EXPONENTIAL FUNCTIONALS OF SPECTRALLY ONE-SIDED LÉVY PROCESSES CONDITIONED TO STAY POSITIVE

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Abstract. We study the properties of the exponential functional $\int_0^{+\infty} e^{-X^{\uparrow}(t)}dt$ where $X^{\uparrow}$ is a spectrally one-sided Lévy process conditioned to stay positive. In particular, we study finiteness, self-decomposability, existence of finite exponential moments, asymptotic tail at 0 and smoothness of the density.

1. Introduction

We consider a spectrally negative Lévy process V which is not the opposite of a subordinator and does not oscillate. We denote its Laplace exponent by Ψ_V :

$$\forall t, \lambda \geq 0,\ \mathbb{E}\left[e^{\lambda V(t)}\right] = e^{t\Psi_V(\lambda)}.$$

In the case where V drifts to $-\infty$, it is well known that its Laplace exponent admits a non trivial zero that we denote here by κ, $\kappa := \inf\{\lambda > 0,\ \Psi_V(\lambda) = 0\}$. If V does not drift to $-\infty$, then 0 is the only zero of Ψ_V so we put $\kappa := 0$ in this case. We denote by (Q, γ, ν) the generating triplet of V so Ψ_V can be expressed as

$$\Psi_V(\lambda) = \frac{Q}{2}\lambda^2 - \gamma\lambda + \int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{|x|<1})\nu(dx). \tag{1.1}$$

In the end of the paper, we also consider Z, a spectrally positive Lévy process drifting to $+\infty$.

We are interested in the basic exponential functionals of V and Z conditioned to stay positive,

$$I(V^{\uparrow}) := \int_0^{+\infty} e^{-V^{\uparrow}(t)}dt \quad \text{and} \quad I(Z^{\uparrow}) := \int_0^{+\infty} e^{-Z^{\uparrow}(t)}dt.$$

For both we study finiteness, exponential moments and the asymptotic tail at 0. For $I(V^{\uparrow})$, we also get self-decomposability, more precise estimates on the asymptotic tail at 0 and a condition for smoothness of the density.

Our first motivation is to extend to spectrally one-sided Lévy processes conditioned to stay positive the general study of the exponential functionals of Lévy processes. Those functionals have been widely studied because of their importance in probability theory. For example they are fundamental to the study of diffusions in random environments and appear in many applications such as mathematical finance, see [4] for a survey on those functionals and their applications. For a general Lévy process, equivalent conditions for the finiteness of the exponential functional are given in [4], the asymptotic tail at $+\infty$ of the functional is studied in [10], the absolute continuity is proved in [9] and properties of the density (such as regularity) are studied in [11]

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under some hypothesis on the jumps of the Lévy process and in [12]. In this paper we also obtain, as a by-product of our approach, some new results on the exponential functionals of spectrally one-sided Lévy processes.

Our second motivation is the possibility to apply our results to the study of diffusions in a spectrally negative Lévy environment. Such processes, introduced by Brox [6] when the environment is given by a brownian motion have been specifically studied for the spectrally negative Lévy case by Singh [15]. In [2], they prove that the supremum of local time $\mathcal{L}^*_X$ of a diffusion in a drifted brownian environment converges in law and they express the limit law in term of a subordinator and an exponential functional of the environment conditioned to stay positive. In order to generalize their result to a diffusion in a spectrally negative Lévy environment, knowledge on the exponential functionals involved is needed. These are precisely exponential functionals of the environment (which is spectrally negative) and its dual (which is spectrally positive) conditioned to stay positive.

Finally, we have hints that the almost sure asymptotic behavior of $\mathcal{L}^*_X$, for a diffusion in the spectrally negative Lévy environment V, is crucially linked to the right and left tails of the distribution of $I(V^\uparrow)$. This is why we study these tails here and give for the left tail a precise asymptotic estimate when it is possible, in particular, when $\Psi_V(\lambda) \sim c\lambda^\alpha$, for some constant c and $\alpha \in]1, 2]$. For the right tail, we are only interested in the existence of some finite exponential moments. The application of the present work to diffusions in random environment is a work in preparation by the author [16].

We start with some facts about $V^\uparrow$, that is, V conditioned to stay positive.

V beeing spectrally negative, the Markov family $(V_x^\uparrow, x \geq 0)$ may be defined as in [3], Chapter VII, Section 3. For any $x \geq 0$, the process $V_x^\uparrow$ must be seen as V conditioned to stay positive and starting from x. We denote $V^\uparrow$ for the process $V_0^\uparrow$. It is known that $V_x^\uparrow$ converges in the Skorokhod space to $V^\uparrow$ when x goes to 0.

For X a positive random variable, we denote $V_X^\uparrow$ for the process V conditioned to stay positive and starting from the random variable X. More rigorously, $V_X^\uparrow$ is the Markov process that conditionally on $\{X = x\}$ has law $V_x^\uparrow$.

For any positive x, we have from Markov property and the absence of positive jumps that the process $V^\uparrow$, shifted at its first passage time at x, is equal in law to $V_x^\uparrow$, and in the case where V drifts to $+\infty$, it is known from [3], Chapter VII, Section 3, that $V_x^\uparrow$ has the same law as V_x conditioned in the usual sense to remain positive. This property, interesting for our study, is unfortunately not true when V drifts to $-\infty$.

In the case where V drifts to $-\infty$, we define $V^\sharp$ to be "V conditioned to drift to $+\infty$", as in [3], page 193. The Laplace exponent $\Psi_{V^\sharp}$ of $V^\sharp$ satisfies $\Psi_{V^\sharp} = \Psi_V(\kappa + .)$ where κ is the non-trivial zero of Ψ_V. As a consequence $\Psi'_{V^\sharp}(0) > 0$, so $V^\sharp$ drift to infinity (this is deduced thanks to Corollary 2 page 190 in [3]) and it is also proven that $V^\uparrow = (V^\sharp)^\uparrow$. Therefore, for $x > 0$, $V_x^\uparrow$ is only $V_x^\sharp$ conditioned in the usual sense to remain positive. $V^\sharp$ is not defined in the case where V oscillates, this is why we had to exclude this case. Then, either V drifts to $-\infty$, either V drifts to $+\infty$, and in both cases $V^\uparrow$ can be obtain from a spectrally negative Lévy process drifting to $+\infty$ and conditioned to stay positive.

In order to do our proofs in a systematic way, we often work with $V^\sharp$ which is "V conditioned to drift to $+\infty$" in the case where V drifts to $-\infty$ and "only V" in the case where V drifts to $+\infty$. As a consequence, $V^\sharp$ always denotes a spectrally negative Lévy process drifting to $+\infty$.

For A a process, we denote

$$\begin{aligned}\tau(A,x) &:= \inf\{t\geq 0,\ A(t)\geq x\},\\ \mathcal{R}(A,x) &:= \sup\{t\geq 0,\ A(t)\leq x\}.\end{aligned}$$

Also let $\underline{A}(t) := \inf\{A(s),\ s\in[0,t]\}$ be the infimum process of A. If A is Markovian and $x\in\mathbb{R}$ we denote A_x for the process A starting from x. For A_0 we shall only write A. For any (possibly random) time $T>0$, we write A^T for the process A shifted and centered at time T : $\forall s\geq 0,\ A^T(s) := A(T+s)-A(T)$.

1.1. **Results.** In the special case of the exponential functional of a drifted brownian motion conditioned to stay positive, all the properties that are established here are already known and sometimes more explicitly. We discuss this case in the next subsection.

Our first result is the finiteness of $I(V^\uparrow)$, which turns out to be an easy consequence of the finiteness of $I(V^\sharp)$, given by Theorem 1 in [4].

Theorem 1.1. *The random variable $I(V^\uparrow)$ is almost surely finite.*

Remark 1.2. *When V oscillates, we believe the exponential functional $I(V^\uparrow)$ to be finite but this, and the other properties we establish here for $I(V^\uparrow)$ when V drifts to $\pm\infty$, can not be extended to the oscillating case with our approach. This is due to the fact that we always need to consider $V^\uparrow$ as a spectrally negative Lévy process drifting to $+\infty$ and conditioned to stay positive ($V^\uparrow = (V^\sharp)^\uparrow$), but this is impossible in the oscillating case.*

Under an extra assumption on V, we prove the stronger result that $I(V^\uparrow)$ also admits some finite exponential moments. The result can be stated as follows :

Theorem 1.3. *If either V drifts to $-\infty$ or $V(1)$ admits a finite negative exponential moment, that is,*

$$V(t)\underset{t\to+\infty}{\longrightarrow}-\infty \quad or \quad \exists\gamma>0,\ \mathbb{E}\left[e^{-\gamma V(1)}\right]<+\infty, \tag{1.2}$$

then the random variable $I(V^\uparrow)$ admits some finite exponential moments, that is

$$\exists\lambda>0,\ \mathbb{E}\left[e^{\lambda I(V^\uparrow)}\right]<+\infty. \tag{1.3}$$

A fundamental point of our study is Proposition 3.1 which says that for any positive y, $I(V^\uparrow)$ satisfies the random affine equation

$$I(V^\uparrow)\overset{\mathfrak{L}}{=}A^y+e^{-y}I(V^\uparrow), \tag{1.4}$$

where A^y is independent of the second term and will be specified later. We see that $I(V^\uparrow)$ is a positive self-decomposable random variable and is therefore absolutely continus and unimodal. It is well known (see for example expression (1.10) in [13]) that the exponential functional $I(V)$ of a spectrally negative Lévy process V is also self-decomposable (as long as it is finite), it can be seen by splitting the trajectory at $\tau(V,y)$, the first passage time at y. An other consequence of (1.4) is that for any positive y, $I(V^\uparrow)$ can be written as the random series

$$I(V^\uparrow)\overset{\mathfrak{L}}{=}\sum_{k\geq 0}e^{-ky}A_k^y,$$

where the random variables A^y_k are *iid* and have the same law as A^y. This decomposition is a very useful tool for the study of the random variable $I(V^\uparrow)$ and is also the base of the proofs of the results we present below.

Our next results make a link between the properties of $I(V^\uparrow)$ and the asymptotic behavior of Ψ_V. Before stating them we need to define some quantification of this asymptotic behavior, so we define, as in [3], page 94,

$$\sigma := \sup\left\{\alpha \geq 0,\ \lim_{\lambda\to+\infty} \lambda^{-\alpha}\Psi_V(\lambda) = \infty\right\},$$
$$\beta := \inf\left\{\alpha \geq 0,\ \lim_{\lambda\to+\infty} \lambda^{-\alpha}\Psi_V(\lambda) = 0\right\}.$$

Recall that $\Psi_{V^\sharp}(.) = \Psi_V(\kappa + .)$, so σ and β are identical whether they are defined from $\Psi_{V^\sharp}$ or Ψ_V.

If Ψ_V has α-regular variations for $\alpha \in [1, 2]$ (for example if V is a drifted α-stable Lévy process with no positive jumps), we have $\sigma = \beta = \alpha$. Recall that Q is the brownian component of V. It is well known that $\Psi_V(\lambda)/\lambda^2$ converges to $Q/2$ when λ goes to infinity so, when $Q > 0$, Ψ_V has 2-regular variations, and when $Q = 0$,

$$1 \leq \sigma \leq \beta \leq 2,$$

where $1 \leq \sigma$ comes from the convexity of Ψ_V.

Remark 1.4. *When V has bounded variations, we know (see for example [3] page 15) that the brownian component of V is null, the Lévy measure ν of V satisfies $\int_{-1}^0 |x|\nu(dx) < +\infty$ and $-\gamma - \int_{-1}^0 x\nu(dx)$, the factor of λ in the expression of $\Psi_V(\lambda)$, is positive (otherwise V would be the opposite of a subordinator). It is thus easy to see that in this case $\Psi_V(\lambda)/\lambda$ converges to $-\gamma - \int_{-1}^0 x\nu(dx)$ when λ goes to infinity, so $\sigma = \beta = 1$. In the remaining, we sometimes assume that $\sigma > 1$, the reader should be aware that it excludes the case where V has bounded variations. However, this case is quite easy and shall be treated in the remarks.*

We are now ready to state our results on the asymptotic tails at 0 of $I(V^\uparrow)$:

Theorem 1.5. *We have*

$$\forall l < \frac{1}{\beta - 1},\ \lim_{x\to 0} x^l \log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) = -\infty,$$

$$\textit{If } \sigma > 1,\ \forall l > \frac{1}{\sigma - 1},\ \lim_{x\to 0} x^l \log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) = 0,$$

and the same is true for $I(V^\sharp)$ instead of $I(V^\uparrow)$.

Theorem 1.5 gives for $\mathbb{P}(I(V^\uparrow) \leq x)$ and $\mathbb{P}(I(V^\sharp) \leq x)$ a lower bound involving σ and an upper bound involving β. In the case of α-regular variations we can expect, under some extra hypothesis, to get a stronger result. We indeed have :

Theorem 1.6. *We assume that there are two positive constants $c < C$ and $\alpha \in]1, 2]$ such that $c\lambda^\alpha \leq \Psi_V(\lambda) \leq C\lambda^\alpha$ for λ large enough, then there exist two positive constants $K_1 < K_2$ such that for x small enough,*

$$\exp\left(-\frac{K_2}{x^{\frac{1}{\alpha-1}}}\right) \leq \mathbb{P}\left(I(V^\uparrow) \leq x\right) \leq \exp\left(-\frac{K_1}{x^{\frac{1}{\alpha-1}}}\right),$$

and the same is true for $I(V^\sharp)$ instead of $I(V^\uparrow)$.

Recall that we put $V^\sharp := V$ in the case where V drifts to $+\infty$ (which is the case where $I(V)$ is finite, according to Theorem 1 in [4]). The version for $I(V^\sharp)$ of Theorems 1.5 and 1.6 are thus pure results about exponential functionals of spectrally negative Lévy processes (which, to our knowledge, were not known yet) and they are examples of how the study of the exponential functional of the Lévy process conditioned to stay positive can be useful for the study of the exponential functional of the corresponding Lévy process. However, the versions for $I(V^\uparrow)$ of these results seem to be of greater interest for the applications to diffusions in random environment.

Remark 1.7. *The case where V has bounded variations is not contained in Theorem 1.6 but we prove in this case a more precise result : $\mathbb{P}(I(V^\uparrow) \leq x)$ and $\mathbb{P}(I(V^\sharp) \leq x)$ are null for x small enough.*

We already mentioned that the law of $I(V^\uparrow)$ is absolutely continus but we do not know how smooth the density is in general. The following theorem provides a condition for smoothness :

Theorem 1.8. *If $\sigma > 1$ and β are such that*

$$2\beta^2 - 3\sigma\beta + \sigma + \beta - 1 < 0, \tag{1.5}$$

then the density of $I(V^\uparrow)$ is of class $\mathcal{C}^\infty$ and all its derivatives converge to 0 at $+\infty$ and 0. If in addition $I(V^\uparrow)$ has moments of any positive order (this happens for example in the context of Theorem 1.3), then the density of $I(V^\uparrow)$ belongs to Schwartz space.

This theorem admits the following corollary :

Corollary 1.9. *If V is a spectrally negative Lévy process drifting to $+\infty$ such that $\sigma > 1$ and (1.5) is satisfied, then the density of $I(V)$ is of class $\mathcal{C}^\infty$ and all its derivatives converge to 0 at $+\infty$ and 0.*

Here again, we have obtained, as a consequence of our study, a pure result about exponential functionals of spectrally negative Lévy processes.

Remark 1.10. *If Ψ_V has α-regular variations with $\alpha > 1$, then $\sigma = \beta = \alpha$ so the condition (1.5) becomes $-(\alpha-1)^2 < 0$, but this is always true for $\alpha > 1$, so Theorem 1.8 and Corollary 1.9 apply. In other words, α-regular variations for the Laplace exponent of V imply smoothness of the densities for $I(V^\uparrow)$ and $I(V)$ when $\alpha > 1$.*

In the spectrally positive case, the finiteness of the exponential functional is quite easy to obtain, but our argument also yields the existence of some finite exponential moments. We can state the result as follows :

Theorem 1.11. *The random variable $I(Z^\uparrow)$ is almost surely finite and admits some finite exponential moments.*

Note that this time, we do not need extra hypothesis for the existence of finite exponential moments.

We also obtain a lower bound for the asymptotic tail at 0 of both $I(Z)$ and $I(Z^\uparrow)$. This tail is heavier than those given in Theorems 1.5 and 1.6 and this comes from the positive jumps.

Theorem 1.12. *If Z has unbounded variations and non-zero Lévy measure then, there is a positive constant c such that*

$$e^{-c(\log(x))^2} \leq \mathbb{P}\left(I(Z) \leq x\right) \leq \mathbb{P}\left(I(Z^\uparrow) \leq x\right).$$

The lower bound for $\mathbb{P}(I(Z) \leq x)$ does not require the hypothesis of unbounded variations.

