Controllability of a 2 x 2 parabolic system by one force with space-dependent coupling term of order one

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Abstract

This paper is devoted to the controllability of linear systems of two coupled parabolic equations when the coupling involves a space dependent first order term. This system is set on an bounded interval $I \subset\subset \mathbb{R}$, and the first equation is controlled by a force supported in a subinterval of I or on the boundary. In the case where the intersection of the coupling and control domains is nonempty, we prove null controllability at any time. Otherwise, we provide a minimal time for null controllability. Finally we give a necessary and sufficient condition for the approximate controllability. The main technical tool for obtaining these results is the moment method.

1 Introduction and main results

Let $T > 0$, $\omega := (a, b) \subseteq (0, \pi)$ and $Q_T := (0, \pi) \times (0, T)$. We consider in the present paper the following distributed control system

$$\begin{cases} \partial_t y_1 - \partial_{xx} y_1 = \mathbb{1}_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + p(x)\partial_x y_1 + q(x) y_1 = 0 & \text{in } Q_T, \\ y_1(0, \cdot) = y_1(\pi, \cdot) = y_2(0, \cdot) = y_2(\pi, \cdot) = 0 & \text{on } (0, T), \\ y_1(\cdot, 0) = y_1^0,\ y_2(\cdot, 0) = y_2^0 & \text{in } (0, \pi) \end{cases} \tag{1.1}$$

and boundary control system

$$\begin{cases} \partial_t z_1 - \partial_{xx} z_1 = 0 & \text{in } Q_T, \\ \partial_t z_2 - \partial_{xx} z_2 + p(x)\partial_x z_1 + q(x) z_1 = 0 & \text{in } Q_T, \\ z_1(0, \cdot) = u,\ z_1(\pi, \cdot) = z_2(0, \cdot) = z_2(\pi, \cdot) = 0 & \text{on } (0, T), \\ z_1(\cdot, 0) = z_1^0,\ z_2(\cdot, 0) = z_2^0 & \text{in } (0, \pi), \end{cases} \tag{1.2}$$

where $y^0 := (y_1^0, y_2^0) \in L^2(0, \pi)^2$ and $z^0 := (z_1^0, z_2^0) \in H^{-1}(0, \pi)^2$ are the initial conditions, $v \in L^2(Q_T)$ and $u \in L^2(0, T)$ are the controls, $p \in W^1_\infty(0, \pi)$, $q \in L^\infty(0, \pi)$.

It is known (see [20] (resp. [16])) that for given initial data $y^0 \in L^2(0, \pi)^2$ (resp. $z^0 \in H^{-1}(0, \pi)^2$) and a control $v \in L^2(Q_T)$ (resp. $u \in L^2(0, T)$) System (1.1) (resp. (1.2)) has a unique solution $y = (y_1, y_2)$ (resp. $z = (z_1, z_2)$) in

$$L^2(0, T; H_0^1(0, \pi)^2) \cap \mathcal{C}([0, T]; L^2(0, \pi)^2)$$
$$(resp.\ L^2(Q_T)^2 \cap \mathcal{C}([0, T]; H^{-1}(0, \pi)^2)\ ),$$

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which depends continuously on the initial data and the control, that is

$$\|y\|_{L^2(0,T;H_0^1(0,\pi)^2)} + \|y\|_{\mathcal{C}([0,T];L^2(0,\pi)^2)} \leqslant C_T(\|y_0\|_{L^2(0,\pi)^2} + \|v\|_{L^2(Q_T)})$$
$$(resp.\ \|z\|_{L^2(Q_T)^2} + \|z\|_{\mathcal{C}([0,T];H^{-1}(0,\pi)^2)} \leqslant C_T(\|z_0\|_{H^{-1}(0,\pi)^2} + \|u\|_{L^2(0,T)})\ ).$$

Let us introduce the notion of null and approximate controllability for this kind of systems.

- System (1.1) (resp. System (1.2)) is *null controllable* at time T if for every initial condition $y^0 \in L^2(0,\pi)^2$ (resp. $z^0 \in H^{-1}(0,\pi)^2$) there exists a control $v \in L^2(Q_T)$ (resp. $u \in L^2(0,T)$) such that the solution to System (1.1) (resp. System (1.2)) satisfies

$$y(T) \equiv 0 \qquad (\text{resp. } z(T) \equiv 0) \qquad \text{in } \ (0,\pi).$$

- System (1.1) (resp. System (1.2)) is *approximately controllable* at time T if for all $\varepsilon > 0$ and all $y^0,\ y^T \in L^2(0,\pi)^2$ (resp. $z^0,\ z^T \in H^{-1}(0,\pi)^2$) there exists a control $v \in L^2(Q_T)$ (resp. $u \in L^2(0,T)$) such that the solution to System (1.1) (resp. System (1.2)) satisfies

$$\|y(T) - y^T\|_{L^2(0,\pi)^2} \leqslant \varepsilon \qquad (\text{resp. } \|z(T) - z^T\|_{H^{-1}(0,\pi)^2} \leqslant \varepsilon).$$

The main goal of this article is to provide a complete answer to the null and approximate controllability issues for System (1.1) and (1.2). For a survey and some applications in physics, chemistry or biology concerning the controllability of this kind of systems, we refer to [6]. In the last decade, many papers studied this problem, however most of them are relating to some parabolic systems with zero order coupling terms. Without first order coupling terms, some Kalman coupling conditions are made explicit in [3], [4] and [16] for distributed null controllability of systems of more than two equations with constant matrices and in higher space dimension and, in the case of time dependent matrices, some Silverman-Meadows coupling conditions are given in [3].

Concerning the null and approximate controllability of Systems (1.1) and (1.2) in the case $p \equiv 0$ and $q \not\equiv 0$ in $(0,\pi)$, a partial answer is given in [1, 2, 13, 23] under the sign condition

$$q \leqslant 0 \ \text{ or } \ q \geqslant 0 \ \text{ in } (0,\pi).$$

These results are obtained as a consequence of controllability results of a hyperbolic system using the transmutation method (see [21]). One can find a necessary and sufficient condition in [7] when

$$\int_0^\pi q(x)dx \neq 0.$$

Finally in a recent work [8, 9], a complete study for any $q \in L^\infty(0,\pi)$ is given.

Let us now remind known results concerning null controllability for systems of the following more general form. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and ω_0 an arbitrary nonempty subset of Ω. We denote by $\partial\Omega$ the boundary of Ω. Consider the system of two coupled linear parabolic equations

$$\begin{cases} \partial_t y_1 = \Delta y_1 + g_{11}\cdot\nabla y_1 + g_{12}\cdot\nabla y_2 + a_{11}y_1 + a_{12}y_2 + \mathbb{1}_{\omega_0}v & \text{in } \Omega\times(0,T), \\ \partial_t y_2 = \Delta y_2 + g_{21}\cdot\nabla y_1 + g_{22}\cdot\nabla y_2 + a_{21}y_1 + a_{22}y_2 & \text{in } \Omega\times(0,T), \\ y = 0 & \text{on } \partial\Omega\times(0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $y^0 \in L^2(\Omega)^2$, $g_{ij} \in L^\infty(\Omega\times(0,T))^N$ and $a_{ij} \in L^\infty(\Omega\times(0,T))$ for all $i,j \in \{1,2\}$.

As a particular case of the result in [17] (see also [5]), System (1.3) is null controllable whenever

$$g_{21} \equiv 0 \ \text{ and } \ (a_{21} > C \text{ or } a_{21} < -C) \quad \text{in } \omega_1 \subseteq \omega_0, \tag{1.4}$$

for a positive constant C.

In [18], the author supposes that $a_{11},\ g_{11},\ a_{22},\ g_{22}$ are constant and the first order coupling operator $g_{21}\cdot\nabla+a_{21}$ can be written as

$$g_{21}\cdot\nabla+a_{21}=P_1\circ\theta \text{ in } \Omega\times(0,T), \tag{1.5}$$

where $\theta\in\mathcal{C}^2(\overline{\Omega})$ satisfies $|\theta|>C$ in $\omega_1\subseteq\omega_0$ for a positive constant C and P_1 is given by

$$P_1:=m_0\cdot\nabla+m_1,$$

for some $m_0,m_1\in\mathbb{R}$. Moreover the operator P_1 satisfies

$$\|u\|_{H^1(\Omega)}\leqslant C\|P_1^*u\|_{L^2(\Omega)}\quad \forall u\in H_0^1(\Omega).$$

Under these assumptions, the author proves the null controllability of System (1.3) at any time.

In [10], the authors prove that the same property holds true for System (1.3) if we assume that $a_{ij}\in\mathcal{C}^4(\overline{\Omega\times(0,T)})$, $g_{ij}\in\mathcal{C}^1(\overline{\Omega\times(0,T)})^N$ for all $i,j\in\{1,2\}$, $g_{21}\in\mathcal{C}^3(\overline{\Omega\times(0,T)})$ and the geometrical condition

$$\begin{cases}\partial\omega\cap\partial\Omega \text{ contains a nonempty open subset } \gamma \text{ s.t. } \mathring{\gamma}\neq\varnothing,\\ \exists x_0\in\gamma \text{ s.t. } g_{21}(t,x_0)\cdot\nu(x_0)\neq0 \text{ for all } t\in[0,T],\end{cases} \tag{1.6}$$

where ν represents the exterior normal unit vector to the boundary $\partial\Omega$.

Lastly, for constant coefficients, it is proved in [14] that System (1.3) is null/approximately controllable at any time T if and only if

$$g_{21}\neq0 \quad\text{or}\quad a_{21}\neq0.$$

In [14], the authors give also a condition of null/approximate controllability in dimension one which can be written for system (1.1) as: $p\in\mathcal{C}^2(\omega_0)$, $q\in\mathcal{C}^3(\omega_0)$ and

$$\begin{gathered}-4\partial_x(q)\partial_x(p)p+\partial_{xx}(q)p^2+2q\partial_x(q)p-3pq\partial_{xx}p+6q(\partial_xp)^2-2q^2\partial_xp\\ -\partial_{xxx}(p)p^2+5\partial_x(p)\partial_{xx}(p)p-4(\partial_xp)^3\neq0 \text{ in } \omega_0\end{gathered}$$

for a subinterval ω_0 of ω.

Now let us go back to Systems (1.1) and (1.2) for which we will provide a complete description of the null and approximate controllability. Our first and main result is the following

Theorem 1.1. *Let us suppose that $p\in W^1_\infty(0,\pi)\cap W^2_\infty(\omega)$, $q\in L^\infty(0,\pi)\cap W^1_\infty(\omega)$ and*

$$(\mathrm{Supp}(p)\cup\mathrm{Supp}(q))\cap\omega\neq\varnothing. \tag{1.7}$$

Then System (1.1) is null controllable at any time T.

Let us compare this result with the previously described results to highlight our main contribution:

1. Even though System (1.1) is considered in one space dimension, we remark first that our coupling operator has a more general form than the one in (1.5) assumed by Guerrero [18]. Moreover unlike [14], its coefficients are non-constant with respect to the space variable.

2. We do not have the geometrical restriction (1.6) assumed in [10] by A. Benabdallah and al. More precisely we do not require the control support to be a neighbourhood of a part of the boundary.

For all $k \in \mathbb{N}^*$, we denote by $\varphi_k : x \mapsto \sqrt{\frac{2}{\pi}} \sin(kx)$ the eigenvector of the Laplacian operator, with Dirichlet boundary condition, and consider the two following quantities

$$\begin{cases} I_{a,k}(p,q) := \displaystyle\int_0^a \left(q - \tfrac{1}{2}\partial_x p\right) \varphi_k^2, \\ I_k(p,q) := \displaystyle\int_0^\pi \left(q - \tfrac{1}{2}\partial_x p\right) \varphi_k^2, \end{cases} \tag{1.8}$$

for all $k \in \mathbb{N}^*$. Combined with the Hautus test ([15, Cor. 3.3] or Th. 5.1 in the present paper), Theorem 1.1 leads to the following characterization:

Theorem 1.2. *Let us suppose that $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$ and $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$. System* (1.1) *is approximately controllable at time T if and only if*

$$(\operatorname{Supp}(p) \cup \operatorname{Supp}(q)) \cap \omega \neq \varnothing \tag{1.9}$$

or

$$|I_k(p,q)| + |I_{a,k}(p,q)| \neq 0 \text{ for all } k \in \mathbb{N}^*. \tag{1.10}$$

This last result recovers the case $p \equiv 0$ studied in [11] for Supp $(q) \cap \omega = \varnothing$, where the authors use also the Hautus test. In [19], the authors prove the approximate controllability at any time T of System (1.1) under the condition $p \equiv 0$ and $q \equiv \mathbb{1}_{\omega_0}$ with ω_0 a nonempty open subset of $(0,\pi)$, which implies (1.10).

Remark 1. We will see in the prove of Theorems 1.1 and 1.2 that only the following regularity are needed for p and q

$$\begin{cases} p \in W^1_\infty(0,\pi) \cap W^2_\infty(\widetilde{\omega}), \\ q \in L^\infty(0,\pi) \cap W^1_\infty(\widetilde{\omega}), \end{cases}$$

for an open subinterval $\widetilde{\omega}$ of ω. These hypotheses are used in Definition (1.8) of $I_k(p,q)$ and $I_{a,k}(p,q)$ and the change of unknown described in Section 3.2. For more general coupling terms, these control problems are open.

