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A Decision-theoretic Approach to Robust Optimization in Multivalued Graphs

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Abstract This paper is devoted to the search of robust solutions in finite graphs when costs depend on scenarios. We first point out similarities between robust optimization and multiobjective optimization. Then, we present axiomatic requirements for preference compatibility with the intuitive idea of robustness in a multiple scenarios decision context. This leads us to propose the Lorenz dominance rule as a basis for robustness analysis. Then, after presenting complexity results about the determination of Lorenz optima, we show how the search can be embedded in algorithms designed to enumerate k best solutions. Then, we apply it in order to enumerate Lorenz optimal spanning trees and paths. We investigate possible refinements of Lorenz dominance and we propose an axiomatic justification of OWA operators in this context. Finally, the results of numerical experiments on randomly generated graphs are provided. They show the numerical efficiency of the suggested approach.

Key words robust optimization, multicriteria optimization, Lorenz optima, k best solutions, minimum spanning tree, shortest path

1 Introduction

The applications of operations research techniques and decision aiding methodologies in real decision contexts have shown the importance of the problem structuring stage in a decision process, see e.g., Roy (1996). The analyst has to elaborate a formal model capturing most significant features of the problem. In particular, he has to define precisely the set of feasible solutions (the alternatives) but also to collect or elicit the relevant preference information so as to evaluate and compare the potential solutions. This evaluation task becomes more and more complex as both the size of the solutions space and the sophistication of preferences increase. Thus, in combinatorial decision problems, the set of feasible solutions is only implicitly known which often leads the analyst to consider very simple preferences structures as those induced by a scalar-valued cost function. For example, in classical graph problems like the *shortest path problem* or the *minimal spanning tree problem*, preferences over paths or trees directly derive from the sum of the costs of their components (arcs or edges). This makes it possible to define efficient constructive algorithms (Bellman, 1954; Kruskal, 1956) to determine the min-cost solution.

The assessment of such elementary costs during the structuring stage is critical because it entirely determines which solutions are preferred. However, when facing real-world problems, the adequacy of a unique and single-valued cost function can be questioned. It is indeed often unrealistic to assume the existence of a unique plausible view on the decision context, and several plausible scenarios should be considered instead. This makes the determination of scalar costs functions uneasy. For this reason, an alternative approach for the analyst consists in explicitly considering several views of the world (called here scenarios) leading to several plausible cost functions. The initial optimization problem is then reformulated as a *robust optimization problem* where the aim is to determine a solution which remains good (in some sense) whatever scenario is considered.

Note that several ideas of robustness have been studied in decision aiding (Roy, 2002). Initially, the robustness concept was introduced in OR by Rosenhead et al. in the context of dynamic planning. Quickly robustness appeared very appealing a concept in many different contexts, but taking into account specificities of the latter led to very different definitions of robustness. For example, the term "robust" has been used to qualify:

– a flexible strategy, preserving nice perspectives with respect to the various possible evolutions of the decision context (Gupta and Rosenhead, 1968);
– a prudent solution, remaining satisfactory in all possible instances of a same problem (Kouvelis and Yu, 1997; Vincke, 1999b);
– a stable conclusion, which remains valid for multiple realistic configurations of the parameters of a decision model (Roy, 1998; Vincke, 1999a).

More specifically, concerning combinatorial problems, robust discrete optimization has been first studied by Kouvelis and Yu. The underlying idea is that of prudence. The authors suggest modelling uncertainty via a set S of scenarios, defined implicitly or explicitly. They define a *robust solution* as the one which has the best performance in its worst case. They propose also another formulation of robustness based on the min-max regret criterion.

We can distinguish two main approaches to take into account the ambiguity about the costs in combinatorial problems:

– a set of possible costs is associated to each edge or arc of the graph. In this case the set S is defined implicitly as the cartesian product of these sets of costs.
– a vector of possible costs is associated to each edge or arc of the graph, one per scenario in a predefined list S of scenarios, given explicitly.

