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Abstract. The renowned Riemann Hypothesis is considered from the perspective of symmetry in the functional equation for the Riemann zeta function developed by Bernhard Riemann. The validity of the Riemann Hypothesis is shown by considering the complex conjugations of the leading terms of this equation.

1. Introduction

$\zeta(s)$ is defined as the series $\zeta(s) = \sum_{n=1}^{\infty} \dfrac{1}{n^s}$ where $s = \alpha + \beta i$ with α, β real and n a positive integer, and where $\zeta(s)$ is convergent for $\alpha > 1$. The Riemann zeta function is an analytical continuation of this series by Bernhard Riemann which is holomorphic for all s except at $s = 1$ [3]. Using a natural symmetry in the result Riemann [3, 2] obtained the functional equation:

$$\zeta(s) = \aleph(s)\zeta(1-s) = 2^s \pi^{s-1} \sin \frac{s\pi}{2} \Gamma(1-s)\zeta(1-s) \tag{1}$$

$$\text{with } \Gamma(1-s) = \int_0^{\infty} e^{-z} z^{-s} dz.$$

The renowned Riemann Hypothesis (RH) is that non-trivial zeroes of the Riemann zeta function $\zeta(s)$ all have real part on the 'critical line' $Re(s) = \alpha = \frac{1}{2}$ [1]-[4].

My explanation for the validity of RH will focus on the complex conjugations of this equation. Starting with $\zeta(s) = 0$ and using the fact that $\aleph(s) \neq 0$ on the critical strip $0 < \alpha < 1$ where all non-trivial zeroes are known to occur [1]-[4] it is shown that $\zeta(s) = 0 \Rightarrow \aleph(s).\overline{\aleph(s)} = 1$ and why this is only possible if $\alpha = 0.5$.

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2. Details

Here I will define the complex functions resulting from (1) as:

$$\zeta(s) = u + vi, \zeta(1-s) = w + ti, \aleph(s) = x + yi \tag{2}$$

where u, v, w, t, x, y are all real. Thus,

$$(u + vi) = (x + yi)(w + ti) \tag{3}$$

resulting in

$$x = \frac{uw + vt}{w^2 + t^2}, y = \frac{vw - ut}{w^2 + t^2}, x^2 + y^2 = \frac{u^2 + v^2}{w^2 + t^2} > 0 \tag{4}$$

Note in (4) $x^2 + y^2 \neq 0$ and exists on the critical strip $0 < \alpha < 1$ where all non-trivial zeroes are known to occur [1]-[4] because: $2^s \neq 0$, $\pi^{s-1} \neq 0$, $\int_0^\infty e^{-z} z^{-s} dz \neq 0$, and $\sin \frac{s\pi}{2} \neq 0$ for $0 < \alpha < 1$, and all of these are holomorphic for the same range, $\Rightarrow \aleph(s) \neq 0$ and is holomorphic for $0 < \alpha < 1$. Thus, using (1), $\zeta(s) = 0 \Rightarrow \zeta(1-s) = 0 \Rightarrow \zeta(s) = \zeta(1-s) \Rightarrow u = w, v = t \Rightarrow x = 1, y = 0 \Rightarrow x^2 + y^2 = 1$ from (4).

Thus the RH is true if

$$\aleph(s).\overline{\aleph(s)} = x^2 + y^2 = 1 \iff \alpha = 0.5. \tag{5}$$

Theorem 1. $\aleph(s).\overline{\aleph(s)} = 1 \iff \alpha = 0.5$ *for* $0 < \alpha < 1$ *and* $\beta > 10$.