When the supports of the control and the coupling terms are disjoint in System (1.1), following the ideas in [9] where the authors studied the case $p \equiv 0$, we obtain a minimal time of null controllability:

Theorem 1.3. *Let $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$. Suppose that Condition* (1.10) *holds and*

$$(\operatorname{Supp}(p) \cup \operatorname{Supp}(q)) \cap \omega = \varnothing. \tag{1.11}$$

Let $T_0(p,q)$ be given by

$$T_0(p,q) := \limsup_{k\to\infty} \frac{\min(-\log|I_k(p,q)|, -\log|I_{a,k}(p,q)|)}{k^2}. \tag{1.12}$$

One has

1. *If $T > T_0(p,q)$, then System* (1.1) *is null controllable at time T.*
2. *If $T < T_0(p,q)$, then System* (1.1) *is not null controllable at time T.*

Concerning the boundary controllability, in [22, Th. 3.3], using the Hautus test, the author proves that System (1.2) is approximately controllable at time T if and only if

$$I_k(p,q) \neq 0 \text{ for all } k \in \mathbb{N}^*. \tag{1.13}$$

About null controllability of System (1.2), we can again generalize the results given in [9] to obtain a minimal time:

Theorem 1.4. *Let $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$ and suppose that Condition* (1.13) *is satisfied. Let us define*

$$T_1(p,q) := \limsup_{k\to\infty} \frac{-\log|I_k(p,q)|}{k^2}. \tag{1.14}$$

One has

1. *If $T > T_1(p,q)$, then System* (1.2) *is null controllable at time T.*

2. *If $T < T_1(p,q)$, then System* (1.2) *is not null controllable at time T.*

Remark 2. A simple computation leads to the convergence of the tow sequences $(I_k(p,q))_{k\in\mathbb{N}^*}$ and $(I_{a,k}(p,q))_{k\in\mathbb{N}^*}$, more precisely

$$\lim_{k\to\infty} I_k(p,q) = I(p,q) := \frac{1}{\pi}\int_0^\pi (q - \tfrac{1}{2}\partial_x p) \text{ and } \lim_{k\to\infty} I_{a,k}(p,q) = I_a(p,q) := \frac{1}{\pi}\int_0^a (q - \tfrac{1}{2}\partial_x p).$$

Thus, we remark that $T_0(p,q) = T_1(p,q) = 0$ when

$$\int_0^\pi q \neq \frac{1}{2}\int_0^\pi \partial_x p$$

and in particular under the condition

$$q > \frac{1}{2}\partial_x p \text{ in } (0,\pi) \text{ or } q < \frac{1}{2}\partial_x p \text{ in } (0,\pi).$$

This article is organized as follows. In the first section we present some preliminary results useful to reduce the null controllability issues to the moment problem. In the second and third sections we study the null controllability issue of System (1.1) in the two cases when the intersection of the coupling and control supports is empty or not. Then we give the proof of Theorems 1.2 and 1.4 in Section 4 and 5. We finish with some comments and open problem in Section 6.

2 Preliminary results

Consider the differential operator

$$\begin{array}{rrcl} L: & D(L)\subset L^2(0,\pi)^2 & \to & L^2(0,\pi)^2 \\ & f & \mapsto & -\partial_{xx} f + A_0(p\partial_x f + qf), \end{array}$$

where the matrix A_0 is given by

$$A_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the domain of L and its adjoint L^* is given by $D(L) = D(L^*) = H^2(0,\pi)^2 \cap H^1_0(0,\pi)^2$. In section 2.1, we will first establish some properties of the operator L that will be useful for the moment method and, in section 2.2, we will recall some characterizations of the approximate and null controllability of system (1.1).

2.1 Biorthogonal basis

Let us first analyze the spectrum of the operators L and L^*.

Proposition 2.1. *For all $k \in \mathbb{N}^*$ consider the two vectors*

$$\Phi_{1,k}^* := \begin{pmatrix} \psi_k^* \\ \varphi_k \end{pmatrix}, \Phi_{2,k}^* := \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix},$$

where ψ_k^* is defined for all $x \in (0,\pi)$ by

$$\begin{cases} \psi_k^*(x) = \alpha_k^* \varphi_k(x) - \dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) + \partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]d\xi, \\ \alpha_k^* = \dfrac{1}{k}\displaystyle\int_0^\pi \int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) + \partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\varphi_k(x)d\xi dx. \end{cases}$$

One has

1. *The spectrum of L^* is given by $\sigma(L^*) = \{k^2 : k \in \mathbb{N}^*\}$.*
2. *For $k \geqslant 1$, the eigenvalue k^2 of L^* is simple if and only if $I_k(p,q) \neq 0$. In this case, $\Phi_{2,k}^*$ and $\Phi_{1,k}^*$ are respectively an eigenfunction and a generalized eigenfunction of the operator L^* associated with the eigenvalue k^2, more precisely*
$$\begin{cases} (L^* - k^2 Id)\Phi_{1,k}^* = I_k \Phi_{2,k}^*, \\ (L^* - k^2 Id)\Phi_{2,k}^* = 0. \end{cases} \tag{2.1}$$
3. *For $k \geqslant 1$, the eigenvalue k^2 of L^* is double if and only if $I_k(p,q) = 0$. In this case, $\Phi_{1,k}^*$ and $\Phi_{2,k}^*$ are two eigenfunctions of the operator L^* associated with the eigenvalue k^2, that is for $i = 1, 2$*
$$(L^* - k^2 Id)\Phi_{i,k}^* = 0.$$

Proof. The adjoint operator L^* of L is given by

$$\begin{array}{rccl} L^* : & D(L) \subset L^2(0,\pi)^2 & \to & L^2(0,\pi)^2 \\ & f & \mapsto & -\partial_{xx} f + A_0(-\partial_x(pf) + qf). \end{array}$$

We can remark first that the inverse of L^* is compact. Thus the spectrum of L^* reduces to its point spectrum. The eigenvalue problem associated with the operator L^* is

$$\begin{cases} -\partial_{xx}\psi - \partial_x(p(x)\varphi) + q(x)\varphi = \lambda\psi & \text{in } (0,\pi), \\ -\partial_{xx}\varphi = \lambda\varphi & \text{in } (0,\pi), \\ \varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0, \end{cases} \tag{2.2}$$

where $(\psi,\varphi) \in D(L^*)$ and $\lambda \in \mathbb{C}$. For $\varphi \equiv 0$ in $(0,\pi)$ and $\psi = \varphi_k$ in $(0,\pi)$, $\lambda = k^2$ is an eigenvalue of L^* and the vector $\Phi_{2,k}^* := (\varphi_k, 0)$ is an associated eigenfunction. If now $\varphi \not\equiv 0$ in $(0,\pi)$, then $\lambda = k^2$ is an eigenvalue and $\varphi = \kappa\varphi_k$ with $\kappa \in \mathbb{R}^*$. We remark that System (2.2) has a solution if and only if $I_k(p,q) = 0$. If $I_k(p,q) = 0$, $\Phi_{1,k}^* := (\psi_k^*, \varphi_k)$ is a second eigenfunction of L^* linearly independent of $\Phi_{2,k}^*$, where, applying the Fredholm alternative, ψ_k^* is the unique solution to the non-homogeneous Sturm-Liouville problem

$$\begin{cases} -\partial_{xx}\psi - k^2\psi = f \text{ in } (0,\pi), \\ \psi(0) = \psi(\pi) = 0, \end{cases} \tag{2.3}$$

with

$$f := \partial_x(p(x)\varphi_k) - q(x)\varphi_k$$

and is such that

$$\int_0^\pi \psi(x)\varphi_k(x)\,dx = 0. \tag{2.4}$$

A solution to System (2.3) can be written for all $x \in (0,\pi)$ as

$$\psi(x) = \alpha\varphi_k(x) - \frac{1}{k}\int_0^x \sin(k(x-\xi))f(\xi)d\xi,$$

with $\alpha \in \mathbb{R}$. Under Condition (2.4), we obtain the expression of ψ_k^* given in Proposition 2.1. Thus, in the case $I_k(p,q) = 0$, $\lambda = k^2$ is a double eigenvalue of L^*. Items 1 and 3 are now proved.

Let us now suppose that $I_k(p,q) \neq 0$. The eigenvalue $\lambda = k^2$ is simple, $\Phi_{2,k}^* := (\varphi_k, 0)$ is an eigenfunction and a solution $\Phi_{1,k}^* := (\psi,\varphi)$ to $(L^* - k^2 Id)\Phi_{1,k}^* = I_k(p,q)\Phi_{2,k}^*$, that is

$$\begin{cases} -\partial_{xx}\psi - \partial_x(p(x)\varphi) + q(x)\varphi = k^2\psi + I_k(p,q)\varphi_k & \text{in } (0,\pi), \\ -\partial_{xx}\varphi = k^2\varphi & \text{in } (0,\pi), \\ \varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0, \end{cases} \tag{2.5}$$

is a generalized eigenfunction of L^*. We deduce that $\varphi = \kappa\varphi_k$ in $(0,\pi)$ for a constant $\kappa \in \mathbb{R}^*$. Again System (2.5) has a solution if and only if $\kappa = 1$. Then ψ is solution to the Sturm-Liouville problem (2.3) with

$$f = I_k(p,q)\varphi_k + \partial_x(p(x)\varphi_k) - q(x)\varphi_k$$

and satisfying (2.4). Again, using (2.4), we obtain the expression of ψ_k^* given in Proposition 2.1. □

The function ψ_k^* given in Proposition 2.1 will play an important role in this paper and we will need the following straightforward property

Lemma 2.1. *There exists a positive constant C such that*

$$|\alpha_k^*| \leqslant \frac{C}{k}, \quad \|\psi_k^*\|_{L^\infty(0,\pi)} \leqslant \frac{C}{k}, \quad \|\partial_x\psi_k^*\|_{L^\infty(0,\pi)} \leqslant C, \quad \forall k \in \mathbb{N}^*. \tag{2.6}$$

Since the eigenvalues of the operator L^* are real, we deduce that L and L^* have the same spectrum and the associated eigenspaces have the same dimension. The eigenfunctions and the generalized eigenfunctions of L can be found as previously.

Proposition 2.2. *For all $k \in \mathbb{N}^*$ consider the two vectors*

$$\Phi_{1,k} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \Phi_{2,k} := \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix},$$

where ψ_k is defined for all $x \in (0,\pi)$ by

$$\begin{cases} \psi_k(x) := \alpha_k\varphi_k(x) - \dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) - p(\xi)\partial_x(\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]d\xi, \\ \alpha_k := \dfrac{1}{k}\displaystyle\int_0^\pi\int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) - p(\xi)\partial_x(\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\varphi_k(x)d\xi dx, \end{cases}$$

One has

1. *The spectrum of L is given by $\sigma(L) = \sigma(L^*) = \{k^2 : k \in \mathbb{N}^*\}$.*
2. *For $k \geqslant 1$, the eigenvalue k^2 of L is simple if and only if $I_k(p,q) \neq 0$. In this case, $\Phi_{1,k}$ and $\Phi_{2,k}$ are an eigenfunction and a generalized eigenfunction of the operator L associated with the eigenvalue k^2, more precisely*

$$\begin{cases} (L - k^2 Id)\Phi_{1,k} = 0, \\ (L - k^2 Id)\Phi_{2,k} = I_k\Phi_{1,k}. \end{cases} \tag{2.7}$$

3. *For $k \geqslant 1$, the eigenvalue k^2 of L is double if and only if $I_k(p,q) = 0$. In this case, $\Phi_{1,k}$ and $\Phi_{2,k}$ are two eigenfunctions of the operator L associated with the eigenvalue k^2, that is for $i = 1, 2$*
$$(L - k^2 Id)\Phi_{i,k} = 0.$$

Lemma 2.3 and Corollary 2.6 in [9] can be adapted easily to prove the following property.

