is hyperbolic, and its translation axis is

$$X(\varphi) = \bigcup_{k\in\mathbb{Z}} (\pi \cup e).\varphi^k,$$

where π is the geodesic path between e and the image $e.\varphi$. That is, $X(\varphi)$ is the bi-infinite geodesic path

$$X(\varphi) = (\ldots, G_2 t_n^{-1}\sigma_n^{-1}, G_1\sigma_n^{-1}, G_2, G_1, G_2 t_1, G_1\sigma_1 t_1, \ldots, G_2 t_n \sigma_{n-1}\cdots t_1, G_1\varphi, G_2 t_1 \varphi, \ldots). \tag{2.9}$$

In particular, we have $\ell(\varphi) = 2n$. (See Figure 2.5.)

Proof. We have to prove that the path (2.9) is geodesic. That is, we have to prove that there is no backtracking, which is the same as proving that any two vertices on it are distinct. This can be verified directly from the uniqueness of the normal form (Lemma 2.9 and Remark 2.10), noticing that the normal form of a power φ^k is

$$(\sigma_n t_n \sigma_{n-1} t_{n-1} \cdots \sigma_1 t_1) \cdots (\sigma_n t_n \sigma_{n-1} t_{n-1} \cdots \sigma_1 t_1),$$

with the $(\sigma_n t_n \sigma_{n-1} t_{n-1} \cdots \sigma_1 t_1)$ repeated k times. □

Remark 2.26. For any $g \in G$ and φ of the form (2.8), the translation axis of the conjugate $\psi = g\varphi g^{-1}$ is $X(\psi) = X(\varphi).g^{-1}$.

[1]Notice that the group generated by h, k is not normal, so the cyclical rewriting may take g out of this group. However this has no influence on the rest of the proof.

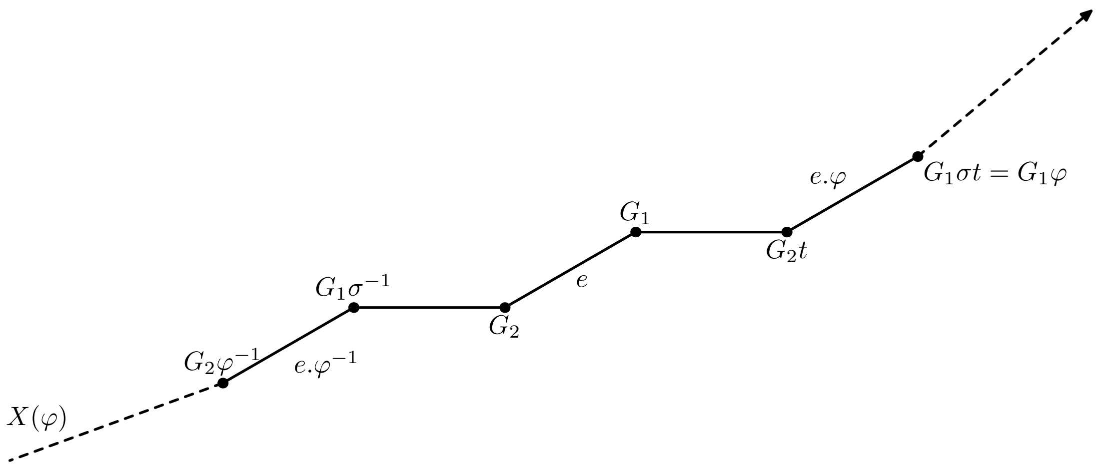

Figure 2.5: The translation axis of an element $\varphi = \sigma t$, with $t \in G_1 \setminus Z, \sigma \in G_2 \setminus Z$.

2.7 Step 2: An exponential lower bound for the sum of derivatives

Using the normal form of elements in an amalgamated product, we will use a tool developed in [11] for free groups. The aim of this step is to find a sequence of subsets $A(n)$ with an exponential lower bound for $S_{A(n)}$. We actually prove more: the exponential lower bound for the sum of the derivatives holds *at every point* $x \in \mathbf{S}^1$. This turns out to be very useful, since it gives exponential lower bounds for the sum $S_{\psi A(n)\psi^{-1}}$ associated to each conjugate set $\psi A(n)\psi^{-1}$ of $A(n)$, where $\psi \in G$.

We start by noticing that, since G_1 acts minimally, the proof of [11, Proposition 2.5] combined with a compactness type argument immediately yields:

Proposition 2.27. *For every $M > 0$, there exists $R_1 \in \mathbb{N}$ such that for every $x \in S^1$ we have*

$$\sum_{g \in B_1(R_1)} g'(x) > M, \tag{2.10}$$

where $B_1(R_1)$ is the ball of radius R_1 in G_1.

Using the previous proposition, we next prove:

Lemma 2.28. *For every $M' > 0$, there exists $R_1' \in \mathbb{N}$ such that*

$$\sum_{t \in B_1^{\times}(R_1') \cap T_1} t'(x) > M',$$

where $B_1^{\times}(R_1)$ is the ball $B_1(R_1)$ in G_1, but with the identity excluded.

Proof. Let $c_0 = |Z| \cdot \sup_{\gamma \in Z} \|\gamma'\|_0$. Take $M > c_0(1 + M')$ and fix the associated R_1 given by Proposition 2.27. Decomposing the sum (2.10) using the transversal set, we write

$$\sum_{g \in B_1(R_1)} g'(x) = \sum_{\gamma \in Z,\, t \in T_1 \,:\, \gamma t \in B_1(R_1)} (\gamma t)'(x). \tag{2.11}$$

Observe that, by the triangular inequality, one has the inclusion

$$\{g = \gamma t \mid \gamma \in Z,\, t \in T_1 \text{ such that } \gamma t \in B_1(R_1)\} \subset \{g = \gamma t \mid \gamma \in Z,\, t \in B_1(R_1 + \rho(Z)) \cap T_1\},$$

(recall that $\rho(Z)$ denotes the outer radius of the set Z). Thus the sum (2.11) is bounded from above by the same sum but over the larger set:

$$\sum_{g\in B_1(R_1)} g'(x) \leq \sum_{\gamma\in Z}\left(\sum_{t\in B_1(R_1+\rho(Z))\cap T_1} (\gamma t)'(x)\right).$$

Next, using the chain rule and taking care of the identity element, we obtain:

$$\begin{aligned}
M \leq \sum_{g\in B_1(R_1)} g'(x) \leq \sum_{\gamma\in Z} &\left(\sum_{t\in B_1(R_1+\rho(Z))\cap T_1} \gamma'(t(x))t'(x)\right) \\
&\leq |Z|\cdot \sup_{\gamma\in Z}\|\gamma'\|_0 \left(1+\sum_{t\in B_1^{\times}(R_1+\rho(Z))\cap T_1} t'(x)\right) \\
&= c_0\left(1+\sum_{t\in B_1^{\times}(R_1+\rho(Z))\cap T_1} t'(x)\right).
\end{aligned}$$

Setting $R_1' = R_1+\rho(Z)$, this closes the proof. □

If we now consider repeat alternate products by representatives in T_2, it is easy to construct a sequence of sets $A(n)$ with an exponential lower bound for the sum of the derivatives. Actually, it is enough to fix an element $\sigma\in T_2\setminus\{id\}$, and define the product set

$$A(n) = \sigma\left(B_1^{\times}(R_1')\cap T_1\right)\cdots\sigma\left(B_1^{\times}(R_1')\cap T_1\right),$$

where the product of $\sigma\left(B_1^{\times}(R_1')\cap T_1\right)$ is repeated n times and R_1' is appropriately chosen. Notice that $A(n)$ is contained in the ball of radius $n(R_1'+d_{\mathcal{G}}(id,\sigma))$ in G, so the outer radius of $A(n)$ grows at most linearly on n. Indeed, $\rho(A(n))\leq n(R_1'+d_{\mathcal{G}}(id,\sigma))$.

Lemma 2.29. *There exists $a>1$ such that for all $n\in\mathbb{N}$ and every $x\in\mathbf{S}^1$,*

$$\sum_{g\in A(n)} g'(x) \geq a^{\rho(A(n))}.$$

Proof. Take $M' > (\inf\sigma')^{-1}$ and the associated R_1' from Lemma 2.28. Let us consider all the products σt_1, with $t_1\in B_1^{\times}(R_1')\cap T_1$. We define $\overline{M} = M'\cdot\inf\sigma'$, which is larger than 1 by assumption. With this choice, we have

$$\begin{aligned}
\sum_{g\in A(n)} g'(x) \ &= \sum_{t_1,\dots,t_n\in B_1^{\times}(R_1')\cap T_1} (\sigma t_n\cdots\sigma t_1)'(x) \\
&\geq \overline{M}\cdot \sum_{t_1,\dots,t_{n-1}\in B_1^{\times}(R_1')\cap T_1} (\sigma t_{n-1}\cdots\sigma t_1)'(x).
\end{aligned}$$

Proceeding inductively, we get $S_{A(n)}(x)\geq \overline{M}^n$. Letting $a=\overline{M}^{1/\left(R_1'+d_{\mathcal{G}}(id,\sigma)\right)}$, we obtain the desired exponential lower bound. □

Finally, we have:

Proposition 2.30. *For any $\psi \in G$, there exists a constant $C(\psi)$ such that*

$$S_{\psi A(n)\psi^{-1}} \geq C(\psi)\, a^{\rho\left(\psi A(n)\psi^{-1}\right)}.$$

Proof. For $\psi \in G$, let $\lambda = \|\psi\|$ denote its length in the generating system $\mathcal{G}$. Then for any $n \in \mathbb{N}$, we have

$$\rho\left(\psi A(n)\psi^{-1}\right) \leq \rho(A(n)) + 2\lambda.$$

We can easily compare the sum $S_{\psi A(n)\psi^{-1}}$ with the sum of the derivatives of elements in $A(n)$:

$$\begin{aligned} S_{\psi A(n)\psi^{-1}} &= \sum_{g \in \psi A(n)\psi^{-1}} g'(x_0) \\ &= \sum_{h \in A(n)} (\psi h \psi^{-1})'(x_0) \\ &\geq \inf \psi' \cdot \sum_{h \in A(n)} h'(\psi^{-1}(x_0)) \cdot (\psi^{-1})'(x_0). \end{aligned}$$

Hence, by Lemma 2.29, we have the inequality

$$S_{\psi A(n)\psi^{-1}} \geq \left(\inf \psi' \cdot (\psi^{-1})'(x_0)\right) a^{\rho(A(n))}.$$

The proof is finished by letting $C(\psi) = a^{-2\lambda}\,(\psi^{-1})'(x_0) \inf \psi'$. □

Now, let us set $E(n) = \{id\} \cup A(n)$ and $F(n) = E(n)^{-1}E(n)$. In order to close the second step, it remains to estimate the quantity $c_{\psi E(n)\psi^{-1}}$, which gives an upper bound for the number of overlaps of the intervals $g(J_{\psi F(n)\psi^{-1}})$, for $g \in \psi E(n)\psi^{-1}$.

Proposition 2.31. *For any $\psi \in G$, the function*

$$c_{\psi E(n)\psi^{-1}} = \max_{h \in \psi E(n)\psi^{-1}} \#\left(\psi E(n)\psi^{-1} \cap h\, \mathrm{Stab}_G(x_0)\right)$$

grows at most linearly in terms of the outer radius $\rho\left(\psi E(n)\psi^{-1}\right)$. More precisely, there exists a constant $L \in \mathbb{N}$ such that $c_{\psi E(n)\psi^{-1}} \leq L\,\rho\left(\psi E(n)\psi^{-1}\right)$.

Proof. If the stabilizer $\mathrm{Stab}_G(x_0)$ is trivial, clearly c_E is always 1, no matter what E is. Hence, we can suppose that the stabilizer $\mathrm{Stab}_G(x_0)$ is infinite cyclic and generated by some element $\varphi \in G$. Here we distinguish two cases, depending on whether φ is *undistorted* in G or not. If φ is undistorted, then the quantity $c_{\psi E(n)\psi^{-1}}$ grows at most linearly in terms of the outer radius $\rho(\psi E(n)\psi^{-1})$. If φ is distorted, Lemma 2.18 implies that φ belongs to a conjugate factor $g^{-1}G_i g$. Without loss of generality, we can suppose that if φ is distorted then $\varphi \in g^{-1}G_1 g$, for some $g \in G$. Indeed, only a subgroup acting minimally can contain distorted elements, since virtually free groups do not have distorted elements of infinite order (Lemma 2.19). Let us show that in this case $c_{\psi E(n)\psi^{-1}}$ is bounded in n.

