Elementary Considerations on Prime Numbers and on the Riemann Hypothesis

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The main purpose of this paper: an elementary disproof of the Riemann Hypothesis.

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COMBINATORICS ON PARTIAL SUM OF THE LIOUVILLE FUNCTION UP TO A PRIME SUPERIOR QUOTA, LIMITING AND SUBSIDIARY COMPLEMENTS

The Liouville function $\lambda(n)$ depending on the variable $n \in \mathbb{N} = \{1, 2, 3, 4, \cdots\}$ (in this paper, $0 \notin \mathbb{N}$) is given by:

$$\lambda(n) = (-1)^{\omega(n)}, \tag{1}$$

where:

$$\omega(n) = \textit{ Number of prime factors of } n, \tag{2}$$

being these prime fators not necessarily distinct, counted with multiplicity. Hence, the image of $\lambda(n)$ is $\{1, -1\}$. E.g.: $\lambda(1) = 1$, since 1 has not got any prime factors ($\omega(1) = 0$); $\lambda(2) = -1$, since 2 has got just one prime factor (itself, since 2 is a prime number), from which $\omega(2) = 1$; $\lambda(20) = -1$, since, counted with multiplicity, 20 has got 3 prime factors ($20 = 2 \times 2 \times 5$), from which $\omega(20) = 3$. Of course, all the numbers $n \in \mathbb{N}$, except the number 1, will have prime factors, being such quantity of prime factors, exactly $\omega(n)$, either odd or even. A prime number has got itself as its prime factor, hence $\lambda(p) = -1$, i.e., being $p \in \mathbb{N}$ prime: $\omega(p) = 1$.

By virtue of these considerations, one may be interested in a generic factorization with an even number of prime factors and also be interested in a factorization with an odd number of prime numbers. The presence of 1 as a factor in a given factorization does not matter, as well as for any quantity of the number 1 as a factor, except for the unique case of the factorization of the number 1, this latter being trivial and unique, once a consideration of 1 factor(s) in a given natural number $n \neq 1$ contributes nothing to $\omega(n)$, viz.:

$$\omega(n) = \omega(1 \times 1 \times \cdots \times 1 \times \cdots \times 1 \times n), \tag{3}$$

by the very elementary fact that we just count the quantity of prime factors of an n. Also, once a factorization is unique, except for the order of the prime factors of a given natural number, a given even (or odd) factorization with $2k$ (or $2k-1$), $k \in \mathbb{N}$, factors turns out to be unique under a combinatorial (combination) consideration with repetition with $2k$ (or $2k-1$) elements. As a matter of fact and clarification, here, we should start to put these assertions under a more mathematically generalized sound.

Let $\mathbb{P}_N = \{p_1, p_2, \cdots, p_N\}$ be the set of all the first N prime numbers, so that $\mathbb{P} = \mathbb{P}_\infty$ is the set of the prime numbers. One may consider $2k$, $k \in \mathbb{N}$, slots filled with any p_l element, $l \in \{1, 2, \cdots, N\}$, from the $\mathbb{P}_N$ set, e.g.:

$$\underbrace{\boxed{p_3} \times \boxed{p_5} \times \boxed{p_5} \times \boxed{p_2} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_8} \times \cdots \times \boxed{p_2}}_{(2k)-slots} \tag{4}$$

Of course, any permutation of (4) generates the very same configuration, generates the very same number due to the multiplication ($\times$ between slots). One should infer that repetition is allowed, viz., the elements in (4) do not need to be different. However, for an instantaneously fixed order of (4), a change in a given slot element (choosing a different one from $\mathbb{P}_N$), with the elements within the remaining slots not changed, would lead to a new configuration for (4). A permutation of this latter new configuration does not change it. A given configuration represents a unique natural number with an even number of prime factors, since, in spite of permutation, its factorization (configuration) is unique. Simmilarly, One may consider $2k-1$, $k \in \mathbb{N}$, slots filled with any p_l element, $l \in \{1, 2, \cdots, N\}$, from the $\mathbb{P}_N$ set, e.g.:

$$\underbrace{\boxed{p_3} \times \boxed{p_5} \times \boxed{p_5} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_8} \times \cdots \times \boxed{p_2}}_{(2k-1)-slots} \tag{5}$$

Of course, any permutation of (5) generates the very same configuration, generates the very same number due to the multiplication ($\times$ between slots). One should infer that repetition is allowed, viz., the elements in (5) do not need to be different. However, for an instantaneously fixed order of (5), a change in a given slot element (choosing a different one from $\mathbb{P}_N$), with the elements within the remaining slots not changed, would lead to a new configuration for (5). A permutation of this latter new configuration does not change it. A given configuration represents a unique natural number with an odd number of prime factors, since, in spite of permutation, its factorization (configuration) is unique.