Property 2.1. *Consider the families*

$$\mathcal{B} := \{\Phi_{1,k}, \Phi_{2,k} : k \in \mathbb{N}^*\} \quad \text{and} \quad \mathcal{B}^* := \left\{\Phi^*_{1,k}, \Phi^*_{2,k} : k \in \mathbb{N}^*\right\}.$$

Then

1. *The sequences $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal Riesz bases of $L^2(0,\pi)^2$.*
2. *The sequence $\mathcal{B}^*$ is a Schauder basis of $H^1_0(0,\pi)^2$ and $\mathcal{B}$ is its biorthogonal basis in $H^{-1}(0,\pi)$.*

2.2 Duality

As it is well known, the controllability has a dual concept called *observability* (see for instance [6], [12, Th. 2.44, p. 56–57]). Consider the dual system associated with System (1.1)

$$\begin{cases} -\partial_t \theta - \partial_{xx}\theta + A_0^*(-\partial_x(p(x)\theta) + q(x)\theta) = 0 & \text{in } Q_T, \\ \theta(0,\cdot) = \theta(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,T) = \theta^0 & \text{in } (0,\pi), \end{cases} \tag{2.8}$$

where $\theta^0 \in L^2(0,\pi)^2$. Let B the matrix given by

$$B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The approximate controllability is equivalent to a *unique continuation property*:

Proposition 2.3. 1. *System (1.1) is approximately controllable at time T if and only if for all initial condition $\theta^0 \in L^2(0,\pi)^2$ the solution to System (2.8) satisfies the unique continuation property*
$$\mathbb{1}_\omega B^*\theta \equiv 0 \text{ in } Q_T \;\Rightarrow\; \theta \equiv 0 \text{ in } Q_T. \tag{2.9}$$

2. *System (1.2) is approximately controllable at time T if and only if for all initial condition $\theta^0 \in H^1_0(0,\pi)^2$ the solution to System (2.8) satisfies the unique continuation property*
$$B^*\partial_x\theta(0,t) \equiv 0 \text{ in } (0,T) \;\Rightarrow\; \theta \equiv 0 \text{ in } Q_T. \tag{2.10}$$

The null controllability is characterized by an *observability inequality*:

Proposition 2.4. 1. *System (1.1) is null controllable at time T if and only if there exists a constant C_{obs} such that for all initial condition $\theta^0 \in L^2(0,\pi)^2$ the solution to System (2.8) satisfies the observability inequality*
$$\|\theta(0)\|^2_{L^2(0,\pi)^2} \leqslant C_{obs} \iint_{Q_T} |\mathbb{1}_\omega B^*\theta|^2 dxdt. \tag{2.11}$$

2. *System (1.1) is null controllable at time T if and only if there exists a constant C_{obs} such that for all initial condition $\theta^0 \in H^1_0(0,\pi)^2$ the solution to System (2.8) satisfies the observability inequality*
$$\|\theta(0)\|^2_{H^1_0(0,\pi)^2} \leqslant C_{obs} \int_0^T |B^*\partial_x\theta(0,t)|^2 dt. \tag{2.12}$$

3 Proof of Theorem 1.1

In this section, we first establish the moment problem related to the null controllability for System (1.1) and then we will solve it in section 3.2. The strategy involves finding an equivalent system (see Definition 3.1) to System (1.1), which has a associated quantity I_k satisfying "some good properties".

3.1 The moment problem

Let $y^0 := (y_1^0, y_2^0) \in L^2(0,\pi)^2$. For $i \in \{1,2\}$ and $k \in \mathbb{N}^*$, if we consider $\theta^0 := \Phi_{i,k}^*$ in the dual System (2.8), we get after an integration by part

$$\iint_{Q_T} v(x,t)\mathbb{1}_\omega B^*\theta(x,t)dxdt = \langle y(T), \Phi_{i,k}^*\rangle_{L^2(0,\pi)^2} - \langle y^0, \theta(0)\rangle_{L^2(0,\pi)^2}.$$

Since $\mathcal{B}^*$ is a Riesz basis of $L^2(0,\pi)^2$, System (1.1) is null controllable if and only if for all $y^0 \in L^2(0,\pi)^2$, there exists a control $v \in L^2(Q_T)$ such that for all $k \in \mathbb{N}^*$ and $i \in \{1,2\}$ the solution y to System (1.1) satisfies the following equality

$$\iint_{Q_T} v(x,t)\mathbb{1}_\omega B^*\theta_{i,k}(x,t)\,dx\,dt = -\langle y^0, \theta_{i,k}(0)\rangle_{L^2(0,\pi)^2}, \tag{3.1}$$

where $\theta_{i,k}$ is the solution to the dual system (2.8) with the initial condition $\theta^0 := \Phi_{i,k}^*$.

In the moment problem (3.1), we will look for a control v of the form

$$v(x,t) := f^{(1)}(x)v^{(1)}(T-t) + f^{(2)}(x)v^{(2)}(T-t) \text{ for all } (x,t) \in Q_T, \tag{3.2}$$

with $v^{(1)}, v^{(2)} \in L^2(0,T)$ and $f^{(1)}, f^{(2)} \in L^2(0,\pi)$ satisfying

$$\operatorname{Supp}\left(f^{(1)}\right),\ \ \operatorname{Supp}\left(f^{(2)}\right) \subseteq \omega.$$

The solutions $\theta_{1,k}$ and $\theta_{2,k}$ to the dual System (2.8) with the initial condition $\Phi_{1,k}^*$ and $\Phi_{2,k}^*$ are given for all $(x,t) \in Q_T$ by

$$\left\{\begin{array}{l} \theta_{1,k}(x,t) = e^{-k^2(T-t)}\left(\Phi_{1,k}^*(x) - (T-t)I_k(p,q)\Phi_{2,k}^*(x)\right), \\ \theta_{2,k}(x,t) = e^{-k^2(T-t)}\Phi_{2,k}^*(x). \end{array}\right. \tag{3.3}$$

Plugging (3.2) and (3.3) in the moment problem (3.1), we get for all $k \geqslant 1$

$$\left\{\begin{array}{l} \widetilde{f}_k^{(1)}\displaystyle\int_0^T v^{(1)}(t)e^{-k^2t}\,dt + \widetilde{f}_k^{(2)}\int_0^T v^{(2)}(t)e^{-k^2t}\,dt \\ \qquad\qquad -I_k(p,q)f_k^{(1)}\displaystyle\int_0^T v^{(1)}(t)te^{-k^2t}\,dt - I_k(p,q)f_k^{(2)}\int_0^T v^{(2)}(t)te^{-k^2t}\,dt \\ \qquad\qquad\qquad\qquad = -e^{-k^2T}\left\{y_{1,k}^0 - TI_k(p,q)y_{2,k}^0\right\}, \\ f_k^{(1)}\displaystyle\int_0^T v^{(1)}(t)e^{-k^2t}\,dt + f_k^{(2)}\int_0^T v^{(2)}(t)e^{-k^2t}\,dt = -e^{-k^2T}y_{2,k}^0, \end{array}\right.$$

where $f_k^{(i)}$, $\widetilde{f}_k^{(i)}$ and $y_{i,k}^0$ are given for all $i \in \{1,2\}$ and $k \in \mathbb{N}^*$ by

$$f_k^{(i)} := \int_0^\pi f^{(i)}(x)\varphi_k(x)dx, \quad \widetilde{f}_k^{(i)} := \int_0^\pi f^{(i)}(x)\psi_k^*(x)dx, \tag{3.4}$$

and

$$y_{i,k}^0 := \langle y^0, \Phi_{i,k}^*\rangle_{L^2(0,\pi)}. \tag{3.5}$$

In [16], the authors prove that the family $\left\{e_{1,k} := e^{-k^2 t}, e_{2,k} := te^{-k^2 t}\right\}_{k\geq 1}$ admits a biorthogonal family $\{q_{1,k}, q_{2,k}\}_{k\geq 1}$ in the space $L^2(0,T)$, *i.e.* a family satisfying

$$\int_0^T e_{i,k} q_{j,l}(t)\, dt = \delta_{ij}\delta_{kl}, \quad \forall k,l \geq 1, \quad 1 \leq i,j \leq 2. \tag{3.6}$$

Moreover for all $\varepsilon > 0$ there exists a constant $C_{\varepsilon,T} > 0$ such that

$$\|q_{i,k}\|_{L^2(0,T)} \leq C_{\varepsilon,T}\, e^{\varepsilon k^2}, \quad \forall k \geq 1, \quad i = 1,2. \tag{3.7}$$

We will look for $v^{(1)}$ and $v^{(2)}$ of the form

$$v^{(i)}(t) = \sum_{k\geqslant 1}\{v^{(i)}_{1,k} q_{1,k}(t) + v^{(i)}_{2,k} q_{2,k}(t)\}, \quad i = 1,2. \tag{3.8}$$

Thus the moment problem (3.1) can be written as

$$A_{1,k}V_{1,k} + A_{2,k}V_{2,k} = F_k, \quad \text{for all } k \geqslant 1, \tag{3.9}$$

with for all $k \in \mathbb{N}^*$

$$A_{1,k} = \begin{pmatrix} \widetilde{f}^{(1)}_k & \widetilde{f}^{(2)}_k \\ f^{(1)}_k & f^{(2)}_k \end{pmatrix}, \quad A_{2,k} = \begin{pmatrix} -I_k(p,q) f^{(1)}_k & -I_k(p,q) f^{(2)}_k \\ 0 & 0 \end{pmatrix}, \tag{3.10}$$

$$V_{1,k} := \begin{pmatrix} v^{(1)}_{1,k} \\ v^{(2)}_{1,k} \end{pmatrix}, \quad V_{2,k} := \begin{pmatrix} v^{(1)}_{2,k} \\ v^{(2)}_{2,k} \end{pmatrix} \tag{3.11}$$

and

$$F_k = \begin{pmatrix} -e^{-k^2 T}\left(y^0_{1,k} - T I_k(p,q) y^0_{2,k}\right) \\ -e^{-k^2 T} y^0_{2,k} \end{pmatrix}. \tag{3.12}$$

The next sections will be devoted to solving problem (3.9) and prove that the corresponding solution $v^{(1)},\ v^{(2)}$ belongs to $L^2(0,T)$.

3.2 Resolution of the moment problem

In this section, we will prove the null controllability of System (1.1) at any time T when the supports of p or q intersects the control domain ω. In [17], the authors obtain the null controllability of System (1.1) at any time under Condition (1.4), so we will not consider this case and we will always suppose that $|p| > C$ in $\widetilde{\omega}$ for a positive constant C and an open subinterval $\widetilde{\omega}$ of ω.

Let us first introduce the following notion of equivalent systems.

Definition 3.1. Let $p_1,\ p_2 \in W^1_\infty(0,\pi)$ and $q_1,\ q_2 \in L^\infty(0,\pi)$. Consider the systems given for $i \in \{1,2\}$ by

$$\begin{cases} \text{For given } y^0 \in L^2(0,\pi)^2,\ v \in L^2(Q_T), \\ \text{Find } y := (y_1,y_2) \in W(0,T)^2 \text{ such that :} \\ \partial_t y_1 - \partial_{xx} y_1 = \mathbb{1}_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + p_i(x)\partial_x y_1 + q_i(x) y_1 = 0 & \text{in } Q_T, \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi). \end{cases} \tag{$\mathcal{S}_i$}$$

We say that System $(\mathcal{S}_1)$ is *equivalent* to System $(\mathcal{S}_2)$ if System $(\mathcal{S}_1)$ is null controllable at time T if and only if System $(\mathcal{S}_2)$ is null controllable at time T.

Let us present the main technique used all along this section. Suppose that System (1.1) is null controllable at time T. Let v a control such that the solution y to System (1.1) verifies $y(T)=0$ in $(0,\pi)$ and $\omega_0:=(\alpha,\beta)$ a subinterval of $\omega=(a,b)$. Consider a function $\theta\in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta\equiv\kappa_1 & \text{in } (0,\alpha),\\ \theta\equiv\kappa_2 & \text{in } (\beta,\pi),\\ |\theta|>\kappa_3 & \text{in } (0,\pi), \end{cases} \tag{3.13}$$

with $\kappa_1,\ \kappa_2,\ \kappa_3\in\mathbb{R}^*_+$. Thus if we consider the change of unknown

$$\widehat{y}:=(\widehat{y}_1,y_2)\ \text{ with }\ \widehat{y}_1:=\theta^{-1}y_1, \tag{3.14}$$

then $\widehat{y}$ is solution in $L^2(0,T;H^1_0(0,\pi)^2)\cap\mathcal{C}([0,T];L^2(0,\pi)^2)$ to the system

$$\begin{cases} \partial_t\widehat{y}_1-\partial_{xx}\widehat{y}_1=\mathbb{1}_\omega\widehat{v} & \text{in } Q_T,\\ \partial_t y_2-\partial_{xx}y_2+\widehat{p}\partial_x\widehat{y}_1+\widehat{q}\widehat{y}_1=0 & \text{in } Q_T,\\ \widehat{y}(0,\cdot)=\widehat{y}(\pi,\cdot)=0 & \text{on } (0,T),\\ \widehat{y}(\cdot,0)=\widehat{y}_0 & \text{in } (0,\pi), \end{cases} \tag{3.15}$$

where the initial condition is $\widehat{y}_0:=(\theta^{-1}y^0_1,y^0_2)\in L^2(0,\pi)^2$, the control is $\widehat{v}:=-\partial_{xx}(\theta^{-1})y_1-2\partial_x(\theta^{-1})\partial_x y_1+\theta^{-1}v\in L^2(Q_T)$ and the coupling terms are given by

$$\begin{cases} \widehat{p}:=p\theta,\\ \widehat{q}:=p\partial_x\theta+q\theta. \end{cases} \tag{3.16}$$

Since θ is constant in $(0,\pi)\backslash\omega_0$, we have

$$\operatorname{Supp}\,\widehat{v}\subseteq\omega\times(0,T).$$

Since y is controlled, then $\widehat{y}$ also. The converse is clearly true: starting from the controlled System (3.15) the same process leads to the construction of a controlled solution of System (1.1). Thus through the change of unknown (3.14), following Definition 3.1, Systems (1.1) and (3.15) are equivalent.