Notice first that the quantity

$$c_{\psi E(n)\psi^{-1}} = \max_{h \in \psi E(n)\psi^{-1}} \#\left(\psi E(n)\psi^{-1} \cap h\, \mathrm{Stab}_G(x_0)\right)$$

is also equal to

$$\max_{h \in E(n)} \#\left(E(n) \cap h\, \psi^{-1}\mathrm{Stab}_G(x_0)\psi\right),$$

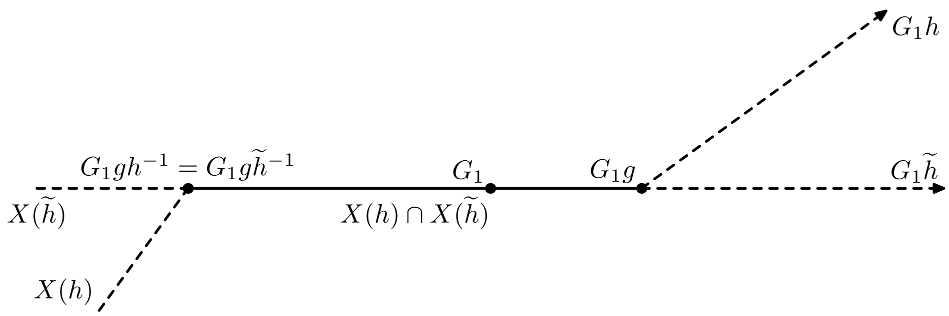

Figure 2.6: The translation axes of the two elements in the proof of Proposition 2.31

therefore up to replacing g above with $g\psi$, it is enough to find a uniform bound for $c_{E(n)}$.

Let us fix $\widetilde{h} \in A(n)$, and suppose that there exists some $h \in E(n)$, and $\ell \in \mathbb{Z}$ such that $h = \widetilde{h}\,\varphi^\ell$. After Lemma 2.25, the elements h and $\widetilde{h}$ are hyperbolic isometries, with translation length $2n$ and their translation axes intersect, as they both contain the segment $(G_1\sigma^{-1}, G_2, G_1)$.

Consider the point G_1g in the Bass-Serre tree, which is fixed by $\varphi \in g^{-1}G_1g$. Because of the equality $\widetilde{h}^{-1}h = \varphi^\ell \in g^{-1}G_1g$, the images G_1gh^{-1} and $G_1g\widetilde{h}^{-1}$ are the same (recall that the action on the Bass-Serre tree is naturally a right action). Applying the formula (2.5) for the distances of the images, we find

$$
\begin{aligned}
d(G_1g, G_1g\widetilde{h}^{-1}) &= 2n + 2d(G_1g, X(\widetilde{h})),\\
d(G_1g, G_1gh^{-1}) &= 2n + 2d(G_1g, X(h))
\end{aligned}
$$

and by equality of the images, we must have $d(G_1g, X(\widetilde{h})) = d(G_1g, X(h))$.

Suppose first that this distance is not zero. Since the two axes intersect, the geodesic segments from G_1g to $X(\widetilde{h})$ and $X(h)$ respectively, must be the same: indeed, if this was not the case, these segments would give a nontrivial geodesic path connecting $X(h)$ and $X(\widetilde{h})$; then the union of such a path and the intersection of the axes would give a nontrivial loop in the tree. Call this geodesic segment γ, which goes from G_1g to the intersection $X(h) \cap X(\widetilde{h})$.

We claim that if γ has more than one vertex, then the quotient $\widetilde{h}^{-1}h = \varphi^\ell$ fixes it: indeed, we repeat the previous argument and get that the geodesic paths from $G_1gh^{-1} = G_1g\widetilde{h}^{-1}$ to $X(h)$ and $X(\widetilde{h})$ coincide, and this common path is exactly the image $\gamma.h^{-1} = \gamma.\widetilde{h}^{-1}$.

From the claim we deduce that φ^ℓ belongs to a conjugate of the edge group Z. However φ has infinite order, so this case is impossible.

Thus the vertex G_1g belongs to both axes $X(h)$ and $X(\widetilde{h})$. Since the images G_1gh^{-1} and $G_1g\widetilde{h}^{-1}$ are the same, we have that the intersection $X(h) \cap X(\widetilde{h})$ contains the whole segment of length $2n$ between G_1g and its image G_1gh^{-1}.

On the one side $\widetilde{h}^{-1}h = \varphi^\ell$ fixes exactly one point, while on the other side the two elements act like translations by $2n$ on their own translation axes. If the intersection was containing more than $2n$ points, then we would get that the quotient $\widetilde{h}^{-1}h = \varphi^\ell$ fixes at least two points, absurd.

Thus the intersection $X(h) \cap X(\widetilde{h})$ actually coincides with the segment from $G_1gh^{-1} = G_1g\widetilde{h}^{-1}$ to G_1g. The vertex G_1 belongs to this intersection. The situation is cartooned in Figure 2.6.

The elements h and $\widetilde{h}$ are in $A(n)$: there exist t_i's and $\widetilde{t}_i$'s in $B_1^\times(R_1') \cap T_1$, $i = 1, \dots n$, such that

$$
h = \sigma t_n \cdots \sigma t_1, \quad \widetilde{h} = \sigma \widetilde{t}_n \cdots \sigma \widetilde{t}_1. \tag{2.12}
$$

Given the explicit expression (2.9) for the translation axes of elements in $A(n)$, we deduce that there exists $0 \leq k \leq n$ such that

$$G_1 g = G_1 \sigma \widetilde{t}_k \cdots \sigma \widetilde{t}_1$$

(with abuse of notation, the case $k = 0$ corresponds to $G_1 g = G_1$). Let us write temporarily $\widetilde{g} = \sigma \widetilde{t}_k \cdots \sigma \widetilde{t}_1$ and observe that $g^{-1} G_1 g = \widetilde{g}^{-1} G_1 \widetilde{g}$. Therefore we can suppose that g is the initial part of $\widetilde{h}$, that is $g = \widetilde{g} = \sigma \widetilde{t}_k \cdots \sigma \widetilde{t}_1$. We also write $\varphi^\ell = g^{-1} x^\ell g$, with $x \in G_1$.

The product $\widetilde{h} \varphi^\ell = \widetilde{h} g^{-1} x^\ell g$ has therefore a "cyclic simplification":

$$\widetilde{h} \varphi^\ell = \sigma \widetilde{t}_n \cdots \sigma \left(\widetilde{t}_{k+1} x^\ell \right) \sigma \widetilde{t}_k \cdots \sigma \widetilde{t}_1, \tag{2.13}$$

with $\left(\widetilde{t}_{k+1} x^\ell \right)$ belonging to G_1. Now, this product $\widetilde{h} \varphi^\ell$ equals h, so we compare the expression (2.13) above to the first expression in (2.12). From Remark 2.10, we deduce that the quotient $\left(\widetilde{t}_{k+1} x^\ell \right) t_{k+1}^{-1}$ is in Z and thus $x^\ell \in B_1(R_1') Z B_1(R_1')$. This is a finite set that does not depend on n: there is only a finite number of possible choices for ℓ. $\square$

As a consequence of the results of § 2.5, we obtain the following key fact:

Corollary 2.32. *Given $\varepsilon_0 > 0$ and $\psi \in G$, there exists $n = n(\psi)$ such that the element $g_{\psi F(n) \psi^{-1}}$ is locally ε_0-close to the identity in the C^0 topology when restricted to a certain complex neighbourhood of $x_0 \in \mathrm{NE}$.*

Proof. Given $\psi \in G$, consider the constants $C = C(\psi)$ and L from Propositions 2.30 and 2.31 respectively. Then the quantity

$$\rho \left(\psi E(n) \psi^{-1} \right) \frac{c_{\psi E(n) \psi^{-1}}}{S_{\psi E(n) \psi^{-1}}} \leq \frac{L}{C} \rho \left(\psi E(n) \psi^{-1} \right)^2 a^{-\rho \left(\psi E(n) \psi^{-1} \right)}$$

is certainly $o(1)$ as n goes to ∞. Thus Proposition 2.6 applies and the sequence $g_{\psi F(n) \psi^{-1}}$ for $F(n) = E(n)^{-1} E(n)$ converges C^0 to the identity over a complex disc of size $o(1/\rho\left(\psi E(n) \psi^{-1}\right))$ around x_0. $\square$

2.8 Step 3: Chain of commutators

Strategy – As we have already explained, Proposition 2.5 implies that if two diffeomorphisms f_1, f_2 in G are ε_0-close to the identity over a small interval, then the sequence of commutators $f_{k+2} = [f_{k+1}, f_k]$ must be eventually trivial, since G is locally discrete. We want to get a contradiction, finding two elements f_1 and f_2 which are locally ε_0-close to id, generating a free subgroup in G. The main result in this third step is the following:

Proposition 2.33. *Given $\varepsilon_0 > 0$, there exists $\psi_1, \psi_2 \in G$ and n such that the elements $f_1 = g_{\psi_1 F(n) \psi_1^{-1}}$ and $f_2 = g_{\psi_2 F(n) \psi_2^{-1}}$ satisfy the following two properties:*

1. *they are both ε_0-close to the identity in the C^0 topology when restricted to a certain complex neighbourhood of $x_0 \in \mathrm{NE}$,*

2. *the elements $f_3 = [f_1, f_2]$ and $f_4 = [f_2, f_3]$ generate a free group.*

Before starting the proof, let us describe the general strategy. By Corollary 2.32, for any $\psi_1, \psi_2 \in G$ there exists n such that the elements $f_1 = g_{\psi_1 F(n) \psi_1^{-1}}$ and $f_2 = g_{\psi_2 F(n) \psi_2^{-1}}$ are both locally ε_0-close to the identity in the C^0 topology when restricted to some complex neighbourhood of $x_0 \in \mathrm{NE}$. By the ping-pong Proposition 2.24, if f_1 and f_2 have disjoint invariant sets, then f_3 and f_4 generate a free subgroup in G, and the proof is over.

Reduced forms for elements in $F(n)$ – Here we consider elements in the set

$$F(n) = A(n) \cup A(n)^{-1} \cup A(n)^{-1}A(n).$$

Each element in $A(n)$ can be written in the reduced form (2.8). Also, if an element is in $A(n)^{-1}$, then its inverse is in $A(n)$. It remains to describe the elements in $A(n)^{-1}A(n)$.

Lemma 2.34. *Let $g \in A(n)^{-1}A(n)$ be an element which does not belong to the ball $B_1(3R_1')$ of radius $3R_1'$ in G_1. Then there exist elements $s, t \in G_1$ and an element $w \in G$ such that:*

- *$s, t \in B_1(R_1') \setminus Z$,*
- *a reduced form representing w starts and ends with a letter in $G_2 \setminus Z$,*
- *$g = swt$.*

Proof. As g belongs to $A(n)^{-1}A(n)$, we can write g as

$$g = s_1^{-1}\sigma^{-1} \cdots s_n^{-1}\sigma^{-1}\sigma t_n \cdots \sigma t_1, \tag{2.14}$$

with $s_i, t_i \in B_1(R_1')$ and $\sigma \in G_2$ our fixed element. The problem is that the expression (2.14) is not reduced: clearly the subword $\sigma^{-1}\sigma$ in the middle represents the identity, but there could be further central simplifications. For this, after erasing $\sigma^{-1}\sigma$, we look at the new middle subword $s_n^{-1}t_n$. It represents an element in G_1; if it does not belong to Z, then the expression

$$g = s_1^{-1}\sigma^{-1} \cdots \sigma^{-1}(s_n^{-1}t_n)\sigma \cdots \sigma t_1,$$

is already reduced; otherwise the subword $\sigma^{-1}s_n^{-1}t_n\sigma$ represents an element in G_2, and we have similar further cases to analyze. Proceeding in this way, we end up with a word w such that $g = s_1^{-1}wt_1$, and there are two possibilities:

1. the element w is not in Z, and in this case we have that a reduced form representing it starts and ends with a letter in $G_2 \setminus Z$,
2. or $w \in Z$ and thus $g = s_1^{-1}wt_1 \in B_1(R_1')ZB_1(R_1')$ belongs to the ball $B_1(3R_1')$ (the choice of the radius R_1' implies in particular that $B_1(R_1') \supset Z$).

After our assumption on g, only the first possibility may happen, whence we get the properties of the statement, with $s = s_1^{-1}$ and $t = t_1$. □

Conjugation – After the work done in the previous paragraph, we can determine the position of the elements in $\psi F(n)\psi^{-1}$ for suitable choices of ψ.