To consider the totality of numbers in $\mathbb{N}$, one needs to impose $N \to \infty$, $N \in \mathbb{N}$, hence $\mathbb{P}_N \to \mathbb{P}_\infty = \mathbb{P}$, and to consider the totality of $2k$ slots and to consider the totality of $2k-1$ slots, these latter totalities by completely considering k (viz.: $\forall k \in \mathbb{N}$).

Since a given configuration does not change with a permutation of its elements, we are dealing with a problem of combination (the order does not matter). However, elements may be repeated, hence the combinatorics involved is neither $C_{N,2k}$ nor $C_{N,2k-1}$. To uniquely represent a configuration, we may define a convention. In fact, considering an hypothetical factorization with $q \in \mathbb{N}$ factors, e.g., as represented below:

$$\underbrace{\boxed{p_N} \times \boxed{p_5} \times \boxed{p_1} \times \boxed{p_2} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_8} \times \cdots \times \boxed{p_2}}_{(q)-slots}, \tag{6}$$

it may rearranged respecting the order of the indexes:

$$\underbrace{\boxed{p_1} \times \boxed{p_2} \times \boxed{p_2} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_5} \times \boxed{p_8} \times \cdots \times \boxed{p_N}}_{(q)-slots}, \tag{7}$$

which is the very same factorization preserving its same elements, now written in increasing order of indexes, which, to uniquely be represented, may have each index increased by an amount exactly equal to the quantity of previous elements (which is a unique characteristic per element), also changing the label to x, viz.:

$$\underbrace{\boxed{x_1} \times \boxed{x_3} \times \boxed{x_4} \times \boxed{x_6} \times \boxed{x_7} \times \boxed{x_{10}} \times \boxed{x_{14}} \times \cdots \times \boxed{x_{N+q-1}}}_{(q)-slots} \tag{8}$$

With this convention, each q-combination turns out to have got all of its elements with different indexes, with the maximum index ocurring when a q-combination has got its last slot accupied by p_N, from which we turn out to have a problem of simple q-combination of $N+q-1$ elements. Hence, the quantity Q_q^N of unique factorizations having q prime, with multiplicity allowed, factors taken from $\mathbb{P}_N$ is:

$$Q_q^N = C_{N+q-1,\,q} = \frac{(N+q-1)\,!}{(N-1)\,!\,q\,!}, \tag{9}$$

and the quantity $Q_q^{\mathbb{N}}$ of unique factorizations having q prime factors turns out to be:

$$Q_q^{\mathbb{N}} = \lim_{N\to\infty} C_{N+q-1,\,q} = \lim_{N\to\infty} \frac{(N+q-1)\,!}{(N-1)\,!\,q\,!} = |\mathbb{N}_q|\,, \tag{10}$$

the totality of distinct numbers belonging to $\mathbb{N}$ and having got q prime, with multiplicity allowed, factors. The totality of numbers belonging to $\mathbb{N}$ turns out to be:

$$\begin{aligned} Q^{\mathbb{N}} &= 1 + \sum_{q=1}^{\infty} Q_q^{\mathbb{N}} \\ &= 1 + \sum_{q=1}^{\infty} \lim_{N\to\infty} C_{N+q-1,\,q} \\ &= 1 + \sum_{q=1}^{\infty} \lim_{N\to\infty} \frac{(N+q-1)\,!}{(N-1)\,!\,q\,!} \\ &= 1 + \sum_{q=1}^{\infty} |\mathbb{N}_q| = |\mathbb{N}| = \aleph_0. \end{aligned} \tag{11}$$

We will be interested in the infinite sum:

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda\,(n) &= \lambda\,(1) + \sum_{n=2}^{\infty} \lambda\,(n) \\ &= 1 + \sum_{n=2}^{\infty} \lambda\,(n) \\ &= 1 + \sum_{k=1}^{\infty} Q_{2k}^{\mathbb{N}} - \sum_{k=1}^{\infty} Q_{2k-1}^{\mathbb{N}}, \end{aligned} \tag{12}$$

where:

$$\begin{aligned} Q_{2k}^{\mathbb{N}} &= \lim_{N\to\infty} C_{N+2k-1,\,2k} \\ &= \lim_{N\to\infty} \frac{(N+2k-1)\,!}{(N-1)\,!\;(2k)\,!}, \end{aligned} \tag{13}$$

and:

$$\begin{aligned} Q_{2k-1}^{\mathbb{N}} &= \lim_{N\to\infty} C_{N+(2k-1)-1,\,2k-1} \\ &= \lim_{N\to\infty} \frac{(N+2k-2)\,!}{(N-1)\,!\;(2k-1)\,!}, \end{aligned} \tag{14}$$