Remark 1.13. *If the Lévy measure of Z is the zero measure then it is known, from the Lévy-Khintchine formula, that Z is a drifted brownian motion. The exact asymptotic tail at 0 of $I(Z^{\uparrow})$ is then given in Proposition 1.14 of the next subsection and is lighter than the one provided by Theorem 1.12. The existence of jumps thus plays an important role for the asymptotic tail at 0 of the exponential functional and the proof of Theorem 1.12 indeed crucially relies on this hypothesis.*

The study of the spectrally positive case does not go as far as the study of the spectrally negative case. The reason for this is twofold. First, we do not have, in the spectrally positive case, a decomposition of the law of $I(Z^{\uparrow})$ as in (1.4), which deprives us of an important tool for the study. Secondly we do not need, in the applications, the results on the exponential functional to be as precise, in the spectrally positive case, as in the spectrally negative case. Indeed, in the study of a diffusion in a spectrally negative Lévy environment V drifting to $-\infty$, a random variable $\mathcal{R}$ appears. Its law is the convolution of the laws of $I(V^{\uparrow})$ and $I(\hat{V}^{\uparrow})$, where $\hat{V} := -V$ is the dual process of V and is thus spectrally positive. The combination of the above theorems shows that the behavior of $I(V^{\uparrow})$ dominates in the study of $\mathcal{R}$. In particular, the asymptotic tail at 0 of $\mathcal{R}$ is the same as the one of $I(V^{\uparrow})$.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.1 and preliminary results on $V^{\uparrow}$. In Section 3 we establish Proposition 3.1 and prove Theorem 1.3. In Section 4 we establish a connection between the tails at 0 of the exponential functionals $I(V^{\uparrow})$ and $I(V^{\sharp})$, we then prove an upper bound for the tail of $I(V^{\sharp})$ and a lower bound for the tail of $I(V^{\uparrow})$, Theorems 1.5 and 1.6 then follow. In Section 5 we prove Theorem 1.8 and Corollary 1.9 via a study of excursions. Section 6 is devoted to the spectrally positive case and the proofs of Theorems 1.11 and 1.12.

1.2. **The example of drifted brownian motion conditioned to stay positive.** The most simple case is the intersection of the spectrally positive and the spectrally negative case, that is, when V is a drifted brownian motion. All the results mentioned here are already known in this case. We define the κ-drifted brownian motion by $W_\kappa(t) := W(t) - \frac{\kappa}{2}t$. It is known that the two processes $W_\kappa^{\uparrow}$ and $W_{-\kappa}^{\uparrow}$ are equal in law. This follows, for example, from the expression of the generator of $W_\kappa^{\uparrow}$, or from the fact that for positive κ, the Laplace exponent of $W_\kappa^{\sharp}$ is equal to the Laplace exponent of $W_{-\kappa}$, so the processes conditioned to stay positive have the same law. We thus only consider positive κ.

It is known (see (4.6) in [1]) that $I(W_k^{\uparrow})$ is almost surely finite and has Laplace transform

$$\mathbb{E}[e^{-\lambda I(W_k^{\uparrow})}] = \frac{\frac{1}{2^\kappa \Gamma(1+\kappa)}(2\sqrt{2\lambda})^\kappa}{I_\kappa(2\sqrt{2\lambda})},$$

where I_κ is a modified Bessel function. This expression can also be written

$$\mathbb{E}[e^{-\lambda I(W_k^{\uparrow})}] = \frac{1}{\Gamma(1+\kappa)} \frac{1}{\sum_{j=0}^{+\infty} \frac{(2\lambda)^j}{j!\Gamma(1+j+\kappa)}}, \tag{1.6}$$

and it is easy to see that it can be analytically extended in a neighborhood of 0, so the random variable $I(W_k^{\uparrow})$ admits some finite exponential moments.

An easy calculation on the asymptotic of this expression when λ goes to infinity yields

$$-\log\left(\mathbb{E}[e^{-\lambda I(W_k^{\uparrow})}]\right) \underset{\lambda \to +\infty}{\sim} 2\sqrt{2\lambda},$$

and using De Bruijn's Theorem (see Theorem 4.12.9 in [5]) we get

Proposition 1.14.

$$-\log\left(\mathbb{P}\left(I(W_k^{\uparrow})\leq x\right)\right)\underset{x\to 0}{\sim}\frac{2}{x}.$$

This proposition precises Theorem 1.6 in the case of drifted brownian motion.

As the expression (1.6) extends to a neighborhood of 0, we get the expression of the characteristic function of $I(W_k^{\uparrow})$ which can be proved, using estimates on modified Bessel functions, to belong to Schwartz space. Therefore, the density of $I(W_k^{\uparrow})$, which is the Fourier transform of its characteristic function, belongs to Schwartz space, but this is already included in Theorem 1.8.

2. Preliminary results on $V^{\uparrow}$ and finiteness of $I(V^{\uparrow})$

2.1. **Exponential functionals and excursions theory.** We fix $y>0$. In this subsection, we use excursions to prove that the integral of exponential $V^{\uparrow}(\tau(V^{\uparrow},y)+.)$ or $V^{\sharp}$ stopped at there last passage time at y and 0 respectively are equal in law to some subordinators stopped at independent exponential random variables.

It is easy to see that regularity of $\{y\}$ for the markovian processes $V_y^{\sharp}$ and $V_y^{\uparrow}$ is equivalent to the regularity of $\{0\}$ for V (or $V^{\sharp}$) which in turn, according to Corollary 5 page 192 in [3], is equivalent to the fact that V has unbounded variations. The property of $\{y\}$ being instantaneous for $V_y^{\sharp}$ and $V_y^{\uparrow}$ is equivalent to the same property of $\{0\}$ for V, but this is a well known property of spectrally negative Lévy processes. $\{y\}$ is thus always instantaneous for $V_y^{\sharp}$ and $V_y^{\uparrow}$ and the only alternative is wether it is regular or not, which corresponds to the fact that V has or not unbounded variations.

We apply excursions theory away from y (see [3]). Let us denote by $L_y^{\uparrow}$ (respectively $L_y^{\sharp}$) a local time at y of the process $V_y^{\uparrow}$ (respectively $V_y^{\sharp}$) and $\eta_y^{\uparrow}$ (respectively $\eta_y^{\sharp}$) the associated excursions measure. We denote $\eta^{\sharp}$ for $\eta_0^{\sharp}$. The inverse of the local time $L_y^{\uparrow,-1}$ (respectively $L_y^{\sharp,-1}$) is a subordinator and $V_y^{\uparrow}$ (respectively $V_y^{\sharp}$) can be represented as a Poisson point process on the set of excursions, with intensity measure $\eta_y^{\uparrow}$ (respectively $\eta_y^{\sharp}$). Note that this is also true in the irregular case (when V has bounded variations) if the local time $L_y^{\uparrow}$ (respectively $L_y^{\sharp}$) is defined artificially as in [3], Chapter IV, Section 5. In this case, the excursion measure is proportional to the law of the first excursion and in particular the total mass of the excursion measure is finite.

We consider IP, FP and N three subsets that make a partition of the excursions of $V^{\sharp}$ away from y. These three subsets are respectively : the set of excursions of infinite length that stay positive, the set of excursions of finite length that stay positive, the set of excursions that reach $]-\infty,0]$. $\eta_y^{\sharp}(IP)$ and $\eta_y^{\sharp}(N)$ are always finite whereas $\eta_y^{\sharp}(FP)$ is infinite in the regular case (when V has unbounded variations).

Given $\xi:[0,\zeta]\to\mathbb{R}$ an excursion away from y, we define $\zeta(\xi)$ to be its life-time, $H(\xi):=\sup_{[0,\zeta(\xi)]}\xi-y$ its height and $G(\xi):=\int_0^{\zeta(\xi)}e^{-\xi(t)}dt$.

Lemma 2.1. *Let y be positive and let S be a pure jump subordinator with Lévy measure $G\eta_y^{\sharp}(.\cap FP)$, the image measure of $\eta_y^{\sharp}(.\cap FP)$ by G. Let T be an exponential random variable with parameter $\eta_y^{\sharp}(IP)+\eta_y^{\sharp}(N)$, independent of S. We have*

$$\int_{\tau(V^{\uparrow},y)}^{\mathcal{R}(V^{\uparrow},y)}e^{-V^{\uparrow}(t)}dt\overset{\mathfrak{L}}{=}S_T,\tag{2.7}$$

where $\tau(.,.)$ and $\mathcal{R}(.,.)$ are defined in the introduction.

Proof. $V^{\uparrow}(\tau(V^{\uparrow},y)+.)$ has the same law as $V_y^{\uparrow}$, from Markov property applied to $V^{\uparrow}$ at time $\tau(V^{\uparrow},y)$ and the absence of positive jumps. Then, as it is mentioned in the introduction, $V_y^{\uparrow}$ is only $V_y^{\sharp}$ conditioned in the usual sense to remain positive.

$V_y^{\sharp}$ can be built from the Poisson point process on the set of excursions with intensity measure $\eta_y^{\sharp}$. $V_y^{\uparrow}$ can be built from this same process, conditioned not to have jumps in N before its first jump in IP. In other words, we build $V_y^{\uparrow}$ from the process of jumps in FP stopped at the independent exponential time (that has parameter $\eta_y^{\sharp}(IP)+\eta_y^{\sharp}(N)$) at which occurs the first jump in $IP\cup N$ and conditionally to the fact that this jump belongs to IP. Then, the process of jumps in FP and in $IP\cup N$ are independent and by a property of Poisson point processes, the fact that the first jump in $IP\cup N$ belongs to IP is independent of the time when this jump occurs. As a consequence, $V_y^{\uparrow}$ is built from the Poisson point process with intensity measure $\eta_y^{\sharp}(.\cap FP)$, until an independent exponential time of parameter $\eta_y^{\sharp}(IP)+\eta_y^{\sharp}(N)$ where we pick, independently, a jump following the law $\eta_y^{\sharp}(.\cap IP)/\eta_y^{\sharp}(IP)$.

The excursion measure of $V_y^{\uparrow}$ away from y can therefore be expressed as follows :

$$\eta_y^{\uparrow}(.)=\eta_y^{\sharp}(.\cap FP)+\frac{\eta_y^{\sharp}(IP)+\eta_y^{\sharp}(N)}{\eta_y^{\sharp}(IP)}\eta_y^{\sharp}(.\cap IP). \tag{2.8}$$

The points process in IP and the points process in FP are independent. Denoting by T the instant of the first point into IP, we have that T is an exponential random variable with parameter $\eta_y^{\sharp}(IP)+\eta_y^{\sharp}(N)$, independent of the points process in FP.

$\int_{\tau(V^{\uparrow},y)}^{\mathcal{R}(V^{\uparrow},y)} e^{-V^{\uparrow}(t)}dt$ is the sum of the images by G of the finite excursions of $V_y^{\uparrow}$ away from y. From above, if $(p_s)_{s\geq 0}$ is a Poisson point process on in FP with measure $\eta_y^{\sharp}(.\cap FP)$ and if T is an independent exponential random variable with parameter $\eta_y^{\sharp}(IP)+\eta_y^{\sharp}(N)$, we have

$$\int_{\tau(V^{\uparrow},y)}^{\mathcal{R}(V^{\uparrow},y)} e^{-V^{\uparrow}(t)}dt \overset{\mathfrak{L}}{=} \sum_{0<s<T} G(p_s). \tag{2.9}$$

By properties of Poisson point processes, the process in the right hand side, $\sum_{0<s<.} G(p_s)$, is the sum of the jumps of a Poisson point process on $\mathbb{R}_+$, with intensity measure $G\eta_y^{\sharp}(.\cap FP)$. Thus, from the Lévy-Ito decomposition, it has the same law as the subordinator S, which yields the result.

$\square$

Remark 2.2. *In the case where V has bounded variations, the total mass of $\eta_y^{\sharp}$ is finite so S is only a compound Poisson process. In particular S_T can be then null with positive probability.*

We now consider I and F, respectively the subset of excursions with infinite and finite length. A similar proof as for Lemma 2.1 gives the following lemma.

Lemma 2.3. *Let S be a pure jump subordinator with Lévy measure $G\eta^{\sharp}(.\cap F)$, the image measure of $\eta^{\sharp}(.\cap F)$ by G. Let T be an exponential random variable with parameter $\eta^{\sharp}(I)$ which is independent of S. We have*

$$\int_0^{\mathcal{R}(V^{\sharp},0)} e^{-V^{\sharp}(t)}dt \overset{\mathfrak{L}}{=} S_T. \tag{2.10}$$

2.2. $V^{\uparrow}$ **and** $V^{\sharp}$ **shifted at a last passage time.** To obtain decomposition (1.4) of the law of $I(V^{\uparrow})$, we split $V^{\uparrow}$ at its last passage time at a point y and obtain two independent trajectories that we can identify.

Lemma 2.4. *(Corollary 19 page 207 of [3])*

For any positive y, the two trajectories

$$\left(V^{\uparrow}(t),\ 0 \le t \le \mathcal{R}(V^{\uparrow}, y)\right) \text{ and } \left(V^{\uparrow}(t + \mathcal{R}(V^{\uparrow}, y)) - y,\ t \ge 0\right)$$

are independent and the second is equal in law to $V^{\uparrow}$.

Even if this result is proved in [3] for V a general spectrally negative Lévy process, we give in our setting a proof based on excursions. The interest of our proof is that it is easily extandable to $V^{\sharp}$ for which we need the same kind of result.

Proof. We fix $y > 0$. Since $V^{\uparrow}(\tau(V^{\uparrow}, y)) = y$ and $\tau(V^{\uparrow}, y)$ is a stopping time we have

$$\left(V^{\uparrow}(t),\ 0 \le t \le \tau(V^{\uparrow}, y)\right) \perp\!\!\!\perp \left(V^{\uparrow}(t + \tau(V^{\uparrow}, y)),\ t \ge 0\right), \tag{2.11}$$

$(V^{\uparrow}(t+\tau(V^{\uparrow}, y)),\ t \ge 0)$ has the same law as $V_y^{\uparrow}$ and we denote by $(e_y(s),\ s \ge 0)$ its excursions process away from y. Let $0 < a < b$ and define for $h > 0$, $E_h := \{\xi,\ H(\xi) \ge h\}$, the set of excursions of $V_y^{\uparrow}$ away from y that are higher than h, $T_\infty := \inf\{s \ge 0,\ e_y(s) \in I\}$ is the time when occurs the first infinite excursion and ξ_∞ is this first infinite excursion.

Decomposing $V^{\uparrow}$, after time $\tau(V^{\uparrow}, y)$, as its excursions away from y, we see that $\mathcal{R}(V^{\uparrow}, y)$ is the instant when begins the first infinite excursion, so

$$\left(V^{\uparrow}(t + \mathcal{R}(V^{\uparrow}, y)) - y,\ t \ge 0\right) = (\xi_\infty(t) - y,\ t \ge 0). \tag{2.12}$$

$(V^{\uparrow}(t + \tau(V^{\uparrow}, y)),\ 0 \le t \le \mathcal{R}(V^{\uparrow}, y) - \tau(V^{\uparrow}, y))$ is thus a function of $(e_y(s)\mathbb{1}_{e_y(s)\in F},\ 0 \le s \le T_\infty)$ while $(V^{\uparrow}(t + \mathcal{R}(V^{\uparrow}, y)) - y,\ t \ge 0)$ is a function of ξ_∞. By properties of Poisson point processes, T_∞ is an exponential random variable independent of ξ_∞ and the process of finite excursions $(e_y(s)\mathbb{1}_{e_y(s)\in F},\ s \ge 0)$ is also independent of ξ_∞. Therefore the objects $(e_y(s)\mathbb{1}_{e_y(s)\in F},\ 0 \le s \le T_\infty)$ and ξ_∞ are independent. From this independence we deduce that

$$\left(V^{\uparrow}(t + \tau(V^{\uparrow}, y)),\ 0 \le t \le \mathcal{R}(V^{\uparrow}, y) - \tau(V^{\uparrow}, y)\right) \perp\!\!\!\perp \left(V^{\uparrow}(t + \mathcal{R}(V^{\uparrow}, y)) - y,\ t \ge 0\right),$$

which combined with (2.11) gives the independence. It only remains to prove that the right hand side in (2.12) has the same law as $V^{\uparrow}$.

Using Markov property at time $\tau(\xi_a, y + a)$, for an excursion $\xi_a \in E_a$, we have that $\xi_a(. + \tau(\xi_a, y + a))$ equals in law $V_{a+y}^{\uparrow}$ killed when it ever reaches y.

Since $E_b \subset E_a$ we can apply this to an excursion $\xi_b \in E_b$ and get that $(\xi_b(t + \tau(\xi_b, y + a)),\ 0 \le t \le \tau(\xi_b, y + b) - \tau(\xi_b, y + a))$ is equal in law to $(V_{a+y}^{\uparrow}(t),\ 0 \le t \le \tau(V_{a+y}^{\uparrow}, b + y))$ conditioned to reach $b + y$ before y.

From the fact that $V_{a+y}^{\uparrow}$ has the same law has $V_{a+y}^{\sharp}$ conditioned in the usual sense to remain positive and Markov property at the hitting time of $b + y$, we get that $(V_{a+y}^{\uparrow}(t),\ 0 \le t \le \tau(V_{a+y}^{\uparrow}, b + y))$ has the same law as $(V_{a+y}^{\sharp}(t),\ 0 \le t \le \tau(V_{a+y}^{\sharp}, b + y))$ conditioned to reach $b + y$ before 0. Using this and the fact that $y > 0$, we easily obtain that $(V_{a+y}^{\uparrow}(t),\ 0 \le t \le \tau(V_{a+y}^{\uparrow}, b+y))$

conditioned to reach $b+y$ before y is equal in law to $(V^\sharp_{a+y}(t),\ 0\le t\le \tau(V^\sharp_{a+y}, y+b))$ conditioned to reach $b+y$ before y.

Combining the above two paragraphs and substracting y we get that $(\xi_b(t+\tau(\xi_b,y+a))-y,\ 0\le t\le \tau(\xi_b,y+b)-\tau(\xi_b,y+a))$ is equal in law to $(V^\sharp_a(t),\ 0\le t\le \tau(V^\sharp_a,b))$ conditioned to reach b before 0, which we know to be equal in law to $(V^\uparrow_a(t),\ 0\le t\le \tau(V^\uparrow_a,b))$. Therefore :

$$(\xi_b(t+\tau(\xi_b,y+a))-y,\ 0\le t\le \tau(\xi_b,y+b)-\tau(\xi_b,y+a)) \overset{\mathfrak{L}}{=} \left(V^\uparrow_a(t),\ 0\le t\le \tau(V^\uparrow_a,b)\right).$$

Since $\tau(\xi_b,y+a)$ converges almost surely to 0 when a goes to 0 and $V^\uparrow_a$ converges in law to $V^\uparrow$ according to Proposition 14 page 201 in [3], we can let a go to 0 in both members and get

$$(\xi_b(t)-y,\ 0\le t\le \tau(\xi_b,y+b)) \overset{\mathfrak{L}}{=} \left(V^\uparrow(t),\ 0\le t\le \tau(V^\uparrow,b)\right).$$

This equality in law is true for excursions in E_b so in particular, it is for ξ_∞ :

$$(\xi_\infty(t)-y,\ 0\le t\le \tau(\xi,y+b)) \overset{\mathfrak{L}}{=} \left(V^\uparrow(t),\ 0\le t\le \tau(V^\uparrow,b)\right),$$

and since b is arbitrary,

$$(\xi_\infty(t)-y,\ t\ge 0) \overset{\mathfrak{L}}{=} \left(V^\uparrow(t),\ t\ge 0\right),$$

which combined with (2.12) gives the result.