Proposition 2.35. *Fix $x \in G_1 \setminus Z$ and $y \in G_1 \setminus B_1(2R_1')$. Consider the element $\psi = x\sigma y$. Then for any element $g \in \psi\,(F(n) \setminus B_1(3R_1'))\,\psi^{-1}$, the first letter of g is in Zx^{-1}.*

In other words, if π denotes the geodesic path going from the vertex G_1 to G_1g in the Bass-Serre tree of G, then the first edge of π is (G_1, G_2x^{-1}).

Proof. As $g \in \psi F(n)\psi^{-1}$, there exists an element $h \in F(n)$ such that $g = \psi h\psi^{-1}$. We separate our discussion into two cases:

1. the element h is in $A(n) \cup A(n)^{-1}$,

2. the element h is in $A(n)^{-1}A(n)$.

Suppose we are in the first situation, and suppose $h \in A(n)$ (the other case being similar). We write

$$h = \sigma t_n \cdots \sigma t_1,$$

thus

$$\begin{aligned} g &= \psi h \psi^{-1} \\ &= x\sigma y \sigma t_n \cdots \sigma t_1 y^{-1} \sigma^{-1} x^{-1}. \end{aligned}$$

We look at the subword $t_1 y^{-1}$ appearing in the last expression: after our assumption on y, we have that the product $t_1 y^{-1}$ is in G_1, but it does not belong to Z, otherwise we would have $t_1 y^{-1} \in Z \subset B_1(R_1')$ and thus $y^{-1} \in B_1(R_1')B_1(R_1') \subset B_1(2R_1')$, against our assumption.

Hence the writing

$$g = x\sigma y \sigma t_n \cdots \sigma (t_1 y^{-1}) \sigma^{-1} x^{-1}$$

is in reduced form, and it clearly starts with x^{-1}. If we consider another reduced form representing g, then we can replace the letter x^{-1} by another letter in Zx^{-1} (see Remark 2.10).

If we are in the second situation, the previous Lemma 2.34 says that we can write $h = swt$, with $s, t \in B_1(R_1') \setminus Z$. Hence

$$g = \psi g \psi^{-1} = x\sigma y \, swt \, y^{-1} \sigma^{-1} x^{-1}.$$

Arguing as before, we get that both subwords ys, ty^{-1} are in $G_1 \setminus Z$. Therefore g is represented by the reduced form

$$g = \psi g \psi^{-1} = x\sigma (ys) w (ty^{-1}) \sigma^{-1} x^{-1},$$

and we conclude as in the previous situation.

The last statement about the geodesic π is now a direct consequence of Remark 2.10. □

Corollary 2.36. *Take $y \in G_1 \setminus B_1(2R_1')$. If $x_1, x_2 \in G_1 \setminus Z$ are such that $G_2 x_1^{-1} \neq G_2 x_2^{-1}$, then letting*

$$\psi_1 = x_1 \sigma y, \quad \psi_2 = x_2 \sigma y,$$

for any

$$g_1 \in \psi_1 \left(F(n) \setminus B_1(3R_1') \right) \psi_1^{-1}, \quad g_2 \in \psi_2 \left(F(n) \setminus B_1(3R_1') \right) \psi_2^{-1},$$

the invariant sets $X(g_1)$, $X(g_2)$ are disjoint.

Proof. It follows directly from Propositions 2.35 and 2.15. □

End of the proof – We are now in position to prove Proposition 2.33.

Proof of Proposition 2.33. Consider two elements $\psi_1, \psi_2 \in G$ given by Corollary 2.36. Given $\varepsilon_0 > 0$ we take n such that the elements $f_1 = g_{\psi_1 F(n) \psi_1^{-1}}$ and $f_2 = g_{\psi_2 F(n) \psi_2^{-1}}$ are both ε_0-close to the identity in the C^0 topology when restricted to a certain complex neighbourhood of x_0, which exists after Corollary 2.32. Since the sequences $g_{\psi_i F(m) \psi_i^{-1}}$ do not belong to a finite set (the lengths $\ell_{\psi_i F(m) \psi_i^{-1}}$ go to zero), up to consider a larger n, we can suppose that $f_i \notin \psi_i G_1 \psi_i^{-1}$, $i = 1, 2$: indeed it is easy to see that the intersection $F(n) \cap G_1$ is contained in $B_1(3R_1')$ and hence is finite (see Lemma 2.34).

Similarly, up to consider a larger n (or ε_0 smaller), we can suppose that the orders of f_1 and f_2 is at least $3|Z|$ (possibly infinite): if a periodic element locally converges to the identity, its order must go to infinity (*cf.* [19, Lemma 10]).

Corollary 2.36 guarantees that the invariant sets $X(f_1)$ and $X(f_2)$ are disjoint. Then, by applying the ping-pong Proposition 2.24 (the group G is $|Z|$-bounded, as in the action on its Bass-Serre tree, stabilizers of edges are conjugates of Z), we deduce that $f_3 = [f_1, f_2]$ and $f_4 = [f_2, [f_1, f_2]]$ generate a free group of rank two, as desired. □

This also completes the proof of Theorem A, as we now recall.

Summary of the proof of Theorem A – We start with $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ a locally discrete, finitely generated subgroup with infinitely many ends, and a point $x_0 \in \mathrm{NE}$. We suppose by way of contradiction that no nontrivial element in G fixes x_0.

By Stallings' theorem, G has a Stallings' decomposition. Without loss of generality, we may suppose $G = G_1 *_Z G_2$. In § 2.3 we have seen how to rule out the case when both no factor acts minimally. Therefore we consider the case when G_1 acts minimally. Under this assumption, Proposition 2.33 ensures the existence of elements $f_1, f_2 \in G$ such that:

1. they are both ε_0-close to the identity in the C^0 topology, when restricted to a certain complex neighbourhood of x_0,
2. no iterated commutators $f_{k+2} = [f_k, f_{k+1}]$ is trivial.

Then we apply Proposition 2.5 and get that the sequence f_k converges to the identity in the C^1 topology when restricted to a fixed neighbourhood of x_0. This contradicts the hypothesis that the group G is locally discrete.

3 Theorem B: Duminy revisited

3.1 Proof of Theorem B from Theorem C

The main purpose of this Section is to give the proof of Theorem C which is a version of Duminy's theorem in the context of minimal actions satisfying property $(\star)$. Theorem B will be then an easy consequence:

Proof of Theorem B. Let $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a group with property $(\star)$. If the set of non-expandable points $\mathrm{NE} = \mathrm{NE}(G)$ is empty, then Deroin's Theorem 1.28 implies that G is C^ω conjugate to a finite central extension of a cocompact Fuchsian group.

If G has an exceptional minimal set, then Ghys' Theorem 1.21 implies that G is virtually free.

Therefore, we are left to suppose that G acts minimally with non-expandable points. In this case we apply Theorems C and D: this gives that the groupoid of one-sided germs $\mathcal{G}_{x_0}$ has infinitely many ends.

On the other hand, the analyticity of the action implies that any germ uniquely determines an element of the group G: the groupoid of germs $\mathcal{G}_{x_0}$ coincides with the group G. Thus if $\mathcal{G}_{x_0}$ has infinitely many ends, so does G. Since G has property $(\star)$ and NE is not empty, we apply Theorem A and get that in the latter case G is virtually free. □

3.2 Warm up: Duminy's theorem in analytic regularity

Duminy's result deals with pseudogroups of class C^2 that act on the circle with exceptional minimal sets (a proof can be found in [35, §3]). Here we discuss the case of finitely generated groups of real-analytic diffeomorphisms. In this context, we provide a relatively short proof which illustrates the core of the proof of Theorem C.

Theorem 3.1 (Duminy – C^ω case)**.** *Let $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a finitely generated group acting on $\mathbf{S}^1$ with an exceptional minimal set Λ. Let J_0 be a connected component of $\mathbf{S}^1 \setminus \Lambda$ (a "gap"). Then the Schreier graph $\mathrm{Sch}(X, \mathcal{G})$ of the orbit of gaps $X = G \cdot J_0$ has infinitely many ends.*

This implies that the group G itself has infinitely many ends.

Proof. We will prove that if the conclusion fails to be true, then G preserves an *affine structure* on $\mathbf{S}^1$. This is done by using control of the affine distortion of well chosen maps. The relevant tool to do this is the *nonlinearity* of a diffeomorphism of the line: If $f : I \to J$ is a C^2 diffeomorphism of one dimensional manifolds, let

$$\mathcal{N}(f) = \frac{f''}{f'}.$$

The nonlinearity of a map vanishes if and only if the map is affine. Moreover, this nonlinearity operator satisfies the cocycle relation

$$\mathcal{N}(f \circ g) = g' \mathcal{N}(f) \circ g + \mathcal{N}(g). \tag{3.1}$$

The first step of the proof is to use the nonlinearity to find a criterion for distinguishing different ends in the Schreier graph $\mathrm{Sch}(X, \mathcal{G})$. Remind that the stabilizer of J_0 is generated by some $h \in G$ (*cf.* Theorem 1.2). We set $b = \int_{J_0} \mathcal{N}(h)$.

Proposition - Definition. *Assume we are under the hypotheses of Theorem 3.1. The function*

$$\begin{array}{rccc} N : & X & \longrightarrow & \mathbb{R}/b\mathbb{Z} \\ & g(J_0) & \longmapsto & \int_{J_0} \mathcal{N}(g) \end{array} \tag{3.2}$$

is well defined along the orbit X, and verifies

$$N(f(J)) = N(J) + \int_J \mathcal{N}(f) \quad \textit{for all } J \in X \textit{ and all } f \in G. \tag{3.3}$$

Proof. If two elements g_1 and g_2 are such that $g_1(J_0) = g_2(J_0)$, then there exists some $k \in \mathbb{Z}$ such that $g_2 = g_1 h^k$. To verify that the function N is well defined, we have to show that for a fixed $g \in G$, all the integrals $\int_{J_0} \mathcal{N}(gh^k)$ are equal modulo $b\mathbb{Z}$.

Using the cocycle relation (3.1) and the change of variable formula, we have

$$\begin{aligned} \int_{J_0} \mathcal{N}(gh^k) &= \int_{J_0} (h^k)' \mathcal{N}(g) \circ h^k + \sum_{i=0}^{k-1} \int_{J_0} (h^i)' \mathcal{N}(h) \circ h^i \\ &= \int_{h^k(J_0)} \mathcal{N}(g) + \sum_{i=0}^{k-1} \int_{h^i(J_0)} \mathcal{N}(h), \end{aligned}$$

which is equal to $\int_{J_0} \mathcal{N}(g) + k \int_{J_0} \mathcal{N}(h) = \int_{J_0} \mathcal{N}(g) + kb$. This proves the first assertion. The relation (3.3) can be verified in a similar way. $\square$

If f is written in the form $f = g_n \cdots g_1$ in the generating system $\mathcal{G}$, then similarly to (2.2) we obtain the bound

$$|N(f(J)) - N(J)| \le C_{\mathcal{G}} \sum_{i=0}^{n-1} |g_i \cdots g_1(J)| \tag{3.4}$$

with respect to the same constant $C_{\mathcal{G}} := \max_{g \in \mathcal{G} \cup \mathcal{G}^{-1}} \sup_{\mathbf{S}^1} |g''/g'|$. From this fact it is not difficult to prove the following lemma which provides a criterion to distinguish ends of the Schreier graph of J_0. It is close to the original ideas of Duminy; see for example [35, Lemma 3.4.2]. Recall that the ends, and the fact that a sequence converges to a certain end, are independent of the finite system of generators of the group.

Lemma 3.2. *Assume we are under the hypotheses of Theorem 3.1.*

i. *If $(J_n)_{n\in\mathbb{N}}$ is a sequence of gaps which goes to an end in the Schreier graph $\mathrm{Sch}(X, \mathcal{G})$, then $\lim_{n\to\infty} N(J_n)$ exists.*

ii. *If $(I_n)_{n\in\mathbb{N}}$ and $(J_n)_{n\in\mathbb{N}}$ determine the same end in the Schreier graph $\mathrm{Sch}(X, \mathcal{G})$, then*

$$\lim_{n\to\infty} N(I_n) = \lim_{n\to\infty} N(J_n).$$

Proof. It is enough to prove the first assertion for $J_n = g_n \cdots g_1(J_0)$, where $(g_n)_{n\in\mathbb{N}}$ is a sequence of elements of the (symmetric) system of generators of $\mathcal{G}$. To do this, notice that (3.4) easily shows that the sequence $(N(J_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and hence converges.