The first approach has been mainly investigated in the case where each edge or arc is valued by an interval of possible costs and S is the cartesian product of these intervals, see for instance Yaman et al. (2001), Karaşan et al. (2003), Montemanni and Gambardella (2004). The second approach has been mainly investigated in the case where a finite set of scenarios is defined, and treated in a multiobjective setting as a *min-max* or *min-max regret* optimization problem (Warburton, 1985; Hamacher and Ruhe, 1994; Yu and Yang, 1998; Murthy and Her, 1992). Conversely, Sayin and Kouvelis (2002) have shown how these approaches can be used to identify the efficient solutions of multiple objective discrete optimization problems, thus strengthening the links between robust and multicriteria optimization. In all these approaches, the robust solution is obtained by optimizing a function quantifying the robustness of solutions. This implicitly assumes that all pairs of solutions are comparable in terms of robustness. In this paper, we focus on the second approach but without necessarily assuming complete comparability of solutions.

We consider a finite set of scenarios $S = \{s_1, \ldots, s_p\}$. Therefore, the initial optimization problem can be recast in a multicriteria setting using p cost functions (one per scenario). Thus, the impact of each solution x in terms of cost is completely described by a cost vector $(x_1,\ldots,x_p)$ where x_j is an integer representing the cost of solution x if scenario s_j occurs. Hence, the comparison of solutions is reduced to the comparison of such cost vectors. In this respect, the problem can be understood as a standard multicriteria optimization problem (see also Hites et al., 2003) with three important specific features:

– all criteria share the same evaluation scale (the various criteria represent different possible views on the consequences of solutions in terms of cost).
– the criteria play a symmetric role in such problems because 1) preferences in terms of cost do not depend on the scenario considered 2) scenarios are seen as equally plausible and there is no reason to attach more importance to some of them.
– the overall objective is to determine a solution (called "robust") which is fairly well evaluated on every criterion.

Note that the first specific feature is of particular interest because many aggregation operations (over criterion values) which are meaningless in the general case become meaningful here due to the commensurateness of criteria. In this paper we are going to exploit these specificities in order to propose multicriteria decision models refining Pareto-dominance while including the idea of robustness informally introduced above. These models will receive an axiomatic characterization relying on inequality measurement and decision theory. Then we will investigate the use of such models for robust optimization in graphs. More precisely, we will provide algorithmic solutions to multicriteria discrete optimization problems with commensurate objectives, focusing on classical problems like the shortest path problem and the minimal spanning tree problem.

The paper is organized as follows: in Section 2, after an informal introduction to the robustness concept via simple examples of the robust path problem and the robust spanning tree problem, we interpret the notion of robustness in terms of equity and show the interest of generalized Lorenz dominance as a preliminary multicriteria comparison rule. In Section 3, we establish complexity results concerning the search of Lorenz optima; then, we propose a general approach based on an early scalarization of costs vectors to find robust solutions. This approach is illustrated on robust paths and robust trees problems. In Section 4, we investigate a possible refinement of the notion of robustness and provide an axiomatic justification of the Ordered Weighted Average aggregation function to define a measure of the relative robustness of solutions. Finally, in Section 5, we provide the results of numerical experiments on randomly generated instances.

2 From multicriteria analysis to robust optimization

In this section, we introduce a new concept to characterize robustness of a solution in combinatorial problems. We first present the "robust" variations of two classical problems, namely the *robust spanning tree* problem and the *robust shortest path* problem. We see them as special instances of multiobjective problems. Then, we formalize an intuitive idea of robustness by putting the emphasis on the notion of equity of a solution with respect to the different scenarios.

2.1 Motivation

We present here two examples that are representative of the intuitive idea underlying the notion of robustness.