□

Lemma 2.5.

- *The two trajectories*

$$\left(V^\sharp(t),\ 0\le t\le \mathcal{R}(V^\sharp,0)\right) \text{ and } \left(V^\sharp(t+\mathcal{R}(V^\sharp,0)),\ t\ge 0\right)$$

are independent and the second is equal in law to $V^\uparrow$.

- *For $y>0$, let $\mathcal{R}^y(V^\sharp,0)$ be the last passage time at 0 before $\tau(V^\sharp,y)$:*

$$\mathcal{R}^y(V^\sharp,0):=\sup\{t\in[0,\tau(V^\sharp,y)],\ V^\sharp(t)=0\},$$

then the two trajectories $(V^\sharp(t),\ 0\le t\le \mathcal{R}^y(V^\sharp,0))$

$$\text{and } \left(V^\sharp(t+\mathcal{R}^y(V^\sharp,0)),\ 0\le t\le \tau(V^\sharp,y)-\mathcal{R}^y(V^\sharp,0)\right)$$

are independent and the second is equal in law to $(V^\uparrow(t),\ 0\le t\le \tau(V^\uparrow,y))$. As a consequence we have $\tau(V^\uparrow,y)\overset{\mathfrak{L}}{=}\tau(V^\sharp,y)-\mathcal{R}^y(V^\sharp,0)\le \tau(V^\sharp,y)$.

Proof. For the first point, dealing with excursions of $V^\sharp$ away from 0 instead of excursions of $V^\uparrow$ away from y, the proof follows the same steps as the one we have given for Lemma 2.4 and is even simpler. For the second point, we proceed as before, looking at the first excursion in $\{H\ge y\}$ instead of the first infinite excursion.

□

The finiteness of $I(V^\uparrow)$ is a consequence of this Lemma 2.5 :

Proof. of Theorem 1.1

From the first statement of Lemma 2.5, we have

$$I(V^\uparrow)\overset{\mathfrak{L}}{=}\int_0^{+\infty} e^{-V^\sharp(t+\mathcal{R}(V^\sharp,0))}dt=\int_{\mathcal{R}(V^\sharp,0)}^{+\infty} e^{-V^\sharp(t)}dt\le \int_0^{+\infty} e^{-V^\sharp(t)}dt=I(V^\sharp),$$

and $V^\sharp$ drifts to $+\infty$, so Theorem 1 in [4] ensures that $I(V^\sharp)$ is almost surely finite which yields the result.

$\square$

3. Decomposition and exponential moments

3.1. **Decomposition of the law of $I(V^\uparrow)$.** In this subsection, we prove that the law of $I(V^\uparrow)$ is solution of the random affine equation (1.4) and we give a decomposition of its non-trivial coefficient A^y. This is the basis of our analysis of the law of $I(V^\uparrow)$.

Proposition 3.1. *For any $y > 0$, the law of $I(V^\uparrow)$ satisfies the random affine equation*

$$I(V^\uparrow) \stackrel{\mathfrak{L}}{=} \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)}dt + S_T + e^{-y}I(V^\uparrow), \tag{3.13}$$

where the three terms of the right hand side are independent and S_T is as in Lemma 2.1.

We define

$$A^y := \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)}dt + S_T$$

to lighten notations.

As a consequence, $I(V^\uparrow)$ has the same law as the sum of a random series :

$$I(V^\uparrow) \stackrel{\mathfrak{L}}{=} \sum_{k\geq 0} e^{-ky}A_k^y, \tag{3.14}$$

where the random variables A_k^y are iid and have the same law as A^y.

Remark 3.2. *The almost sure convergence of the random series in (3.14) is a consequence of the almost sure finiteness, given by Theorem 1.1, of the positive random variable $I(V^\uparrow)$. Also, it is a well known fact on random power series with iid coefficients that their radius of convergence is almost surely equal to a constant belonging to $\{0, 1\}$. Since this constant, in the case of the power series in (3.14), has been proved to be greater that e^{-y}, we deduce that it equals 1.*

Proof. of Proposition 3.1

We fix $y > 0$. As $V^\uparrow$ has no positive jumps and go to infinity we have $\tau(V^\uparrow, y) \leq \mathcal{R}(V^\uparrow, y) < +\infty$ and $V^\uparrow(\tau(V^\uparrow, y)) = V^\uparrow(\mathcal{R}(V^\uparrow, y)) = y$.

We write :

$$\begin{aligned}
I(V^\uparrow) &= \int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)}dt + \int_{\mathcal{R}(V^\uparrow,y)}^{+\infty} e^{-V^\uparrow(t)}dt \\
&= \int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)}dt + e^{-y}\int_0^{+\infty} e^{-\left(V^\uparrow(t+\mathcal{R}(V^\uparrow,y))-y\right)}dt \\
&\stackrel{\mathfrak{L}}{=} \int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)}dt + e^{-y}I(V^\uparrow),
\end{aligned}$$

where we used Lemma 2.4 for the last equality in which the two terms are independent. We now decompose :

$$\int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)}dt = \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)}dt + \int_{\tau(V^\uparrow,y)}^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)}dt \tag{3.15}$$

Since $V^{\uparrow}(\tau(V^{\uparrow},y)) = y$, combining with Markov property at time $\tau(V^{\uparrow},y)$, the two terms in the right hand side of (3.15) are independent :

$$\int_0^{\tau(V^{\uparrow},y)} e^{-V^{\uparrow}(t)}dt \perp\!\!\!\perp \int_{\tau(V^{\uparrow},y)}^{\mathcal{R}(V^{\uparrow},y)} e^{-V^{\uparrow}(t)}dt.$$

Now, thanks to Lemma 2.1, the second term has the same law as S_T with S_T as in the lemma. This achieves the proof.

$\square$

Remark 3.3. *When V oscillates, we see, using the argument of this proof and Corollary 19 page 207 of [3], that $I(V^{\uparrow})$ satisfies for every positive y an equality of the kind of (1.4), for some independent random variable A^y equal in law to $\int_0^{\mathcal{R}(V^{\uparrow},y)} e^{-V^{\uparrow}(t)}dt$. As a consequence, when V is a general spectrally negative Lévy process, we have that $I(V^{\uparrow})$ is self-decomposable as long as it is finite. This is the same property that we have for $I(V)$.*

3.2. **Exponential moments : Proof of Theorem 1.3.** In this subsection only, we need that the large negative jumps of $V^{\uparrow}$ are not too frequent, otherwise $V^{\uparrow}$ could become quite small after long times, which could imply $I(V^{\uparrow})$ to be a little too large to have finite exponential moments. In fact, in the case where V drifts to $-\infty$, we do not need hypothesis on the jumps of V, but when V drifts to $+\infty$, we have to assume that $V(1)$ admits a finite negative exponential moment, that is

$$\exists\lambda > 0,\ \mathbb{E}\left[e^{-\lambda V(1)}\right] < +\infty, \tag{3.16}$$

which, according to Theorem 35.3 page 159 in [14], is equivalent to the fact that the restriction of the Lévy measure ν of V to $]-\infty,1]$ admits a finite negative exponential moment :

$$\exists\lambda > 0,\ \int_{-\infty}^{1} e^{-\lambda x}\nu(dx) < +\infty.$$

This is in fact a crucial point of our reasoning and this assumption turns out to be always true for $V^{\sharp}$ when V drifts to $-\infty$, indeed, from the definition of the law of $V^{\sharp}$ in [3], we have

$$\forall\lambda > -\kappa,\ \mathbb{E}\left[e^{\lambda V^{\sharp}(1)}\right] = \mathbb{E}\left[e^{(\lambda+\kappa)V(1)}\right] < +\infty, \tag{3.17}$$

so the spectrally negative Lévy process $V^{\sharp}$ has its Laplace transform, as well as its Laplace exponent $\Psi_{V^{\sharp}}$, defined on the half-plane $\{\mathcal{R}(z) > -\kappa\}$.

To summarize, the hypothesis that we assume on V in this subsection are that it is a spectrally negative Lévy process which is not the opposite of a subordinator and does not oscillate and, if V drifts to $+\infty$, we assume in addition that (1.2) is fulfilled. We note that thanks to (3.17), $V^{\sharp}$ always satisfies hypothesis (1.2) in the case where V drifts to $-\infty$. As a consequence, in this subsection, $V^{\sharp}$ always denotes a spectrally negative Lévy process having some finite negative exponential moments and drifting to $+\infty$.

The main idea of the proof of Theorem 1.3 is to use the decomposition (3.14) of the law of $I(V^{\uparrow})$. The coefficient A^y appearing in that proposition is the sum of two independent random variables, each one being controlled by one of the lemmata below.

Lemma 3.4. *There are two positive constants c_1 and c_2 such that*

$$\forall y, s > 0,\ \mathbb{P}\left(\tau(V^{\uparrow},y) > s\right) \le e^{c_1 y - c_2 s}.$$

Proof. Fix y and $s > 0$. From the second point of Lemma 2.5 we have

$$\mathbb{P}\left(\tau(V^{\uparrow}, y) > s\right) \leq \mathbb{P}\left(\tau(V^{\sharp}, y) > s\right). \tag{3.18}$$

Then, $V^{\sharp}$ is a spectrally negative Lévy process, so, according to Theorem 1 page 189 in [3], the process $\tau(V^{\sharp}, .)$ is a subordinator which Laplace exponent $\Phi_{V^{\sharp}}$ is defined for $\lambda \geq 0$ by

$$\Phi_{V^{\sharp}}(\lambda) := -\log\left(\mathbb{E}\left[e^{-\lambda\tau(V^{\sharp},1)}\right]\right),$$

and we have $\Phi_{V^{\sharp}} = \Psi_{V^{\sharp}}^{-1}$.

From the introduction we know that $V^{\sharp}$ has its Laplace transform, as well as its Laplace exponent $\Psi_{V^{\sharp}}$, defined in a neighborhood of 0. Then, since $\Psi'_{V^{\sharp}}(0) > 0$, the holomorphic local inversion theorem tells us that $\Psi_{V^{\sharp}}^{-1}$, that is $\Phi_{V^{\sharp}}$, extends in a neighborhood of 0.

Therefore, the subordinator $\tau(V^{\sharp}, .)$ has a Laplace transform defined in a neighborhood of 0. From Markov inequality we get, for a positive c_2 in this neighborhood,

$$\mathbb{P}\left(\tau(V^{\sharp}, y) > s\right) \leq e^{-c_2 s}\mathbb{E}\left[e^{c_2\tau(V^{\sharp},y)}\right] = e^{-y\Phi_{V^{\sharp}}(-c_2)}e^{-c_2 s},$$

Note that $c_1 := -\Phi_{V^{\sharp}}(-c_2)$ is positive. Combining with (3.18) we get the result.

□

The study of the Lévy measure of the subordinator appearing in the second term of A^y is the object of the lemma below. It provides the existence of finite exponential moments for this Lévy measure.

Lemma 3.5. *For any $y > 0$, the restriction to $[1, +\infty[$ of measure $G\eta_y^{\sharp}(. \cap FP)$ admits some finite exponential moments :*

$$\exists\lambda > 0, \int_1^{+\infty} e^{\lambda x} G\eta_y^{\sharp}(. \cap FP)(dx) < +\infty. \tag{3.19}$$

Proof. We fix $y > 0$. We have

$$G\eta_y^{\sharp}(. \cap FP)(]t, +\infty[) = \eta_y^{\sharp}\left(\xi \in FP,\ \int_0^{\zeta(\xi)} e^{-\xi(s)}ds > t\right) \leq \eta_y^{\sharp}\left(\xi \in FP,\ \zeta(\xi) > t\right), \tag{3.20}$$

because for all $\xi \in FP$ we have $e^{-\xi(s)} < 1$, $\forall s \in [0, \zeta(\xi)]$. We thus only need to control the length of excursions. It is easy to see that there exist three types of excursions in FP : those that stay in $]0, y]$, those that stay in $[y, +\infty[$, and those having a first part in $[y, +\infty[$ and a second part in $]0, y]$. The absence of positive jumps forbids the existence of any other kind of excursions.

For any excursion, we look at its part in $[y, +\infty[$ and its part in $]0, y]$, considering that excursions of the first and second type have respectively there part in $[y, +\infty[$ and there part in $]0, y]$ of length 0.

We start the proof by controlling the length of the part in $[y, +\infty[$,

$\eta_y^{\sharp}\left(\xi \in FP,\ \inf\{s > 0, \xi(s) \leq y\} > t\right)$ is equal to

$$\eta_y^{\sharp}\left(\xi \in FP,\ \inf\{s > 0, \xi(s) \leq y\} > t,\ \tau(\xi, y+1) \leq t/2\right) \tag{3.21}$$

$$+\eta_y^{\sharp}\left(\xi \in FP,\ \inf\{s > 0, \xi(s) \leq y\} > t,\ \tau(\xi, y+1) > t/2\right), \tag{3.22}$$

where it should be noted that $y+1$ might actually never be reached in the case $\tau(\xi, y+1) > t/2$. For fixed $t > 2$, (3.21) is less than

$$\begin{aligned}
&\eta^\sharp\left(\xi,\ \tau(\xi,1) < +\infty,\ t/2 < \inf\left\{s > 0, \xi^{\tau(\xi,1)}(s) \le 0\right\} < +\infty\right)\\
=&\eta^\sharp\left(\xi,\ \tau(\xi,1) < +\infty\right) \times \mathbb{P}\left(t/2 < \tau(V_1^\sharp,]-\infty, 0]) < +\infty\right)\\
\le&\eta^\sharp\left(\xi,\ \tau(\xi,1) < +\infty\right) \times \mathbb{P}\left(\inf_{[t/2,+\infty[} V_1^\sharp < 0\right)\\
=&\eta^\sharp\left(\xi,\ \tau(\xi,1) < +\infty\right) \times \mathbb{P}\left(1 + V^{\sharp,1}(t/2) + \underline{V}^{\sharp,2}(+\infty) < 0\right),
\end{aligned} \tag{3.23}$$

from the independence of increments, and where $V^{\sharp,1}$ and $V^{\sharp,2}$ are two independent copies of $V^\sharp$. The Laplace transform of the random variables $V^\sharp(t/2)$ and $\underline{V^\sharp}(+\infty)$ are defined for positive λ and respectively equal to

$$\mathbb{E}\left[e^{\lambda V^\sharp(t/2)}\right] = e^{t\Psi_{V^\sharp}(\lambda)/2}, \qquad \mathbb{E}\left[e^{\lambda \underline{V^\sharp}(+\infty)}\right] = \frac{\lambda}{\Psi_{V^\sharp}(\lambda)}. \tag{3.24}$$

The second expression is given page 192 of [3]. Since, from our assumptions, $\Psi_{V^\sharp}$ is defined (and analytic) on a neighborhood of 0, the properties of Laplace transform imply that the first equality in (3.24) extends in a neighborhood of 0. Then, since $\Psi_{V^\sharp}(0) = 0$ and $\Psi'_{V^\sharp}(0) > 0$, we have that $\Psi_{V^\sharp}(\lambda)/\lambda$ is positive for λ in a neighborhood of 0 which implies that the second of the above equalities extends in this neighborhood. We choose a negative $\tilde{\lambda}$ such that the expressions in (3.24) are true (and finite) at this particular $\tilde{\lambda}$. From Markov inequality we have

$$\mathbb{P}\left(1 + V^{\sharp,1}(t/2) + \underline{V}^{\sharp,2}(+\infty) < 0\right) \le e^{\tilde{\lambda}} \times e^{t\Psi_{V^\sharp}(\tilde{\lambda})/2} \times \frac{\tilde{\lambda}}{\Psi_{V^\sharp}(\tilde{\lambda})},$$

where we note that $\Psi_{V^\sharp}(\tilde{\lambda}) < 0$. Combining with (3.23) we get that (3.21) is less than

$$\eta^\sharp\left(\xi,\ \tau(\xi,1) < +\infty\right) \times \frac{\tilde{\lambda}}{\Psi_{V^\sharp}(\tilde{\lambda})} \times e^{t\Psi_{V^\sharp}(\tilde{\lambda})/2}. \tag{3.25}$$

We now turn to (3.22), it is less than

$$\begin{aligned}
&\eta^\sharp\left(\inf\{s > 0, \xi(s) \le 0\} > t,\ \tau(\xi,1) > t/2\right)\\
\le&\int_0^1 \mathbb{P}\left(V_z^\sharp(t/2-1) < 1\right) \times \eta^\sharp\left(\zeta(\xi) > 1,\ \xi(1) \in dz\right),
\end{aligned}$$

where we used Markov property at time 1 in the excursions,

$$\begin{aligned}
\le&\int_0^1 \mathbb{P}\left(V^\sharp(t/2-1) < 1\right) \times \eta^\sharp\left(\zeta(\xi) > 1,\ \xi(1) \in dz\right)\\
\le&\eta^\sharp\left(\zeta(\xi) > 1\right) \times e^{-\tilde{\lambda}}\mathbb{E}\left[e^{\tilde{\lambda}V^\sharp(t/2-1)}\right],
\end{aligned}$$

where we used Markov inequality and chose some negative $\tilde{\lambda}$ for which the Laplace exponent $\Psi_{V^\sharp}$ of $V^\sharp(1)$ is finite and negative at $\tilde{\lambda}$ (this is possible for $\tilde{\lambda}$ small enough since $\Psi_{V^\sharp}(0) = 0$ and $\Psi'_{V^\sharp}(0) > 0$). We thus get that (3.22) is less than

$$\eta^\sharp\left(\zeta(\xi) > 1\right) \times e^{-\tilde{\lambda}-\Psi_{V^\sharp}(\tilde{\lambda})} \times e^{t\Psi_{V^\sharp}(\tilde{\lambda})/2}. \tag{3.26}$$

Now combining estimates (3.25) and (3.26) we get the existence of positive constants c_1, c_2 such that

$$\forall t > 2,\ \eta_y^\sharp\left(\xi \in FP,\ \inf\{s > 0,\ \xi(s) \le y\} > t\right) \le c_1 e^{-c_2 t}. \tag{3.27}$$

It remains to control the length of the part of excursions in $]0, y]$. In order to apply Lemma 3.4, we deal with $V^{\uparrow}$ instead of $V^{\sharp}$. We can do this because the measures $\eta^{\uparrow}(. \cap FP)$ and $\eta^{\sharp}(. \cap FP)$ are the same according to (2.8).