To show the second assertion, given $\varepsilon > 0$, let n_0 be such that $\sum_{J \notin X(n_0)} |J| < \varepsilon$, where $X(n_0)$ denotes the set of those $x \in X$ at distance no greater than n_0 to J_0 for the word distance in X. If n is large enough, there exists a path linking I_n and J_n which avoids $X(n_0)$. A direct application of (3.4) yields $|N(I_n) - N(J_n)| < \varepsilon C_{\mathcal{G}}$. Since ε is arbitrary, this concludes the proof. □

From now on, we suppose that $\mathrm{Sch}(X, \mathcal{G})$ has only one end and look for a contradiction. The general case when the Schreier graph has finitely many ends can be treated similarly, as we detail along the proof of Lemma 3.18.

The second step relies on Sacksteder's theorem: there exists a local hyperbolic contraction, *i.e.* $f \in G$, $I \subset \mathbf{S}^1$ and $p \in I$ with $f' < 1$ on I and $f(p) = p$. Using Sternberg's (or in this case Kœnigs-Poincaré's) linearization theorem, we can make a C^ω change of coordinates on I and suppose that f is a homothety of ratio $\mu = f'(p)$.

A way to describe an end of $\mathrm{Sch}(X, \mathcal{G})$ is to pick some gap $J \subset I \cap G \cdot J_0$ and iterate it by f. Using the cocycle relation (3.3), we find

$$\lim_{n\to\infty} N(f^n(J)) = N(J), \tag{3.5}$$

for f is affine and thus its nonlinearity is 0.

We want to prove that if there is one only end in $\mathrm{Sch}(X, \mathcal{G})$, in this chart we have *affine holonomy*: every element $\gamma \in G$ satisfying $I_\gamma = \gamma^{-1}(I) \cap I \neq \emptyset$ has to be an affine map. Note that by minimality of Λ, the union of gaps $I_\gamma \cap G \cdot J_0$ is dense in I_γ. So let $J \subset I_\gamma \cap G \cdot J_0$ be a gap. Since $J \subset I_\gamma$, we also have $\gamma(J) \subset I$.

If $\gamma \in G$ maps J inside I and is not a power of f, then the iterates of $\gamma(J)$ by f also go towards the one only end of $\mathrm{Sch}(X, \mathcal{G})$ and after (3.5) we must have $N(J) = N(\gamma(J))$. Using (3.3) again, the latter implies $\int_J \mathcal{N}(\gamma) = 0$ (supposing the gap J sufficiently small, *cf.* the proof of Lemma 3.18).

We have just shown that the mean nonlinearity of γ over every sufficiently small gap in $I_\gamma \cap G \cdot J_0$ vanishes. By continuity, there is a point x_J in every such gap J, at which the nonlinearity $\mathcal{N}(\gamma)$ is zero. Observe that the points x_J accumulate on $\Lambda \cap I_\gamma$. By the analytic continuation principle, γ is affine on $\mathbf{S}^1$.

We remark that a subgroup of automorphisms of some affine structure on $\mathbf{S}^1$ must have a finite number of globally periodic points and thus cannot preserve a Cantor set, leading to a contradiction. Therefore, the Schreier graph $\mathrm{Sch}(X,\mathcal{G})$ has infinitely many ends.

It remains to show that the group itself has infinitely many ends. This requires some additional work: the map $\pi : g \in G \mapsto g(J_0) \in X$ defines a non-regular covering from G (actually from its Cayley graph) to the the Schreier graph $\mathrm{Sch}(X,\mathcal{G})$. The number of leaves usually does not well-behave when passing to covering spaces, unless the nontrivial monodromy of the covering is compactly supported.

The following lemma is somehow classical in foliation theory (see [4, Corollary 4.8]):

Lemma 3.3. *Assume we are under the hypotheses of Theorem 3.1. There exists $\varepsilon > 0$ such that the following holds. Consider a gap J in the orbit of J_0. Let $g \in G$ be an element that stabilizes J and suppose that g can be written in the form $g = g_n \cdots g_1$ in the generating system $\mathcal{G}$. Suppose that the intermediate images of the gap satisfy*

$$\sum_{i=0}^{n-1} |g_i \cdots g_1(J)| < \varepsilon.$$

Then g is the identity.

Finally, arguing as in [21, Corollaire 2.6], we can deduce that the group G has infinitely many ends. Indeed, consider the class of the loop defined by the stabilizer $h \in \mathrm{Stab}_G(J_0)$ in the fundamental group $\pi_1(\mathrm{Sch}(X,\mathcal{G}), J_0)$. After Lemma 3.3, it defines a nontrivial element in the image of the natural morphism $H^1_c(\mathrm{Sch}(X,\mathcal{G}),\mathbb{Z}) \to H^1(\mathrm{Sch}(X,\mathcal{G}),\mathbb{Z})$. The covering $\pi : G \to \mathrm{Sch}(X,\mathcal{G})$ is exactly the covering associated with this element. We deduce that G has infinitely many ends. □

Strategy of the proof of Theorem C – In the setting of minimal actions with non-expandable points, the strategy we adopt is similar to that of the proof of Duminy's theorem described above.

However, in our setting, it is not an invariant affine structure, but an invariant *projective structure* that we intend to build. The relevant quantity is no longer the nonlinearity, but the *Schwarzian derivative* of diffeomorphisms of one-dimensional manifolds.

The first step of the proof will be to use a control of the *projective* distortion. Instead of using gaps of Cantor sets, we substitute them by considering the orbit of a non-expandable point $x_0 \in \mathrm{NE}$. The control of the distortion is ensured by taking advantage of the Markov partition for groups acting minimally with property $(\star)$, whose construction we recall in § 3.3.

This allows to define a function Q on the Schreier graph X of x_0, that we call the *Schwarzian energy* and is analogue to (3.2). As for the function N, the Schwarzian energy has a well-defined extension to the space of ends $e(X)$ of the Schreier graph of x_0 (Lemma 3.16).

Secondly, we first suppose that the Schwarzian energy takes only finitely many values on $e(X)$. We obtain an intermediate result, that it is interesting on its own: the group is C^r conjugate to a subgroup of some finite covering of $\mathrm{PSL}(2,\mathbb{R})$ (Theorem 3.17). The strategy follows the lines of our proof of Duminy's Theorem. As above we take an element with a hyperbolic fixed point; using Sternberg's linearization theorem, this allows one to construct a chart with projective holonomy (see Lemma 3.18). Using the minimality of the action, we extend this chart to a projective structure: this is Lemma 3.19. Finally, relying on Kuiper-Goldman's classification of the automorphisms groups of a projective structure on $\mathbf{S}^1$, we find that the group is virtually a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, with non-expandable points, and thus virtually free.

Finally, we put all the pieces together and prove Theorem C in § 3.6

Strategy of the proof of Theorem D – We suppose that the orbit of any non-expandable point has infinitely many ends. We prove the analogue of Lemma 3.3 for groups with $(\star)$, and deduce that if the Schreier graph of the orbit of x_0 has infinitely many ends, then also the groupoid of one-sided germs G_{x_0} does.

3.3 Markov partition and expansion procedure

Markov partition – We recall one result of [18] in the case of minimal actions:

Theorem 3.4 (Filimonov, Kleptsyn)**.** *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group whose action is minimal and with property $(\star)$. Let ℓ be the number of non-expandable points of G, and write $\mathrm{NE} = \{x_1, \ldots, x_\ell\}$. Then there exists a partition of the circle $\mathbf{S}^1$ into finitely many open intervals*

$$\mathcal{I} = \left\{ I_1, \ldots, I_k, I_1^+, I_1^-, \ldots, I_\ell^+, I_\ell^- \right\},$$

an expansion constant $\lambda > 1$ and elements $g_I \in G$, $I \in \mathcal{I}$ such that:

- **i.** *for every $I \in \mathcal{I}$, the image $g_I(I)$ is a union of intervals in $\mathcal{I}$;*
- **ii.** *we have $g_I'|_I \geq \lambda$ for every $I = I_1, \ldots, I_k$;*
- **iii.** *the intervals I_i^+ and I_i^- are adjacent respectively on the right and on the left to the non-expandable x_i, which is the unique fixed point, topologically repelling, for $g_{I_i^+}$ (resp. $g_{I_i^-}$) on the interval I_i^+ (resp. I_i^-); moreover x_i is the unique non-expandable point in $g_{I_i^\pm}(I_i^\pm)$;*
- **iv.** *for every $I = I_1^\pm, \ldots, I_\ell^\pm$, set*
$$k_I : I \longrightarrow \mathbb{N}$$
to be the function $k_I(x) = \min\{k \in \mathbb{N} \mid g_I^k(x) \notin I\}$ and
$$j : I \longrightarrow \{1, \ldots, k\}$$
defined by the condition $g_I^{k_I(x)}(x) \in I_{j(x)}$. Then for every $x \in I$, $\left(g_{I_{j(x)}} \circ g_I^{k_I(x)}\right)'(x) \geq \lambda$.

Remark 3.5. If we assume moreover that G is in Diff^ω_+, then iv above can be reformulated as follows: if $k_I(x) = \min\{k \in \mathbb{N} \mid g_I^k(x) \notin I\}$, then for every $x \in I$ one has $\left(g_I^{k_I(x)}\right)'(x) \geq \lambda$.

Indeed, as g_I is a parabolic stabilizer one of the endpoints x_I (say the leftmost one) of the interval I, there exist $A, B > 0$ and $n \geq 1$ an integer such that

$$g_I(x) = x\left(1 + A(x - x_I)^n + o((x - x_I)^n)\right) \quad \text{for every } x \in I, \text{ as } x \to x_I \tag{3.6}$$

and

$$g_I'(x) = 1 + B(x - x_I)^n + o((x - x_I)^n) \quad \text{for every } x \in I, \text{ as } x \to x_I. \tag{3.7}$$

Therefore the derivative of g_I is never less than one on a small right neighbourhood of x_I. This fact will be crucial in our proof of Theorem D.

Remark 3.6. It is worthwhile to observe that Theorem B was first conjectured in [19] as a moral consequence of Theorem 3.4: the (non-uniformly) expanding maps g_I's give a way to decompose the Schreier graphs of all but finitely many orbits into a finite number of trees [18], thus suggesting freeness in the structure.

Magnification maps – From now on, we fix a Markov partition

$$\mathcal{I} = \left\{ I_1, \ldots, I_k, I_1^+, I_1^-, \ldots, I_\ell^+, I_\ell^- \right\},$$

an expansion constant $\lambda > 1$ and elements $g_I \in G$, $I \in \mathcal{I}$ given by Theorem 3.4. We also denote by Δ_0 the set of endpoints of atoms of the partition $\mathcal{I}$. We introduce a first *magnification map* $\mathcal{R} : \mathbf{S}^1 \setminus \Delta_0 \to \mathbf{S}^1$ defined as

$$\mathcal{R}|_I = g_I \quad \text{for any } I \in \mathcal{I}, \tag{3.8}$$

and its modification $\widetilde{\mathcal{R}} : \mathbf{S}^1 \setminus \Delta_0 \to \mathbf{S}^1$ defined as

$$\widetilde{\mathcal{R}}|_I : x \in I \mapsto \begin{cases} g_I(x) & \text{if } I \in \{I_1, \ldots, I_k\} \\ g_{I_{j(x)}} g_I^{k_I(x)} & \text{if } I \in \left\{ I_1^\pm, \cdots, I_\ell^\pm \right\} \end{cases}, \quad \text{for any } I \in \mathcal{I}, \tag{3.9}$$

which, after Theorem 3.4.iv above, is *uniformly expanding*: $\widetilde{\mathcal{R}}'(x) \geq \lambda$ for any $x \in \mathbf{S}^1 \setminus \Delta_0$.

The following result will be very helpful during the proof of Theorem D:

Lemma 3.7. *Assume we are under the hypotheses of Theorem 3.4 and suppose moreover that $G \subset \mathrm{Diff}_+^\omega(\mathbf{S}^1)$. Then the magnification map $\mathcal{R}$ can be chosen to be everywhere expanding:*

$$\mathcal{R}'(x) > 1 \quad \textit{for every } x \in \mathbf{S}^1 \setminus \Delta_0.$$

Proof. The magnification map is piecewise defined by (3.8).

However, it depends on the construction of the collection $\mathcal{I}$ in Theorem 3.4. The proof in [19] starts first by fixing neighbourhoods $I_j^\pm$ of the non expandable points $\{x_1, \ldots, x_\ell\}$, then subdividing the rest of the circle into intervals I_j. Taking smaller neighbourhoods $I_j^\pm$ has usually the result of decreasing the expansion constant $\lambda > 1$.