Example 1 We now present an example from Vincke (1999b). Consider the complete graph of Figure 1 (representing a transportation network or a communication network for instance) and assume that we are looking for a minimum spanning tree. Two scenarios (representing the uncertainty about the transportation cost or the routing delay for instance) are taken into account, leading to a vector-valued graph. The first component of the vectors is associated to the first scenario and the second component of the vectors is associated to the second scenario. For the the first scenario, the spanning tree $\{((a,b),(a,d),(b,c)\}$ is optimal; for the second scenario, the spanning tree $\{(d,a),(d,b),(d,c)\}$ is optimal. However, none of these two solutions is convenient for the considered problem. Indeed, the first solution yields a cost of 17 in the second scenario, compared to a cost of 9 in the optimal solution. Similarly, the second solution yields a cost of 14 in the first scenario, compared to a cost of 8 in the optimal solution. The tree $\{(a,d),(d,b),(b,c)\}$, which yields a cost of 9 in the first scenario and a cost of 10 in the second scenario, is near the optimal value in both scenarios. It can be seen as the *robust spanning tree* in this problem, i.e. it is a network configuration that hedges against the worst possible contingency in terms of transportation costs (routing delays).

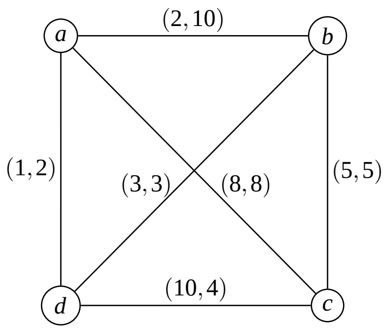

Fig. 1 The robust spanning tree problem.

Another problem of interest is the robust path problem that can be introduced by the following example:

Example 2 Consider a taxi who wants to rush a man from point a to point g in a city network the map of which is represented by the graph of figure 2. We assume the travel times are not perfectly known because they depend on scenarios concerning the traffic. For the sake of simplicity, we consider here only two scenarios s_1, s_2. Hence, each arrow representing a path from a point to another is valued by a time vector of type $(t(s_1), t(s_2))$ where $t(s_j)$ represents the expected time if scenario s_j occurs. The problem is then to determine the best path from a to g. In such a problem, one might be interested in finding a "robust" solution, i.e., a path which remains suitable whatever scenario is considered. This idea of robustness is consistent with the view of Kouvelis and Yu (1997) and Vincke (1999b) but differs from the approach for interval-valued problems. The major difference is that, in our context, costs are linked to scenarios, thus making some combinations impossible. For example, considering Figure 1, the effective cost of path (a,b,c) cannot be 7, because (a,b) and (b,c) cannot get simultaneously costs like 5 and 2 (or 3 and 4) respectively. Considering the graph pictured on Figure 2, the costs vectors of solution-paths are listed in the right table.

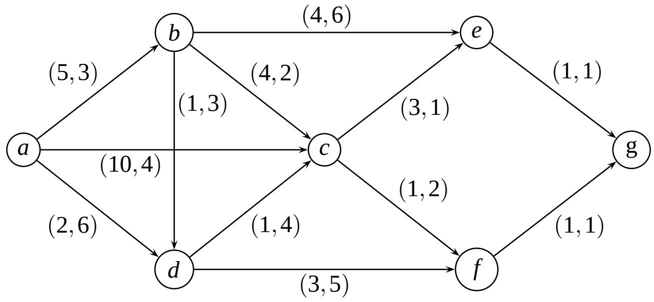

Fig. 2 The robust shortest path problem.