For $\xi \in FP$, let us denote $T(\xi) := \inf\{s > 0,\ \xi(s) \leq y\}$. Then, for $t > 1$, the measure of excursions whose second part is longer than t is $\eta^{\sharp}_y\left(\xi \in FP,\ \inf\{s \geq 0,\ \xi(T(\xi)+s) = y\} > t\right)$ which equals

$$\begin{aligned}&\eta^{\uparrow}_y\left(\xi \in FP,\ \inf\{s \geq 0,\ \xi(T(\xi)+s) = y\} > t\right)\\ &= \int_{FP} \mathbb{1}_{\{\inf\{s\geq 0,\ \xi(T(\xi)+s)=y\}>1\}} \mathbb{P}\left(\tau(V^{\uparrow}_{\xi(T(\xi)+1)}, y) > t-1\right) \eta^{\uparrow}_y(d\xi),\end{aligned}$$

by Markov property applied at time $T(\xi)+1$ in the excursions. Therefore, the above is less than

$$\int_{FP} \mathbb{1}_{\{\inf\{s\geq 0,\xi(T(\xi)+s)=y\}>1\}} \mathbb{P}\left(\tau(V^{\uparrow}, y) > t-1\right) \eta^{\uparrow}_y(d\xi),$$

because, from Markov property, $V^{\uparrow}_a \overset{\mathcal{L}}{=} V^{\uparrow}(\tau(V^{\uparrow}, a) + .)$ for any $a > 0$ and therefore $\tau(V^{\uparrow}_a, y) \leq \tau(V^{\uparrow}, y)$. We thus get

$$\eta^{\sharp}_y\left(\xi \in FP,\ \inf\{s \geq 0, \xi(T(\xi)+s) = y\} > t\right) \leq c_3 \mathbb{P}\left(\tau(V^{\uparrow}, y) > t-1\right), \tag{3.28}$$

where $c_3 := \eta^{\uparrow}_y\left(\xi \in FP,\ \inf\{s \geq 0, \xi(T(\xi)+s) = y\} > 1\right)$. c_3 is finite because it is less than the measure of the set of excursions longer than 1.

Combining (3.28) with Lemma 3.4 we obtain the existence of positive constants $c_4, c_5 > 0$ such that

$$\forall t > 1,\ \eta^{\sharp}_y\left(\xi \in FP,\ \inf\{s \geq 0,\ \xi(T(\xi)+s) = y\} > t\right) \leq c_4 e^{-c_5 t}. \tag{3.29}$$

We can now prove (3.19). Using (3.20), we get that for $t > 4$, $G\eta^{\sharp}_y(. \cap FP)(]t, +\infty[)$ is less than

$$\eta^{\sharp}_y\left(\xi \in FP,\ \inf\{s > 0, \xi(s) \leq y\} > t/2\right) + \eta^{\sharp}_y\left(\xi \in FP,\ \inf\{s \geq 0, \xi(T(\xi)+s) = y\} > t/2\right),$$

because an excursion that is longer than t must have, either its part in $[y, +\infty[$, either its part in $]0, y]$ longer than $t/2$. We then conclude thanks to the expressions (3.27) and (3.29).

□

Lemmata 3.4 and 3.5 combined with Proposition 3.1 allow us to finally prove Theorem 1.3.

Proof. of Theorem 1.3

We fix $y > 0$. The expression (3.14) shows that $I(V^{\uparrow})$ admits some finite exponential moments if and only if the random variable A^y does, and A^y does if and only if the two independent random variables of which it is the sum do.

The random variable $\int_0^{\tau(V^{\uparrow}, y)} e^{-V^{\uparrow}(t)} dt$ is positive and can be trivially bounded by $\tau(V^{\uparrow}, y)$ which admits some finite exponential moments thanks to Lemma 3.4.

We now aim to establish the existence of a Laplace transform for the second term, S_T (S_T being as in Lemma 2.1), on a neighborhood of 0. Let Φ_S be the Laplace exponent of the subordinator S :

$$\forall \lambda > 0,\ \Phi_S(\lambda) := -\log\left(\mathbb{E}\left[e^{-\lambda S_1}\right]\right).$$

We can see that the Laplace transform of the random variable S_T is given by

$$\forall \lambda > 0,\ \mathbb{E}\left[e^{-\lambda S_T}\right] = \frac{\eta_y^\sharp(IP) + \eta_y^\sharp(N)}{\eta_y^\sharp(IP) + \eta_y^\sharp(N) + \Phi_S(\lambda)}. \tag{3.30}$$

As $\eta_y^\sharp(IP) + \eta_y^\sharp(N) > 0$, to prove that S_T has a Laplace transform on a neighborhood of 0, it suffices to show that the fonction Φ_S extends on a neighborhood of 0. To prove this, it suffices to show that S_1 admits a Laplace transform on a neighborhood of 0, and to prove this, it suffices to show, according to Theorem 25.3 page 159 in [14], that the restriction to $[1, +\infty[$ of the Lévy measure of S admits some finite exponential moments, but this is indeed true because of Lemma 3.5.

Then, the random variable A^y indeed admits some finite exponential moments and so does $I(V^\uparrow)$.

□

Remark 3.6. *If $\mathbb{E}[e^{\lambda S_1}] = +\infty$ for some positive λ then we have easily $\mathbb{E}[e^{\lambda S_T}] = +\infty$. If $\mathbb{E}[e^{\lambda S_1}] < +\infty$ for any positive λ, then from the increases in λ and the fact it goes to infinity when λ goes to infinity, there exists λ_0 such that $\Phi_S(-\lambda_0) = -(\eta_y^\sharp(IP) + \eta_y^\sharp(N))$ and we have $\mathbb{E}[e^{\lambda_0 S_T}] = +\infty$ because of the extension of expression (3.30). Therefore, in any case, the Laplace transform of S_T is not finite everywhere, so neither is the Laplace transform of $I(V^\uparrow)$. This is why we can not say better than "$I(V^\uparrow)$ admits **some** finite exponential moments".*

4. Asymptotic tail at 0 : Proof of Theorems 1.5 and 1.6

4.1. Connection between $I(V^\uparrow)$ and $I(V^\sharp)$.

We prove in this subsection a simple connection between the asymptotic tails at 0 of $I(V^\uparrow)$ and $I(V^\sharp)$. This allows us to study sometimes $I(V^\sharp)$ rather than $I(V^\uparrow)$, the first one being sometimes simpler as it is an exponential functional of a Lévy process.

Proposition 4.1. *There is a positive constant c such that for all positive ϵ and x small enough,*

$$\log\left(\mathbb{P}\left(I(V^\sharp) \le x\right)\right) \le \log\left(\mathbb{P}\left(I(V^\uparrow) \le x\right)\right) \le \log\left(\mathbb{P}\left(I(V^\sharp) \le (1+\epsilon)x\right)\right) - \log(c\epsilon x).$$

As in the proof of Proposition 3.1, we decompose $I(V^\sharp)$ as the sum of two independent random variables, one having the same law as a subordinator stopped at an independent exponential time and the other having the same law as $I(V^\uparrow)$. We first need an easy lemma on the asymptotic tail at 0 of subordinators stopped at independent exponential times.

Lemma 4.2. *Let S be a subordinator and T an independent exponential random variable, there exists a positive constant c such that for all x small enough*

$$\mathbb{P}(S_T \le x) \ge cx.$$

Proof. We prove in fact a stronger result : the function $x \mapsto \mathbb{P}(S_T \le x)$ is sub-additive, that is

$$\forall x, y \ge 0,\ \mathbb{P}(S_T \le x + y) \le \mathbb{P}(S_T \le x) + \mathbb{P}(S_T \le y) \tag{4.31}$$

and the lemma follows easily. Let $x, y > 0$ (the case when $x = 0$ or $y = 0$ is obvious), we have

$$\begin{aligned}\mathbb{P}(S_T \le x + y) &= \mathbb{P}(S_T \le x) + \mathbb{P}(T \ge \tau(S, x),\ T \le \tau(S, x + y)) \\ &\le \mathbb{P}(S_T \le x) + \mathbb{P}\left(T \ge \tau(S, x),\ T \le \tau(S, S_{\tau(S,x)} + y)\right)\end{aligned}$$

because $S_{\tau(S,x)} \geq x$ almost surely,

$$= \mathbb{P}\left(S_T \leq x\right) + \mathbb{P}\left(T \geq \tau(S,x)\right) \times \mathbb{P}\left(T \leq \tau(S,y)\right),$$

from the characteristic property of the exponential law and Markov property applied to S at time $\tau(S,x)$. Since $\mathbb{P}\left(T \geq \tau(S,x)\right) \leq 1$ we obtain (4.31).

$\square$

We now prove the proposition

Proof. of Proposition 4.1.

We write :

$$\begin{aligned} I(V^\sharp) &= \int_0^{\mathcal{R}(V^\sharp,0)} e^{-V^\sharp(t)} dt + \int_{\mathcal{R}(V^\sharp,0)}^{+\infty} e^{-V^\sharp(t)} dt \\ &= \int_0^{\mathcal{R}(V^\sharp,0)} e^{-V^\sharp(t)} dt + \int_0^{+\infty} e^{-\left(V^\sharp(t+\mathcal{R}(V^\sharp,0))\right)} dt \\ &\overset{\mathcal{L}}{=} \int_0^{\mathcal{R}(V^\sharp,0)} e^{-V^\sharp(t)} dt + I(V^\uparrow), \end{aligned} \tag{4.32}$$

where we used the first point of Lemma 2.5 in the last inequality in which the two terms $\int_0^{\mathcal{R}(V^\sharp,0)} e^{-V^\sharp(t)} dt$ and $I(V^\uparrow)$ are independent.

We thus have

$$\mathbb{P}\left(I(V^\sharp) \leq x\right) \leq \mathbb{P}\left(I(V^\uparrow) \leq x\right). \tag{4.33}$$

According to Lemma 2.3, the term $\int_0^{\mathcal{R}(V^\sharp,0)} e^{-V^\sharp(t)} dt$ has the same law as S_T where S is a pure jump subordinator with Lévy measure $G\eta^\sharp(. \cap F)$, the image measure by G of $\eta^\sharp(. \cap F)$, and T an independent exponential random variable with parameter $\eta^\sharp(I)$.

For $\epsilon > 0$ and $x \geq 0$, combining the equality in law of lemma 2.3 with (4.32) we obtain

$$\mathbb{P}\left(I(V^\sharp) \leq (1+\epsilon)x\right) \geq \mathbb{P}\left(I(V^\uparrow) \leq x\right) \times \mathbb{P}\left(S_T \leq \epsilon x\right) \geq c\epsilon x \mathbb{P}\left(I(V^\uparrow) \leq x\right),$$

for an appropriate constant $c > 0$, when x is small enough, according to lemma 4.2. Taking the logarithm we get

$$\log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) \leq \log\left(\mathbb{P}\left(I(V^\sharp) \leq (1+\epsilon)x\right)\right) - \log\left(c\epsilon x\right). \tag{4.34}$$

Combining with (4.33) we get the result.

$\square$

4.2. **Tail at 0 of $I(V^\sharp)$.** As it can be suspected from Theorem 3 of [4], the asymptotic tail at 0 of the exponential functional $I(V^\sharp)$ is related to the asymptotic behavior of $\Psi_{V^\sharp}$. This asymptotic behavior is described by the two numbers σ and β defined in the introduction. In the next proposition, we provide for $\log(\mathbb{P}(I(V^\sharp) \leq x))$ an upper bound involving β.

Proposition 4.3.

$$\forall l < \frac{1}{\beta - 1}, \ \lim_{x \to 0} x^l \log\left(\mathbb{P}\left(I(V^\sharp) \leq x\right)\right) = -\infty.$$

Proof. We show that for any $l < 1/(\beta - 1)$, the random variable $(1/I(V^\sharp))^l$ has a Laplace transform. We prove this by studying the sequence of the moments of $1/I(V^\sharp)$ given in [4].

We choose $0 < l < 1/(\beta - 1)$. From the definition of β we have that $\lambda^{-\frac{1}{l}-1}\Psi_{V^\sharp}(\lambda)$ converges to 0 when λ goes to infinity so, using the continuity of $\Psi_{V^\sharp}$, we get the existence of a constant $c \geq 1$ such that

$$\forall \lambda \geq 1,\ \Psi_{V^\sharp}(\lambda) \leq c\lambda^{\frac{1}{l}+1}. \tag{4.35}$$

For $k \geq 1$ we have

$$\mathbb{E}\left[\left(1/I(V^\sharp)\right)^{kl}\right] \leq \left(\mathbb{E}\left[\left(1/I(V^\sharp)\right)^{\lfloor kl\rfloor+1}\right]\right)^{\frac{kl}{\lfloor kl\rfloor+1}} = \left(\Psi'_{V^\sharp}(0)\frac{\Psi_{V^\sharp}(1)\ldots\Psi_{V^\sharp}(\lfloor kl\rfloor)}{\lfloor kl\rfloor!}\right)^{\frac{kl}{\lfloor kl\rfloor+1}},$$

from Jensen inequality and Theorem 3 of [4],

$$\leq \left(\Psi'_{V^\sharp}(0)\ c^{\lfloor kl\rfloor}\left(\lfloor kl\rfloor!\right)^{1/l}\right)^{\frac{kl}{\lfloor kl\rfloor+1}} \leq \Psi'_{V^\sharp}(0)\ c^{\lfloor kl\rfloor}\left(\lfloor kl\rfloor!\right)^{1/l},$$

from (4.35) and then, the fact that $\Psi'_{V^\sharp}(0)\ c^{\lfloor kl\rfloor}\left(\lfloor kl\rfloor!\right)^{1/l}$ is obviously greater than 1 when k is large enough,

$$\leq 2\Psi'_{V^\sharp}(0)\ (2\pi)^{1/2l}c^{kl}\lfloor kl\rfloor^{1/2l}\left(\frac{\lfloor kl\rfloor}{e}\right)^{\frac{\lfloor kl\rfloor}{l}},$$

for k large enough, according to Stirling's Formula,

$$\leq 2\Psi'_{V^\sharp}(0)\ (2\pi)^{1/2l}c^{kl}\lfloor kl\rfloor^{1/2l}\left(\frac{kl}{e}\right)^{k} \leq \mathcal{C}^k k^k,$$

for some positive constant $\mathcal{C}$. We deduce that the power series

$$\sum_{k\geq 1} z^k\frac{1}{k!}\mathbb{E}\left[\left(1/I(V^\sharp)\right)^{kl}\right]$$

has a positive radius of convergence so the random variable $(1/I(V^\sharp))^l$ admits a Laplace transform on a neighborhood of 0. Since this is also true for $l' \in]l, \frac{1}{\beta-1}[$, the Laplace transform of $(1/I(V^\sharp))^l$ is defined on the whole positive real line. For any positive λ we thus have

$$\mathbb{P}\left(I(V^\sharp) \leq x\right) = \mathbb{P}\left(e^{\lambda\left(1/I(V^\sharp)\right)^l} \geq e^{\lambda/x^l}\right) \leq e^{-\lambda/x^l}\mathbb{E}\left[e^{\lambda\left(1/I(V^\sharp)\right)^l}\right].$$

Putting $c_{l,\lambda} := \mathbb{E}\left[e^{\lambda\left(1/I(V^\sharp)\right)^l}\right]$ we get that for all $0 < l < 1/(\beta - 1)$ and $\lambda > 0$, there is $c_{l,\lambda} > 0$ such that

$$\forall x > 0,\ \mathbb{P}\left(I(V^\sharp) \leq x\right) \leq c_{l,\lambda}e^{-\lambda/x^l}.$$

Taking the logarithm and multiplying by x^l we get

$$\limsup_{x\to 0} x^l \log\left(\mathbb{P}\left(I(V^\sharp) \leq x\right)\right) \leq -\lambda,$$

but since λ is as large as we want we get the result.

$\square$

Remark 4.4. *Note that the proof works even if $\beta = 1$ and the result of Proposition 4.3 is then true for any $l \in]0, +\infty[$. When V has bounded variations, we even have a stronger result. Indeed, $V^\sharp$ has also, in this case, bounded variations, so, as it can be seen from Remark 1.4, it is the* *difference of a positive drift $d^\sharp t$ and a pure jump subordinator $S^\sharp_t$: $\forall t>0,\ V^\sharp(t)=d^\sharp t-S^\sharp_t\le d^\sharp t$ so almost surely,*

$$I(V^\sharp):=\int_0^{+\infty}e^{-V^\sharp(t)}dt\ge\int_0^{+\infty}e^{-d^\sharp t}dt=\frac{1}{d^\sharp}>0.$$

We thus get that, when V has bounded variations, the probability $\mathbb{P}(I(V^\sharp)\le x)$ is 0 for $x<1/d^\sharp$.