$k \in \mathbb{N}$. Considering the partial difference:

$$D_k^N = Q_{2k}^N - Q_{2k-1}^N, \tag{15}$$

which, by virtue of the Eq. (9), leads to:

$$
\begin{aligned}
D_k^N &= C_{N+2k-1,\,2k} - C_{N+(2k-1)-1,\,2k-1} = \frac{(N+2k-1)\,!}{(N-1)\,!\,(2k)\,!} - \frac{(N+2k-2)\,!}{(N-1)\,!\,(2k-1)\,!} \\
&= \frac{(N+2k-1)\,! - 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N+2k-1)\,(N+2k-2)\,! - 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N+2k-1-2k)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N-1)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N-1)\,(N+2k-2)\,!}{(N-1)\,(N-2)\,!\,(2k)\,!} \\
&= \frac{(N+2k-2)\,!}{(N-2)\,!\,(2k)\,!} = \frac{[(N-2)+2k]\,!}{(N-2)\,!\,(2k)\,!} = \frac{\overbrace{[(N-2)+2k]\,[(N-2)+2k-1]\times\cdots\times[(N-2)+1]}^{2k\ factors}\,(N-2)\,!}{(N-2)\,!\,(2k)\,!} \\
&= \frac{\overbrace{[(N-2)+2k]\,[(N-2)+2k-1]\times\cdots\times[(N-2)+1]}^{2k\ factors}}{(2k)\,!} \\
&= \frac{1}{(2k)\,!}\prod_{j=1}^{2k}[(N-2)+j] \\
&> d_k^N\,,
\end{aligned} \tag{16}
$$

where:

$$
d_k^N = \frac{(N-2)^{2k}}{(2k)\,!}. \tag{17}
$$

Hence, from the Eqs. (15), (16) and (17):

$$
\begin{aligned}
\sum_{k=1}^{\infty} D_k^N &= \sum_{k=1}^{\infty} Q_{2k}^N - \sum_{k=1}^{\infty} Q_{2k-1}^N \\
&= \sum_{k=1}^{\infty} \frac{1}{(2k)\,!}\prod_{j=1}^{2k}[(N-2)+j] \\
&> \sum_{k=1}^{\infty} d_k^N = \sum_{k=1}^{\infty} \frac{(N-2)^{2k}}{(2k)\,!} = \cosh\,(N-2) - 1.
\end{aligned} \tag{18}
$$

Now, consider the set $\mathbb{N}-\{1\}$, hence $\mathbb{N}-\{1\}$ exhausts any prime factorization. This is so, since a given prime factorization represents one and only one element in $\mathbb{N}-\{1\}$ and a given element of $\mathbb{N}-\{1\}$ has got one and only one prime factorization. Hence, all the factorizations exhaust $\mathbb{N}-\{1\}$. $\mathbb{N}-\{1\}$ is the entire set of possible prime factorizations and the entire set of possible prime factorizations is $\mathbb{N}-1$. The arithmetic progression:

$$
a_n = 1 + (n-1)1 = n \Leftrightarrow a_n = n, \tag{19}
$$

with $a_n \in \mathbb{N}-1$, shows that the number of possible factorizations grows as n grows. Hence, since this is essentially the set $\mathbb{N}-1$, the quantity of possible factorizations exhaustively grows [a given finite set of factorizations will have a factorization, say f, representing a greatest factorized number in this set and being one-to-one to an $a_n = f$ for this set, with the remaining factorizations $(< a_n)$] as $n \to \infty$. Hence:

$$
n \to 1 + \sum_{k=1}^{\infty} \lim_{N\to\infty} Q_{2k}^N + \sum_{k=1}^{\infty} \lim_{N\to\infty} Q_{2k-1}^N, \tag{20}
$$

viz.:

$$
n \to 1 + \sum_{k=1}^{\infty} \left(Q_{2k}^{\mathbb{N}} + Q_{2k-1}^{\mathbb{N}}\right) = \infty. \tag{21}
$$

By virtue of Eq. (21):

$$
n^{\frac{1}{2}+\epsilon} \to \left[1 + \sum_{k=1}^{\infty} \left(Q_{2k}^{\mathbb{N}} + Q_{2k-1}^{\mathbb{N}}\right)\right]^{\frac{1}{2}+\epsilon} = \infty. \tag{22}
$$