Proof. First, for $\aleph(0.5 + \beta i)\overline{\aleph(0.5 + \beta i)}$ using (1):

(i) $2^s \pi^{s-1} \overline{2^s \pi^{s-1}} = 2^{0.5+\beta i} \pi^{-0.5+\beta i} 2^{0.5-\beta i} \pi^{-0.5-\beta i} = \frac{2}{\pi}$

(ii) $\sin \frac{s\pi}{2} \overline{\sin \frac{s\pi}{2}} = (\sin \frac{\alpha\pi}{2} \cos \frac{\beta i\pi}{2} + \cos \frac{\alpha\pi}{2} \sin \frac{\beta i\pi}{2})(\sin \frac{\alpha\pi}{2} \cos \frac{\beta i\pi}{2} - \cos \frac{\alpha\pi}{2} \sin \frac{\beta i\pi}{2})$
$= 0.5(\cos^2 \frac{\beta\pi i}{2} - \sin^2 \frac{\beta\pi i}{2})$ at $\alpha = 0.5$
$= 0.5 \cos(\beta\pi i) = 0.5 \sin(\beta\pi i + \frac{\pi}{2})$

(iii) $\overline{\Gamma(1-s)} = \Gamma(\overline{1-s}) = \Gamma(s)$ at $\alpha = 0.5$ and so $\overline{\Gamma(1-s)}\Gamma(1-s)$
$= \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\beta\pi i + \frac{\pi}{2})}$ by Euler's reflection formula.

Thus, combining these equalities, $\aleph(0.5 + \beta i).\overline{\aleph(0.5 + \beta i)} = 1$.

Second, consider $\aleph(\alpha+\beta i).\overline{\aleph(\alpha+\beta i)}$ in general on the critical strip $0<\alpha<1$ where all non-trivial zeroes are known to occur. To simplify some of what follows define functions f,g,h such that:

(i) $f = 2^s\pi^{s-1}\overline{2^s\pi^{s-1}} = 4^\alpha\pi^{2\alpha-2}$ and $0 < 4^a\pi^{2a-2} < 4$ for $0<\alpha<1$.

(ii) $g = \sin\frac{s\pi}{2}\overline{\sin\frac{s\pi}{2}} = (\sin\frac{\alpha\pi}{2}\cos\frac{\beta i\pi}{2} + \cos\frac{\alpha\pi}{2}\sin\frac{\beta i\pi}{2})(\sin\frac{\alpha\pi}{2}\cos\frac{\beta i\pi}{2} - \cos\frac{\alpha\pi}{2}\sin\frac{\beta i\pi}{2})$

$= (\sin^2\frac{\alpha\pi}{2}\cos^2\frac{\beta i\pi}{2} - \cos^2\frac{\alpha\pi}{2}\sin^2\frac{\beta i\pi}{2})$

$\simeq \cos^2\frac{\beta i\pi}{2}(\cos^2\frac{\alpha\pi}{2}+\sin^2\frac{\alpha\pi}{2})$ for $0<\alpha<1$ and as for β large enough $\cos^2\frac{\beta i\pi}{2} \simeq -\sin^2\frac{\beta i\pi}{2}$

$= \cos^2\frac{\beta i\pi}{2}$ which is constant on $0<\alpha<1$ for each β large enough.

Thus, $\sin\frac{s\pi}{2}\overline{\sin\frac{s\pi}{2}} \simeq \textit{constant}$ for β large enough and $0<\alpha<1$. Even $\beta>10$ suffices which is less than the first known zero of the zeta function (for the same reason this criteria will be used in what follows). Also, the value increasingly converges as β increases.

(iii) $h = \Gamma(1-s)\overline{\Gamma(1-s)}$.

Now $\frac{\partial|\aleph(s)|^2}{\partial\alpha} = f'gh + fg'h + fgh'$ and using the partial derivatives:

(i) $f' = \frac{\partial(4^\alpha\pi^{2\alpha-2})}{\partial\alpha} = 2^{(2\alpha+1)}\pi^{(2\alpha-2)}(\ln(\pi)+\ln(2))$

(ii) $h' = \frac{\partial(\Gamma(1-s)\overline{\Gamma(1-s)})}{\partial\alpha} =$
$-(\Psi(1-\alpha-i\beta)+\Psi(1-\alpha+i\beta))\Gamma(1-\alpha+i\beta)\Gamma(1-\alpha-i\beta)$,
where $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$