In the general case, the gap between σ and β does not allow our method to provide an exact equivalent for $\log(\mathbb{P}(I(V^\sharp)\le x))$, but in the case of α-regular variations, we can improve our result under some hypothesis.

When Ψ_V has α-regular variations for $\alpha\in]1,2]$, we would like to obtain the existence of the Laplace transform in the proof of Proposition 4.3 with $l=1/(\alpha-1)$ and not only $l<1/(\alpha-1)$. We can get this if we assume the existence of a positive constant C such that $\Psi_V(\lambda)\le C\lambda^\alpha$ ultimately. We have therefore the following proposition.

Proposition 4.5. *If there is a positive constant C such that $\Psi_V(\lambda)\le C\lambda^\alpha$ for λ large enough, then the random variable $(1/I(V^\sharp))^{\frac{1}{\alpha-1}}$ admits some finite exponential moments.*

4.3. **Tail at 0 of $I(V^\uparrow)$.** The object of this subsection is to prove for $\log(\mathbb{P}(I(V^\uparrow)\le x))$ a lower bound involving σ. The key tool of our proof is Proposition 3.1. It is thus natural that we need first to study the asymptotic tail at 0 of A^y.

Lemma 4.6. *We fix $y>0$. Let A^y be as in Proposition 3.1, then, if $\sigma>1$,*

$$\forall l>\frac{1}{\sigma-1},\ \lim_{x\to0}x^l\log\left(\mathbb{P}\left(A^y\le x\right)\right)=0.$$

Proof. We use the decomposition of A^y given by Proposition 3.1 :

$$A^y\overset{\mathfrak{L}}{=}\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt+S_T,$$

with S_T as in Lemma 2.1 and the two terms on the right hand side are independent.

Choose $\epsilon>0$ and $z>0$. We have

$$\begin{aligned}\mathbb{P}\left(A^y\le z\right)&\ge\mathbb{P}\left(\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\le(1-\epsilon)z\right)\times\mathbb{P}\left(S_T\le\epsilon z\right)\\&\ge c\epsilon z\mathbb{P}\left(\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\le(1-\epsilon)z\right),\end{aligned}$$

for an appropriate constant $c>0$, when z is small enough, according to Lemma 4.2,

$$\ge c\epsilon z\mathbb{P}\left(\tau(V^\uparrow,y)\le(1-\epsilon)z\right),$$

because $V^\uparrow$ is non-negative,

$$\ge c\epsilon z\mathbb{P}\left(\tau(V^\sharp,y)\le(1-\epsilon)z\right),$$

according to the second point of Lemma 2.5. Now taking the logarithm we get, for z small enough,

$$\log\left(\mathbb{P}\left(A^y\le z\right)\right)\ge\log\left(c\epsilon z\right)+\log\left(\mathbb{P}\left(\tau(V^\sharp,y)\le(1-\epsilon)z\right)\right). \tag{4.36}$$

As we already mentioned, we have, using Theorem 1 page 189 in [3], that $\tau(V^\sharp,.)$ is a subordinator which Laplace exponent $\Phi_{V^\sharp}$ is defined for $\lambda \geq 0$ by

$$\Phi_{V^\sharp}(\lambda) := -\log\left(\mathbb{E}\left[e^{-\lambda\tau(V^\sharp,1)}\right]\right),$$

and $\Phi_{V^\sharp} = \Psi_{V^\sharp}^{-1}$.

We choose $l > 1/(\sigma-1)$. From the definition of σ we have that $\lambda^{-\frac{1}{l}-1}\Psi_{V^\sharp}(\lambda)$ converges to infinity when λ goes to infinity. As a consequence, for any $\delta > 0$, there is $\lambda_0 > 0$ such that

$$\forall\lambda \geq \lambda_0,\ \Psi_{V^\sharp}(\lambda) \geq \left(\frac{y}{\delta}\right)^{1+\frac{1}{l}}\lambda^{\frac{1}{l}+1},$$

so

$$\forall\lambda \geq \lambda_1 := \left(\frac{y}{\delta}\lambda_0\right)^{\frac{1+l}{l}},\ \Psi_{V^\sharp}\left(\frac{\delta}{y}\lambda^{\frac{l}{1+l}}\right) \geq \lambda,$$

and taking the inverse

$$\forall\lambda \geq \lambda_1,\ \Phi_{V^\sharp}(\lambda) \leq \frac{\delta}{y}\lambda^{\frac{l}{1+l}}.$$

$\tau(V^\sharp,y)$ has for Laplace transform $\mathbb{E}[e^{-\lambda\tau(V^\sharp,y)}] = e^{-y\Phi_{V^\sharp}(\lambda)}$ so we have

$$\forall\lambda \geq \lambda_1,\ \mathbb{E}\left[e^{-\lambda\tau(V^\sharp,y)}\right] \geq e^{-\delta\lambda^{\frac{l}{1+l}}}. \tag{4.37}$$

For any $z \in]0, 1/\lambda_1^{\frac{1}{1+l}}[$ we have

$$\begin{aligned}\mathbb{E}\left[e^{-(1/z^{1+l})\tau(V^\sharp,y)}\right] &= \mathbb{E}\left[e^{-(1/z^{1+l})\tau(V^\sharp,y)}\mathbb{1}_{\{\tau(V^\sharp,y)\leq z\}}\right] + \mathbb{E}\left[e^{-(1/z^{1+l})\tau(V^\sharp,y)}\mathbb{1}_{\{\tau(V^\sharp,y)>z\}}\right] \\ &\leq \mathbb{P}\left(\tau(V^\sharp,y) \leq z\right) + e^{-1/z^l},\end{aligned}$$

and combining with (4.37) we get

$$\mathbb{P}\left(\tau(V^\sharp,y) \leq z\right) \geq e^{-\delta/z^l} - e^{-1/z^l} = e^{-\delta/z^l}\left(1 - e^{-(1-\delta)/z^l}\right).$$

Now taking the logarithm and the lim inf we obtain

$$\liminf_{z\to 0}\ z^l\log\left(\mathbb{P}\left(\tau(V^\sharp,y) \leq z\right)\right) \geq -\delta,$$

and since δ is as small as we want we deduce

$$\lim_{z\to 0}\ z^l\log\left(\mathbb{P}\left(\tau(V^\sharp,y) \leq z\right)\right) = 0.$$

Putting in (4.36) we get the result.

□

We can now prove the main estimate of this subsection.

Proposition 4.7. *If $\sigma > 1$ then*

$$\forall l > \frac{1}{\sigma-1},\ \lim_{x\to 0} x^l\log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) = 0.$$

Proof. We fix $l \geq 1/(\sigma-1)$. For some $y>0$, we use the decomposition (3.14) of $I(V^{\uparrow})$ given by Proposition 3.1, with the sequence $(A_k^y)_{k\geq 0}$ as in the proposition. We have

$$\forall x>0,\ \prod_{k=0}^{+\infty} \mathbb{P}\left(A^y \leq e^{ky/2}(1-e^{-y/2})x\right) = \prod_{k=0}^{+\infty} \mathbb{P}\left(e^{-ky}A_k^y \leq e^{-ky/2}(1-e^{-y/2})x\right)$$
$$\leq \mathbb{P}\left(I(V^{\uparrow}) \leq x(1-e^{-y/2})\sum_{k\geq 0} e^{-ky/2}\right)$$
$$= \mathbb{P}\left(I(V^{\uparrow}) \leq x\right).$$

Taking the logarithm we get

$$\forall x>0,\ \log\left(\mathbb{P}\left(I(V^{\uparrow}) \leq x\right)\right) \geq \sum_{k=0}^{+\infty} \log\left(\mathbb{P}\left(A^y \leq e^{ky/2}(1-e^{-y/2})x\right)\right). \tag{4.38}$$

$\eta>0$ being fixed, we choose $z_0>0$ such that

$$\forall z \in]0,z_0[,\ \log\left(\mathbb{P}\left(A^y \leq z\right)\right) \geq -\eta/z^l, \tag{4.39}$$

note that such a z_0 exists according to Lemma 4.6. We also choose $M>z_0$ large enough so that $\forall x \leq \mathbb{P}(A^y>M),\ \log(1-x) \geq -2x$.

For any $x \in]0,z_0[$, we define $n_1(x) := \lfloor 2\log(z_0/x(1-e^{-y/2}))/y \rfloor$ and $n_2(x) := \lfloor 2\log(M/x(1-e^{-y/2}))/y \rfloor$, we thus write

$$\forall x \in]0,z_0[,\ \sum_{k=0}^{+\infty} \log\left(\mathbb{P}\left(A^y \leq e^{ky/2}(1-e^{-y/2})x\right)\right) = T_1(x)+T_2(x)+T_3(x), \tag{4.40}$$

with

$$T_1(x) := \sum_{k=0}^{n_1(x)} \log\left(\mathbb{P}\left(A^y \leq e^{ky/2}(1-e^{-y/2})x\right)\right),$$
$$T_2(x) := \sum_{k=n_1(x)+1}^{n_2(x)} \log\left(\mathbb{P}\left(A^y \leq e^{ky/2}(1-e^{-y/2})x\right)\right),$$
$$T_3(x) := \sum_{k=n_2(x)+1}^{+\infty} \log\left(\mathbb{P}\left(A^y \leq e^{ky/2}(1-e^{-y/2})x\right)\right).$$

We have

$$T_2(x) \geq (n_2(x)-n_1(x))\log\left(\mathbb{P}\left(A^y \leq e^{y(n_1(x)+1)/2}(1-e^{-y/2})x\right)\right)$$
$$\geq \frac{y+2\log(M/z_0)}{y}\log\left(\mathbb{P}\left(A^y \leq z_0\right)\right), \tag{4.41}$$

just by using the definitions of $n_1(x)$ and $n_2(x)$. From the definitions of $n_2(x)$ and M we have

$$T_3(x) \geq -2\sum_{k=n_2(x)+1}^{+\infty} \mathbb{P}\left(A^y > e^{ky/2}(1-e^{-y/2})x\right) \geq -2\sum_{k=0}^{+\infty}\mathbb{P}\left(A^y > e^{ky/2}M\right), \tag{4.42}$$

which does not depend on x. To control $T_3(x)$, we are left to show that the series $\sum_{k=0}^{+\infty} \mathbb{P}(A^y > e^{ky/2}M)$ is finite, but we know from Remark 3.2 that the random series $\sum_{k=0}^{+\infty} e^{-ky/2} A_k^y$ converges almost surely and the required finiteness follows from Borel-Cantelli's 0-1 law.

Because of the definition of $n_1(x)$, all the terms in the sum defining $T_1(x)$ are of the form $\log\left(\mathbb{P}\left(A^y \le z_k\right)\right)$ for $z_k \in]0, z_0[$ and we can thus apply estimate (4.39) to each term :

$$
\begin{aligned}
T_1(x) &\ge -\sum_{k=0}^{n_1(x)} \frac{\eta}{\left[e^{ky/2}(1-e^{-y/2})x\right]^l} \\
&= -\frac{\eta}{\left[(1-e^{-y/2})x\right]^l} \frac{1-e^{-n_1(x)ly/2}}{1-e^{-ly/2}} \\
&\ge -\frac{\eta}{x^l} \times \frac{1}{(1-e^{-ly/2})(1-e^{-y/2})^l}.
\end{aligned} \tag{4.43}
$$

Now putting (4.43), (4.41) and (4.42) in (4.40) and notting that η can be chosen arbitrarily small we get

$$
\lim_{x\to 0} x^l \sum_{k=0}^{+\infty} \log\left(\mathbb{P}\left(A^y \le e^{ky/2}(1-e^{-y/2})x\right)\right) = 0,
$$

and because of (4.38), this yields the result.

□

As in the end of the previous subsection, the problem that naturally arises now is to see how the estimate we have proved can be improved in the case of α-regular variations. When Ψ_V has α-regular variations for $\alpha \in]1, 2]$, we would like to obtain the result of Proposition 4.7 with $l = 1/(\alpha-1)$ and not only $l > 1/(\alpha-1)$. We can get this if we assume the existence of a positive constant c such that $\Psi_V(\lambda) \ge c\lambda^\alpha$ ultimately. We have therefore the following proposition.

Proposition 4.8. *If there is a positive constant c such that $\Psi_V(\lambda) \ge c\lambda^\alpha$ for λ large enough, then there exists a positive constant K such that*

$$
\log\left(\mathbb{P}\left(I(V^\uparrow) \le x\right)\right) \ge -\frac{K}{x^{\frac{1}{\alpha-1}}},
$$

for all x small enough.

Proof. It is less obvious than the generalization in the previous subsection so we give a few details. We fix $y > 0$. With $l := 1/(\alpha-1)$, we deduce (in the same way we established (4.37)) from the hypothesis $c\lambda^\alpha \le \Psi_V(\lambda)$ that

$$
\mathbb{E}\left[e^{-\lambda\tau(V^\sharp,y)}\right] \ge e^{-y(\lambda/c)^{\frac{l}{1+l}}},
$$

for all λ large enough. As a consequence for any $u > 0$ and z small enough we have

$$
\begin{aligned}
\mathbb{E}\left[e^{-(u/z^{1+l})\tau(V^\sharp,y)}\right] &= \mathbb{E}\left[e^{-(u/z^{1+l})\tau(V^\sharp,y)}\mathbb{1}_{\{\tau(V^\sharp,y)\le z\}}\right] + \mathbb{E}\left[e^{-(u/z^{1+l})\tau(V^\sharp,y)}\mathbb{1}_{\{\tau(V^\sharp,y)> z\}}\right] \\
&\le \mathbb{P}\left(\tau(V^\sharp,y) \le z\right) + e^{-u/z^l},
\end{aligned}
$$

and combining with the previous inequality we get

$$
\mathbb{P}\left(\tau(V^\sharp,y) \le z\right) \ge e^{-yu^{\frac{l}{1+l}}(1/c)^{\frac{l}{1+l}}/z^l} - e^{-u/z^l},
$$

and for small z, the second term is negligible with respect to the first if $u > y^{1+l}/c^l$. Taking the logarithm and the $\liminf$ we obtain,

$$\liminf_{z\to 0} \; z^l \log\left(\mathbb{P}\left(\tau(V^\sharp, y) \le z\right)\right) \ge -yu^{\frac{l}{1+l}}(1/c)^{\frac{l}{1+l}}. \tag{4.44}$$

It is now possible to repeat readily the arguments of the proof of Proposition 4.7 with $l := 1/(\alpha-1)$ and estimate (4.44) instead of (4.39). We obtain the result for any $K > yu^{\frac{l}{1+l}}(1/c)^{\frac{l}{1+l}}/(1-e^{-ly/2})(1-e^{-y/2})^l$.

□

4.4. **Synthesis of the results.** Combining the results of the three preceding subsections, we get results for the tails at 0 of $I(V^\sharp)$ and $I(V^\uparrow)$. First, putting together propositions 4.3, 4.7 and 4.1 we get Theorem 1.5. In the case of α-regular variations we can put together propositions 4.5, 4.8 and 4.1 to get Theorem 1.6.

Remark 4.9. *When V has bounded variations, then we have $\mathbb{P}(V^\sharp(s) > 0,\ \forall s > 0) > 0$ and it is easy to see that $V^\uparrow$ is only $V^\sharp$ conditioned in the usual sense to remain positive. Combining with Remark 4.4, $I(V^\uparrow)$ is more than the positive constant $1/d^\sharp$ almost surely so $\mathbb{P}(I(V^\uparrow) \le x)$ is null for $x < 1/d^\sharp$.*

5. Smoothness of the density : Proof of Theorem 1.8

According to Proposition 3.1, $I(V^\uparrow)$ contains, as a convolution factor, the sum of infinitely many independent multiples of random variables having the same law as S_T (S_T being as in Lemma 2.1). We can thus use a condition on the Lévy mesure of S to have the existence of the smooth density for $I(V^\uparrow)$. Actually, the condition that we check for S is the one of Proposition 28.3 page 190 in [14], which is a condition on the Lévy measure of a Lévy process for it to have a $\mathcal{C}^\infty$ density with bounded derivatives.

As a jump of S is the image by the mapping G of an excursion of $V^\sharp$, we start by lemmas on the excursions of $V^\sharp$.