The next main result of this section is Proposition 3.1 that will be introduced after some lemmas. The first of them is the following.

Lemma 3.1. *Let $p\in W^1_\infty(0,\pi)\cap W^2_\infty(\omega)$ and $q\in L^\infty(0,\pi)\cap W^1_\infty(\omega)$ with $|p|>C$ in an open subinterval $\widetilde{\omega}$ of ω for a positive constant C. There exists a subinterval $\omega_0:=(\alpha,\beta)\subset\widetilde{\omega}$ and a function $\theta\in W^2_\infty(0,\pi)$ satisfying (3.13) such that System (1.1) is equivalent to System (3.15) with $\widehat{q}\equiv0$ in ω_0. Moreover for all $\epsilon>0$ the interval ω_0 can be chosen in order to have for all $k\in\mathbb{N}^*$*

$$|I(p,q)-I(\widehat{p},\widehat{q})|\leqslant\varepsilon\quad\text{and}\quad|I_k(p,q)-I_k(\widehat{p},\widehat{q})|\leqslant\varepsilon. \tag{3.17}$$

Proof. Let $\omega_0:=(\alpha,\beta)$ be an interval strictly included in $\widetilde{\omega}:=(\widetilde{a},\widetilde{b})$ and $\theta\in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} p\partial_x\theta+q\theta=0 & \text{in } \omega_0,\\ \theta\equiv1 & \text{in } (0,\pi)\backslash\widetilde{\omega},\\ |\theta|>C & \text{in } (0,\pi), \end{cases} \tag{3.18}$$

for a positive constant C. In the intervals $(\widetilde{a},\alpha]$ and $[\beta,\widetilde{b})$, we can take θ of class $\mathcal{C}^\infty$ in order to have $\theta\in W^2_\infty(0,\pi)$. Thus the function θ verifies (3.13) and, following the change of unknown described in (3.14), System (1.1) is equivalent to System (3.15) with $\widehat{q}\equiv0$ in ω_0 (see (3.16)). The estimates in (3.17) are obtained taking the interval ω_0 small enough. ☐

Let us first study System (1.1) in a particular case.

Lemma 3.2. *Consider $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$ and $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$. Let us suppose that $p \equiv C \in \mathbb{R}^*$ and $q \equiv 0$ in an open subinterval $\widetilde{\omega}$ of ω. Then System (1.1) is equivalent to a system of the form (3.15) with coupling terms $\widehat{p}$, $\widehat{q}$ satisfying*

$$|I_k(\widehat{p},\widehat{q})| > C/k^6,\ \forall k \in \mathbb{N}^*.$$

To prove this result we will need this lemma:

Lemma 3.3. *Let $(u_k)_{k\in\mathbb{N}^*}$ be a real sequence. Then there exists $\kappa \in \mathbb{R}^*_+$ such that for all $k \in \mathbb{N}^*$*

$$|u_k + \kappa| \geqslant 1/k^2.$$

Proof of Lemma 3.3. By contradiction let us suppose that for all $\kappa \in \mathbb{R}^*_+$ there exists $k \in \mathbb{N}^*$ such that for all $k \in \mathbb{N}^*$

$$|u_k + \kappa| < 1/k^2.$$

Then

$$\mathbb{R}^*_+ \subseteq \bigcup_{k\in\mathbb{N}^*} (u_k - 1/k^2, u_k + 1/k^2). \tag{3.19}$$

The convergence of the series $\sum_{k\in\mathbb{N}^*} 1/k^2$ implies that the measure of the set in the right hand-side in (3.19) is finite and leads to the conclusion. □

Proof of Lemma 3.2. Let (α,β) an open subinterval of $\widetilde{\omega}$ with α and β to be determined later, $\kappa \in \mathbb{R}^*_+$ and $\theta \in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta \equiv 1 & \text{in } (0,\pi)\backslash(\alpha,\beta), \\ \theta \equiv 1 + \kappa\xi & \text{in } (\alpha,\beta), \end{cases} \tag{3.20}$$

where

$$\xi = \sin^2\left(\frac{\pi(x-\alpha)}{\beta-\alpha}\right) \text{ in } (\alpha,\beta).$$

In particular, we have $\theta \geqslant 1$ in $(0,\pi)$. Let $k \in \mathbb{N}^*$, $\widehat{y}_1 := \theta^{-1}y_1$ and $\widehat{y} := (\widehat{y}_1, y_2)$ the solution to System (3.15). For System (3.15) the quantity I_k defined in the introduction is given by

$$\begin{array}{rcl} I_k(\widehat{p},\widehat{q}) & = & \displaystyle\int_0^\pi \{\widehat{q} - \frac{1}{2}\partial_x\widehat{p}\}\varphi_k^2 \\ & = & I_k(p,q) + \kappa J_k, \end{array}$$

with $\widehat{p}$, $\widehat{q}$ given in (3.16) and J_k defined by

$$J_k := \frac{1}{2}\int_\alpha^\beta \partial_x(\xi)\varphi_k^2.$$

Then, after a simple calculation, we obtain

$$J_k = \frac{\frac{2\pi}{(\beta-\alpha)^2}}{(2k + \frac{2\pi}{\beta-\alpha})(2k - \frac{2\pi}{\beta-\alpha})} \sin(k(\beta+\alpha))\sin(k(\beta-\alpha)). \tag{3.21}$$

Let $n \in \mathbb{N}^*$ and ℓ an algebraic number of order two satisfying

$$\frac{a}{n} < \ell < \frac{b}{n+1} \text{ and } \ell \neq \frac{\pi}{j} \text{ for all } j \in \mathbb{N}^*.$$

Let us take $\alpha := n\ell$ and $\beta := (n+1)\ell$. Thus $\alpha,\ \beta \in (a,b)$ and

$$k(\beta+\alpha) = k(2n+1)\ell \ \text{ and } \ k(\beta-\alpha) = k\ell. \tag{3.22}$$

Moreover

$$\left|2k + \frac{2\pi}{\beta-\alpha}\right| \times \left|2k - \frac{2\pi}{\beta-\alpha}\right| < Rk^2,$$

with $R > 0$. Since ℓ is an algebraic number of order two, using diophantine approximations it can be proved that

$$\inf_{j\geqslant 1}(j|\sin(j\ell)|) \geqslant \gamma, \tag{3.23}$$

for a positive constant γ (see [9]). The expressions (3.21)-(3.23) give

$$|J_k| \geqslant \frac{2\pi}{(\beta-\alpha)^2}\frac{\gamma^2}{R(2n+1)k^4}. \tag{3.24}$$

Using Lemma 3.3, there exists $\kappa \in \mathbb{R}_+^*$ satisfying

$$\left|\frac{I_k(p,q)}{J_k} + \kappa\right| \geqslant 1/k^2.$$

Combining the last inequality with Estimate (3.24),

$$|I_k(\widehat{p},\widehat{q})| = |I_k(p,q) + \kappa J_k| \geqslant J_k/k^2 \geqslant C/k^6.$$

□

The next lemma is proved in [9] but, for the sake of completeness, we will include the proof in the appendix A.

Lemma 3.4. *There exist functions $f^{(1)}$, $f^{(2)} \in L^2(0,\pi)$ satisfying*

$$\text{Supp}\left(f^{(1)}\right),\ \text{Supp}\left(f^{(2)}\right) \subseteq \omega$$

and such that for all $k \in \mathbb{N}^*$

$$\begin{cases} \min\{|f_k^{(1)}|, |f_k^{(2)}|\} \geqslant \dfrac{C}{k^3}, \\ |B_k| := |\widehat{f}_k^{(1)} f_k^{(2)} - \widehat{f}_k^{(2)} f_k^{(1)}| \geqslant \dfrac{C}{k^5}, \end{cases} \tag{3.25}$$

where for $i \in \{1,2\}$ the terms $f_k^{(i)}$ and $\widehat{f}_k^{(i)}$ are given by

$$f_k^{(i)} := \int_0^\pi f^{(i)}(x)\varphi_k(x)dx \ \text{ and } \ \widehat{f}_k^{(i)} := \int_0^\pi f^{(i)}(x)\cos(kx)dx. \tag{3.26}$$

With the help of Lemma 3.4, we deduce the following proposition:

Proposition 3.1. *Consider $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$ and $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$. Let us suppose that $|p| > C$ in an open subinterval $\widetilde{\omega}$ of ω for a positive constant C. Then System (1.1) is equivalent to a system of the form (3.15) with coupling terms $\widehat{p}$, $\widehat{q}$ satisfying Condition (1.10), $T_0(\widehat{p},\widehat{q}) = 0$ and*

$$|\det A_{1,k}| \geq \frac{C_1}{k^7}\,|I_{a,k}(\widehat{p},\widehat{q})| - \frac{C_2}{k}\,|I_k(\widehat{p},\widehat{q})| \quad \forall k \in \mathbb{N}^*, \tag{3.27}$$

where C_1 and C_2 are two positive constants independent on k (the notion of equivalent systems is defined at the beginning of Section 3.2).

Proof. Using Lemma 3.1, without loss of generality, we can suppose that $q \equiv 0$ and $|p| > C$ in a subinterval $\widehat{\omega}$ of $\widetilde{\omega}$ for a positive constant C. If $\partial_x p \equiv 0$ in $\widehat{\omega}$, Lemma 3.2 leads to

$$|I_k(p,q)| \geqslant C/k^6, \quad \forall k \in \mathbb{N}^*,$$

which implies that Condition (1.10) is satisfied and the right hand-side of inequality (3.27) is negative for some appropriate constants C_1 and C_2. Otherwise, let $(\alpha,\beta) \subseteq \widehat{\omega}$ such that $\partial_x p > C$ in (α,β) or $\partial_x p < -C$ in (α,β) for a positive constant C.

The rest of the proof is divided into three steps:

(i) In a first step we will see that System (1.1) is equivalent to a system with coupling terms $p,\ q$ satisfying
$$I(p,q) := \int_0^\pi \{q - \frac{1}{2}\partial_x p\} \neq 0.$$

(ii) We will show in a second step that we can suppose that System (1.1) is equivalent to a system with coupling terms $p,\ q$ such that for a positive constant C
$$|I_k(p,q)| > C \text{ for all } k \in \mathbb{N}^* \text{ satisfying } p\varphi_k \text{ non-constant.}$$

(iii) Finally, in a third step, we will prove that System (1.1) is equivalent to a system which fulfills the three conditions described in (3.27).

Step 1: Assume that $I(p,q) = 0$ and consider $\theta \in W^2_\infty(0,\pi)$ defined in (3.20), with $\kappa := 1$. We remark that $|\theta| \geqslant 1$. If we consider the change of unknown described in (3.14), then for all $k \in \mathbb{N}^*$, using the definition of I_k, we obtain

$$\begin{array}{rcl} I_k(\widehat{p},\widehat{q}) & = & I_k(p,q) + \displaystyle\int_\alpha^\beta \{\frac{1}{2}\partial_x(\xi)p - \frac{1}{2}\xi\partial_x(p)\}\varphi_k^2 dx \\ & = & I_k(p,q) + J_k(p,q), \end{array}$$

where

$$\begin{array}{rcl} J_k(p,q) & = & \displaystyle\frac{1}{2\pi}\int_\alpha^\beta \{\partial_x(\xi)p - \xi\partial_x(p)\}\{1-\cos(2kx)\}dx \\ & \underset{k\to\infty}{\longrightarrow} & \displaystyle\frac{1}{2\pi}\int_\alpha^\beta \{\partial_x(\xi)p - \xi\partial_x(p)\}dx \\ & = & \displaystyle -\frac{1}{\pi}\int_\alpha^\beta \xi\partial_x(p)dx =: J(p,q). \end{array}$$

Using the definition of ξ, we get

$$\begin{array}{rcl} |J(p,q)| & \geqslant & \displaystyle\frac{1}{\pi}\inf_{(\alpha,\beta)}|\partial_x p|\int_\alpha^\beta \sin^2\left(\frac{\pi(x-\alpha)}{\beta-\alpha}\right)dx \\ & = & \displaystyle\frac{1}{2\pi}\inf_{(\alpha,\beta)}|\partial_x p|\int_\alpha^\beta \{1-\cos\left(\frac{2\pi(x-\alpha)}{\beta-\alpha}\right)\}dx \\ & = & \displaystyle\frac{(\beta-\alpha)}{2\pi}\inf_{(\alpha,\beta)}|\partial_x p| \neq 0. \end{array}$$

We recall that $I_k(p,q) \to I(p,q) = 0$. Thus we obtain $I_k(\widehat{p},\widehat{q}) \to I(\widehat{p},\widehat{q}) \neq 0$.