Considering the partial sum:

$$
s_k^N = Q_{2k}^N + Q_{2k-1}^N, \tag{23}
$$

which, by virtue of the Eq. (9), leads to:

$$
\begin{aligned}
s_k^N &= C_{N+2k-1,\,2k} + C_{N+(2k-1)-1,\,2k-1} = \frac{(N+2k-1)\,!}{(N-1)\,!\,(2k)\,!} + \frac{(N+2k-2)\,!}{(N-1)\,!\,(2k-1)\,!} \\
&= \frac{(N+2k-1)\,! + 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N+2k-1)\,(N+2k-2)\,! + 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N+2k-1+2k)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N+4k-1)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N+4k-1)\,[(N-1)+(2k-1)]\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N+4k-1)\overbrace{[(N-1)+(2k-1)]\,\{(N-1)+[(2k-1)-1]\}\times\cdots\times[(N-1)+1]}^{2k-1\ factors}(N-1)\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N+4k-1)\overbrace{[(N-1)+(2k-1)]\,\{(N-1)+[(2k-1)-1]\}\times\cdots\times[(N-1)+1]}^{2k-1\ factors}}{(2k)\,!} \\
&= \frac{N+4k-1}{(2k)\,!}\prod_{j=1}^{2k-1}[(N-1)+j] < \frac{N+4k-1+1}{(2k)\,!}\,[(N-1)+2k-1]^{2k-1} \\
&< \frac{N+4k}{(2k)\,!}\,[(N-1)+2k-1+2k]^{2k-1} < \frac{N+4k}{(2k)\,!}\,(N+4k-2+2)^{2k-1} \\
&\therefore \\
s_k^N &< S_k^N, \qquad (24)
\end{aligned}
$$

where:

$$S_k^N = \frac{N+4k}{(2k)\,!}\,(N+4k)^{2k-1} = \frac{(N+4k)^{2k}}{(2k)\,!}, \qquad (25)$$

remembering: $k \in \mathbb{N}$. Furthermore:

$$S_k^N < \Xi_k^N, \qquad (26)$$

where:

$$\Xi_k^N = \frac{(N+N/\alpha)^{2k}}{(2k)\,!} = \left(1+\frac{1}{\alpha}\right)^{2k}\frac{N^{2k}}{(2k)\,!}, \qquad (27)$$

$\forall$ *fixed* $\alpha \in \mathbb{R}_+^*$. The Eq. (26) follows from the proposition:

$$4\alpha k < N,\ \forall\,\alpha \in \mathbb{R}_+^*. \qquad (28)$$

Since N is to exhaustively cover the set $\mathbb{P}_\infty = \mathbb{P}$, this latter being the entire set of prime numbers, viz., in:

$$\mathbb{P}_N = \{p_1, p_2, p_3, \cdots, p_N\} = \{2, 3, 5, \cdots, p_N\}, \qquad (29)$$

N has been taken, defined, *to exhaustively cover on demand* the set of prime numbers, as we have defined from the beginning of this paper. Now, to conversely suppose the condition stated by the Eq. (28), i.e.:

$$\exists\ \alpha \in \mathbb{R}_+^* |\ 4\alpha k \geq N, \qquad (30)$$

one turns out to be, by hypothesis, considering an implied superior quota for the existence of prime numbers. Suppose the Eq. (30) is correct. Hence, there exists, by virtue of the Eq. (30), an $\alpha_0 \in \mathbb{R}_+^*$ such that the successive values of N to exhaustively cover the set of prime numbers never exceed $4\alpha_0 k$. Putting such $4\alpha_0 k$, with the obeyer number of the Eq. (30), *a fortiori*:

$$4\alpha_0 k = [4\alpha_0 k] + \{4\alpha_0 k\}, \qquad (31)$$

with $[4\alpha_0 k]$ and $\{4\alpha_0 k\}$ being, respectively, the integer and the fractionary parts of $4\alpha_0 k$, one is led to a superior quota:

$$N = [4\alpha_0 k], \qquad (32)$$

since $N \in \mathbb{N}$. By virtue of this superior quota for the quantity of prime numbers, there exist only finitely many primes $p_1 < p_2 < \cdots < p_N$. Hence, let $\eta = \prod_{f=1}^{N} p_f > 2$, and consider $\eta - 1 \in \mathbb{N}$. Since $\eta - 1$ is a product of primes, it turns out to have a common prime divisor p_f with η, implying this common prime divisor p_f divides $\eta - (\eta - 1) = 1$: an absurd! Henceforth, Eq. (30) is an absurd, the proposition given by the Eq. (28) is correct $\forall$ fixed $k \in \mathbb{N}$ and, by virtue of the Eqs. (23), (24), (26) and (27), one turns out to be led to:

$$s_k^N = Q_{2k}^N + Q_{2k-1}^N < \Xi_k^N = \frac{1}{(2k)\,!}\left[\left(1+\frac{1}{\alpha}\right)N\right]^{2k}. \qquad (33)$$