If I is one of the $I_j^\pm$, then we have seen in Remark 3.5 that $\mathcal{R}|_I = g_I|_I$ is of the form (3.6), and its derivative of the form (3.7). Hence, shrinking I a little in Theorem 3.4, we may assure $(\mathcal{R})'|_I = g_I'|_I > 1$.

On the other hand, if $I \in \mathcal{I}$ is one of the I_j, then we already have a good expansion by construction: $\mathcal{R}'|_I \geq \lambda$ after Theorem 3.4.ii. $\square$

Partitions of higher level – In order to encode the dynamics within the orbit of the set of non-expandable points, it is appropriate to define subpartitions of $\mathcal{I}$. We define the endpoints of the atoms of the partition of level k by the following inductive procedure, starting from the set Δ_0 of endpoints of atoms of the partition $\mathcal{I}$. If Δ_k is constructed, consider $\Delta_k(I) = \Delta_k \cap I$, where $I \in \mathcal{I}$, so that $\Delta_k = \bigcup_{I \in \mathcal{I}} \Delta_k(I)$. We distinguish two possibilities:

- if I is not adjacent to a non-expandable point, set

$$\Delta_{k+1}(I) = g_I^{-1}(\Delta_k \cap g_I(I));$$

- for $I \in \mathcal{I}$ adjacent to one of the non-expandable points, set

$$\Delta_{k+1}(I) = \bigcup_{j=1}^{\infty} g_I^{-j}(\Delta_k \cap (g_I(I) \setminus I)).$$

Definition 3.8. The connected components of $\mathbf{S}^1 \setminus \Delta_k$ form a partition called the *partition of level k* that we denote by $\mathcal{I}_k$.

Expansion of a non-expandable point − We start by the following result describing the orbits of non-expandable points (see for instance [35, Lemma 3.5.14]).

Lemma 3.9. *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group whose action is minimal and satisfies property $(\star)$. Then a point $x \in \mathbf{S}^1$ belongs to the orbit of a non-expandable point if and only if the set $\{g'(x) \mid g \in G\}$ is bounded.*

The tool of the proof is a process of expansion that we describe below. Assume that $x \in G \cdot \mathrm{NE}$. There exists $k(x) \in \mathbb{N} \cup \{\infty\}$ and a sequence of $k(x)$ points $(x_i)_{i=0}^{k(x)} \subset G \cdot \mathrm{NE}$, that we call the *expansion sequence* of x and is defined recursively as follows. First, set $x_0 = x$. Now assume that x_i has been constructed. Then there exists $I \in \mathcal{I}$ such that $x_i \in \bar{I}$ (if x_i is one of the endpoints of I, one can always ask that it is the left one). Then we have three *mutually exclusive* possibilities:

- if $x_i \in \mathrm{NE}$, then the procedure stops and $k(x) = i$;
- if I is not adjacent to a non-expandable point, we set $x_{i+1} = g_{i+1}(x_i)$, where $g_i = g_I$;
- if the right endpoint of I is a non-expandable point we set $x_{i+1} = g_{i+1}(x_i)$, where $g_{i+1} = g_{I_{j(x_i)}} g_I^{k_I(x_i)}$. Here k_I and j are the numbers defined in Theorem 3.4.

In other words, if the point x_i is not non-expandable, we set $x_{i+1} = \widetilde{\mathcal{R}}(x_i)$, where $\widetilde{\mathcal{R}}$ is the expanding magnification map introduced at (3.9).

If the procedure never stops we can set $k(x) = \infty$, though it turns out that this possibility never occurs:

Proposition 3.10. *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group whose action is minimal, satisfies property $(\star)$ and such that $\mathrm{NE} \neq \emptyset$. Let $x \in G \cdot \mathrm{NE}$. Then the following assertions hold true.*

i. *There exists a finite integer $k = k(x)$, called the* level *of x, such that the procedure stops after k steps.*

ii. *Let $\mathbf{g}_x$ denote the composition $g_k\, g_{k-1} \cdots g_1$ (locally equal to $\widetilde{\mathcal{R}}^k$). By construction $\mathbf{g}_x(x) = x_k$ belongs to NE and is the leftmost point of some $I^+_{j(x_k)}$. Define the interval $J^+_x = \mathbf{g}_x^{-1}(I^+_{j(x_k)})$, whose leftmost point is x. Then there exists a number $\kappa = \kappa(x) \geq k$ such that J^+_x is an atom of $\mathcal{I}_\kappa$, the partition of level κ.*

iii. *There exists a constant $C_0 > 0$ which does not depend on $x \in G \cdot \mathrm{NE}$ such that $\varkappa(\mathbf{g}_x, J^+_x) \leq C_0$.*

Proof. We observe that the expanding property of the magnification map $\widetilde{\mathcal{R}}$ imply that the derivatives of the compositions $g_j\, g_{j-1} \cdots g_1 = \widetilde{\mathcal{R}}^j$ are always larger than λ^j. Since $x \in G \cdot \mathrm{NE}$, by Lemma 3.9, $(\widetilde{\mathcal{R}}^j)'(x)$ has to be bounded. This is possible if and only if the expansion procedure described above stops at some step k.

That the intervals J^+_x are atoms of the partition of some level κ is clear from the definition of the two procedures.

The map $\mathbf{g}_x$ is precisely the *expansion map* $\widetilde{\mathcal{R}}^{k(x)}$ of J^+_x, in the sense of [18, Definition 7]. Thus, the third assertion follows from [18, Proposition 2] and because the size of the intervals $g_i \cdots g_1(J^+_x) = \widetilde{\mathcal{R}}^j(J^+_x)$ is uniformly bounded from below. □

Lemma 3.11. *The following assertions hold true.*

i. *The family $(J^+_x)_{k(x)=k}$ consists of disjoint intervals.*

ii. *There exists a constant $C > 1$ which does not depend on $x \in G \cdot \mathrm{NE}$ such that*

$$\frac{C^{-1}}{|J_x^+|} \leq \mathbf{g}_x'(x) \leq \frac{C}{|J_x^+|}.$$

Proof. By Proposition 3.10.ii, each interval J_x^+ is an atom of some partition of level $\kappa(x)$. This implies that two different intervals J_x^+ either are disjoint, or one is contained into the other.

Assume for example that J_x^+ contains J_y^+ for some $x, y \in G \cdot \mathrm{NE}$. Then we claim that $k(x) < k(y)$. Indeed, the maps g_i defined by the expansion procedure of x and y must coincide at least before the procedure stops for x. It stops for x when $i = k$, and $x = x_k$. Then $\mathbf{g}_x(y) = y_k$ lies strictly inside $I^+_{j(x_k)}$, which contains no non-expandable point. Hence, the expansion procedure of y must continue after the k-th step, and we have $k(x) < k(y)$ as desired.

The second assertion directly follows from Proposition 3.10.iii. □

In the final part of the proof of Theorem C, we will also need a second important result from [18]:

Theorem 3.12 (Filimonov, Kleptsyn). *Let $G \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a finitely generated group whose action is minimal and with property $(\star)$. Let $\mathcal{I}$ be the partition of $\mathbf{S}^1$ given by Theorem 3.4, with the associated expanding maps g_I's. There exists a finite number of interval $L_1, \ldots L_N, L_1', \ldots L_N'$ and maps $h_i : L_i \to L_i'$, such that any element $g \in G$ admits the following representation:*

- **i.** *there is a partition of the circle into intervals $J_1, \ldots, J_q$ which depends on g;*
- **ii.** *for any $p = 1, \ldots, q$ there exist intervals L_{i_p}, L_{i_p}' in the expansion sequences of the intervals J_p and $g(J_p)$ respectively. In other words for some n_p, n_p' one has*

$$\mathcal{R}^{n_p}(J_p) = L_{i_p}, \quad \mathcal{R}^{n_p'}(g(J_p)) = L_{i_p}';$$

- **iii.** *The map g equals h_{i_p} under magnification:*

$$g|_{J_p} = \mathcal{R}^{-n_p'} h_{i_p} \mathcal{R}^{n_p}. \tag{3.10}$$

Moreover, the partition $J_1, \ldots, J_q$ can be chosen to be the same for any finite set of elements in G.

Remark 3.13. Observe that the property (3.10) implies that the maps $h_i : L_i \to L_i'$ are the restriction of elements in G. Therefore with abuse of notation, we can consider the maps h_i's to be elements in G, and in particular defined on the whole circle.

3.4 Distinguishing different ends: control of the projective distortion

We assume that G has property $(\star)$ and that there exists $x_0 \in \mathrm{NE}$. Our goal is to show that G has infinitely many ends: here we present a criterion to distinguish two different ends.

Distortion control – From [18, Lemma 5] we have:

Lemma 3.14. *The stabilizer $\mathrm{Stab}_G(x_0)$ (in the C^2 setting, considered as the group of one-sided germs) is an infinite cyclic group, generated by some $h \in G$.*

We introduce a function $\mathcal{E} : X \to (0,1]$, that we will call the *energy* (and which is, in fact, the inverse of the function defined in [18]), defined on the orbit $X = G \cdot x_0$ as

$$\mathcal{E}(g(x_0)) = g'(x_0) \quad \text{for every } g \in G. \tag{3.11}$$

The map is well-defined. Indeed, assume that $x = g_1(x_0) = g_2(x_0)$ for $g_1, g_2 \in G$. Then the element $g_2^{-1}g_1$ fixes x_0. Since this point is non-expandable, we must have $(g_2^{-1}g_1)'(x_0) = 1$, hence $g_1'(x_0) = g_2'(x_0)$.

Lemma 3.15. *The series $\sum_{x\in X} \mathcal{E}(x)^2$ converges.*

Proof. Let $x \in X$, and let $\mathbf{g}_x$ be the map obtained in Proposition 3.10. We have $\mathcal{E}(x) = \mathbf{g}_x'(x)^{-1}$. By Lemma 3.11, the ratio between $\mathcal{E}(x)$ and $|J_x^+|$ is uniformly bounded away from 0 and ∞. Therefore, it is enough to prove that the series $\sum_{x\in X} |J_x^+|^2$ is convergent.

We can decompose this sum as

$$\sum_{k=0}^{\infty} \sum_{k(x)=k} |J_x^+|^2 \le \sum_{k=0}^{\infty} \left(\left(\max_{x\,:\,k(x)=k} |J_x^+| \right) \sum_{k(x)=k} |J_x^+| \right). \tag{3.12}$$

We first note that $|J_x^+|$ can be controlled by a term of the order of $\lambda^{-k(x)}$, because by construction we have $\mathbf{g}_x'(x) \ge \lambda^{k(x)}$.

Using Lemma 3.11, we get the following inequality holding for every $k \in \mathbb{N}$:

$$\sum_{k(x)=k} |J_x^+| \le |\mathbf{S}^1| = 1.$$

This suffices to prove that the upper bound in (3.12) is controlled by a converging geometric sum. □

The Schwarzian energy – If $f \in \mathrm{Diff}_+^3(\mathbf{S}^1)$, we consider its *Schwarzian derivative* given by the classical expression

$$\mathcal{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

We have the following cocycle formula:

$$\mathcal{S}(f \circ g) = (g')^2 \cdot \mathcal{S}(f) \circ g + \mathcal{S}(g). \tag{3.13}$$

Recall that the stabilizer of x_0 is generated by some $h \in G$, which moreover verifies $h'(x_0) = 1$; we set $b = \mathcal{S}(h)(x_0)$. From this we can define a new function on the orbit X of x_0:

Proposition - Definition. *The* Schwarzian energy *is the function*

$$\begin{array}{rccc} Q: & X & \longrightarrow & \mathbb{R}/b\mathbb{Z} \\ & g(x_0) & \longmapsto & \mathcal{S}(g)(x_0) \end{array} \tag{3.14}$$

(where the quotient $\mathbb{R}/b\mathbb{Z}$ can possibly be $\mathbb{R}$, if $b = 0$).

Proof. We follow the arguments previously given for the function N. We have to check that the function Q is well-defined. Assume that $x = g_1(x_0) = g_2(x_0)$ for some $g_1, g_2 \in G$. By Lemma 3.14, we have $g_1 = g_2 h^k$ for some $k \in \mathbb{Z}$. Using the cocycle relation (3.13) and the fact that $h'(x_0) = 1$, we find

$$\mathcal{S}(g_1)(x_0) = \mathcal{S}(g_2)(x_0) + k\,\mathcal{S}(h)(x_0).$$

which is equal to $\mathcal{S}(g_2)(x_0) \pmod b$. □

An immediate corollary of (3.13) is

$$Q(f(x)) = \mathcal{E}(x)^2 \cdot \mathcal{S}(f)(x) + Q(x). \tag{3.15}$$

Extension to the space of ends – The following lemma provides a criterion to distinguish ends of the Schreier graph of the orbit of x_0, that we identified with the orbit X.