Step 2: Let us now assume that $I(p,q) \neq 0$. Using Lemma 3.1, up to the change of unknown (3.14) we can also suppose that $q \equiv 0$ in an open subinterval $\widehat{\omega}$ of $\widetilde{\omega}$. Moreover, by (3.17), the function θ and $\widehat{\omega}$ can be chosen in order to keep the quantity I different of zero. Let $(\alpha,\beta) \subseteq \widehat{\omega}$ such that $|p| > C > 0$ in (α,β). Since $I(p,q) \neq 0$ and $I_k(p,q) \to I(p,q)$, there exists $k_0 \in \mathbb{N}^*$ such that $|I_k(p,q)| > C$ for a constant $C > 0$ and all $k \geqslant k_0$. Let us define the set

$$S_0 := \{k \in \mathbb{N}^* : I_k(p,q) = 0 \text{ and } p\varphi_k \text{ non}-\text{constant in } (\alpha,\beta)\}$$

and $M := \#S_0 < \infty$. Let $\theta \in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta = 1 + \sum_{m=1}^M \xi_m, & \\ \xi_m \in W^2_\infty(0,\pi), & \text{for all } m \in \{1,...,M\}, \\ \mathrm{Supp}(\xi_m) \subseteq (\alpha,\beta), & \text{for all } m \in \{1,...,M\}, \\ |\theta| > C > 0, & \end{cases}$$

where $\xi_1, ..., \xi_M$ are to be determined. Again, if we consider the change of unknown (3.14), then for all $k \in \mathbb{N}^*$, using the definition of I_k, we obtain

$$\begin{aligned} I_k(\widehat{p},\widehat{q}) &= I_k(p,q) + \sum_{m=1}^M \int_\alpha^\beta \{\frac{1}{2}\partial_x(\xi_m)p - \frac{1}{2}\xi_m\partial_x(p)\}\varphi_k^2 dx \\ &=: I_k(p,q) + \sum_{m=1}^M J_{m,k}(p,q). \end{aligned}$$

The goal is to choose the functions $\xi_1, ..., \xi_M$ such that for a constant $C > 0$ we have $|I_k(\widehat{p},\widehat{q})| > C$ for all $k \in \mathbb{N}^*$ satisfying $p\varphi_k$ non-constant in (α,β). We will construct $\xi_1, ..., \xi_M$ from ξ_1 until ξ_M.

Let $k \in S_0$ and consider $(f_1,\xi_1) \in W^1_\infty(\alpha,\beta) \times W^2_\infty(\alpha,\beta)$ a solution to

$$\begin{cases} \frac{1}{2}\partial_x(\xi_1)p - \frac{1}{2}\xi_1\partial_x(p) = f_1 & \text{in } (\alpha,\beta), \\ \xi_1(\alpha) = \xi_1(\beta) = \partial_x\xi_1(\alpha) = \partial_x\xi_1(\beta) = 0. & \end{cases}$$

This system is equivalent to

$$\begin{cases} \xi_1(x) = p(x)\displaystyle\int_\alpha^x \frac{2f_1(s)}{p^2(s)}ds, \quad \text{for all } x \in (\alpha,\beta), \\ \displaystyle\int_\alpha^\beta \frac{2f_1(s)}{p^2(s)}ds = 0, \\ f_1(\alpha) = f_1(\beta) = 0. \end{cases}$$

We remark that we need that $p \in W^2_\infty(\alpha,\beta)$. Finding a function f_1 satisfying

$$f_1(\alpha) = f_1(\beta) = 0, \int_\alpha^\beta \frac{2f_1(s)}{p^2(s)}ds = 0 \text{ and } J_{1,k}(p,q) = \int_\alpha^\beta f_1(s)\varphi_k^2(s)ds \neq 0, \tag{3.28}$$

is equivalent to finding a function $g := 2f_1/p^2$ satisfying

$$g_1(\alpha) = g_1(\beta) = 0, \int_\alpha^\beta g_1(s)ds = 0 \text{ and } \int_\alpha^\beta g_1(s)p^2(s)\varphi_k^2(s)ds \neq 0.$$

Let $\kappa_1 \in \mathbb{R}$ and define for all $j \in \mathbb{N}^*$ and all $x \in (\alpha,\beta)$

$$g_{1,j}(x) := \kappa_1 \sin\left(\frac{2\pi j(x-\alpha)}{\beta-\alpha}\right).$$

Using the fact that $p\varphi_k$ is non-constant in (α,β), without loss of generality, we can suppose that

$$\varphi_k\left(\alpha + \frac{\beta-\alpha}{4}\right)p\left(\alpha + \frac{\beta-\alpha}{4}\right) \neq \varphi_k\left(\alpha + \frac{3(\beta-\alpha)}{4}\right)p\left(\alpha + \frac{3(\beta-\alpha)}{4}\right),$$

otherwise we adapt the interval (α,β) at the beginning of Step 2. We deduce that the function h_k of $L^2(\alpha,\alpha+(\beta-\alpha)/2)$ defined by

$$\begin{array}{llll} h_k: & (\alpha,\alpha+(\beta-\alpha)/2) & \to & \mathbb{R} \\ & s & \mapsto & p^2(s)\varphi_k^2(s)-p^2(\beta+\alpha-s)\varphi_k^2(\beta+\alpha-s) \end{array}$$

is not equal to zero in $(\alpha,\alpha+(\beta-\alpha)/2)$. Since $(g_{1,j})_{j\in\mathbb{N}^*}$ is a Riesz basis of $L^2(\alpha,\alpha+(\beta-\alpha)/2)$, there exists $j_1\in\mathbb{N}^*$ such that

$$\int_\alpha^{\alpha+(\beta-\alpha)/2} g_{1,j_1}(s)\left[p^2(s)\varphi_k^2(s)-p^2(\beta+\alpha-s)\varphi_k^2(\beta+\alpha-s)\right]ds\neq 0.$$

Moreover, using the fact that

$$g_{1,j_1}(s)=g_{1,j_1}(\beta+\alpha-s)\ \forall s\in(\alpha,\alpha+(\beta-\alpha)/2),$$

we have

$$\int_\alpha^{\alpha+(\beta-\alpha)/2} g_{1,j_1}(s)p^2(s)\varphi_k^2(s)ds\neq -\int_{\alpha+(\beta-\alpha)/2}^{\beta} g_{1,j_1}(s)p^2(s)\varphi_k^2(s)ds.$$

Thus

$$\int_\alpha^\beta g_{1,j_1}(s)p^2(s)\varphi_k^2(s)ds\neq 0.$$

Plugging $g_1:=g_{1,j_1}$ and $f_1:=\dfrac{g_{1,j_1}p^2}{2}$ in (3.28), we obtain

$$J_{1,k}(p,q)=\frac{\kappa_1}{2}\int_\alpha^\beta \sin\left(\frac{2\pi j_1(s-\alpha)}{\beta-\alpha}\right)p(s)^2\varphi_k(s)^2ds\neq 0.$$

We have also for all $j\in\mathbb{N}^*$

$$J_{1,j}(p,q)=\frac{\kappa_1}{2}\int_\alpha^\beta \sin\left(\frac{2\pi j_1(s-\alpha)}{\beta-\alpha}\right)p(s)^2\varphi_j(s)^2ds.$$

We fix κ_1 in order to have

$$\sup_{i\in\mathbb{N}^*}|J_{1,i}(p,q)|\leqslant\frac{1}{2}\inf_{i\in\mathbb{N}^*\setminus S_0}|I_i(p,q)|.$$

Let $m\in\{2,...,M\}$ and let us assume that $\xi_1,...,\xi_{m-1}$ are already constructed. Consider the set

$$S_{m-1}:=\{k\in\mathbb{N}^*:I_k(p,q)+\sum_{j=1}^{m-1}J_{j,k}(p,q)=0\text{ and }p\varphi_k\text{ non}-\text{constant in }(\alpha,\beta)\}.$$

If $S_{m-1}=\varnothing$, then we take $\xi_m=0$ in $(0,\pi)$. Otherwise, let $k\in S_{m-1}$ and consider $(f_m,\xi_m)\in W^1_\infty(\alpha,\beta)\times W^2_\infty(\alpha,\beta)$ a solution to

$$\left\{\begin{array}{ll}\frac{1}{2}\partial_x(\xi_m)p-\frac{1}{2}\xi_m\partial_x(p)=f_m & \text{in }(\alpha,\beta),\\ \xi_m(\alpha)=\xi_m(\beta)=\partial_x\xi_m(\alpha)=\partial_x\xi_m(\beta)=0. & \end{array}\right.$$

This system is equivalent to

$$\left\{\begin{array}{l}\xi_m(x)=p(x)\displaystyle\int_\alpha^x\frac{2f_m(s)}{p^2(s)}ds,\qquad\text{for all }x\in(\alpha,\beta),\\ \displaystyle\int_\alpha^\beta\frac{2f_m(s)}{p^2(s)}ds=0,\\ f_m(\alpha)=f_m(\beta)=0.\end{array}\right.$$

Let $\kappa_m > 0$. Again, there exists $j_m \in \mathbb{N}^*$ such that the function f_m given for all $x \in (\alpha, \beta)$ by

$$f_m(x) := \frac{\kappa_m}{2} \sin\left(\frac{2\pi j_m(x-\alpha)}{\beta-\alpha}\right) p(x)^2$$

is solution to this system. Then, we obtain

$$J_{m,j}(p,q) = \frac{\kappa_m}{2} \int_\alpha^\beta \sin\left(\frac{2\pi j_m(s-\alpha)}{\beta-\alpha}\right) p(s)^2 \varphi_j(s)^2 ds.$$

The last quantity is different of zero for $j = k$. Let us fix κ_m in order to have

$$\sup_{i\in\mathbb{N}^*} |J_{m,i}(p,q)| \leqslant \frac{1}{2} \inf_{i\in\mathbb{N}^*\setminus S_{m-1}} \left| I_i(p,q) + \sum_{j=1}^{m-1} J_{j,i}(p,q) \right|.$$

Thus, after constructing the functions $\xi_1, ..., \xi_M$, the obtained functions $\widehat{p}$ and $\widehat{q}$ are such that

$$|I_k(\widehat{p},\widehat{q})| > C \text{ for all } k \in \mathbb{N}^* \text{ satisfying } p\varphi_k \text{ non-constant in } (\alpha,\beta),$$

where C is a positive constant which does not depend on k. If $|I_k(\widehat{p},\widehat{q})| > C$ for all $k \in \mathbb{N}^*$ and a positive constant C, then Condition (1.10) is satisfied and the right hand-side of inequality (3.27) is negative for some appropriate constants C_1 and C_2.

Step 3: Assume that there exists $m \in \mathbb{N}^*$ such that

$$\left\{\begin{array}{l} |I_k(p,q)| > C > 0 \text{ for all } k \in \mathbb{N}^*\setminus\{m\}, \\ I_m(p,q) = 0, \\ p\varphi_m \text{ constant in } (\alpha,\beta). \end{array}\right. \tag{3.29}$$

Again, using Lemma 3.1, up to the change of unknown (3.14) described at the beginning of the section we can also suppose that $q \equiv 0$ in a subinterval (α,β) of $\widetilde{\omega}$. Moreover, using (3.17), this change of unknown can be chosen in order to keep the property: $|I_k(p,q)| > C > 0$ for all $k \in \mathbb{N}^*\setminus\{m\}$. Let $m \in \mathbb{N}^*$ such that $I_m(p,q) = 0$ and $p\varphi_m$ is constant in (α,β), otherwise we argue as in Step 2. Let $\theta \in W^2_\infty(0,\pi)$ satisfying

$$\left\{\begin{array}{l} \theta = 1 + \xi \quad \text{ in } (0,\pi), \\ \xi \in W^2_\infty(0,\pi), \\ \xi \equiv \xi_\alpha \in \mathbb{R}^*_+ \text{ in } (0,\alpha), \\ \xi \equiv 0 \text{ in } (\beta,\pi), \\ |\theta| > C > 0. \end{array}\right.$$

Again, if we consider the change of unknown described in (3.14), then for all $k \in \mathbb{N}^*$

$$\begin{array}{rcl} I_k(\widehat{p},\widehat{q}) & = & I_k(p,q) + \displaystyle\int_0^\beta \{\frac{1}{2}\partial_x(\xi)p + \xi q - \frac{1}{2}\xi\partial_x(p)\}\varphi_k^2 dx \\ & =: & I_k(p,q) + J_k(p,q). \end{array}$$

We will distinguish the cases $I_{\alpha,m}(p,q) = 0$ and $I_{\alpha,m}(p,q) \neq 0$ (see (1.8) for the definition of this quantity) for the new control domain $\omega := (\alpha,\beta)$.