Lemma 5.1. *Assume $\sigma > 1$ and choose σ' such that $1 < \sigma' < \sigma$. For all h small enough we have*

$$\eta^\sharp\left(\xi,\ \zeta(\xi) < +\infty,\ H(\xi) > h\right) \ge h^{-(\sigma'-1)}.$$

Proof. We consider excursions away from 0. Let p_h denote the probability that $V^\sharp$ has no finite excursion that reaches h before its infinite excursion. Since the set of finite and infinite excursions are disjoint, we have, by a property of Poisson point processes

$$p_h = \frac{\eta^\sharp\left(\xi,\ \zeta(\xi) = +\infty\right)}{\eta^\sharp\left(\xi,\ \zeta(\xi) = +\infty\right) + \eta^\sharp\left(\xi,\ \zeta(\xi) < +\infty,\ H(\xi) > h\right)}, \tag{5.45}$$

so we only need to give an upper bound for p_h. Now, note that

$$p_h = \mathbb{P}\left(\forall s \ge \tau(V^\sharp, h),\ V^\sharp(s) \ne 0\right) = \mathbb{P}\left(\forall s \ge 0,\ V_h^\sharp(s) \ne 0\right),$$

from Markov property at time $\tau(V^\sharp, h)$,

$$= \mathbb{P}\left(\inf_{[0,+\infty[} V_h^\sharp > 0\right) = \mathbb{P}\left(\inf_{[0,+\infty[} V^\sharp > -h\right),$$

because, since $V_h^\sharp$ drifts to $+\infty$ and has no positive jumps, it reaches $]-\infty,0]$ if and only if it reaches 0,

$$\leq \mathbb{P}\left(\exp\left(\frac{\underline{V}^\sharp(+\infty)}{h}\right) > e^{-1}\right) \leq e\mathbb{E}\left[\exp\left(\frac{\underline{V}^\sharp(+\infty)}{h}\right)\right] = e\frac{1/h}{\Psi_{V^\sharp}(1/h)},$$

where we composed by exponential, used Markov inequality and the expression of the Laplace transform of $\underline{V}^\sharp(+\infty)$ given page 192 of [3]. Now, from the definition of σ and the fact that $\sigma' < \sigma$, we have $\Psi_{V^\sharp}(\lambda) \geq \lambda^{\sigma'}$ provided λ is large enough. The above expression is thus less than $eh^{\sigma'-1}$ for small h. We thus get for h small enough,

$$p_h \leq eh^{\sigma'-1}, \tag{5.46}$$

and taking the inverse in (5.45),

$$\eta^\sharp\left(\xi,\ \zeta(\xi) < +\infty,\ H(\xi) > h\right) \geq \eta^\sharp\left(\xi,\ \zeta(\xi) = +\infty\right) \times \left(\frac{1}{p_h} - 1\right). \tag{5.47}$$

The combination of (5.46) and (5.47) yields the result. We got rid of the constants since the same result with the same constant is true for σ' increased a little bit. □

We now need the following lemma which states that for a spectrally negative Lévy process, the excursions of a given height can be split into two independents parts of which the law are known.

Lemma 5.2. *For any $h > 0$, assume that the process X follows the law $\eta^\sharp(.|H(.) > h)$. Then we have :*

- $(X(s),\ 0 \leq s \leq \tau(X,h)) \sim (V^\uparrow(s),\ 0 \leq s \leq \tau(V^\uparrow,h))$,
- $(X(s),\ \tau(X,h) \leq s \leq \tau(X(.+\tau(X,h)),0)) \sim (V_h^\sharp(s),\ 0 \leq s \leq \tau(V_h^\sharp,0))$,
- $(X(s),\ 0 \leq s \leq \tau(X,h)) \perp\!\!\!\perp (X(s),\ \tau(X,h) \leq s \leq \tau(X(.+\tau(X,h)),0))$.

Note that the time $\tau(V_h^\sharp,0)$ may possibly be infinite, but this is unlikely when h is small.

Proof. of Lemma 5.2

The first point is a consequence of the second point of Lemma 2.5. The second and third points come from Markov property.

□

We can now prove the main lemma of this section, it will allow us to check the condition on the Lévy mesure of S.

Lemma 5.3. *Assume $\sigma > 1$ and choose σ' and β' such that $1 < \sigma' < \sigma \leq \beta < \beta'$. We also choose $\epsilon > \frac{\beta'-1}{\sigma'-1} - 1$ and fix $C > 0$ an arbitrary constant. Then, for r small enough we have*

$$\eta^\sharp\left(\xi,\ \zeta(\xi) \in]x,r]\right) \geq x^{-(\sigma'-1)/\beta'},\ \forall x \in]0, Cr^{1+\epsilon}].$$

Proof. Let us fix $r > 0$ and $x \in]0, Cr^{1+\epsilon}]$, then

$$\begin{aligned}\eta^\sharp\left(\xi,\ \zeta(\xi) \in]x,r]\right) &\geq \eta^\sharp\left(\xi,\ \zeta(\xi) \in [x,r],\ H(\xi) > x^{1/\beta'}\right)\\ &= \eta^\sharp\left(\xi,\ \zeta(\xi) \in]x,r] \,\middle|\, H(\xi) > x^{1/\beta'}\right) \times \eta^\sharp\left(\xi,\ H(\xi) > x^{1/\beta'}\right),\end{aligned}$$

where $\eta^\sharp(.|H(.) > x^{1/\beta'})$ is the measure of excursions conditioned to be higher than $x^{1/\beta'}$. The last quantity thus equals

$$\left[\eta^\sharp\left(\xi, \zeta(\xi) > x \,\middle|\, H(\xi) > x^{1/\beta'}\right) - \eta^\sharp\left(\xi, \zeta(\xi) > r \,\middle|\, H(\xi) > x^{1/\beta'}\right)\right] \times \eta^\sharp\left(\xi,\ H(\xi) > x^{1/\beta'}\right) \tag{5.48}$$

We now study the three quantities appearing in (5.48) and show that this expression is of the same order as $\eta^\sharp(\xi,\ H(\xi) > x^{1/\beta'})$ for which Lemma 5.1 provides a lower bound. We start by proving that $\eta^\sharp(\xi,\ \zeta(\xi) > r|H(\xi) > x^{1/\beta'})$ converges to 0 uniformly in $x \in]0, Cr^{1+\epsilon}]$ when r goes to 0. First, Lemma 5.2 gives for all x in $]0, Cr^{1+\epsilon}]$ that

$$\eta^\sharp\left(\xi,\ \zeta(\xi) > r \,\middle|\, H(\xi) > x^{1/\beta'}\right) \leq \mathbb{P}\left(\tau(V^\sharp_{x^{1/\beta'}}, 0) > r/2\right) + \mathbb{P}\left(\tau(V^\uparrow, x^{1/\beta'}) > r/2\right). \tag{5.49}$$

The first thing is thus to prove that

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau(V^\sharp_{x^{1/\beta'}}, 0) > r/2\right) \underset{r\to 0}{\longrightarrow} 0. \tag{5.50}$$

Before reaching 0 (if it does) the process $V^\sharp_{x^{1/\beta'}}$ reaches $]-\infty, 0]$ (not necessarily at 0). We define $\tau^N_{x^{1/\beta'}} := \tau(V^\sharp_{x^{1/\beta'}},]-\infty, 0])$. For all x in $]0, Cr^{1+\epsilon}]$ we have

$$\mathbb{P}\left(\tau(V^\sharp_{x^{1/\beta'}}, 0) > r/2\right) \leq \mathbb{P}\left(\tau^N_{x^{1/\beta'}} > r/4\right) + \mathbb{P}\left(\tau^N_{x^{1/\beta'}} \leq r/4,\ \tau(V^\sharp_{x^{1/\beta'}}(. + \tau^N_{x^{1/\beta'}}), 0) > r/4\right) \tag{5.51}$$

Since $(x \mapsto \tau^N_{x^{1/\beta'}})$ is stochastically increasing in the variable x we have

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau^N_{x^{1/\beta'}} > r/4\right) = \mathbb{P}\left(\tau^N_{C^{1/\beta'}r^{(1+\epsilon)/\beta'}} > r/4\right) = \mathbb{P}\left(\underline{V^\sharp}(r/4) > -C^{1/\beta'}r^{(1+\epsilon)/\beta'}\right).$$

We introduce T, an exponential random variable with parameter 1 independent of the process V, and a decreasing function $q :]0, +\infty[\to]0, +\infty[$ that converges to $+\infty$ as r go to 0 and will be specified latter. We have

$$\begin{aligned}
\mathbb{P}\left(\underline{V^\sharp}(r/4) > -C^{1/\beta'}r^{(1+\epsilon)/\beta'}\right) &\leq \mathbb{P}\left(\underline{V^\sharp}(T/q(r)) > -C^{1/\beta'}r^{(1+\epsilon)/\beta'}\right) + \mathbb{P}\left(T/q(r) > r/4\right) \\
&= \mathbb{P}\left(\exp\left(\frac{\underline{V^\sharp}(T/q(r))}{r^{(1+\epsilon)/\beta'}}\right) > \exp(-C^{1/\beta'})\right) + e^{-rq(r)/4} \\
&\leq e^{C^{1/\beta'}}\mathbb{E}\left[\exp\left(\frac{\underline{V^\sharp}(T/q(r))}{r^{(1+\epsilon)/\beta'}}\right)\right] + e^{-rq(r)/4},
\end{aligned}$$

from Markov inequality,

$$= \frac{1/\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} - 1}{\Psi_{V^\sharp}(r^{-(1+\epsilon)/\beta'})/q(r) - 1} \times e^{C^{1/\beta'}} + e^{-rq(r)/4},$$

from the expression of the Laplace transform of the random variable $\underline{V^\sharp}(T/q(r))$ that can be found page 192 of [3]. The last expression goes to 0 if these three conditions are satisfied :

$$\begin{aligned}
&q(r)r \underset{r\to 0}{\longrightarrow} +\infty, \\
&\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} \underset{r\to 0}{\longrightarrow} 0, \\
&q(r)/\Psi_{V^\sharp}(r^{-(1+\epsilon)/\beta'})\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} \underset{r\to 0}{\longrightarrow} 0.
\end{aligned}$$

From the definition of σ and β and the fact that $\sigma' < \sigma \leq \beta < \beta'$, we have $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$, $\Psi_{V^\sharp}(u) \geq u^{\sigma'}$ and $\Phi_{V^\sharp}(u) \geq u^{1/\beta'}$, provided u is large enough. The three conditions can thus be simplified and we only need to have :

$$\begin{aligned}
&q(r)r \underset{r\to 0}{\longrightarrow} +\infty,\\
&(q(r))^{1/\sigma'} r^{(1+\epsilon)/\beta'} \underset{r\to 0}{\longrightarrow} 0,\\
&(q(r))^{1-1/\beta'} r^{(\sigma'-1)(1+\epsilon)/\beta'} \underset{r\to 0}{\longrightarrow} 0.
\end{aligned}$$

Elevating $(q(r))$ to the right power so it makes its exponent disappear, these three conditions become

$$\begin{aligned}
&q(r)r \underset{r\to 0}{\longrightarrow} +\infty,\\
&q(r) r^{\sigma'(1+\epsilon)/\beta'} \underset{r\to 0}{\longrightarrow} 0,\\
&q(r) r^{(\sigma'-1)(1+\epsilon)/(\beta'-1)} \underset{r\to 0}{\longrightarrow} 0.
\end{aligned}$$

Since $\sigma' < \beta'$ we can check that $\sigma'/\beta' > (\sigma'-1)/(\beta'-1)$, so the third condition implies the second. We then only need to verify the first and the third condition, but from the choice of ϵ, we have that $(\sigma'-1)(1+\epsilon)/(\beta'-1) > 1$, so $q(r)$ can be chosen such that the first and the third conditions are satisfied. This yields

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau^N_{x^{1/\beta'}} > r/4\right) \underset{r\to 0}{\longrightarrow} 0. \tag{5.52}$$

We now turn to the second term of (5.51). It is known that the jumps of $V^\sharp$ is a Poisson process with intensity measure $\nu^\sharp$, the Lévy measure of $V^\sharp$. The probability that $V^\sharp$ has a jump smaller than $-r$ before time $r/4$ is thus $1-e^{r\nu^\sharp(]-\infty,-r])/4} \sim r\nu^\sharp(]-\infty,-r])/4 := \gamma(r)$ and this goes to 0 for any Lévy measure. As a consequence, on $\{\tau^N_{x^{1/\beta'}} \le r/4\}$ we have $V^\sharp_{x^{1/\beta'}}(\tau^N_{x^{1/\beta'}}) \in [-r,0]$, exepted on some event having probability less than $\gamma(r)$. We thus get

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau^N_{x^{1/\beta'}} \le r/4,\ \tau(V^\sharp_{x^{1/\beta'}}(.+\tau^N_{x^{1/\beta'}}),0) > r/4\right) \le \mathbb{P}\left(\tau(V^\sharp,r) > r/4\right) + \gamma(r),$$

and we now only need to show that $\tau(V^\sharp,r)/r$ converges to 0 in probability. Now recall that we know from Theorem 1 page 189 in [3] that the Laplace transform of this random variable is given by

$$\mathbb{E}\left[\exp\left(-\lambda\frac{\tau(V^\sharp,r)}{r}\right)\right] = e^{-r\Phi_{V^\sharp}(\lambda/r)}.$$

Since $\Phi_{V^\sharp}(u) \le u^{1/\sigma'}$ for u large enough, the last quantity converges to 1 when r goes to 0 so we indeed have the convergence to 0 in probability of $\tau(V^\sharp,r)/r$ and as a consequence

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau^N_{x^{1/\beta'}} \le r/4,\ \tau(V^\sharp_{x^{1/\beta'}}(.+\tau^N_{x^{1/\beta'}}),0) > r/4\right) \underset{r\to 0}{\longrightarrow} 0.$$

Putting this, together with (5.52), in (5.51) we get (5.50). We now deal with the second term of (5.49), more precisely we prove that

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau(V^\uparrow, x^{1/\beta'}) > r/2\right) \underset{r\to 0}{\longrightarrow} 0. \tag{5.53}$$

Because of the increases of the quantity $\mathbb{P}(\tau(V^{\uparrow}, x^{1/\beta'}) > r/2)$ we can write

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \mathbb{P}\left(\tau(V^{\uparrow}, x^{1/\beta'}) > r/2\right) = \mathbb{P}\left(\tau(V^{\uparrow}, C^{1/\beta'} r^{(1+\epsilon)/\beta'}) > r/2\right)$$
$$\leq \mathbb{P}\left(\tau(V^{\sharp}, C^{1/\beta'} r^{(1+\epsilon)/\beta'}) > r/2\right),$$

according to the second point of Lemma 2.5.

Therefore, (5.53) will follow if we prove that the random variable $\tau(V^{\sharp}, C^{1/\beta'} r^{(1+\epsilon)/\beta'})/r$ converge to 0 in probability as r goes to 0. Again, we see from Theorem 1 page 189 in [3] that the Laplace transform of this random variable is given by

$$\mathbb{E}\left[\exp\left(-\lambda \frac{\tau(V^{\sharp}, C^{1/\beta'} r^{(1+\epsilon)/\beta'})}{r}\right)\right] = e^{-C^{1/\beta'} r^{(1+\epsilon)/\beta'} \Phi_{V^{\sharp}}(\lambda r^{-1})},$$

but, since $\Phi_{V^{\sharp}}(u) \leq u^{1/\sigma'}$ provided λ is large enough, we have that, for small r,

$$0 \leq r^{(1+\epsilon)/\beta'} \Phi_{V^{\sharp}}(\lambda r^{-1}) \leq r^{(1+\epsilon)/\beta' - 1/\sigma'} \lambda^{1/\sigma'}.$$

By the choice of ϵ, we have $1+\epsilon > (\beta'-1)(\sigma'-1) > \beta'/\sigma'$, so the last quantity converges to 0 as r goes to 0. This shows that the Laplace transform of $\tau(V^{\sharp}, C^{1/\beta'} r^{(1+\epsilon)/\beta'})/r$ converges to 1 as r goes to 0 so we get the asserted convergence to 0 in probability as r goes to 0 and (5.53) follows.

Putting (5.50) and (5.53) in (5.49) we get

$$\sup_{x\in]0,Cr^{1+\epsilon}]} \eta^{\sharp}\left(\xi,\ \zeta(\xi) > r \,\middle|\, H(\xi) > x^{1/\beta'}\right) \underset{r\to 0}{\longrightarrow} 0, \tag{5.54}$$

which means that among the excursions of heigh greater than $x^{1/\beta'}$, those of length greater than r are in negligible proportion, and it thus remains to show that those of length greater than x are in non-negligible proportion. We want to show that

$$\liminf_{r\to 0} \inf_{x\in]0,Cr^{1+\epsilon}]} \eta^{\sharp}\left(\xi,\ \zeta(\xi) > x \,\middle|\, H(\xi) > x^{1/\beta'}\right) > 0. \tag{5.55}$$

From the second point of Lemma 5.2 we have for all x in $]0, Cr^{1+\epsilon}]$,

$$\eta^{\sharp}\left(\xi,\ \zeta(\xi) > x \,\middle|\, H(\xi) > x^{1/\beta'}\right) \geq \mathbb{P}\left(\tau(V^{\sharp}_{x^{1/\beta'}}, 0) > x\right) \geq \mathbb{P}\left(\tau^{N}_{x^{1/\beta'}} > x\right). \tag{5.56}$$

As we did before, we introduce T, an exponential random variable with parameter 1 independent of the process $V^{\sharp}$, and a decreasing function $q:]0,+\infty[\to]0,+\infty[$ that converges to $+\infty$ at 0 and will be specified latter.

For any x in $]0, Cr^{1+\epsilon}]$ we have

$$\mathbb{P}\left(\tau^{N}_{x^{1/\beta'}} > x\right) = \mathbb{P}\left(\underline{V^{\sharp}}(x) > -x^{1/\beta'}\right)$$
$$\geq \mathbb{P}\left(\underline{V^{\sharp}}(T/q(x)) > -x^{1/\beta'}\right) - \mathbb{P}\left(T/q(x) < x\right)$$
$$= \mathbb{P}\left(\frac{\underline{V^{\sharp}}(T/q(x))}{x^{1/\beta'}} > -1\right) - 1 + e^{-xq(x)}$$

We choose $q(x) = 1/x$ and, because of (5.56), (5.55) will follow if we prove that the random variable $\underline{V^{\sharp}}(T/q(x))/x^{1/\beta'}$ converges in probability to 0 as x goes to 0. According to [3] p 192, the Laplace transform of this random variable is given by

$$\mathbb{E}\left[\exp\left(\lambda\frac{\underline{V^\sharp}(T/q(x))}{x^{1/\beta'}}\right)\right] = \frac{q(x)\left(\Phi_{V^\sharp}(q(x)) - \lambda/x^{1/\beta'}\right)}{\Phi_{V^\sharp}(q(x))\left(q(x) - \Psi_{V^\sharp}(\lambda/x^{1/\beta'})\right)}.$$

This last quantity converges to 1 when x goes to 0 if

- $\lambda/x^{1/\beta'}$ is negligible compared to $\Phi_{V^\sharp}(q(x))$ when x goes to 0,
- $\Psi_{V^\sharp}(\lambda/x^{1/\beta'})$ is negligible compared to $q(x)$ when x goes to 0.