Lemma 3.16.

i. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X which goes to an end. Then $\lim_{n\to\infty} Q(x_n)$ exists.*

ii. *If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ determine the same end in X, then*

$$\lim_{n\to\infty} Q(x_n) = \lim_{n\to\infty} Q(y_n).$$

Proof. It goes like the proof of Lemma 3.2, but we detail it for the sake of clarity. Consider a sequence of the form $x_n = g_n \cdots g_1(x_0)$, where $(g_n)_{n\in\mathbb{N}}$ is a sequence of elements of the (symmetric) system of generators of $\mathcal{G}$.

Using (3.15), we get

$$Q(x_{n+1}) - Q(x_n) = \mathcal{E}(x_n)^2 \cdot \mathcal{S}(g_{n+1})(x_n).$$

Using Lemma 3.15 and an upper bound for the Schwarzian derivatives of the generators, we easily get that the sequence $(Q(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and hence converges.

We have the convergence of the sequence $(Q(x_n) - Q(y_n))_{n\in\mathbb{N}}$, and we have to prove that the limit is 0 in the case where x_n and y_n converge to the same end. Let $\varepsilon > 0$ and n_0 such that $\sum_{x\notin X(n_0)} \mathcal{E}(x)^2 < \varepsilon$, where $X(n_0)$ denotes the set of those $x \in X$ at distance no greater than n_0 to x_0 for the word distance in X.

Assume that x_n and y_n converge to the same end. When n is large enough, there exists a path linking x_n and y_n which avoids $X(n_0)$. Using the same type of argument as above, we get that $|Q(x_n) - Q(y_n)|$ is smaller than ε times a uniform constant which only depends on the system of generators. Since ε is arbitrary, this concludes the proof of the lemma. □

As a consequence, the function Q defined in (3.14) extends to the space of ends $e(X)$ of X. With abuse of notation, we keep denoting by Q this extension.

3.5 Invariant projective structure

Within this section, *we will assume that the Schwarzian energy Q takes finitely many values on the space of ends $e(X)$*. In particular, this holds if the Schreier graph of x_0 has finitely many ends, but we will show that this is never the case. The goal is to produce a projective structure which is invariant for the action of G.

Theorem 3.17. *Let $G \subset \mathrm{Diff}^r_+(\mathbf{S}^1)$, $r \geq 3$, be a finitely generated group of C^r diffeomorphisms, such that the action of G is minimal, has property $(\star)$ and a non-expandable point $x_0 \in \mathbf{S}^1$. Suppose that the Schwarzian energy Q defined on the Schreier graph $\mathrm{Sch}(X,\mathcal{G})$ of the orbit X of the non-expandable point $x_0 \in \mathrm{NE}$ takes finitely many values on the space of ends of X. Then G is C^r conjugate to a subgroup of some finite covering of $\mathrm{PSL}(2,\mathbb{R})$.*

In particular, the Schreier graph X has infinitely many ends and the group G is virtually free.

A projective chart – We begin by the construction of a single projective chart. We will next use the minimality of the action to construct a projective atlas.

The action of G on $\mathbf{S}^1$ is at least C^2, minimal and does not preserve any probability measure. Then Sacksteder's theorem (Theorem 1.6) applies: the group G acts on $\mathbf{S}^1$ with *hyperbolic holonomy*.

More precisely, there exists a point $p \in \mathbf{S}^1$ and an element $f \in G$ with $f(p) = p$ and $\mu = f'(p) < 1$. Sternberg's linearization theorem [35, Section 3.6.1] provides an interval I about p, as well as a C^r-diffeomorphism $\varphi : (I,p) \to (\mathbb{R}, 0)$, with $\varphi(p) = 0$ and

$$\varphi\, f\, \varphi^{-1} = h_\mu,$$

where h_μ denotes the homothety $x \mapsto \mu x$.

Lemma 3.18 (Projective holonomy)**.** *Assume that the Schwarzian energy Q takes finitely many values on the space of ends of Schreier graph of x_0 has finitely many ends. Then the chart (I, φ) has projective holonomy. More precisely, for every $\gamma \in G$ such that $J = \gamma^{-1}(I) \cap I \neq \emptyset$, the following equality holds on $\varphi(J)$:*

$$\mathcal{S}(\varphi\gamma\varphi^{-1}) = 0.$$

Proof. Assume that Q takes finitely many values on the space of ends of X. It comes from Lemma 3.16 that for every $x \in I \cap X$, the limit $\lim_{n\to\infty} Q(f^n(x))$ exists and there is a finite set $\mathbf{q} = \{q_1, \ldots, q_\ell\}$ such that

$$\lim_{n\to\infty} Q(f^n(x)) \in \mathbf{q} + b\mathbb{Z}.$$

Now let $x = g(x_0) \in I \cap X$. Note that any homothety has zero Schwarzian derivative. Hence, the cocycle relation (3.13) implies the following equality:

$$\begin{aligned} Q(f^n(x)) &= \mathcal{S}(\varphi^{-1}\, h_\mu^n\, \varphi g)(x_0) \\ &= \mu^{2n} (\varphi g)'(x_0)^2 \cdot \mathcal{S}(\varphi^{-1})(\mu^n \varphi g(x_0)) + \mathcal{S}(\varphi g)(x_0). \end{aligned}$$

Letting n go to infinity, we find $\lim_{n\to\infty} Q(f^n(x)) = \mathcal{S}(\varphi g)(x_0)$. The latter shows that for every $g \in G$ satisfying $g(x_0) \in I$, we have that the Schwarzian derivative $\mathcal{S}(\varphi g)(x_0)$ belongs to the discrete set $\mathbf{q} + b\mathbb{Z}$.

Now consider a holonomy map of I, *i.e.* an element $\gamma \in G$ satisfying $J = \gamma^{-1}(I) \cap I \neq \emptyset$. Note that by minimality, the set $J \cap X$ is dense in J. So let $x \in J \cap X$: we can write $x = g(x_0)$ for some $g \in G$. Since $x \in J$, we also have $\gamma g(x_0) = \gamma(x) \in I$. We deduce that both $\mathcal{S}(\varphi g)(x_0)$ and $\mathcal{S}(\varphi\,\gamma g)(x_0)$ are in $\mathbf{q} + b\mathbb{Z}$. By (3.15), their difference is

$$\mathcal{S}(\varphi\,\gamma g)(x_0) - \mathcal{S}(\varphi g)(x_0) = \varphi'(x)^2\, \mathcal{E}(x)^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) \in \mathbf{q} - \mathbf{q} + b\mathbb{Z}.$$

The set $\mathbf{q} - \mathbf{q} + b\mathbb{Z}$ is discrete in $\mathbb{R}$ and contains 0, so there is $\delta > 0$ such that if

$$\left|\varphi'(x)^2\, \mathcal{E}(x)^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x))\right| < \delta$$

then $\varphi'(x)^2\, \mathcal{E}(x)^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) = 0$. Since $\varphi'(x)^2\, \mathcal{E}(x)^2 > 0$, the latter condition implies $\mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) = 0$.

By compactness, there is $M > 0$ such that

$$\sup_J \left|(\varphi')^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1}) \circ \varphi\right| \leq M.$$

Consider the set X' of points $x \in X$ such that $\mathcal{E}(x)^2 < \frac{\delta}{M}$, which contains all but finitely many points of X. The condition that points in $X' \cap J$ verify implies that $\mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) = 0$ for every $x \in X' \cap J$. Since the orbit $X \cap J$ is dense in J, so is $X' \cap J$. Hence, the Schwarzian derivative of $\varphi\,\gamma\,\varphi^{-1}$ vanishes on a dense set of $\varphi(J)$, which implies that $\varphi\,\gamma\,\varphi^{-1}$ is projective on $\varphi(J)$. $\square$

Invariant projective structure – By compactness of $\mathbf{S}^1$ and minimality of the action of G, there exists a finite number of open intervals $(I_j)_{j=1}^m$ and a finite number of elements of the group $(g_j)_{j=1}^m$ such that:

1. the family $(I_j)_{j=1}^m$ is an open cover of $\mathbf{S}^1$,
2. for every $j = 1, \ldots, k$, we have $g_j(I_j) \subset I$.

Lemma 3.19 (Invariant projective structure)**.** *For $j = 1, \ldots, m$, we set $\varphi_j = \varphi \circ g_j : I_j \to \mathbb{R}$.*

i. *The atlas $(I_j, \varphi_j)_{j=1}^m$ defines a projective structure on $\mathbf{S}^1$, i.e. for every j, k with $I_j \cap I_k \neq \emptyset$, we have:*
$$\mathcal{S}(\varphi_k \varphi_j^{-1}) = 0.$$

ii. *The projective structure is G-invariant, i.e. for every $g \in G$ and j, k satisfying $g^{-1}(I_k) \cap I_j \neq \emptyset$, we have:*
$$\mathcal{S}(\varphi_k \, g \, \varphi_j^{-1}) = 0.$$

Proof. For every $g \in G$, when $g^{-1}(I_k) \cap I_j \neq \emptyset$, the map $g_k g g_j^{-1}$ is a holonomy map of I.

Hence, this lemma is a direct application of the fact that (I, φ) has projective holonomy (see Lemma 3.18). □

Projective structures on the circle – On the circle, there is a canonical projective structure which is given by that of $\mathbb{R}\mathbf{P}^1$, and whose group of automorphisms is $\mathrm{PSL}(2, \mathbb{R})$.

For a general projective structure we have the following result originally due to Kuiper, but whose proof contained a little mistake corrected by Goldman (see [24, 25, 29] and [34], it also appears in [22, Lemme 5.1]):

Theorem 3.20 (Kuiper–Goldman)**.** *If the group of orientation preserving automorphisms of a C^r projective structure is not abelian, then it is C^r conjugate to some finite covering of* $\mathrm{PSL}(2, \mathbb{R})$*.*

Let us explain the main lines of the proof. In what follows, we denote by Γ the group of orientation preserving automorphisms of a projective structure on $\mathbf{S}^1$. We also denote by $\widetilde{\mathbf{S}}^1$ and $\widetilde{\mathbb{R}\mathbf{P}}^1$ the universal covers of $\mathbf{S}^1$ and $\mathbb{R}\mathbf{P}^1$ respectively. The central extension
$$0 \to \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \to \Gamma \to 1. \tag{3.16}$$
defines the lift $\widetilde{\Gamma}$ of Γ to the universal cover $\widetilde{\mathbf{S}}^1$. The injective homomorphism $\iota : \mathbb{Z} \to \widetilde{\Gamma}$ is such that the quotient $\widetilde{\mathbf{S}}^1/\iota(\mathbb{Z})$ is diffeomorphic to $\mathbf{S}^1$. Similarly, we have that the universal cover $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of $\mathrm{PSL}(2, \mathbb{R})$, defined by the central extension
$$0 \to \mathbb{Z} \to \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R}) \to 1,$$
acts on $\widetilde{\mathbb{R}\mathbf{P}}^1$.

We defined a C^r projective structure on $\mathbf{S}^1$ as an atlas $(I_j, \varphi_j)_{j=1}^m$ of projective charts. An equivalent way of defining it is by the data of a *developing-holonomy pair* $(\mathsf{dev}, \mathsf{hol})$. Here hol is an injective homomorphism $\mathsf{hol} : \widetilde{\Gamma} \to \widetilde{\mathrm{PSL}}(2, \mathbb{R})$, called the *holonomy representation*, and $\mathsf{dev} : \widetilde{\mathbf{S}}^1 \to \widetilde{\mathbb{R}\mathbf{P}}^1$ is a local diffeomorphism of class C^r, called the *developing map*, which is $\widetilde{\Gamma}$-equivariant: $\mathsf{dev} \circ \gamma = \mathsf{hol}(\gamma) \circ \mathsf{dev}$ for every $\gamma \in \widetilde{\Gamma}$. The developing map, which is well-defined up to a post-composition by an element of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, globalizes the projective charts, and the holonomy representation globalizes the transition maps.