Case 1: Assume that $I_{\alpha,m}(p,q) = 0$. Let $(\xi,h) \in W^2_\infty(\alpha,\beta) \times W^1_\infty(\alpha,\beta)$ be a solution to the system

$$\left\{\begin{array}{ll} \frac{1}{2}\partial_x(\xi)p - \frac{1}{2}\xi\partial_x(p) = h & \text{in } (\alpha,\beta), \\ \xi(\beta) = \partial_x\xi(\alpha) = \partial_x\xi(\beta) = 0, & \\ \xi(\alpha) = \xi_\alpha \in \mathbb{R}^*. & \end{array}\right.$$

This system is equivalent to

$$\begin{cases} \xi(x) = -p(x)\int_x^\beta \frac{2h(s)}{p^2(s)}ds, \qquad \text{for all } x\in(\alpha,\beta),\\ \int_\alpha^\beta \frac{2h(s)}{p^2(s)}ds = \frac{-\xi_\alpha}{p(\alpha)},\\ h(\alpha) = \dfrac{-\xi_\alpha\partial_x p(\alpha)}{2},\\ h(\beta)=0. \end{cases}$$

Taking into account that $I_{\alpha,m}(p,q)=0$, $q\equiv 0$ in (α,β) and $p\varphi_m\equiv\gamma$ in (α,β) for a $\gamma\in\mathbb{R}^*$, one gets

$$J_m(p,q) = \xi_\alpha\int_0^\alpha (q-\frac{1}{2}\partial_x(p))\varphi_m^2 dx + \frac{\gamma^2}{2}\int_\alpha^\beta \partial_x\left(\frac{\xi}{p}\right)dx = -\frac{\gamma^2\xi_\alpha}{2p(\alpha)}\neq 0.$$

Let ξ_α and h be such that

$$\sup_{k\in\mathbb{N}^*}|J_k(p,q)| \leqslant \frac{1}{2}\inf_{k\in\mathbb{N}^*\setminus\{m\}}|I_k(p,q)|.$$

Then $|I_k(\widehat{p},\widehat{q})|>C$ for all $k\in\mathbb{N}^*$ and a positive constant C. Thus Condition (1.10) is satisfied and the right hand-side of inequality (3.27) is negative for some appropriate constants C_1 and C_2.

Case 2: Let us now assume that $I_{\alpha,m}(p,q)\neq 0$. Then Condition (1.10) is verified. In this case, we recall that, in the moment problem described in the last section, we have

$$\det A_{1,m} := \widetilde{f}_m^{(1)}f_m^{(2)} - \widetilde{f}_m^{(2)}f_m^{(1)},$$

where $f_m^{(1)}$, $f_m^{(2)}$, $\widetilde{f}_m^{(1)}$ and $\widetilde{f}_m^{(2)}$ are given in (3.4). Since $p\varphi_m$ is constant in (α,β), the function ψ_m^* of Proposition 2.1 reads for all $x\in(\alpha,\beta)$

$$\begin{aligned}\psi_m^*(x) &= \alpha_m^*\varphi_m - \frac{1}{m}\int_0^\alpha \sin(m(x-\xi))[\partial_x(p(\xi)\varphi_m(\xi)) - q(\xi)\varphi_m(\xi)]d\xi\\ &= \tau_m\varphi_m(x) - \sqrt{\frac{\pi}{2}}\frac{1}{m}I_{\alpha,m}(p,q)\cos(mx),\end{aligned}$$

with

$$\tau_m := \alpha_m^* - \sqrt{\frac{\pi}{2}}\frac{1}{m}\int_0^\alpha \cos(m\xi)[\partial_x(p(\xi)\varphi_m(\xi)) - q(\xi)\varphi_m(\xi)]d\xi.$$

We deduce that

$$\det A_{1,m} = -\sqrt{\frac{\pi}{2}}\frac{1}{m}I_{\alpha,m}(p,q)(\widehat{f}_m^{(1)}f_m^{(2)} - \widehat{f}_m^{(2)}f_m^{(1)}),$$

where $\widehat{f}_m^{(1)}$ and $\widehat{f}_m^{(2)}$ are given in Lemma 3.4. Using Lemma 3.4, we obtain $\det A_{1,m}\neq 0$. Thus, for C_1 small enough (3.27) is true for $k=m$ and, for all $k\neq m$, the right hand-side of (3.27) is negative for C_2 be enough.

We conclude this proof remarking that, in each case, there exists $C>0$ and $k_0\in\mathbb{N}^*$ such that, for all $k\geqslant k_0$, we have

$$|I_k(\widehat{p},\widehat{q})|\geqslant C/k^6,$$

which implies that $T_0(\widehat{p},\widehat{q})=0$. □

We recall that $T_0(p,q)$ is given by

$$T_0(p,q) := \limsup_{k\to\infty} \frac{\min(-\log|I_k(p,q)|, -\log|I_{a,k}(p,q)|)}{k^2}. \tag{3.30}$$

Before to prove Theorem 1.1, we will establish the following proposition which is true not necessary in the case where the coupling region intersects the control domain.

Proposition 3.2. *Assume that Condition* (1.10) *holds,* $T > T_0(p,q)$ *and, for positive constants* C_1 *and* C_2*,*

$$|\det A_{1,k}| \geq \frac{C_1}{k^7}|I_{a,k}(p,q)| - \frac{C_2}{k}|I_k(p,q)|, \tag{3.31}$$

Then System (1.1) *is null controllable at time* T*.*

Proof. We will use the same strategy than [9]. Let $\varepsilon > 0$. Using the definition of the minimal time $T_0(p,q)$ in (3.30), there exists a positive integer k_ε for which

$$\min\left\{\log|I_{a,k}(p,q)|^{-1}, \log|I_k(p,q)|^{-1}\right\} < k^2(T_0(p,q)+\varepsilon), \quad \forall k > k_\varepsilon. \tag{3.32}$$

The goal is to solve the moment problem described in Section 3.1. We recall that we look for a control v of the form (3.2) and (3.8) with $f^{(1)}$ and $f^{(2)}$ defined in Lemma 3.4. We will solve the moment problem (3.9) depending on whether k belongs to Λ_1, Λ_2 or Λ_3, where

$$\begin{cases} \Lambda_1 & := & \{k \in \mathbb{N}^* : I_k(p,q) \neq 0,\ I_{a,k}(p,q) \neq 0\}, \\ \Lambda_2 & := & \{k \in \mathbb{N}^* : I_k(p,q) \neq 0,\ I_{a,k}(p,q) = 0\}, \\ \Lambda_3 & := & \{k \in \mathbb{N}^* : I_k(p,q) = 0,\ I_{a,k}(p,q) \neq 0\}. \end{cases}$$

Case 1 : Consider the case $k \in \Lambda_1$ with $k \leq k_\varepsilon$.
Let us take

$$v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0.$$

The moment problem (3.9) becomes

$$\begin{cases} \widetilde{f}_k^{(1)} v_{1,k}^{(1)} - I_k(p,q) f_k^{(1)} v_{2,k}^{(1)} = -e^{-k^2T}\left(y_{1,k}^0 - TI_k(p,q)y_{2,k}^0\right), \\ f_k^{(1)} v_{1,k}^{(1)} = -e^{-k^2T} y_{2,k}^0. \end{cases}$$

Since $I_k(p,q) \neq 0$ and using the estimate of $f_k^{(1)}$ and $f_k^{(2)}$ in Lemma 3.4, the last system has a unique solution

$$\begin{cases} v_{1,k}^{(1)} = -e^{-k^2T}\frac{y_{2,k}^0}{f_k^{(1)}}, \\ v_{2,k}^{(1)} = \frac{e^{-k^2T}}{I_k(p,q)f_k^{(1)}}\left(y_{1,k}^0 - TI_k(p,q)y_{2,k}^0 - \widetilde{f}_k^{(1)}\frac{y_{2,k}^0}{f_k^{(1)}}\right). \end{cases} \tag{3.33}$$

Moreover, since the set of the k considered in this case is finite, we get the inequality

$$\left|v_{j,k}^{(i)}\right| \leq C_\varepsilon e^{-k^2T}\|y^0\|_{L^2(0,\pi)^2}, \quad i,j = 1,2. \tag{3.34}$$

Case 2: Let $k \in \Lambda_1$ such that $k > k_\varepsilon$ and $|I_k(p,q)|^{-1} \leq e^{k^2(T_0(p,q)+2\varepsilon)}$.
As in the previous case, we take $v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0$ and the moment problem (3.9) has a unique solution, given by (3.33). Thanks to the property of ψ_k^* (see (2.6)) and Lemma 3.4, we get for $i = 1,2$ the following estimates

$$|f_k^{(1)}| \geqslant C/k^3, \quad |\widetilde{f}_k^{(i)}| \leqslant \frac{C}{k}, \quad |y_{i,k}^0| \leqslant C\|y_0\|_{L^2(0,\pi)^2}, \quad \forall k \in \mathbb{N}^*. \tag{3.35}$$

Thus, using the assumptions on k, we obtain

$$\begin{cases} |v_{1,k}^{(1)}| \leq Ck^3 e^{-k^2 T} \|y^0\|_{L^2(0,\pi)^2} \leq C_\varepsilon e^{-(T-\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \\ |v_{1,k}^{(2)}| \leq \dfrac{C_\varepsilon e^{-(T-\varepsilon)k^2}}{I_k(p,q)} \|y^0\|_{L^2(0,\pi)^2} \leq C_\varepsilon e^{-(T-T_0-3\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \end{cases}$$

where C_ε is a constant which is independent on k and y^0.

Case 3: Consider now $k \in \Lambda_1$ such that $k > k_\varepsilon$ and $|I_k(p,q)|^{-1} > e^{k^2(T_0(p,q)+2\varepsilon)}$. This implies with (3.32) that

$$|I_{a,k}(p,q)|^{-1} < e^{k^2(T_0(p,q)+\varepsilon)}. \tag{3.36}$$

The two last inequality lead to

$$|I_k(p,q)| < e^{-\varepsilon k^2} |I_{a,k}(p,q)|.$$

Combined with inequality (3.31), taking k_ε large enough, we get

$$|\det A_{1,k}| > C_\varepsilon e^{-\varepsilon k^2} |I_{a,k}(p,q)|, \tag{3.37}$$

with C_ε independent on k. To solve the moment problem (3.9), we take here

$$v_{2,k}^{(1)} = v_{2,k}^{(2)} = 0.$$

Then the moment problem (3.9) reads $A_{1,k} V_{1,k} = F_k$. Since $\det A_{1,k} \neq 0$, the inverse of $A_{1,k}$ is given by

$$(A_{1,k})^{-1} = (\det\ A_{1,k})^{-1} \begin{pmatrix} f_k^{(2)} & -\widetilde{f}_k^{(2)} \\ -f_k^{(1)} & \widetilde{f}_k^{(1)} \end{pmatrix}.$$

We deduce that the solution to the moment problem (3.9) is

$$\begin{cases} v_{1,k}^{(1)} = \frac{e^{-k^2 T}}{\det\ A_{1,k}} \{-f_k^{(2)} y_{1,k}^0 + (TI_k(p,q) f_k^{(2)} + \widetilde{f}_k^{(2)}) y_{2,k}^0\}, \\ v_{1,k}^{(2)} = \frac{e^{-k^2 T}}{\det\ A_{1,k}} \{f_k^{(1)} y_{1,k}^0 - (TI_k(p,q) f_k^{(1)} + \widetilde{f}_k^{(1)}) y_{2,k}^0\}. \end{cases}$$

The last expression together with (3.36) and (3.37) gives

$$|v_{1,k}^{(i)}| \leq C_\varepsilon e^{-(T-T_0-2\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \quad i = 1, 2. \tag{3.38}$$

Case 4: Let us consider $k \in \Lambda_2$.
If $k \leq k_\varepsilon$, we can argue as in Case 1. Let us suppose that $k > k_\varepsilon$. In this case, $I_{a,k}(p,q) = 0$, $I_k(p,q) \neq 0$ and inequality (3.32) reads $|I_k(p,q)|^{-1} < e^{k^2(T_0(p,q)+\varepsilon)}$. We take here

$$v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0$$

and the solution of moment problem (3.9) is given by (3.33). We get

$$\left| v_{j,k}^{(i)} \right| \leq C_\varepsilon e^{-k^2(T-T_0(p,q)-2\varepsilon)} \|y^0\|_{L^2(0,\pi)^2}, \quad i, j = 1, 2.$$

Case 5: Let us now deal with the case $k \in \Lambda_3$.
We recall that $I_k(p,q) = 0$, $I_{1,k}(p,q) \neq 0$ and inequality (3.32) reads

$$|I_{a,k}(p,q)|^{-1} < e^{k^2(T_0(p,q)+\varepsilon)}. \tag{3.39}$$

The moment problem (3.9) is now $A_{1,k}V_{1,k} = F_k$ with $A_{1,k}$ and F_k given in (3.10) and (3.12), respectively. From (3.31), the matrix $A_{1,k}$ is invertible and

$$\begin{cases} v_{1,k}^{(1)} = \frac{e^{-k^2T}}{\det\ A_{1,k}}\{-f_k^{(2)}y_{1,k}^0 + \widetilde{f}_k^{(2)}y_{2,k}^0\}, \\ v_{1,k}^{(2)} = \frac{e^{-k^2T}}{\det\ A_{1,k}}\{f_k^{(1)}y_{1,k}^0 - \widetilde{f}_k^{(1)}y_{2,k}^0\}. \end{cases}$$

Using inequalities (3.31) and (3.39), we obtain estimate (3.38).