(iii) $g' = \frac{\partial\sin\frac{s\pi}{2}\overline{\sin\frac{s\pi}{2}}}{\partial\alpha} = (\frac{1}{2})cos(\frac{\pi}{2}(\alpha+i\beta))\pi sin((\frac{\pi}{2}(\alpha-i\beta)) + (\frac{1}{2})sin(\frac{\pi}{2}(\alpha+i\beta))cos(\frac{\pi}{2}(\alpha-i\beta))\pi$

$= \pi cos(\frac{\pi}{2}\alpha)cosh(\frac{\pi}{2}\beta)^2 sin(\frac{\pi}{2}\alpha) - \pi sin(\frac{\pi}{2}\alpha)sinh(\frac{\pi}{2}\beta)^2 cos(\frac{\pi}{2}\alpha)$

$= cos(\frac{\pi}{2}\alpha)sin(\frac{\pi}{2}\alpha)\pi$

and $0 < cos(\frac{\pi}{2}\alpha)sin(\frac{\pi}{2}\alpha)\pi < 1.6$ for $0<\alpha<1$,

then for $0<\alpha<1$ and $\beta>10$,
$\frac{\partial|\aleph(s)|^2}{\partial\alpha} = fgh(\frac{h'}{h}+\frac{g'}{g}+\frac{f'}{f}) < fgh(\frac{h'}{h}+\frac{1.6}{g}+\frac{f'}{f}) \simeq fgh(\frac{h'}{h}+\frac{1.6}{\cos^2\frac{\beta i\pi}{2}}+\frac{f'}{f})$
$\simeq fgh(\frac{h'}{h}+\frac{f'}{f})$ because $\cos^2\frac{\beta i\pi}{2}$ will be very large.

So, taking the overall partial derivative and simplifying gives

$$\frac{\partial|\aleph(s)|^2}{\partial\alpha} < -4^\alpha \pi^{(2\alpha-2)} \cosh^2 \frac{\pi\beta}{2}\Gamma(1-\alpha+i\beta)\Gamma(1-\alpha-i\beta) \tag{6}$$
$$(\Psi(1-\alpha-i\beta)+\Psi(1-\alpha+i\beta)-2ln(\pi)-2ln(2))$$

The $\Re((\Psi(\alpha+i\beta)) > 0$ if $\beta > 5/3$ and $\alpha \geq 0$ (see [5] for details). Thus, $\Re(\Psi(1-\alpha-i\beta)+\Psi(1-\alpha+i\beta)) > 0$ on $0 < \alpha < 1$ and $\beta > 5/3$. Now $(\Psi(1-\alpha-i\beta)+\Psi(1-\alpha+i\beta)-2ln(\pi)-2ln(2)) > 0$ for $\beta > 10$ on $0 < \alpha < 1$ because the first two terms are then at least > 0.3676, the approximate absolute value of the last two terms.
The term $4^\alpha \pi^{(2\alpha-2)} \cosh^2 \frac{\pi\beta}{2}$ is a very large positive real number for all $0 < \alpha < 1$ and $\beta > 10$.
The remaining term $\Gamma(1-\alpha+i\beta)\Gamma(1-\alpha-i\beta)$ is by definition a positive real number.

From these points, and considering (6), we have $\frac{\partial|\aleph(s)|^2}{\partial\alpha} < 0$ and so $|\aleph(s)|^2$ is strictly monotonic in terms of α on $0 < \alpha < 1$ for each $\beta > 10$.

Now, as $\aleph(\alpha+\beta i).\overline{\aleph(\alpha+\beta i)}$ is strictly monotonic, this function can equal 1 at only a single unique value of α for each $\beta > 10$ on the critical strip $0 < \alpha < 1$. It is already shown above that this occurs at $\alpha = 0.5$.

Thus, $\aleph(s).\overline{\aleph(s)} = 1 \Longrightarrow \alpha = 0.5$ for $\beta > 10$. □

3. Summary

In this simple way, as $\zeta(s) = 0 \Rightarrow \aleph(s).\overline{\aleph(s)} = 1$ and $\aleph(s).\overline{\aleph(s)} = 1 \Longleftrightarrow \alpha = 0.5$, the Riemann Hypothesis is shown to be true. If these results are correct then Riemann had already laid the foundation for the proof of his hypothesis by establishing the functional equation in his original paper [3].

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