Path	*Vertices*	*Costs*
1	(a,b,e,g)	(10,10)
2	(a,b,c,e,g)	(13,7)
3	(a,b,c,f,g)	(11,8)
4	(a,b,d,c,e,g)	(11,12)
5	(a,b,d,c,f,g)	(9,13)
6	(a,b,d,f,g)	(10,12)
7	(a,c,e,g)	(14,6)
8	(a,c,f,g)	(12,7)
9	(a,d,c,e,g)	(7,12)
10	(a,d,c,f,g)	(5,13)
11	(a,d,f,g)	(6,12)

Facing such problems, simple scalarizations of cost-vectors do not lead to convincing results. For instance, using the average of the costs yields, among others, path 10 which is the worst solution if scenario s_2 occurs. Performing a weighted sum of the costs does not solve this problem either. Indeed, by geometrical arguments, it can easily be shown that solutions 1 and 3 cannot be obtained by minimizing a weighted sum of costs (they do not belong to the boundary of the convex hull of the points representing paths in the criteria space). Finally, focusing only on the worst cost over the scenarios (minimax criterion) is not really satisfactory due to overpessimistic evaluation. For example solution 3 cannot be obtained by the minimax criterion despite its promising costs due to the presence of the -indeed interesting- solution 1. Note that the dominance order is not more adequate since it yields too many solutions (paths 10, 11, 1, 3, 8, 7). These observations show the inadequacy of standard decision criteria to account for the idea of robustness as introduced above. Thus, the aim of the paper is to propose an axiomatic framework for robustness and a formal definition of robust solutions, and to introduce new algorithms to determine robust solutions in spanning trees and shortest paths problems.

2.2 Formalization

Let $G=(V,U)$ be a graph (oriented or not), where V is a set of vertices and $U \subseteq V \times V$ is a set of "elementary components" linking vertices (arcs in shortest path problems, edges in spanning tree problems). Considering a finite set of scenarios $S=\{s_1,\dots,s_p\}$, each elementary component in $u \in U$ is valued by a vector $(u_1,\dots,u_p)$ in $\mathbb{Z}_+^p$, where the i^{th} component represents the cost of the elementary component in scenario s_i. The feasible set of a multicriteria combinatorial problem is defined as a subset $\mathscr{F} \subseteq 2^U$ of the power set of U. Hence, assuming costs are additive, we can associate a cost vector $x \in \mathbb{Z}_+^p$ to any subset $X \in \mathscr{F}$ by setting:

$$x_i = \sum_{u \in X} u_i, \quad i = 1,\dots,p$$

Hence, the comparison of any pair (X,Y) of subsets in $\mathscr{F}$ in terms of costs amounts to comparing the vectors $(x_1,\dots,x_p)$ and $(y_1,\dots,y_p)$. In this framework, the classical dominance notions used in multiobjective optimization apply:

Definition 1 *The Weak-Pareto dominance relation (WP-dominance for short) on cost-vectors of $\mathbb{Z}_+^p$ is defined, for all $x,y \in \mathbb{Z}_+^p$ by:*

$$x \succsim_P y \iff [\forall i \in \{1,\dots,p\}, x_i \le y_i)]$$

The Pareto dominance relation (P-dominance for short) on cost-vectors of $\mathbb{Z}_+^p$ is defined as the asymmetric part of $\succsim_P$:

$$x \succ_P y \iff [x \succsim_P y \text{ and } not(y \succsim_P x)]$$

Definition 2 *Within a set A any element x is said to be* P-dominated *when $y \succ_P x$ for some y in A, and* P-efficient *when there is no y in A such that $y \succ_P x$.*

In order to decide whether a path is better than another, we want to define a transitive preference relation $\succsim$ on cost-vectors capturing both the aim of cost-minimization and the idea of robustness. For this reason, the preference relation is expected to satisfy the following axioms:

P-Monotonicity. For all $x, y \in \mathbb{Z}_+^p$, $x \succsim_P y \Rightarrow x \succsim y$ and $x \succ_P y \Rightarrow x \succ y$,

where $\succ$ is the strict preference relation defined as the asymmetric part of $\succsim$. This natural unanimity principle says that, if path x has a lower cost than path y whatever the scenario considered, then x is preferred to y, and this preference is strict as soon as $x \neq y$. In addition, the idea of robustness refers to equity in cost distribution among scenarios which can be expressed by the following axiom:

Transfer Principle. Let $x \in \mathbb{Z}_+^p$ such that $x_i > x_j$ for some i, j. Then for all ε such that $0 \leq \varepsilon \leq x_i - x_j$, $x - \varepsilon e_i + \varepsilon e_j \succsim x$ where e_i (resp. e_j) is the vector whose i^{th} (resp. j^{th}) component equals 1, all others being null.