Conclusion:

We have constructed a control v of the form (3.2) and (3.8), which satisfies

$$\left|v_{j,k}^{(i)}\right| \leq C_\varepsilon \sum_{k\geqslant 1} e^{-k^2(T-T_0(p,q)-3\varepsilon)}\|y^0\|_{L^2(0,\pi)^2}, \quad i,j=1,2,\ k\in\mathbb{N}^*.$$

The last inequality, the estimate (3.7) of $q_{i,k}$ and the expression (3.8) of $v^{(i)}$ $(i=1,2)$ lead to

$$\left\|v^{(i)}\right\|_{L^2(0,T)} \leq C_{\varepsilon,T}e^{-k^2(T-T_0(p,q)-4\varepsilon)}, \quad i=1,2.$$

Thus, taking

$$\varepsilon \in \left(0, \frac{T-T_0(p,q)}{4}\right),$$

we have the absolute convergence of the series defining $v^{(1)}$ and $v^{(2)}$ in $L^2(0,T)$. This ends the proof. □

Proof of Theorem 1.1. Using Proposition 3.1, System (1.1) is equivalent to a system with coupling terms $\widehat{p}$ and $\widehat{q}$ satisfying Condition (1.10) and (3.31). Proposition 3.2 leads to the null controllability of System (1.1) when $T > T_0(\widehat{p},\widehat{q})$. We end the proof of Theorems 1.1 remarking that $T_0(\widehat{p},\widehat{q}) = 0$. □

4 Proof of Theorem 1.3

4.1 Positive null controllability result

Before studying the case where the intersection of the coupling and control domains is empty, we will first rewrite the function ψ_k^* given in Proposition 2.1.

Lemma 4.1. *Let $k\in\mathbb{N}^*$. Consider the function ψ_k^* defined in Proposition 2.1. If we suppose that Condition* (1.11) *holds, then for all $x\in\omega$*

$$\psi_k^*(x) = \tau_k\varphi_k(x) + g_k(x) \text{ for all } x\in\omega,$$

where

$$\tau_k := \alpha_k^* - \sqrt{\frac{\pi}{2}}\frac{1}{k}\int_0^a \cos(k\xi)[\partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]d\xi$$

and

$$g_k(x) := -\frac{I_k(p,q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)d\xi - \sqrt{\frac{\pi}{2}}\frac{1}{k}I_{a,k}(p,q)\cos(kx).$$

Proof. Since $p=q\equiv 0$ in ω, we get for all $x\in\omega$,

$$\begin{array}{rcl}\psi_k^*(x) &=& \alpha_k^*\varphi_k(x) - \dfrac{I_k(p,q)}{k}\displaystyle\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\,d\xi \\ && \qquad -\dfrac{1}{k}\displaystyle\int_0^a \sin(k(x-\xi))[\partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]d\xi.\end{array}$$

□

Proof of Theorem 1.3. We will follow the strategy of [9]. Assume that Conditions (1.10) and (1.11) hold. Consider the functions $f^{(1)}$ and $f^{(2)}$ defined in Lemma 3.4 and the matrix $A_{1,k}$ given in (3.10). Let $k \in \mathbb{N}^*$. We recall that, in this case, we have

$$\det A_{1,k} := \widetilde{f}_k^{(1)} f_k^{(2)} - \widetilde{f}_k^{(2)} f_k^{(1)},$$

where, for $i = 1, 2$, $f_k^{(i)}$ and $\widetilde{f}_k^{(i)}$ are defined in (3.4). Since $\operatorname{Supp}\left(f^{(i)}\right) \subseteq \omega$, using the expression of ψ_k^* given in Lemma 4.1, we obtain

$$\widetilde{f}_k^{(i)} = \tau_k f_k^{(i)} + \int_0^\pi f^{(i)}(x)\, g_k(x)\, dx,$$

where for all $x \in \omega$

$$g_k(x) = -\frac{I_k(p,q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)d\xi - \sqrt{\frac{\pi}{2}}\frac{1}{k} I_{a,k}(p,q)\cos(kx).$$

We deduce that

$$\begin{aligned}
\det A_{1,k} &= f_k^{(2)}\int_0^\pi f^{(1)}(x)\, g_k(x) dx - f_k^{(1)}\int_0^\pi f^{(2)}(x)\, g_k(x)\, dx\\
&= -\frac{I_k(p,q)}{k}\left(f_k^{(2)}\int_0^\pi\int_0^x f^{(1)}(x)\sin(k(x-\xi))\varphi_k(\xi)d\xi\, dx\right.\\
&\qquad\left. -f_k^{(1)}\int_0^\pi\int_0^x f^{(2)}(x)\sin(k(x-\xi))\varphi_k(\xi)d\xi\, dx\right)\\
&\qquad -\sqrt{\frac{\pi}{2}}\frac{1}{k} I_{a,k}(p,q)\left(\widehat{f}_k^{(1)} f_k^{(2)} - \widehat{f}_k^{(2)} f_k^{(1)}\right),
\end{aligned}$$

where $\widehat{f}_k^{(i)}$ are defined in (3.26). Since the integrals

$$\int_0^\pi\int_0^x f^{(i)}(x)\sin(k(x-\xi))\varphi_k(\xi)\, d\xi\, dx$$

are uniformly bounded with respect to k and i, we conclude with the help of Lemma 3.4.

We deduce that Condition (3.31) holds. Thus, using Proposition 3.2, System (1.1) is null controllable at time T.

□

4.2 Negative null controllability result

Let us now prove the negative part of Theorem 1.3 with the strategy used in [9]. Suppose that $T < T_0(p,q)$ and System (1.1) is null controllable at time T. Using Proposition 2.4, there exists a constant $C_{obs} > 0$ such that for all $\theta^0 \in L^2(0,\pi)^2$, the solution to System (2.8) satisfies the observability inequality

$$\|\theta(0)\|^2_{L^2(0,\pi)^2} \leqslant C_{obs}\iint_{Q_T} |\mathbb{1}_\omega B^*\theta|^2 dxdt. \tag{4.1}$$

Using the Definition of $T_0(p,q)$ (see (1.12)) there exists a sequence $(k_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{N}$ satisfying:

$$T_0(p,q) = \lim_{n\to\infty}\frac{\min\left(\log\left|I_{a,k_n}(p,q)^{-1}\right|, \log\left|I_{k_n}(p,q)^{-1}\right|\right)}{k_n^2}. \tag{4.2}$$

Let us fix $n \geq 1$ and $\theta^{0n} := a_n\Phi^*_{1,k_n} + b_n\Phi^*_{2,k_n}$ with $(a_n, b_n) \in \mathbb{R}^2$ to be determined later and Φ^*_{2,k_n}, Φ^*_{1,k_n} the eigenfunction and generalized eigenfunction associated with k_n^2 given in Proposition 2.1. If we denote by θ^n the solution to the dual System (2.8) for initial data θ^{0n}, then

$$\theta^n(x,t) = e^{-k_n^2(T-t)}\{a_n\Phi^*_{1,k_n} + (b_n - (T-t)I_{k_n}(p,q)a_n)\Phi^*_{2,k_n}\},$$

thus we have

$$A_{1,n} := \|\theta^n(0)\|^2_{L^2(0,\pi)^2} = e^{-2k_n^2 T}\left\{|a_n|^2|\psi_{k_n}|^2 + |a_n|^2 + (b_n - TI_{k_n}(p,q)a_n)^2\right\}$$

and

$$A_{2,n} := \iint_{Q_T} |\mathbb{1}_\omega B^*\theta^n|^2 dxdt = \int_0^T \int_\omega e^{-2k_n^2 t}\left|a_n\psi_{k_n}(x) + (b_n - tI_{k_n}(p,q)a_n)\varphi_{k_n}(x)\right|^2 dx\,dt.$$

The observability inequality (4.1) reads

$$A_{1,n} \leqslant C_{obs} A_{2,n}. \tag{4.3}$$

By choosing $a_n := 1$ and $b_n := -\tau_{k_n}$, we get

$$A_{1,n} \geqslant e^{-2k_n^2 T} \tag{4.4}$$

and the expression of $\psi^*_{k_n}(x)$ given in Lemma 4.1 leads to

$$\begin{aligned} A_{2,n} &= \int_0^T \int_\omega e^{-2k_n^2 t}\left|-\sqrt{\frac{\pi}{2}}\frac{1}{k_n} I_{a,k_n}(p,q)\cos(k_n x)\right. \\ &\qquad \left. -I_{k_n}(p,q)\frac{1}{k_n}\int_0^x \sin\left(k_n(x-\xi)\right)\varphi_{k_n}(\xi)\,d\xi - tI_{k_n}(p,q)\varphi_{k_n}(x)\right|^2 dx\,dt \\ &\leqslant C(I_{a,k_n}(p,q)^2 + I_{k_n}(p,q)^2). \end{aligned}$$

Let $\varepsilon > 0$. Equality (4.2) implies that there is $k_\varepsilon \in \mathbb{N}^*$ such that for all $k_n \geqslant k_\varepsilon$

$$\max\left(|I_{a,k_n}(p,q)|^2, |I_{k_n}(p,q)|^2\right) \leqslant e^{-2k_n^2(T_0(p,q)-\varepsilon)}.$$

We deduce that for $\varepsilon := (T_0(p,q) - T)/2$, we get

$$A_{2,n} \leqslant Ce^{-2k_n^2(T+\varepsilon)}. \tag{4.5}$$

Thus estimates (4.4) and (4.5) are in contradiction with inequality (4.3) for n large enough.

5 Approximate controllability of system (1.1)

Theorem 1.2 can be proved with the same argument as in [9], but we propose to use here the Hautus test given below, as in [11] for example.

Theorem 5.1 (see [15], Cor. 3.3)**.** *System* (1.1) *is approximatively controllable at time T if and only if for any $s \in \mathbb{C}$ and for any $u \in \mathcal{D}(L^*)$ we have*

$$\left.\begin{array}{r} L^*u = su \text{ in } (0,\pi) \\ B^*u = 0 \text{ in } \omega \end{array}\right\} \Rightarrow u = 0.$$

Proof of Theorem 1.2.

Necessary condition: Let us suppose that Conditions (1.9)-(1.10) do not hold *i.e.* there exists $k_0 \in \mathbb{N}^*$ such that

$$I_{k_0}(p,q) = I_{a,k_0}(p,q) = 0$$

and

$$(\mathrm{Supp}(p) \cup \mathrm{Supp}(q)) \cap \omega = \varnothing.$$

Then the function $\psi^*_{k_0}$ of Lemma 4.1 is given by $\psi^*_{k_0} = \tau_{k_0}\varphi_{k_0}$. Moreover

$$\Phi^*_{1,k_0} - \tau_{k_0}\Phi^*_{2,k_0} = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}$$

is an eigenfunction associated with the eigenvalue k_0^2 of the operator L^* and satisfies

$$B^*(\Phi^*_{1,k_0} - \tau_{k_0}\Phi^*_{2,k_0}) \equiv 0 \text{ in } \omega.$$

Thus, using Theorem 5.1 System (1.1) is not approximately controllable at time T.

Sufficient condition: Let us suppose that Conditions (1.9)-(1.10) hold. If $(\mathrm{Supp}(p)\cup\mathrm{Supp}(q))\cap\omega \neq \varnothing$ we conclude using Theorem 1.1. Let us now suppose that

$$|I_k(p,q)| + |I_{a,k}(p,q)| \neq 0 \text{ for all } k \in \mathbb{N}^*$$
$$\text{and}$$
$$(\mathrm{Supp}(p)\cup\mathrm{Supp}(q))\cap\omega = \varnothing.$$

If $I_k(p,q) \neq 0$, the set of the eigenvectors associated with the eigenvalue k^2 of L^* is generated by $\Phi^*_{2,k}$ (see Proposition 2.1). In this case, we remark that for all $k \in \mathbb{N}^*$

$$B^*\Phi^*_{2,k} = \varphi_k \not\equiv 0 \text{ in } \omega. \tag{5.1}$$

If $I_k(p,q) = 0$, the eigenvectors associated with the eigenvalue k^2 of L^* are linear combinations of $\Phi^*_{1,k}$ and $\Phi^*_{2,k}$. Let $\alpha,\beta \in \mathbb{R}$ and $\Phi^* := \alpha\Phi^*_{1,k} + \beta\Phi^*_{2,k}$ satisfying

$$B^*\Phi^* \equiv 0 \text{ in } \omega. \tag{5.2}$$

Using Lemma 4.1, it is equivalent to

$$(\alpha + \beta\tau_k)\varphi_k(x) - \beta\sqrt{\frac{\pi}{2}}\frac{1}{k}I_{a,k}(p,q)\cos(kx) = 0 \text{ for all } x \in \omega.$$

Since $I_{a,k}(p,q) \neq 0$, we deduce that $\beta = 0$. Then $\alpha = 0$. We conclude using Theorem 5.1.