Back to the interest carried from the Eqs. (20), (21) and (22), now, we consider the consequence to the sum:

$$\sum_{k=1}^{\infty} s_k^N = \sum_{k=1}^{\infty} Q_{2k}^N + \sum_{k=1}^{\infty} Q_{2k-1}^N, \qquad (34)$$

i.e.:

$$\begin{aligned}\sum_{k=1}^{\infty} s_k^N &= \sum_{k=1}^{\infty} Q_{2k}^N + \sum_{k=1}^{\infty} Q_{2k-1}^N \\ &< \sum_{k=1}^{\infty} \Xi_k^N = \sum_{k=1}^{\infty} \frac{1}{(2k)!}\left[\left(1+\frac{1}{\alpha}\right)N\right]^{2k} = \cosh\left[\left(1+\frac{1}{\alpha}\right)N\right] - 1 \end{aligned} \tag{35}$$

From the reasonings we have carried throughout the march that has been led from that Eq. (12), accomplished from its previous combinatorics, to the Eq. (35), one reaches, *a fortiori*, the following implied consequence:

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &= \lim_{n\to\infty}\frac{1+\sum_{l=2}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} \\ &> \lim_{N\to\infty}\frac{1+\left[\cosh(N-2)-1\right]}{\left\{1+\cosh\left[\left(1+\frac{1}{\alpha}\right)N\right]-1\right\}^{\frac{1}{2}+\epsilon}} = \lim_{N\to\infty}\frac{\cosh(N-2)}{\left\{\cosh\left[\left(1+\frac{1}{\alpha}\right)N\right]\right\}^{\frac{1}{2}+\epsilon}} \\ &\therefore \\ \lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &> \frac{2^{\epsilon}}{\sqrt{2}}\lim_{N\to\infty}\frac{e^{(N-2)}+e^{(2-N)}}{\left[e^{(1+1/\alpha)N}+e^{-(1+1/\alpha)N}\right]^{\frac{1}{2}+\epsilon}} = \frac{2^{\epsilon}}{\sqrt{2}}\lim_{N\to\infty}\frac{e^{(N-2)}\left(1+e^{-2(N-2)}\right)}{\left[e^{(1+1/\alpha)N}\left(1+e^{-2(1+1/\alpha)N}\right)\right]^{\frac{1}{2}+\epsilon}} \\ &\therefore \\ \lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &> \frac{2^{\epsilon}}{e^{2}\sqrt{2}}\lim_{N\to\infty}\frac{e^{N}\left(1+e^{-2(N-2)}\right)}{\left[e^{(1+1/\alpha)N}\left(1+e^{-2(1+1/\alpha)N}\right)\right]^{\frac{1}{2}+\epsilon}} \\ &\therefore \\ \lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &> \frac{2^{\epsilon}}{e^{2}\sqrt{2}}\lim_{N\to\infty}\frac{1+e^{-2(N-2)}}{\left(1+e^{-2(1+1/\alpha)N}\right)^{\frac{1}{2}+\epsilon}}e^{N[1-(\epsilon+1/2)(1+1/\alpha)]}. \end{aligned} \tag{36}$$

Since the Riemann Hypothesis is true *if and only if*:

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} = 0,\ \forall\ \textit{fixed}\ \epsilon > 0, \tag{37}$$

it follows that the convergence of the right-hand side of the Eq. (36), $\forall$ *fixed* $\epsilon > 0$, is a necessary condition for the validity of the Riemann Hypothesis. It follows, then, that:

- *If there exists some* $\epsilon > 0$ *such that the right-hand side of the Eq. (36) diverges, then, the Riemann Hypothesis turns out to be false.*

Supposing the Riemann Hypothesis is true, the condition:

$$1-(\epsilon+1/2)(1+1/\alpha) < 0 \tag{38}$$

holds $\forall$ *fixed* $\epsilon > 0$. Hence:

$$(\epsilon+1/2)(1+1/\alpha) > 1, \tag{39}$$

$\forall$ *fixed* $\epsilon > 0$. But this latter condition is an absurd, since, for $\epsilon = 1/8$ and $\alpha = 2$, e.g.:

$$(\epsilon+1/2)(1+1/\alpha)\big|_{(\alpha,\epsilon)=(2,1/8)} = \frac{15}{16} < 1, \tag{40}$$

implying the Riemann hypothesis turns out to be false.

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