This axiom captures the idea of robustness as follows: if $x_i > x_j$ for some cost-vector $x \in \mathbb{Z}_+^p$, slightly improving (here decreasing) component x_i to the detriment of x_j while preserving the mean of the costs would produce a better distribution of costs, and consequently a more robust solution. Hence, path 1 should be as least as good as path 7 in Example 1 because there is an admissible transfer of size 4 between vectors (14, 6) and (10, 10). Note that using a similar transfer of size greater than 8 would increase inequality in terms of costs. This explains why the transfers must have a size $\varepsilon \leq x_i - x_j$. Such transfers are said to be *admissible* in the following. They are known as *Pigou-Dalton transfers* in Social Choice Theory, where they are used to reduce inequality in the income distribution over a population (see Sen, 1997 for a survey).

Since elementary permutations of the vector $(x_1, \ldots, x_p)$ that just interchange two coordinates can be achieved using an admissible transfer, and since any permutation of $\{1, \ldots, p\}$ is the product of such elementary permutations, the Transfer Principle implies the following axiom:

Symmetry. For all $x \in \mathbb{Z}_+^p$, for all permutations π of $\{1, \ldots, p\}$,
$(x_{\pi(1)}, \ldots, x_{\pi(p)}) \sim (x_1, \ldots, x_p)$,

where $\sim$ is the indifference relation defined as the symmetric part of $\succsim$. This axiom is natural in our context. Since no information about the likelihood of scenarios is available, they must be treated equivalently.

Note that the transfer principle possibly provides arguments to discriminate between vectors having the same average-cost but does not apply in the comparison of vectors having different average-costs. However, the possibility of discriminating is improved when combining the Transfer Principle with P-monotonocity. For example, consider paths 7 and 8 in Table 1 whose cost vectors are $(14, 6)$ and $(12, 7)$ respectively. Although P-dominance cannot discriminate between these two vectors, the discrimination is possible for any preference relation $\succsim$ satisfying both the Transfer Principle and the P-monotonocity axiom. Indeed, on the one hand, $(12, 7) \succ_P (13, 7)$ and therefore $(12, 7) \succ (13, 7)$ thanks to P-monotonicity; on the other hand, $(13, 7) \succsim (14, 6)$ thanks to the Transfer Principle applied to the transfer $(14 - 1, 6 + 1) = (13, 7)$. Hence, we get: $(12, 7) \succ (14, 6)$ by transitivity. In order to better characterize those vectors that can be compared using such combination of the P-monotonicity and the Transfer Principle we recall the definition of Lorenz vectors and related concepts (for more details see e.g. Marshall and Olkin, 1979; Shorrocks, 1983):

Definition 3 *For all $a \in \mathbb{Z}_+^p$, the* Generalized Lorenz Vector *associated to x is the vector:*

$$L(x) = (x_{(1)}, x_{(1)} + x_{(2)}, \ldots, x_{(1)} + x_{(2)} + \ldots + x_{(p)})$$

where $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(p)}$ represents the components of x sorted by decreasing order. The k^{th} component of $L(x)$ is $L_k(x) = \sum_{i=1}^{k} x_{(i)}$.

Definition 4 *The Generalized Lorenz dominance relation (L-dominance for short) on $\mathbb{Z}_+^p$ is defined by:*

$$\forall x, y \in \mathbb{Z}_+^p,\ x \succsim_L y \iff L(x) \succsim_P L(y)$$

The notion of Lorenz dominance was initially introduced to compare vectors with the same average cost and its link to the Transfer Principle was established by Hardy et al. (1934). The generalized version of L-dominance considered here is classical (see e.g. Marshall and Olkin, 1979) and allows any pair of vectors in $\mathbb{Z}_+^p$ to be compared. This notion has been used recently for characterizing equitable solutions in multicriteria optimization (Kostreva and Ogryczak, 1999; Kostreva et al., 2004).