Observe that since $\iota(\mathbb{Z})$ is central in $\widetilde{\Gamma}$, the centralizer of $\mathsf{hol} \circ \iota(\mathbb{Z})$ in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ contains the whole image $\mathsf{hol}(\widetilde{\Gamma})$. Moreover we have the following elementary fact:

Lemma 3.21. *The centralizer of a non-central element of* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *is abelian.*

One deduces that if $\widetilde{\Gamma}$ is not abelian, the element $\mathsf{hol}\circ\iota(1)$ is central in $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and so it must be an automorphism of the universal covering $\widetilde{\mathbb{R}\mathbf{P}}^1 \to \mathbb{R}\mathbf{P}^1$. Finally one has $\mathsf{dev}(\widetilde{\mathbf{S}}^1) = \widetilde{\mathbb{R}\mathbf{P}}^1$, and dev descends to a diffeomorphism between $\mathbf{S}^1$ and some k-fold covering of $\mathbb{R}\mathbf{P}^1$ that conjugates Γ to $\mathrm{PSL}^{(k)}(2,\mathbb{R})$. In order to see that the conjugacy is C^r, notice that it is given by the developing map, which is C^r because the projective charts are of class C^r.

Proof of Theorem 3.17 – The projective structure we constructed in Lemma 3.19 cannot have an abelian group of automorphism, since G realizes as a subgroup and is not abelian. Hence, the group of automorphism of our invariant projective structure has to be conjugate to some finite covering $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$. We conclude that G is C^r conjugate to a subgroup of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$, and this subgroup is discrete in $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ for G is locally discrete. This immediately gives the desired conclusion: by assumption, there are non-expandable points, which means that there are parabolic elements in G, hence G is virtually the fundamental group of a hyperbolic surface with non-empty boundary and so virtually free and with infinitely many ends.

3.6 Proof of Theorem C

Here we summarize all the work done so far in this section and prove Theorem C. Consider a finitely generated subgroup $G \subset \mathrm{Diff}^r_+(\mathbf{S}^1)$, $r \geq 3$, which acts minimally, possesses property $(\star)$, and has at least one non-expandable point x_0. Consider the Schwarzian energy Q defined on the Schreier graph $\mathrm{Sch}(X,\mathcal{G})$ of the non-expandable point x_0, as in (3.14). Recall that Lemma 3.16 ensures that the function Q has a well-defined extension on the space of ends $e(X)$ of the Schreier graph of X.

If Q takes only finitely many values on $e(X)$, we deduce from Theorem 3.17 that X has infinitely many ends and the group G is virtually free. Otherwise, Q takes infinitely many values and this implies that X has infinitely many ends.

3.7 Ends of the groupoid of germs

The main purpose of this section is to prove Theorem D about the number of ends of the grupoid of germs G_{x_0}. The proof relies on the following analogue to Lemma 3.3 (even though we have to "discard" some ends of X):

Proposition 3.22. *Assume we are under the hypotheses of Theorem D. There exists a finite set* $X' \subset X$ *such that for at least an infinite connected component* $\mathcal{C}$ *of the complement* $X \setminus X'$ *the following holds.*

Consider a point $x \in \mathcal{C}$ *in the orbit of* $x_0 \in \mathrm{NE}$. *Let* $g \in G$ *be an element that fixes* x *and suppose that* g *can be written in the form* $g = g_n \cdots g_1$ *with respect to the generating system* $\mathcal{G}$. *Suppose that the intermediate images of* x *satisfy* $g_i \cdots g_1(x) \in \mathcal{C}$, *for any* $i = 1, \ldots, n$. *Then* g *is the identity in restriction to a neighbourhood of* x.

We postpone the proof of Proposition 3.22, since we first need a few preliminary lemmas.

Remark 3.23. It is possible that within the standing assumption of real-analytic regularity, the proof can be largely simplified. Here we want to provide a strategy that relies on this assumption as least as possible, hoping that Theorem D can be generalized to C^2 regularity (our Conjecture 1.22). The essential property we use of C^ω regularity is given by Lemma 3.7, namely the magnification map $\mathcal{R}$ is always expanding.

A particular case – We begin by proving the proposition under the stronger assumption that the sum of the energies is sufficiently small: this condition is analogue to the condition of small sum of length in Lemma 3.3.

Lemma 3.24. *There exists $\delta > 0$ with the following property: let $g \in G$ be an element that fixes x and suppose that g can be written in the form $g = g_n \cdots g_1$ with respect to the generating system $\mathcal{G}$. Suppose that the intermediate images of x satisfy*

$$\sum_{i=0}^{n-1} \mathcal{E}(g_i \cdots g_1(x)) < \delta. \tag{3.17}$$

Then g is the identity in restriction to a neighbourhood of x.

Proof. Note that if (3.17) is satisfied then for the sum of the intermediate derivatives we have

$$\sum_{i=0}^{n-1} (g_i \cdots g_1)'(x) = \frac{\sum_{i=0}^{n-1} \mathcal{E}(g_i \cdots g_1(x))}{\mathcal{E}(x)} < \frac{\delta}{\mathcal{E}(x)}.$$

We consider the restriction of g to right neighbourhood J_x^+ in the partition of level $\kappa(x)$ given by Proposition 3.10.ii (we can proceed in a similar way with the left neighbourhood J_x^-).

Recall from Lemma 3.11 and the beginning of the proof of Lemma 3.15 that the length $|J_x^+|$ is of the same order of magnitude as $\mathcal{E}(x)$: there exists $C > 0$, which does not depend on x, such that

$$C^{-1}|J_x^+| \le \mathcal{E}(x) \le C|J_x^+|. \tag{3.18}$$

We can apply Lemma 2.7 to have an arbitrarily good control of distortion for g on the interval J_x^+: for any $\delta < \log 2/4C_{\mathcal{G}}C$, Lemma 2.7 gives

$$\varkappa(g; J_x^+) \le 4C_{\mathcal{G}}C\delta.$$

Now, $g'(x) = 1$ because x is in the orbit of a non-expandable point. Hence g' is close to 1 uniformly on J_x^+. In particular, there exists a constant $K > 0$, which does not depend on x, such that for every $z \in J_x^+$ we have

$$|g(z) - z| \le K|J_x^+|\delta.$$

Similarly as for to Proposition 3.10.iii, for a sufficiently small δ we have an arbitrarily good control of distortion for $\mathbf{g}_x$ on $[z, g(z)]$, hence the ratio

$$\frac{\mathbf{g}_x'(g(z))}{\mathbf{g}_x'(z)}$$

is uniformly close to 1. We conclude that the element $\widetilde{g} = \mathbf{g}_x\, g\, \mathbf{g}_x^{-1}$ has derivative close to 1 on $\mathbf{g}_x(J_x^+)$. By the definition of the expanding map $\mathbf{g}_x$, the interval $\mathbf{g}_x(J_x^+)$ is in the finite collection $\mathcal{I}$ of Theorem 3.4. The element $\widetilde{g}$ fixes the leftmost point of this interval.

For any interval I_i^+ in the collection $\mathcal{I}$, we know from Lemma 3.14 that the stabilizer of its leftmost point is cyclic, generated by the germ of some element h_i. Choosing the constant δ such that $\widetilde{g}$ is closer to the identity on the macroscopic interval than all the h_i's on their corresponding I_i^+, we can conclude that $\widetilde{g}$ must be the identity on a right neighbourhood of $\mathbf{g}_x(x)$, and so is g on a right neighbourhood of x, as desired. □

Magnification – The next lemma allows to lift our study of the stabilizer to a well-chosen macroscopic level, without losing control of distortion.

Lemma 3.25. *There exists a finite set $\widetilde{\mathcal{G}} \subset G$ and a constant C_0 for which the following holds. For any $\varepsilon > 0$ there exist a finite subset $X' \subset X$ such that:*

i. *for any $x \in X'$, one has $\mathcal{E}(x) \leq \varepsilon$;*

ii. *$\sum_{x \in X'} \mathcal{E}(x) < C_0$;*

iii. *for any y belonging to an infinite connected component of $X \setminus X'$ and for any $g \in \widetilde{\mathcal{G}}$ there exist $n, m \in \mathbb{N}$ and $h \in \widetilde{\mathcal{G}}$ such that*

$$\mathcal{R}^n(y), \mathcal{R}^m(g(y)) \in X', \text{ and } h(\mathcal{R}^n(y)) = \mathcal{R}^m(g(y));$$

Proof of Lemma 3.25. We will choose and fix a sufficiently large constant C and look for X' of the form

$$X' = X_\varepsilon := \left\{ x \in X \mid C^{-3}\varepsilon \leq \mathcal{E}(x) \leq \varepsilon \right\}. \tag{3.19}$$

Note that as a consequence of Lemma 3.15, X' is a finite set. With this choice we clearly ensure property i and we will prove that the remaining two properties are satisfied.

We first choose the finite set $\widetilde{\mathcal{G}}$ and focus on property iii. Take as $\widetilde{\mathcal{G}}$ the union of the generating set $\mathcal{G}$ with the elements g_I's given by Theorem 3.4 (locally defining $\mathcal{R}$) and the elements h_i's given by Theorem 3.12 (see also Remark 3.13):

$$\widetilde{\mathcal{G}} = \mathcal{G} \cup \{g_I\} \cup \{h_i\}.$$

Set

$$C_1 := \max_{g \in \widetilde{\mathcal{G}} \cup \widetilde{\mathcal{G}}^{-1}} \|g'\|_0.$$

We consider a partition $J_1, \ldots, J_q$ which verifies the properties of described in Theorem 3.12, for any $g \in \widetilde{\mathcal{G}} \cup \widetilde{\mathcal{G}}^{-1}$. That is, for every $g \in \widetilde{\mathcal{G}}$ and $p = 1, \ldots, q$ the restriction $g|_{J_p}$ is equal to some h_{i_p} under magnification:

$$g|_{J_p} = \mathcal{R}^{-n'_p} h_{i_p} \mathcal{R}^{n_p},$$

with i_p, n_p, n'_p depending on g. Let N be an integer larger than any power n_p, n'_p in these magnifications for the elements in $\widetilde{\mathcal{G}} \cup \widetilde{\mathcal{G}}^{-1}$. Choose

$$C := \max \left(C_1, \max_{j \leq N} \|(\mathcal{R}^j)'\|_0 \right)$$

to be the constant appearing in the definition (3.19). We still have to fix the good ε.

Let y be a point in an infinite connected component of $X \setminus X_\varepsilon$. In particular we have $\mathcal{E}(y) < C^{-3}\varepsilon$. Take $g \in \widetilde{\mathcal{G}}$. There exist n_1, m_1, i_1 such that $g(y) = \mathcal{R}^{-m_1} h_{i_1} \mathcal{R}^{n_1}(y)$. By our choice, the element h_{i_1} belongs to the finite set $\widetilde{\mathcal{G}}$, so it also admits a magnification on an interval containing $\mathcal{R}^{n_1}(y)$: there exist n_2, m_2, i_2 such that $h_{i_1}\mathcal{R}^{n_1}(y) = \mathcal{R}^{-(m_1+m_2)} h_{i_2} \mathcal{R}^{n_2+n_1}(y)$. Iterating this process, we obtain a tower of magnifications that we schematically represent by the following diagram:

$$\begin{tikzcd}
\mathcal{R}^n(y) \arrow[r, "h"] & \mathcal{R}^m(g(y)) \\
\dots \arrow[u] & \dots \arrow[u] \\
\mathcal{R}^{n_2}\mathcal{R}^{n_1}(y) \arrow[u] \arrow[r, "h_2"] & \mathcal{R}^{m_2}\mathcal{R}^{m_1}(y) \arrow[u] \\
\mathcal{R}^{n_1}(y) \arrow[u, "\mathcal{R}^{n_2}"] \arrow[r, "h_1"] & \mathcal{R}^{m_1}(y) \arrow[u, "\mathcal{R}^{m_2}"'] \\
y \arrow[u, "\mathcal{R}^{n_1}"] \arrow[r, "g"'] & g(y) \arrow[u, "\mathcal{R}^{m_1}"']
\end{tikzcd}$$

The final step is chosen so that for

$$n = n_1 + n_2 + \dots, \quad m = m_1 + m_2 + \dots$$

we have $\mathcal{E}(\mathcal{R}^n(y)) \in [C^{-2}\varepsilon, C^{-1}\varepsilon]$. This choice is always possible because on the one hand the energy increases when applying $\mathcal{R}$ (since in real-analytic regularity $\mathcal{R}$ is expanding, Lemma 3.7), but on the other it cannot grow too fast. Indeed, for any $z \notin \Delta_{n-1}$ and $j \leq n$ we have $\mathcal{E}(\mathcal{R}^j(z)) = (\mathcal{R}^j)'(z) \cdot \mathcal{E}(z)$ and thus

$$C^{-1}\mathcal{E}(z) \leq \mathcal{E}(\mathcal{R}^j(z)) \leq C\mathcal{E}(z).$$

We also have

$$\mathcal{E}(\mathcal{R}^m(g(y))) \in [C^{-3}\varepsilon, \varepsilon]. \tag{3.20}$$

Indeed, we know that $\mathcal{R}^m g = h\mathcal{R}^n$ and $\mathcal{E}(\mathcal{R}^n(y)) \in [C^{-2}\varepsilon, C^{-1}\varepsilon]$; we also have $C^{-1} \leq \|h'\|_0 \leq C$, by our choice of C, therefore from the equality

$$\mathcal{E}(\mathcal{R}^m(g(y))) = \mathcal{E}(h\mathcal{R}^n(y)) = h'(\mathcal{R}^n(y)) \cdot \mathcal{E}(\mathcal{R}(y))$$

we can easily deduce (3.20). In particular, both $\mathcal{R}^n(y)$ and $\mathcal{R}^m(g(y))$ are in X_ε. This proves iii.