These two conditions can be written

$$x^{1/\beta'}\Phi_{V^\sharp}(q(x)) \underset{r\to 0}{\longrightarrow} +\infty,$$
$$\Psi_{V^\sharp}(\lambda/x^{1/\beta'})/q(x) \underset{r\to 0}{\longrightarrow} 0.$$

Now, because of the definition of β, because $\beta' > \beta$, and because $q(x) = 1/x$, it is easy to see that these two conditions are satisfied. This shows that the Laplace transform of the random variable $\underline{V^\sharp}(T/q(x))/x^{1/\beta'}$ converges to 1 as x goes to 0, so this random variable converges to 0 in probability and (5.55) follows.

For the factor $\eta^\sharp(\xi,\ H(\xi) > x^{1/\beta'})$ of (5.48) we note that it is trivially more than $\eta^\sharp(\xi,\ \zeta(\xi) < +\infty,\ H(\xi) > x^{1/\beta'})$ and we can use Lemma 5.1 which yields

$$\eta^\sharp\left(\xi,\ H(\xi) > x^{1/\beta'}\right) \geq x^{-(\sigma'-1)/\beta'}, \tag{5.57}$$

for x small enough. Putting (5.54), (5.55) and (5.57) in (5.48), we get the asserted result. Here again, we actually obtain the result up to a multiplicative constant, but since the result is still true if, for example, we increase σ' a little bit, we can get rid of the constant.

□

We can now prove Theorem 1.8.

Proof. of Theorem 1.8

We make the assumption that (1.5) is satisfied so, by continuity of the left hand side of (1.5) in σ and β, we can choose σ' and β' to be as in Lemma 5.3, but close enough to respectively σ and β so that they also satisfy (1.5). We also choose ϵ as in Lemma 5.3.

We fix $y > 0$. Let S is a pure jump subordinator with Lévy mesure $\mu(.) := G\eta^\sharp_y(.\cap FP)$, the image measure of $\eta^\sharp_y(.\cap FP)$ by the mapping G. From the Lévy-Khintchine formula, the characteristic function of $S(t)$ is

$$\mathbb{E}\left[e^{i\xi S(t)}\right] = e^{t\Phi_S(\xi)},$$

where

$$\forall\xi\in\mathbb{R},\ \Phi_S(\xi) := \int_0^{+\infty}(e^{i\xi x} - 1)\mu(dx),$$

so, taking the real part,

$$\begin{aligned}\forall \xi \in \mathbb{R},\ |\mathcal{R}\left(\Phi_S(\xi)\right)| &= \int_0^{+\infty} \left(1-\cos(\xi x)\right)\mu(dx)\\ &\geq \int_0^{\pi/|\xi|} \left(1-\cos(\xi x)\right)\mu(dx)\\ &\geq \frac{2\xi^2}{\pi^2}\int_0^{\pi/|\xi|} x^2\mu(dx). \end{aligned} \tag{5.58}$$

We now prove that the measure μ satisfies the hypothesis of Proposition 28.3 page 190 of [14]. We have for any $r \in]0,1[$:

$$\int_0^r x^2\mu(dx) = 2\int_0^r x\mu([x,r[)dx \geq 2\int_0^{r^{1+\epsilon}} x\mu([x,r[)dx. \tag{5.59}$$

We thus need to minorate $\mu([x,r[)$ for $x \in]0, r^{1+\epsilon}]$:

$$\begin{aligned}\mu([x,r[) &= \eta_y^\sharp(\{\xi,\ G(\xi) \in [x,r[\} \cap FP)\\ &= \eta_y^\sharp\left(\left\{\xi,\ \int_0^{\zeta(\xi)} e^{-\xi(t)}dt \in [x,r[\right\} \cap FP\right),\end{aligned}$$

from the definition of G,

$$\begin{aligned}&\geq \eta_y^\sharp\left(\{\xi,\ \zeta(\xi) \in [e^{2y}x, r[,\ \sup\xi \leq 2y\} \cap FP\right)\\ &\geq \eta_y^\sharp\left(\{\xi,\ \zeta(\xi) \in [e^{2y}x, r[\}\right) - \eta_y^\sharp\left(\{\xi,\ \sup\xi > 2y\}\right) - \eta_y^\sharp\left(FP^c\right)\\ &= \eta^\sharp\left(\{\xi,\ \zeta(\xi) \in [e^{2y}x, r[\}\right) - \eta_y^\sharp\left(\{\xi,\ \sup\xi > 2y\}\right) - \eta_y^\sharp\left(IP\right) - \eta_y^\sharp\left(N\right)\\ &= \eta^\sharp\left(\{\xi,\ \zeta(\xi) \in [e^{2y}x, r[\}\right) - c,\end{aligned}$$

where we have put $c := \eta_y^\sharp\left(\{\sup\xi > 2y\}\right) + \eta_y^\sharp\left(IP\right) + \eta_y^\sharp\left(N\right)$. Note that c is well defined because the quantities $\eta_y^\sharp\left(\{\sup\xi > 2y\}\right)$, $\eta_y^\sharp\left(IP\right)$ and $\eta_y^\sharp\left(N\right)$ are finite.

We now apply Lemma 5.3 (taking $C = e^{2y}$) and we deduce that for r small enough,

$$\forall x \in]0, r^{1+\epsilon}],\ \mu([x,r[) \geq e^{-2y(\sigma'-1)/\beta'}x^{-(\sigma'-1)/\beta'} - c.$$

Combining this with (5.59), we get that whenever r is small enough :

$$\begin{aligned}\int_0^r x^2\mu(dx) &\geq 2e^{-2y(\sigma'-1)/\beta'}\int_0^{r^{1+\epsilon}} x^{1-(\sigma'-1)/\beta'}dx - 2c\int_0^{r^{1+\epsilon}} xdx\\ &\geq \frac{2e^{-2y(\sigma'-1)/\beta'}}{2-(\sigma'-1)/\beta'}r^{(1+\epsilon)(2-(\sigma'-1)/\beta')} - cr^{2(1+\epsilon)}.\end{aligned}$$

If

$$(1+\epsilon)(2-(\sigma'-1)/(3\beta'-1)) < 2, \tag{5.60}$$

then, choosing $\delta \in]0, 2-(1+\epsilon)(2-(\sigma'-1)/\beta')[$ and combining the above estimate with (5.58), we get that

$$|\Phi_S(\xi)| \geq |\mathcal{R}\left(\Phi_S(\xi)\right)| \geq |\xi|^\delta, \tag{5.61}$$

whenever $|\xi|$ is large enough. As the only assumption on ϵ is that it is greater than $(\beta'-1)/(\sigma'-1)-1$, we can choose ϵ such that (5.60) is satisfied if and only if

$$\frac{\beta'-1}{\sigma'-1}(2-(\sigma'-1)/(3\beta'-1)) < 2,$$

which is equivalent to the fact that σ' and β' satisfy (1.5). Therefore, we have proved that there exists $\delta > 0$ such that (5.61) is true.

Let T be an exponential random variable with parameter $p := \eta_y^\sharp(IP) + \eta_y^\sharp(N)$ which is independent of S, the Fourier transform of S_T is

$$\mathbb{E}\left[e^{i\xi S_T}\right] = \int_0^{+\infty} p e^{t\Phi_S(\xi)} e^{-pt} dt = \frac{p}{p - \Phi_S(\xi)} \underset{|\xi|\to+\infty}{=} \mathcal{O}\left(|\xi|^{-\delta}\right), \tag{5.62}$$

because of (5.61).

Proposition 3.1 gives the decomposition

$$I(V^\uparrow) = \sum_{k\geq 0} e^{-ky} B_k^y + \sum_{k\geq 0} e^{-ky} C_k^y$$

where all the random variables in the two series are mutually independent, and where each term B_k^y has the same law as S_T. Therefore, the characteristic function of $I(V^\uparrow)$ is the product of a characteristic function bounded by 1 and of $\prod_{k\geq 0} \mathbb{E}[e^{ie^{-ky}\xi S_T}]$ which, thanks to (5.62), goes to 0 faster than any negative power of $|\xi|$. This proves that the density of $I(V^\uparrow)$ is of class $\mathcal{C}^\infty$ and that all its derivatives converge to 0 when x goes to $+\infty$. The derivatives of the density of $I(V^\uparrow)$ also converge to 0 when x goes to 0 since this density is of class $\mathcal{C}^\infty$ and null on $]-\infty, 0[$.

We now assume that $I(V^\uparrow)$ admits moments of any positive order. To prove that $\phi_{I(V^\uparrow)}$ actually belongs to Schwartz space, we have to study a little more deeply the infinite product. Let us denote by ψ the characteristic function of $\sum_{k\geq 0} e^{-ky} C_k^y$. Then

$$\phi_{I(V^\uparrow)}(\xi) = \psi(\xi) \prod_{k\geq 0} \mathbb{E}\left[e^{ie^{-ky}\xi S_T}\right]. \tag{5.63}$$

For any $n \in \mathbb{N}$ and $m > n$, we can see by induction that the n^{th} derivative

$$P_{m,n} := \left(\xi \mapsto \prod_{k=0}^{m} \mathbb{E}\left[e^{ie^{-ky}\xi S_T}\right]\right)^{(n)}$$

is a finite sum of products. In each of these products, there are at least $m - n$ factors of the form $\mathbb{E}[e^{ie^{-ky}\xi S_T}]$ for some integers k and the other factors are derivatives at some orders of the functions $(\xi \mapsto \mathbb{E}[e^{ie^{-ky}\xi S_T}])$ for some integers k.

As the random variable S_T admits moments of any positive order (because it is a convolution factors of $I(V^\uparrow)$), the derivative at any order of functions of the kind of $(\xi \mapsto \mathbb{E}[e^{ie^{-ky}\xi S_T}])$, for integers k, are bounded. Therefore, from (5.62), we deduce that

$$P_{m,n}(\xi) \underset{|\xi|\to+\infty}{=} \mathcal{O}\left(|\xi|^{-(m-n)\delta}\right). \tag{5.64}$$

We decompose (5.63) in

$$\phi_{I(V^\uparrow)}(\xi) = \left(\prod_{k=0}^{m} \mathbb{E}\left[e^{ie^{-ky}\xi S_T}\right]\right) \times R_m(\xi),$$

where

$$R_m(\xi) := \psi(\xi) \prod_{k\geq m+1} \mathbb{E}\left[e^{ie^{-ky}\xi S_T}\right].$$

From the Leibniz formula applied to the product, we have

$$\phi^{(n)}_{I(V^{\uparrow})}(\xi) = \sum_{k=0}^{n} C_n^k P_{m,k}(\xi)\ R_m^{(n-k)}(\xi).$$

R_m is the Fourrier transform of a random variable that admits moments of any positive order (because it is a convolution factors of $I(V^{\uparrow})$), so its derivatives at any order are bounded. From (5.64) we thus get that

$$\phi^{(n)}_{I(V^{\uparrow})}(\xi) = \underset{|\xi|\to+\infty}{\mathcal{O}} \left(|\xi|^{-(m-n)\delta}\right).$$

As m is arbitrary, $\phi^{(n)}_{I(V^{\uparrow})}$ goes to 0 faster than any negative power of $|\xi|$. Therefore, $\phi_{I(V^{\uparrow})}$ belongs to Schwartz space and so does the density of $I(V^{\uparrow})$, since Schwartz space is stable by Fourier transform.

□

Remark 5.4. *The case where V has bounded variation is not contained in Theorem 1.8. Moreover, Remark 2.2 shows that the law of S_T (S_T being as in Lemma 2.1) has an atom at 0 if V has bounded variations, so there is no hope to generalize our proof of Theorem 1.8 to this case.*

We also prove Corollary 1.9.

Proof. of Corollary 1.9

Since V drifts to $+\infty$ we have $V^{\sharp} = V$ so the expression (4.32) in the proof of Proposition 4.1 tells us that $I(V^{\uparrow})$ is a convolution factor of $I(V)$. Now, under the assumptions of the corollary, Theorem 1.8 applies and we get the regularity of the density of $I(V)$ thanks to the boundedness of the derivatives of the density of $I(V^{\uparrow})$ and the differentiation under the integral sign theorem. We get the convergence to 0 at $+\infty$ of the derivatives of the density of $I(V)$ thanks to the boundedness of the derivatives of the density of $I(V^{\uparrow})$ and the dominated convergence theorem. The convergence to 0 at 0 of the derivatives of the density of $I(V)$ comes from the fact that this density is of class C^{∞} and null on $]-\infty, 0[$.

□

6. The spectrally positive case

We now make a brief study of the exponential functional of $Z^{\uparrow}$ where Z is a spectrally positive Lévy process drifting to $+\infty$. If Z is a subordinator, then it stays positive and $I(Z^{\uparrow})$ is only $I(Z)$ which is already known to be finite and have some finite exponential moments (see for example Theorem 2 in [4]), so Theorem 1.11 is already known in this case.

We thus assume that Z is not a subordinator. Since, in this case, $-Z$ is spectrally negative and not the opposite of a subordinator (then, we denote by κ the non-trivial zero of Ψ_{-Z}), it is regular for $]0, +\infty[$ according to Theorem 1 page 189 in [3], so Z is regular for $]-\infty, 0[$. Moreover, Z drifts to $+\infty$. We can thus define the Markov family $(Z_x^{\uparrow},\ x \geq 0)$ as in [7], Chapter 8. It can be seen from there that the processes such defined are Markov, have infinite life-time (this is where we need the hypothesis that Z drifts to $+\infty$) and that $Z_0^{\uparrow}$, that we denote by $Z^{\uparrow}$, is indeed well defined.

Here again, for any $x \geq 0$, the process $Z_x^{\uparrow}$ must be seen as Z conditioned to stay positive and starting from x. Note that, since Z converges almost surely to infinity, for $x > 0$, $Z_x^{\uparrow}$ is only Z_x conditioned in the usual sense to remain positive.

6.1. **Finiteness, exponential moments : Proof of Theorem 1.11.** The idea is that adding a small term of negative drift to $Z^{\uparrow}$ does not change its convergence to $+\infty$. It makes $Z^{\uparrow}$ ultimately greater than a deterministic linear function for which the exponential functional is defined and deterministically bounded. The key point is thus to control the time taken by $Z^{\uparrow}$ to become greater than the linear function once and for good. We start with the following lemma.

Lemma 6.1. *For any $y > 0$, there exists $\epsilon > 0$ and positive constants c_1 and c_2 such that*

$$\forall s > 0,\ \mathbb{P}\left(\mathcal{R}(Z_y^{\uparrow}(.) - (y + \epsilon\ .), 0) > s\right) \leq c_1 e^{-c_2 s}.$$

Proof. We fix $y > 0$. From Corollary 2 page 190 in [3], a spectrally negative Lévy process X drifts to $-\infty$ if and only if $\mathbb{E}[X(1)] < 0$. Z is a spectrally positive Lévy process drifting to $+\infty$ so taking the dual in the theorem we get $\mathbb{E}[Z(1)] > 0$. Now $\mathbb{E}[(Z - \epsilon.)(1)] = \mathbb{E}[Z(1)] - \epsilon$ which is positive for ϵ chosen small enough. Still taking the dual in Corollary 2 page 190 in [3], this implies that $Z - \epsilon.$ is also a spectrally positive Lévy process that dirfts to $+\infty$ and which is not a subordinator. We have

$$\begin{aligned}\mathbb{P}\left(\mathcal{R}(Z_y^{\uparrow}(.) - (y + \epsilon\ .), 0) > s\right) &= \mathbb{P}\left(\inf_{t\in[s,+\infty[} Z_y^{\uparrow}(t) - (y + \epsilon t) < 0\right)\\ &= \mathcal{C}\ \mathbb{P}\left(\inf_{t\in[s,+\infty[} Z_y(t) - (y + \epsilon t) < 0,\ \inf_{[0,+\infty[} Z_y > 0\right),\end{aligned}$$

where $\mathcal{C} := 1/\mathbb{P}(\inf_{[0,+\infty[} Z_y > 0) = 1/(1 - e^{-\kappa y})$. This comes from the fact that $Z_y^{\uparrow}$ is only Z_y conditioned to stay positive in the usual sense. Now, noting that $Z_y = y + Z$, we bound the above quantity by

$$\begin{aligned}&\mathcal{C}\ \mathbb{P}\left(\inf_{t\in[s,+\infty[} Z(t) - \epsilon t < 0\right) = \mathcal{C}\ \mathbb{P}\left(\sup_{t\in[s,+\infty[} -Z(t) + \epsilon t > 0\right)\\ &\leq \mathcal{C}\ \mathbb{P}\left(\sup_{t\in[s,+\infty[} -Z(t) + \epsilon t > 0,\ -Z(s) + \epsilon s \leq 0\right) + \mathcal{C}\ \mathbb{P}\left(-Z(s) + \epsilon s > 0\right)\\ &= \mathcal{C}\ \mathbb{P}\left(\sup_{t\in[0,+\infty[} -Z^s(t) + \epsilon t > -\left(-Z(s) + \epsilon s\right), -Z(s) + \epsilon s \leq 0\right) + \mathcal{C}\ \mathbb{P}\left(-Z(s) + \epsilon s > 0\right).\end{aligned} \tag{6.65}$$

From the independency of increments, the process $-Z^s + \epsilon.$ is equal in law to $-Z + \epsilon.$ and independent from $-Z(s) + \epsilon s$. From Theorem 1 page 189 in [3], the supremum over $[0, +\infty[$ of the process $-Z^s + \epsilon.$ follows an exponential law with parameter α, where α is the non-trivial zero of $\Psi_{-Z+\epsilon.}$. From this, combined with the independence from $-Z(s) + \epsilon s$, (6.65) becomes

$$\begin{aligned}&\mathcal{C}\ \mathbb{E}\left(e^{\alpha(-Z(s)+\epsilon s)}\mathbb{1}_{\{-Z(s)+\epsilon s\leq 0\}}\right) + \mathcal{C}\ \mathbb{P}\left(-Z(s) + \epsilon s > 0\right)\\ &\leq \mathcal{C}\ \mathbb{E}\left(e^{\alpha(-Z(s)+\epsilon s)/2}\mathbb{1}_{\{-Z(s)+\epsilon s\leq 0\}}\right) + \mathcal{C}\ \mathbb{P}\left(e^{\alpha(-Z(s)+\epsilon s)/2} > 1\right),\end{aligned}$$

from the decreases of negative exponential and composing by function $x \mapsto \exp(\frac{\alpha}{2}x)$ in the probability of the second term,

$$\leq\ \mathcal{C}\ \mathbb{E}\left(e^{\alpha/2(-Z(s)+\epsilon s)}\right) + \mathcal{C}\ \mathbb{P}\left(e^{\alpha(-Z(s)+\epsilon s)/2} > 1\right) \leq\ 2\mathcal{C}\ \mathbb{E}\left(e^{\alpha/2(-Z(s)+\epsilon s)}\right),$$

where we used Markov inequality in the second term,

$$= 2\mathcal{C}\ e^{s\Psi_{-Z+\epsilon.}(\alpha/2)}.$$

As $\Psi_{-Z+\epsilon.}$ is negative on $]0, \alpha[$, we get the result with $c_1 = 2\mathcal{C}$ and $c_2 = -\Psi_{-Z+\epsilon.}(\alpha/2)$.