Within a set X, any element x is said to be *L-dominated* when $y \succ_L x$ for some y in X, and *L-efficient* when there is no y in X such that $y \succ_L x$. For illustration purposes, the L-dominance cone in the bi-scenario case is depicted in Figure 3 (the subset of points L-dominated by vector $(7,5)$). In order to establish the link between Generalized Lorenz dominance and preferences satisfying combination of P-Monotonocity and Transfer Principle we recall a result of Chong (1976):

Theorem 1 *For any pair of distinct vectors $x, y \in \mathbb{Z}_+^p$, if $x \succsim_P y$, or if x is obtained from y by a Pigou-Dalton transfer, then $x \succsim_L y$. Conversely, if $x \succsim_L y$, then there exists a sequence of admissible transfers and/or Pareto-improvements to transform y into x.*

This theorem establishes $\succsim_L$ as the minimal transitive relation (with respect to set inclusion) satisfying simultaneously P-Monotonicity and the Transfer Principle. As a consequence, the subset of L-efficient elements appears as a very natural solution to choice problems with multiple scenarios, as far as robustness is concerned. For this reason, we investigate in the next section the generation of the set of L-efficient paths in a graph.

3 Seeking robust path and trees

3.1 Complexity issues

3.1.1 Robust shortest path problem. We investigate here the computational complexity of the search of the set of L-efficient paths. Note first that the L-efficient solutions are a subset of the P-efficient solutions which might be very numerous. We wish to evaluate the extent to which focusing on L-efficient solutions (rather than P-efficient solutions) reduces the size of the solutions space. In this respect, the study of the pathological instance introduced in Hansen (1980) for the multi-objective shortest path problem is interesting (one looks for the set of P-efficient paths from a source node to a destination node, see Figure 4). In that bivalued graph, all the paths from node 1 to node $n = 2q + 1$ have the same average-cost (whose value is $(2^q - 1)/2$) but distinct costs on the first component (due to the uniqueness

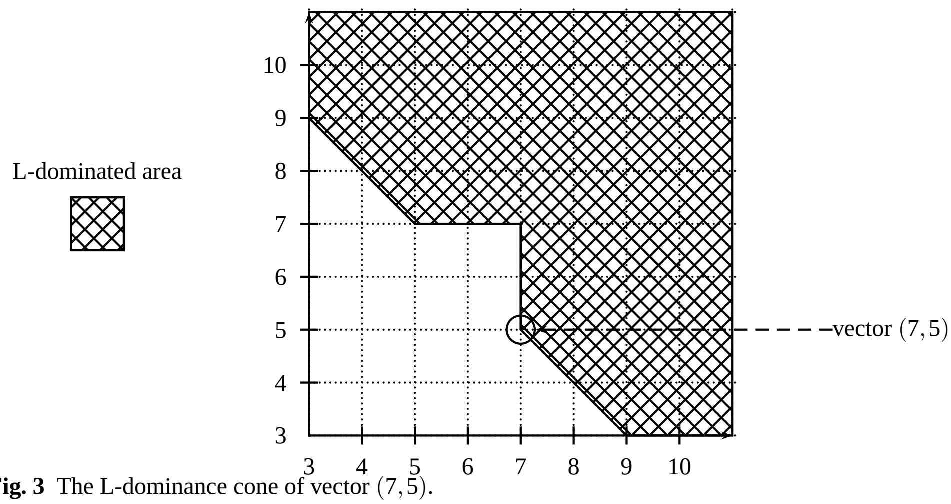

Fig. 3 The L-dominance cone of vector $(7,5)$.

of the binary representation of an integer). The resulting set of cost-vectors is $\{(x, 2^q - 1 - x), x \in \{0, \ldots, 2^q - 1\}\}$, which contains only P-efficient elements by construction. Notice that the cardinality of this set is exponential in the size of the graph. However, due to the Transfer Principle, there exists only two L-efficient cost vectors (those minimizing the difference between their components).