We now proceed to the proof of ii: let us estimate $\sum_{x \in X_\varepsilon} \mathcal{E}(x)$. We will show that it does not exceed some universal constant C_0 that does not depend on ε.

We consider the intervals J_x^+ given by Proposition 3.10. Any two distinct intervals J_x^+, J_y^+ either are disjoint, or one is contained into the other. As in the proof of Lemma 3.15, we observe that in the latter case, the ratio of the lengths is larger than $\lambda > 1$ (due to control of distortion).

We claim that there exists a uniform C_2 such that for any $\varepsilon > 0$, any point of the circle is covered by at most C_2 intervals J_x^+, with $x \in X_\varepsilon$.

Indeed, let $z \in \mathbf{S}^1$ be any point and denote by $J_{x_1}^+ \subset \dots \subset J_{x_d}^+$ all the intervals containing z, given by points $x_i \in X_\varepsilon$, ordered by inclusion. On the one hand, we must have

$$\frac{|J_{x_d}^+|}{|J_{x_1}^+|} \geq \lambda^d.$$

On the other hand, using (3.18), there exists a constant $c > 0$ such that for any $x \in X$ one has

$$c^{-1} \cdot \mathcal{E}(x) \leq |J_x^+| \leq c \cdot \mathcal{E}(x),$$

whence

$$\frac{|J^+_{x_d}|}{|J^+_{x_1}|} \leq \frac{c \cdot \mathcal{E}(x_d)}{c^{-1} \cdot \mathcal{E}(x_1)} \leq c^2 C^3,$$

for $x_1, x_d \in X_\varepsilon$. Thus we have a uniform bound for the number of overlaps d given by $\lambda^d \leq c^2\, C^3$. Therefore it is enough to take C_2 such that

$$\lambda^{C_2} > c^2\, C^3.$$

We deduce the inequality

$$\sum_{x \in X_\varepsilon} |J^+_x| \leq C_2,$$

Thus, using (3.18), there exists C_0 such that

$$\sum_{x \in X_\varepsilon} \mathcal{E}(x) \leq C_0,$$

as desired. This proves ii. □

Decomposition of the stabilizer – Fix $\varepsilon > 0$ and the corresponding finite set $X' \subset X$ from Lemma 3.25. From now on we increase the set of generators $\mathcal{G}$, adding all the elements in $\widetilde{\mathcal{G}}$ to it. The Schreier graph $\mathrm{Sch}(X, \mathcal{G})$ is considered with respect to this increased generating system. For an infinite connected component $\mathcal{C}$ of $X \setminus X'$, denote by $X_{\mathcal{C}}$ the part of X' which is adjacent to $\mathcal{C}$.

Observe that any element $g \in G$ fixing a point of the orbit X defines a loop in the Schreier graph $\mathrm{Sch}(X, \mathcal{G})$ (and *viceversa*). By a local conjugation by a sufficiently large power of the magnification $\mathcal{R}$, we bring any loop contained in an infinite connected component of $X \setminus X'$, to the finite set X':

Lemma 3.26. *Let $\mathcal{C}$ be an infinite connected component of $X \setminus X'$. Consider a point $x \in \mathcal{C}$ and an element $g \in G$ that fixes x and that can be written in the form $g = g_m \cdots g_1$ with respect to the generating set $\mathcal{G}$, such that $g_i \cdots g_1(x) \in \mathcal{C}$ for any $i \leq m$.*

Then there exists $N \in \mathbb{N}$ and an element $h \in G$ such that the following properties are satisfied:

- **i.** *there exists a neighbourhood U of x such that $g|_U = \mathcal{R}^{-N} h \mathcal{R}^N|_U$,*
- **ii.** *$\mathcal{R}^N(x) \in X_{\mathcal{C}}$ and $\mathcal{R}^j(x) \in \mathcal{C}$ for any $j < N$,*
- **iii.** *with respect to the generating system $\mathcal{G}$, h can be written in the form $h = s_k \cdots s_1$, with $s_i \cdots s_1(\mathcal{R}^N(x)) \in X_{\mathcal{C}}$ for any $i \leq k$.*

Proof. Let $x \in \mathcal{C}$ and $g = g_m \cdots g_1 \in G$ be such as in the lemma. We let x_k denote $g_k \cdots g_1(x)$. The magnification procedure yields intervals J_{i_k} such that $g_k|_{J_{i_k}} = \mathcal{R}^{-n'_k} h_{i_k} \mathcal{R}^{n_k}$ with $h_{i_k} \in \mathcal{G}$ and $n_k, n'_k \in \mathbb{N}$ satisfying $\mathcal{R}^{n_k}(x_k), \mathcal{R}^{n'_k}(x_{k+1}) \in X_{\mathcal{C}}$ and $\mathcal{R}^i(x_k), \mathcal{R}^j(x_{k+1}) \in \mathcal{C}$ for every $i < n_k$ and $j < n'_k$.

Let us fix U, a neighbourhood of x such that for every $k \in \{1, \cdots m\}$ $g_k \cdots g_1(U) \subset J_k$. In restriction to U, g has the following decomposition

$$g = \mathcal{R}^{-n_1} \left(\mathcal{R}^{n_1} \mathcal{R}^{-n'_m} h_{i_m} \mathcal{R}^{n_m} \cdots \mathcal{R}^{-n'_2} h_{i_2} \mathcal{R}^{n_2} \mathcal{R}^{-n'_1} h_{i_1} \right) \mathcal{R}^{n_1}.$$

We claim that $N = n_1$ and $h = \mathcal{R}^{n_1} \mathcal{R}^{-n'_m} h_{i_m} \mathcal{R}^{n_m} \cdots \mathcal{R}^{-n'_2} h_{i_2} \mathcal{R}^{n_2} \mathcal{R}^{-n'_1} h_{i_1}$ satisfy the conclusions of the lemma. The first property holds by construction. The second one holds by definition of n_1. Let us prove that the third one holds as well.

Set $y_k = \mathcal{R}^{n_k}(x_k)$ and $z_k = h_{i_k}(y_k)$. By Lemma 3.25 we have $y_k, z_k \in X'$. Let us describe the loop based at y_1 defined by h. It links y_k to z_k via h_{i_k} and z_k to y_{k+1} via some power of $\mathcal{R}$. We claim that this last path must stay inside $X_{\mathcal{C}}$.

Indeed, its two endpoints belong to $X_{\mathcal{C}} \subset X'$, meaning that their energies are between $C^{-3}\varepsilon$ and ε. Using Lemma 3.7, we have that in real-analytic regularity, the map $\mathcal{R}$ is expanding, so the energy is monotone along the path. Therefore the path is contained in X', and hence in $X_{\mathcal{C}}$.

This concludes the proof of the lemma. □

Next, we decompose a loop contained in $X_{\mathcal{C}}$ into finitely many *simple* loops inside $X_{\mathcal{C}}$:

Lemma 3.27. *Let $h \in G$ be an element fixing a point $y \in X_{\mathcal{C}}$, such that h can be written in the form $h = h_m \cdots h_1$ with respect to the generating system $\mathcal{G}$, and such that*

$$h_i \cdots h_1(y) \in X_{\mathcal{C}} \quad \text{for every } i \leq m.$$

Then there exists elements $\gamma_1, \ldots, \gamma_\alpha$ such that the following properties are satisfied:

- **i.** $h = \gamma_\alpha \cdots \gamma_1$,
- **ii.** *any γ_j fixes a point x_j in $X_{\mathcal{C}}$,*
- **iii.** *any γ_j can be written in the form $\gamma_j = t_k \cdots t_1$ with respect to the generating system $\mathcal{G}$, in such a way that for any $i = 1, \ldots, k$, the points $t_i \cdots t_1(x_j)$ belong to $X_{\mathcal{C}}$ and all are distinct.*

Proof. As we said, we take the loop based at y representing h, which is contained in $X_{\mathcal{C}}$, and we decompose it into simple loops contained in $X_{\mathcal{C}}$. □

Proof of Proposition 3.22 – By assumption, the Schreier graph of the orbit X has infinitely many ends, thus for any K there exists $\varepsilon > 0$ such that in the complement of the set X' given by Lemma 3.25 there are at least K infinite connected components. As above, for any such component $\mathcal{C}$ we denote by $X_{\mathcal{C}}$ the subset of X' that is adjacent to $\mathcal{C}$.

Lemma 3.25 implies that for at most $\left\lfloor \frac{3C_0}{\delta} \right\rfloor$ connected components $\mathcal{C}$, the sum of energies $S_{\mathcal{C}}$ satisfies

$$S_{\mathcal{C}} := \sum_{x \in X_{\mathcal{C}}} \mathcal{E}(x) \geq \frac{\delta}{3}.$$

Take K sufficiently large such that $\varepsilon < \frac{\delta}{3}$ and $K > \frac{3C_0}{\delta}$; in this way we ensure at least one infinite connected component $\mathcal{C}$ of $X \setminus X'$ satisfying $S_{\mathcal{C}} < \frac{\delta}{3}$. We show that with this condition, the component $\mathcal{C}$ satisfies the requirements in the statement.

Let $x \in \mathcal{C}$ and $g \in \mathrm{Stab}_G(x)$ be as in the statement. Using Lemma 3.26, we find a power N of $\mathcal{R}$ that locally conjugates g to an element h that fixes a point in $X_{\mathcal{C}}$ and defines a loop contained in $X_{\mathcal{C}}$. By applying Lemma 3.27, we decompose h into a product $h = \gamma_\alpha \cdots \gamma_1$, with every γ_i defining a simple loop in $X_{\mathcal{C}}$. Therefore there exists a neighbourhood U of x such that we have the decomposition

$$g|_U = \mathcal{R}^{-N} \gamma_\alpha \cdots \gamma_1 \mathcal{R}^N|_U. \tag{3.21}$$

The loops defined by the γ_j's are simple: we can write $\gamma_j = t_k \cdots t_1$, such that all the points $t_i \cdots t_1(x_j)$ belong to $X_{\mathcal{C}}$ and all are distinct. Hence we have the upper bound

$$\sum_{i=0}^{k-1} \mathcal{E}(t_i \cdots t_1(x_j)) \leq S_{\mathcal{C}} < \delta,$$

Then Lemma 3.24 implies that the γ_j's are trivial. Because of the decomposition (3.21), also g is trivial, as desired.

Proof of Theorem D − We proceed as for Theorem 3.1. Consider the class of the loop defined by the stabilizer $h \in \mathrm{Stab}_G(x_0)$ in the fundamental group $\pi_1(\mathrm{Sch}(X,\mathcal{G}), x_0)$. The holonomy covering $\pi : G_{x_0} \to \mathrm{Sch}(X,\mathcal{G})$ is exactly the covering associated with this element.

After Proposition 3.22, there exists a finite set X' and an infinite connected component of $X \setminus X'$ such that any loop contained in $\mathcal{C}$ can be lifted to G_{x_0}. This implies that the pre-image of $\mathcal{C}$ of the holonomy covering is homeomorphic to $\mathcal{C} \times \mathbb{Z}$. Denote by ϕ the homeomorphism $\phi : \mathcal{C} \times \mathbb{Z} \to \pi^{-1}(\mathcal{C})$.

For any $n > 0$, Let Y_n be a finite set contained in $\pi^{-1}(X')$ such that $G_{x_0} \setminus Y_n$ contains $\phi(\mathcal{C} \times [1,n])$. We deduce that $G_{x_0} \setminus Y_n$ contains at least n infinite connected components. Letting n go to infinity, we deduce that G_{x_0} has infinitely many ends, as desired.

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