□

6 Null controllability of System (1.2)

As in Section 3.1, System (1.2) is null controllable at time T if and only if for all $y^0 \in H^{-1}(0,\pi)^2$, $k \in \mathbb{N}^*$ and $i \in \{1,2\}$ the solution $\theta_{i,k}$ to the dual System (2.8) for the initial data $\Phi^*_{i,k}$ satisfies

$$\int_0^T u(t)B^*\partial_x\theta_{i,k}(0,t)dt = -\langle y^0, \theta_{i,k}(\cdot,0)\rangle_{H^{-1},H^1_0}. \tag{6.1}$$

We recall that, for all $k \in \mathbb{N}^*$, $\theta_{1,k}$ and $\theta_{2,k}$ are given for all $(x,t) \in Q_T$ by

$$\begin{cases} \theta_{1,k}(x,t) = e^{-k^2(T-t)}\left(\Phi^*_{1,k}(x) - (T-t)I_k(p,q)\Phi^*_{2,k}(x)\right), \\ \theta_{2,k}(x,t) = e^{-k^2(T-t)}\Phi^*_{2,k}(x). \end{cases}$$

Proof of Theorem 1.4. Again, we will follow the strategy used in [9]. Assume that $T > T_1$ and $I_k(p,q) \neq 0$ for all $k \in \mathbb{N}^*$. We will look for the control u under the form

$$u(t) := \sum_{k\in\mathbb{N}^*}\{u_{1,k}q_{1,k}(T-t) + u_{2,k}q_{2,k}(T-t)\}, \tag{6.2}$$

for all $t \in (0,T)$, where $q_{1,k}$ and $q_{2,k}$ are defined in Section 3.1. Plugging the expressions of u, $\theta_{1,k}$ and $\theta_{2,k}$ in Equality (6.1), we obtain the *moment problem*

$$\begin{cases} u_{1,k} = -e^{-k^2T}\dfrac{\langle y_1^0,\varphi_k\rangle_{H^{-1},H_0^1}}{\partial_x\varphi_k(0)}, \\ u_{2,k} = \dfrac{e^{-k^2T}}{I_k\partial_x\varphi_k(0)}\{\langle y_1^0,\psi_k\rangle_{H^{-1},H_0^1} + \langle y_2^0,\varphi_k\rangle_{H^{-1},H_0^1} - \left(I_kT + \dfrac{\partial_x\psi_k(0)}{\partial_x\varphi_k(0)}\right)\langle y_1^0,\varphi_k\rangle_{H^{-1},H_0^1}\}. \end{cases}$$

Let $\varepsilon > 0$. Using the definition of T_1 (see (1.14)), we have $I_k(p,q) > C_\varepsilon e^{-k^2(T_1+\varepsilon)}$ for all $k \in \mathbb{N}^*$. Then, using the estimates (2.6) and (3.35), we get

$$|u_{1,k}| + |u_{2,k}| \leqslant Ce^{-k^2(T-T_1-2\varepsilon)}\|y^0\|_{H^{-1}(0,\pi)^2}.$$

Thus for $\varepsilon < (T-T_1)/2$, the control u defined in (6.2) is an element of $L^2(0,T)$.

Assume now that $T < T_1$ and $I_k(p,q) \neq 0$ for all $k \in \mathbb{N}^*$. By contradiction let us suppose that there exists a constant C_{obs} such that for all $\theta^0 \in H_0^1(0,\pi)^2$ the solution to the dual System (2.8) satisfies

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} \leqslant C_{obs}\int_0^T |B^*\partial_x\theta(0,t)|^2dt. \tag{6.3}$$

Let $\varepsilon = (T_1 - T)/2$. Using the definition of T_1, there exists a sequence $(k_n)_{n\in\mathbb{N}^*}$ such that

$$I_{k_n}(p,q) < e^{-k_n^2(T+\varepsilon)}. \tag{6.4}$$

Let $\theta_n^0 := a_n\Phi^*_{1,k_n} + b_n\Phi^*_{2,k_n}$ with $(a_n,b_n) \in \mathbb{R}^2$. We recall that

$$\theta^n(x,t) = e^{-k_n^2(T-t)}\{a_n\Phi^*_{1,k_n} + (b_n - (T-t)I_k(p,q)a_n)\Phi^*_{2,k_n}\}.$$

Then, after calculation, we get

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} = e^{-2k_n^2T}(a_n^2\|\psi_{k_n}\|^2_{H_0^1} + a_n^2k_n^2 + (b_n - TI_k(p,q)a_n)^2k_n^2)$$

and

$$\int_0^T |B^*\partial_x\theta(0,t)|^2dt = \int_0^T e^{-2k_n^2(T-t)}|a_n\partial_x\psi_{k_n}(0) + (b_n - (T-t)I_{k_n}(p,q)a_n)k_n|^2dt.$$

For $a_n := 1$ and $b_n := -\partial_x\psi_{k_n}(0)/k_n$, taking into account inequality (6.4) and using the estimate (2.6), we obtain

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} \geqslant k_n^2e^{-2k_n^2T} \quad \text{and} \quad \int_0^T |B^*\partial_x\theta(0,t)|^2dt \leqslant Ck_n^2e^{-2k_n^2(T+\varepsilon)}.$$

Thus for n large enough we get a contradiction with observability inequality (6.3).

□

7 Comments and open problems

When the control domain and the support of the coupling coefficients p and q is disjoint in the system

$$\begin{cases} \partial_t y_1 - \partial_{xx}y_1 = \mathbb{1}_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_{xx}y_2 + p(x)\partial_x y_1 + q(x)y_1 = 0 & \text{in } Q_T, \\ y_1(0,\cdot) = y_1(\pi,\cdot) = y_2(0,\cdot) = y_2(\pi,\cdot) = 0 & \text{on } (0,T), \\ y_1(\cdot,0) = y_1^0,\ y_2(\cdot,0) = y_2^0 & \text{in } (0,\pi) \end{cases} \tag{7.1}$$

(resp. system (1.2)), it is legitimate to ask if the minimal time T_1 (resp. T_0) given in Theorem 1.3 (resp. Theorem 1.4) can be different of zero and finite. For $p \equiv 0$ in $(0,\pi)$, it is proved in [9] that for any $\tau_0 \in [0,\infty]$ there exists a function $q \in L^\infty(0,\pi)$ such that the minimal time of null controllability $T_0(p,q)$ associated with System (1.1) is given by

$$T_0(p,q) = \tau_0.$$

The authors give explicit functions and one can easily adapt them to the case $p \not\equiv 0$ in $(0,\pi)$. In the other hand, the null controllability in the cases $T = T_0$ in Theorem 1.3 and $T = T_1$ in Theorem 1.4 are open problems.

In higher space dimension, even for this simplified system (7.1) (resp. system (1.2)), distributed and boundary controllability are also open problems. Considering the different results described in the introduction of the present paper, we can conjecture that the system of two coupled linear parabolic equations

$$\begin{cases} \partial_t y_1 = \Delta y_1 + g_{11}\cdot\nabla y_1 + g_{12}\cdot\nabla y_2 + a_{11}y_1 + a_{12}y_2 + \mathbb{1}_\omega v & \text{in } \Omega\times(0,T),\\ \partial_t y_2 = \Delta y_2 + g_{21}\cdot\nabla y_1 + g_{22}\cdot\nabla y_2 + a_{21}y_1 + a_{22}y_2 & \text{in } \Omega\times(0,T),\\ y = 0 & \text{on } \partial\Omega\times(0,T),\\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \tag{7.2}$$

is null controllable at time $T > 0$ if there exists an open nonempty subset ω_0 of ω such that

$$|a_{21}| > C \text{ in } \omega_0\times(0,T) \text{ or } |g_{21}^k| > C \text{ in } \omega_0\times(0,T), \tag{7.3}$$

for a $k \in \{1, ..., N\}$.

It seems that the main difficulty is to prove a Carleman estimate for the adjoint problem of system (7.2) under condition (7.3) when the coupling term is a differential operator (see for instance [10, 18] and also [14] for a different approach). In the one-dimensional case, we were not able to adapt the strategy developed in this paper in this general setting.

A Proof of Lemma 3.4

Proof of Lemma 3.4. Let $k \in \mathbb{N}^*$. Consider $a_1, b_1, a_2, b_2 \in \omega$ to be determined later with $a_i < b_i$ for $i \in \{1,2\}$ and define the functions

$$\begin{cases} f^{(1)} := \mathbb{1}_{(a_1,b_1)},\\ f^{(2)} := \mathbb{1}_{(a_2,b_2)}. \end{cases}$$

Thus, for $i = 1,2$, we obtain

$$f_k^{(i)} = \int_0^\pi f^{(i)}(x)\varphi_k(x)\,dx = \frac{2}{k}\sqrt{\frac{2}{\pi}}\sin\left(k\frac{a_i+b_i}{2}\right)\sin\left(k\frac{b_i-a_i}{2}\right)$$

and

$$\widehat{f}_k^{(i)} = \int_0^\pi f^{(i)}(x)\cos(kx)\,dx = \frac{2}{k}\cos\left(k\frac{a_i+b_i}{2}\right)\sin\left(k\frac{b_i-a_i}{2}\right).$$

A simple computation leads to

$$|B_k| = \frac{4}{k^2}\sqrt{\frac{2}{\pi}}\left|\sin\left(k\frac{b_1-a_1}{2}\right)\sin\left(k\frac{b_2-a_2}{2}\right)\sin\left(k\frac{a_2+b_2-a_1-b_1}{2}\right)\right|. \tag{A.1}$$

Let $n \in \mathbb{N}^*$ and ℓ an algebraic number of order two satisfying

$$a/2n < \ell < b/(2n+3).$$

Let us take $a_1 := 2n\ell$, $b_1 := a_1 + 2\ell$, $a_2 := a_1 + \ell$ and $b_2 := a_2 + 2\ell$. Thus $a_1, b_1, a_2, b_2 \in \omega$ and $a_i < b_i$ for $i = 1, 2$. Furthermore

$$\begin{cases} b_1 - a_1 = b_2 - a_2 = a_2 + b_2 - a_1 - b_1 = 2\ell, \\ a_1 + b_1 = (4n+2)\ell, \\ a_2 + b_2 = (4n+4)\ell. \end{cases} \tag{A.2}$$

Since ℓ is an algebraic number of order 2, as said in the proof of Proposition 3.2, using diophantine approximations, there exists a constant $\gamma > 0$ such that

$$\inf_{j\geqslant 1} \left(j\,|\sin(j\ell)|\right) \geqslant \gamma. \tag{A.3}$$

Combining (A.1)-(A.3), we deduce that

$$|B_k| \geqslant \sqrt{\frac{2}{\pi}}\frac{4\gamma^3}{k^5},\ |f_k^{(1)}| \geqslant \sqrt{\frac{2}{\pi}}\frac{2\gamma^2}{(2n+1)k^3} \text{ and } |f_k^{(2)}| \geqslant \sqrt{\frac{2}{\pi}}\frac{\gamma^2}{(n+1)k^3},$$

for all $k \in \mathbb{N}^*$. □

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References

[1] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled systems under geometric conditions. *Comptes Rendus Mathematique*, 349(7):395–400, 2011.

[2] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled wave-type systems and applications. *J. Math. Pures Appl.*, 99:544–576, 2013.

[3] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems. *Differ. Equ. Appl.*, 1(3):427–457, 2009.

[4] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A Kalman rank condition for the localized distributed controllability of a class of linear parbolic systems. *J. Evol. Equ.*, 9(2):267–291, 2009.

[5] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and I. Kostin. Null-controllability of some systems of parabolic type by one control force. *ESAIM Control Optim. Calc. Var.*, 11(3):426–448 (electronic), 2005.

[6] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Recent results on the controllability of linear coupled parabolic problems: a survey. *Math. Control Relat. Fields*, 1(3):267–306, 2011.

[7] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Controllability of some system of parabolic equations. In *Procedings of the II Encuentro RSME-SMM*, 2012.

[8] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Minimal time of controllability of two parabolic equations with disjoint control and coupling domains. *C. R. Math. Acad. Sci. Paris*, 352(5):391–396, 2014.

[9] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence. *Submitted*, 2015.

[10] A. Benabdallah, M. Cristofol, P. Gaitan, and L. De Teresa. Controllability to trajectories for some parabolic systems of three and two equations by one control force. *Math. Control Relat. Fields*, 4(1):17–44, 2014.

[11] F. Boyer and G. Olive. Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients. *Math. Control Relat. Fields*, 4(3):263–287, 2014.

[12] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[13] B. Dehman and M. Léautaud. Controllability of two coupled wave equations on a compact manifold. *Arch. Rational Mech. Anal.*, 211:113–187, 2014.

[14] M. Duprez and P. Lissy. Indirect controllability of some linear parabolic systems of m equations with m - 1 controls involving coupling terms of zero or first order. *Submitted*, 2015.

[15] H. O. Fattorini. Some remarks on complete controllability. *SIAM J. Control*, 4:686–694, 1966.

[16] E. Fernández-Cara, M. González-Burgos, and L. de Teresa. Boundary controllability of parabolic coupled equations. *J. Funct. Anal.*, 259(7):1720–1758, 2010.

[17] M. González-Burgos and L. de Teresa. Controllability results for cascade systems of m coupled parabolic PDEs by one control force. *Port. Math.*, 67(1):91–113, 2010.

[18] S. Guerrero. Null controllability of some systems of two parabolic equations with one control force. *SIAM J. Control Optim.*, 46(2):379–394, 2007.

[19] O. Kavian and L. de Teresa. Unique continuation principle for systems of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 16(2):247–274, 2010.

[20] J.-L. Lions. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Avant propos de P. Lelong. Dunod, Paris, 1968.

[21] L. Miller. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim.*, 45(2):762–772 (electronic), 2006.

[22] G. Olive. Boundary approximate controllability of some linear parabolic systems. *Evol. Equ. Control Theory*, 3(1):167–189, 2014.

[23] L. Rosier and L. de Teresa. Exact controllability of a cascade system of conservative equations. *C. R. Math. Acad. Sci. Paris*, 349(5-6):291–296, 2011.