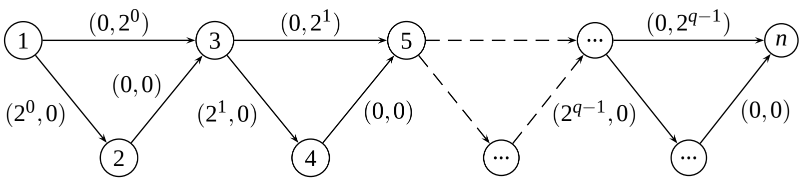

Fig. 4 An instance by Hansen where all paths are P-efficient.

Unfortunately, it is possible to adapt the previous instance so as to get a pathological instance for our problem. Starting from the graph of Hansen, we add an arc valued in such a way that the set of points associated to the paths of the graph shifts above the bisecting line (so that the symmetry axiom cannot apply), and we slightly modify the cost-vectors of the arcs in order to change the angle between the alignment of the points and the bisecting line (so that the more well-balanced is a point, the more costly is its value, and therefore the transfer principle cannot apply). The resulting instance is given on Figure 5 (where every arc without cost-vectors is actually valued $(0,0)$). Let us show that all the paths from node 0 to node $n = 2q + 1$ have distinct Lorenz vectors and are L-efficient. The set of cost vectors associated with the paths of the graph is $\{(2x, 3 \times 2^q - x), x \in \{0, \ldots, 2^q - 1\}\}$. Note that the second component is always greater than the first component for $x \in \{0, \ldots, 2^q - 1\}$. Consequently, the corresponding set of Lorenz vectors writes $\{(3 \times 2^q - x, 3 \times 2^q + x), x \in \{0, \ldots, 2^q - 1\}\}$. All Lorenz vectors have the same average-cost and distinct values on the first component. Moreover, the size of that set is exponential in the size of the graph.

Due to the potentially exponential number of L-efficient paths, we get the following proposition:

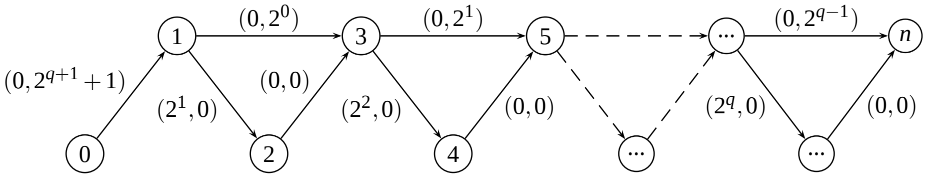

Fig. 5 An instance where all paths are L-efficient.

Proposition 1 *The problem of finding L-efficient paths in a graph is, in worst case, intractable, i.e. requires for some problems a number of operations which grows exponentially with the size of the problem.*

In other respects, one may be interested in the complexity of deciding whether there exists a path whose cost distribution L-dominates a given cost-vector. The following result establishes that this decision problem cannot be solved in polynomial time unless $P = NP$:

Proposition 2 *Deciding whether there exists a path whose cost distribution L-dominates a given cost-vector is an NP-complete decision problem.*

Proof We reduce the partition problem to our problem.

instance: Finite set $A = \{a_1, \ldots, a_q\}$ and a size $s(a) \in \mathbb{Z}_+$ for each $a \in A$.

question: Is there a subset $A' \subseteq A$ such that $\sum_{a\in A'} s(a) = \sum_{a\in A-A'} s(a)$?