$\square$

We can now prove Theorem 1.11.

Proof. of Theorem 1.11

We fix $y > 0$. Let m_y be the point where the process $Z_y^\uparrow$ reaches its infimum, $m_y := \sup\{s \geq 0,\ Z_y^\uparrow(s-) \wedge Z_y^\uparrow(s) = \inf_{[0,+\infty[} Z_y^\uparrow\}$. Note that from the absence of negative jumps the infimum is always reached at least at m_y- so $Z_y^\uparrow(m_y-) = \inf_{[0,+\infty[} Z_y^\uparrow$. In order to get $Z^\uparrow$ from $Z_y^\uparrow$, we use the decomposition given by Theorem 24 in [7], that is :

- The two processes $\left(Z_y^\uparrow(m_y+s) - Z_y^\uparrow(m_y-),\ s \geq 0\right)$ and $\left(Z_y^\uparrow(s),\ 0 \leq s < m_y\right)$ are independent,
- $\left(Z_y^\uparrow(m_y+s) - Z_y^\uparrow(m_y-),\ s \geq 0\right)$ is equal in law to $Z^\uparrow$.

Now,

$$\begin{aligned} I(Z_y^\uparrow) &= \int_0^{m_y} e^{-Z_y^\uparrow(u)} du + \int_{m_y}^{+\infty} e^{-Z_y^\uparrow(u)} du \\ &= \int_0^{m_y} e^{-Z_y^\uparrow(u)} du + e^{-Z_y^\uparrow(m_y-)} \int_0^{+\infty} e^{-(Z_y^\uparrow(m_y+u) - Z_y^\uparrow(m_y-))} du \\ &\geq e^{-y} \int_0^{+\infty} e^{-(Z_y^\uparrow(m_y+u) - Z_y^\uparrow(m_y-))} du, \end{aligned}$$

because almost surely $Z_y^\uparrow(m_y-) \leq y$,

$$\stackrel{\mathcal{L}}{=} e^{-y} I(Z^\uparrow),$$

because of the above decomposition. We thus get

$$I(Z^\uparrow) \stackrel{sto}{\leq} e^y I(Z_y^\uparrow), \tag{6.66}$$

so we only need to prove the result for $I(Z_y^\uparrow)$. We now choose $\epsilon > 0$ as in Lemma 6.1. We have

$$\begin{aligned} 0 \leq I(Z_y^\uparrow) &= \int_0^{\mathcal{R}\left(Z_y^\uparrow(.)-(y+\epsilon\ .),0\right)} e^{-Z_y^\uparrow(t)} dt + \int_{\mathcal{R}\left(Z_y^\uparrow(.)-(y+\epsilon\ .),0\right)}^{+\infty} e^{-Z_y^\uparrow(t)} dt \\ &\leq \mathcal{R}\left(Z_y^\uparrow(.) - (y+\epsilon\ .), 0\right) + \int_{\mathcal{R}\left(Z_y^\uparrow(.)-(y+\epsilon\ .),0\right)}^{+\infty} e^{-Z_y^\uparrow(t)} dt, \end{aligned}$$

but for $t \geq \mathcal{R}(Z_y^\uparrow(.) - (y+\epsilon\ .), 0)$ we have $Z_y^\uparrow(t) \geq y + \epsilon t$, so

$$\begin{aligned} 0 \leq I(Z_y^\uparrow) &\leq \mathcal{R}\left(Z_y^\uparrow(.) - (y+\epsilon\ .), 0\right) + \int_{\mathcal{R}\left(Z_y^\uparrow(.)-(y+\epsilon\ .),0\right)}^{+\infty} e^{-y-\epsilon t} dt \\ &\leq \mathcal{R}\left(Z_y^\uparrow(.) - (y+\epsilon\ .), 0\right) + \int_0^{+\infty} e^{-y-\epsilon t} dt \\ &= \mathcal{R}\left(Z_y^\uparrow(.) - (y+\epsilon\ .), 0\right) + e^{-y}/\epsilon. \end{aligned}$$

From Lemma 6.1, this is almost surely finite and admits some finite exponential moments. Thanks to (6.66), we have the same for $I(Z^\uparrow)$, which is the expected result.

□

6.2. **Tails at** 0 **of** $I(Z^\uparrow)$ **: Proof of Theorem 1.12.** We need an analogous of Lemma 2.5 in order to compare, as we did in subsection 4.1, the exponential functionals $I(Z^\uparrow)$ and $I(Z)$. We define m, the point where the process Z reaches its infimum : $m := \sup\{s \geq 0,\ Z(s-) \wedge Z(s) = \inf_{[0,+\infty[} Z\}$. Here again, from the absence of negative jumps, the infimum is always reached at least at $m-$ so $Z(m-) = \inf_{[0,+\infty[} Z$.

Lemma 6.2. *If Z has unbounded variations, then $Z(m+.) - Z(m-)$ has the same law as $Z^\uparrow$.*

Proof. $Z(m+.) - Z(m-)$ is only the infinite excursion of the post-infimum process $Z - \underline{Z}$, so we only need to prove that this infinite excursion has the same law as $Z^\uparrow$, and for this we want to apply Proposition 4.7 of [8]. We already know that, because it is spectrally positive, Z is regular for $]-\infty,0[$. Taking the dual of the process in Corollary 5 page 192 in [3], we get the regularity of $\{0\}$ and $]0,+\infty[$ for Z, thanks to the hypothesis of unbounded variations. The hypothesis of the proposition in [8] are thus fulfilled.

Let $\mathcal{N}$ denote the excursion measure of the Markov process $Z - \underline{Z}$ and (L^{-1}, U) the ladder process of Z : L^{-1} is the inverse of the local time at 0 of $Z - \underline{Z}$ and for any positive t, $U(t) = \underline{Z}(L^{-1}(t))$.

We denote by $\mathcal{U}$ the potential measure of U and, since $-Z$ is spectrally negative, the formula page 191 in [3] applies and yields that $\mathcal{U}(]-x,0]) = (1-e^{-\kappa x})/\kappa$ for any $x \geq 0$. Proposition 4.7 of [8] tells us that for any positive measurable function G defined on the space of càd-làg functions from $[0,+\infty[$ to $\mathbb{R}$ with finite lifetime, we have

$$\begin{aligned}\mathbb{E}\left[G\left((Z^\uparrow(s))_{0\leq s\leq \tau(Z^\uparrow,h)}\right)\right] &= \mathcal{N}\left(G\left((\xi(s))_{0\leq s\leq\tau(\xi,h)}\right)\mathcal{U}(]-\xi(\tau(\xi,h)),0])\,|\tau(\xi,h)<\infty\right)\\ &\quad\times\mathcal{N}\left(\xi,\tau(\xi,h)<\infty\right)\\ &= c_h\mathcal{N}\left((1-e^{-\kappa\xi(\tau(\xi,h))})G\left((\xi(s))_{0\leq s\leq\tau(\xi,h)}\right)|\tau(\xi,h)<\infty\right),\end{aligned} \tag{6.67}$$

replacing $\mathcal{U}$ by its expression and where we set $c_h := \mathcal{N}\left(\xi,\tau(\xi,h)<\infty\right)/\kappa$. Let ξ_∞ denote the infinite excursion of $Z - \underline{Z}$, then, for any positive measurable function F, we get that

$$\begin{aligned}\mathbb{E}\left[F\left((\xi_\infty(s))_{0\leq s\leq\tau(\xi_\infty,h)}\right)\right] &= \mathcal{N}\left(\mathbb{P}\left(\inf_{[0,+\infty[} Z_{\xi(\tau(\xi,h))}>0\right)F\left((\xi(s))_{0\leq s\leq\tau(\xi,h)}\right)|\tau(\xi,h)<\infty\right)\\ &= \mathbb{E}\left[\frac{\mathbb{P}\left(\inf_{[0,+\infty[} Z_{Z^\uparrow(\tau(Z^\uparrow,h))}>0\right)}{c_h(1-e^{-\kappa Z^\uparrow(\tau(Z^\uparrow,h))})}F\left((Z^\uparrow(s))_{0\leq s\leq\tau(Z^\uparrow,h)}\right)\right],\end{aligned} \tag{6.68}$$

where we used (6.67) with

$$G\left((\xi(s))_{0\leq s\leq\tau(\xi,h)}\right) := \frac{\mathbb{P}\left(\inf_{[0,+\infty[} Z_{\xi(\tau(\xi,h))}>0\right)}{c_h(1-e^{-\kappa\xi(\tau(\xi,h))})}\times F\left((\xi_\infty(s))_{0\leq s\leq\tau(\xi_\infty,h)}\right).$$

Il follows from (6.68) and from $Z^\uparrow(\tau(Z^\uparrow,h)) \geq h$ that, $\mathbb{P}(\inf_{[0,+\infty[} Z_{\xi(\tau(\xi,h))} > 0)/c_h(1 - e^{-\kappa\xi(\tau(\xi,h))})$ is a bounded martingale with respect to the filtration $\mathcal{F}_h := \sigma(Z^\uparrow(s),\ 0 \leq s \leq \tau(Z^\uparrow,h))$ and that it converges almost surely to some constant. As a consequence, this quantity is almost surely equal to 1 for any positive h, hence,

$$\forall h>0,\ \mathbb{E}\left[F\left((\xi_\infty(s))_{0\leq s\leq\tau(\xi_\infty,h)}\right)\right] = \mathbb{E}\left[F\left((Z^\uparrow(s))_{0\leq s\leq\tau(Z^\uparrow,h)}\right)\right],$$

so the infinite excursion of $Z - \underline{Z}$ indeed has the same law as $Z^{\uparrow}$.

$\square$

We can now prove Theorem 1.12.

Proof. of Theorem 1.12

We have

$$\begin{aligned}
I(Z) &= \int_0^m e^{-Z(t)}dt + \int_m^{+\infty} e^{-Z(t)}dt \\
&= \int_0^m e^{-Z(t)}dt + e^{-Z(m-)}\int_m^{+\infty} e^{-(Z(m+t)-Z(m-))}dt \\
&\geq \int_m^{+\infty} e^{-(Z(m+t)-Z(m-))}dt,
\end{aligned}$$

because almost surely $Z(m-) \leq 0$. Since Z has unbounded variations, we can use Lemma 6.2 which tells us that the last term is equal in law to $I(Z^{\uparrow})$. We thus get

$$\mathbb{P}(I(Z) \leq x) \leq \mathbb{P}(I(Z^{\uparrow}) \leq x), \tag{6.69}$$

so we only need to prove the result for $I(Z)$. Obtaining (6.69) is the only thing for which we need the hypothesis of unbounded variation in this proof. The result that we now prove for $I(Z)$ is thus true without this hypothesis.

Let (Q, γ, ν) be the generating triplet of Z in the Lévy-Khintchine representation. Since ν is non zero, there exist $0 < \gamma_1 < \gamma_2 < +\infty$ such that $\nu([\gamma_1, \gamma_2[) > 0$. Then, for $\eta \in]0, 1[$ we define

$$\nu^{\eta,1} := \eta\nu(. \cap [\gamma_1, \gamma_2[) \quad \text{and} \quad \nu^{\eta,2} := \nu - \eta\nu(. \cap [\gamma_1, \gamma_2[),$$

and set $Z^{\eta,1}$ and $Z^{\eta,2}$ to be two independent Lévy processes which generating triplets are respectively $(0, 0, \nu^{\eta,1})$ and $(Q, \gamma, \nu^{\eta,2})$. We have $\nu = \nu^{\eta,1} + \nu^{\eta,2}$ so according to the Lévy-Khintchine formula,

$$Z \overset{\mathcal{L}}{=} Z^{\eta,1} + Z^{\eta,2}.$$

According to Corollary 2 page 190 in [3], a spectrally negative Lévy process X drifts to $-\infty$ if and only if $\mathbb{E}[X(1)] < 0$. Z is a spectrally positive Lévy process drifting to $+\infty$ so taking the dual in the theorem we get $\mathbb{E}[Z(1)] > 0$. Now since $\mathbb{E}[Z(1)] = \mathbb{E}[Z^{\eta,1}(1)] + \mathbb{E}[Z^{\eta,2}(1)]$ and $\mathbb{E}[Z^{\eta,1}(1)] < \gamma_2\eta\nu([\gamma_1, \gamma_2[)$, we have that $\mathbb{E}[Z^{\eta,2}(1)]$ is positive for η small enough. Still taking the dual in Corollary 2 page 190 in [3], this implies that $Z^{\eta,2}$ drifts to $+\infty$ for η small enough. We thus choose such an $\eta_0 \in]0, 1[$ and denote by m_2 the point at which $Z^{\eta_0,2}$ reaches its minimum.

$Z^{\eta_0,1}$ is a compound Poisson process, we define N the counting process of its jumps : $N(t) := \sharp\left\{s \in [0, t],\ Z^{\eta_0,1}(s) - Z^{\eta_0,1}(s-) > 0\right\}$. N is thus a standard Poisson process with parameter $c_0 := \eta_0\nu([\gamma_1, \gamma_2[)$ and it is independent of $Z^{\eta,2}$. We have :

$$\begin{aligned}
I(Z) &= \int_0^{+\infty} e^{-\left(Z^{\eta_0,1}(t)+Z^{\eta_0,2}(t)\right)}dt \\
&\leq e^{-Z^{\eta_0,2}(m_2)}\int_0^{+\infty} e^{-Z^{\eta_0,1}(t)}dt \\
&\leq e^{-Z^{\eta_0,2}(m_2)}\int_0^{+\infty} e^{-\gamma_1 N(t)}dt \\
&= e^{-Z^{\eta_0,2}(m_2)}I(\gamma_1 N),
\end{aligned}$$

so, from the independence between the two factors :

$$\mathbb{P}\left(I(\gamma_1 N) \leq x/2\right) \times \mathbb{P}\left(e^{-Z^{\eta_0,2}(m_2)} \leq 2\right) \leq \mathbb{P}\left(I(Z) \leq x\right). \tag{6.70}$$

We put $c_1 := \mathbb{P}(e^{-Z^{\eta_0,2}(m_2)} \leq 2) > 0$. Now by a property of standard Poisson processes, it is easy to see that $I(\gamma_1 N)$ has the same law as

$$\frac{1}{c_0}\sum_{k=0}^{+\infty} e^{-\gamma_1 k} e_k,$$

where $(e_k)_{k\in\mathbb{N}}$ is a sequence of *iid* exponential random variable with parameter 1. This allows us to compute the Laplace transform of $I(\gamma_1 N)$:

$$\forall \lambda \geq 0,\ \mathbb{E}\left[e^{-\lambda I(\gamma_1 N)}\right] = \prod_{k=0}^{+\infty} \frac{1}{1+\frac{\lambda}{c_0}e^{-\gamma_1 k}}.$$

We put $K(\lambda) := \min\{k\in\mathbb{N},\ \lambda e^{-\gamma_1 k} \leq 1\}$ and taking the logarithm we get

$$\begin{aligned}
\log\left(\mathbb{E}\left[e^{-\lambda I(\gamma_1 N)}\right]\right) &= -\sum_{k=0}^{+\infty} \log\left(1+\frac{\lambda}{c_0}e^{-\gamma_1 k}\right) \\
&\geq -K(\lambda)\log\left(1+\frac{\lambda}{c_0}\right) - \sum_{k\geq K(\lambda)} \log\left(1+\frac{\lambda}{c_0}e^{-\gamma_1 k}\right) \\
&\geq -K(\lambda)\log\left(1+\frac{\lambda}{c_0}\right) - \sum_{k\geq 0} \log\left(1+\frac{1}{c_0}e^{-\gamma_1 k}\right).
\end{aligned}$$

Now, since $K(\lambda) \underset{\lambda\to+\infty}{\sim} \log(\lambda)/\gamma_1$, we get that

$$\log\left(\mathbb{E}\left[e^{-\lambda I(\gamma_1 N)}\right]\right) \geq -2\left(\log(\lambda)\right)^2/\gamma_1,.$$

for λ large enough. Now, reasoning as in the proof of Lemma 4.6 (where, from the estimate (4.37) on the Laplace transform of $\tau(Z^\sharp, y)$, we deduced a lower bound for its asymptotic tail at 0) we get

$$e^{-c_2(\log(x))^2} \leq \mathbb{P}\left(I(\gamma_1 N) \leq x\right),$$

for some positive constant c_2. Combining with (6.69) and (6.70), we get the sought result.

□

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