That problem is proved NP-complete (see e.g. Garey and Johnson, 1979). One constructs -in polynomial time- a graph as indicated on Figure 6 (where every arc without cost-vectors is actually valued $(0,0)$). Deciding whether there exists a path from node 1 to node $n = 2q+1$ such that its vector-cost L-dominates the vector $(\frac{\sum_{a\in A} s(a)}{2}, \frac{\sum_{a\in A} s(a)}{2})$ amounts to solve the partition problem. ■

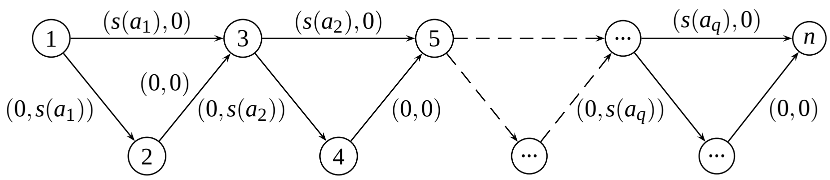

Fig. 6 Reduction from partition problem to robust shortest path.

3.1.2 Robust spanning tree problem. We investigate here the computational complexity of the search of the set of L-efficient spanning trees. Similarly to the study of the multiobjective shortest path problem, Emelichev and Perepelitsa (1988) and Hamacher and Ruhe (1994) have constructed instances of the multi-objective spanning tree problem for which the n^{n-2} spanning trees of a complete graph with n vertices are distinct and $\succsim_P$-efficient. The authors consider complete graphs with n vertices and a set $E = \{e_1, \ldots, e_m\}$ of edges valued by $(2^{i-1}, 2^m - 2^{i-1})$ $(i = 1, \ldots, m)$. Consequently, the average-cost of an edge is 2^{m-1} for all $e_i \in E$ and therefore the average-cost of a tree is $(n-1)2^{m-1}$ for any spanning tree. Once more, by the uniqueness of the binary representation of an integer, all the spanning trees have distinct costs on the first component. Thus, the n^{n-2} spanning trees of the graph (Cayley, 1889) are P-efficient and are associated with distinct cost-vectors. However, as in the robust shortest path problem, there exists only two L-efficient cost vectors (those minimizing the difference between their components).

This instance can be easily transformed to get a pathological instance for our robust spanning tree problem, by slightly modifying the cost-vectors of the edges. We consider complete graphs with n vertices and a set $E = \{e_1, \ldots, e_m\}$ of edges valued by $(2^{i-1} + 2^m, 2^m - 2^i)$ $(i = 1, \ldots, m)$. The set of cost vectors associated with the spanning trees of the graph is a subset of $\{(\sum_{i \in X} 2^{i-1} + (n-1)2^m, (n-1)2^m - \sum_{i \in X} 2^i) : X \subseteq \{1, \ldots, m\}$ and $|X| = n-1\}$. Note that the first component is always greater than the second component. Consequently, the corresponding set of Lorenz vectors writes $\{(\sum_{i \in X} 2^{i-1} + (n-1)2^m, 2(n-1)2^m - \sum_{i \in X} 2^{i-1}) : X \subseteq \{1, \ldots, m\}$ and $|X| = n-1\}$. All Lorenz vectors have the same average-cost (whose value is $3(n-1)2^{m-1}$) and distinct values on the first component. Moreover, the size of the set of spanning trees is exponential in the size of the graph.

In other respects, once again, one may be interested in the complexity of deciding whether there exists a spanning tree whose cost distribution L-dominates a given cost-vector. The following result establishes that this decision problem cannot be solved in polynomial time unless $P = NP$:

Proposition 3 *Deciding whether there exists a spanning tree whose cost distribution L-dominates a given cost-vector is an NP-complete decision problem.*

Proof Similarly to the robust shortest path problem, we reduce the partition problem to our problem. One constructs -in polynomial time- a grid graph as indicated on Figure 7. Deciding whether there exists a spanning tree such that its vector-cost L-dominates the vector $(\frac{\sum_{a \in A} s(a)}{2}, \frac{\sum_{a \in A} s(a)}{2})$ amounts to solve the partition problem. ■

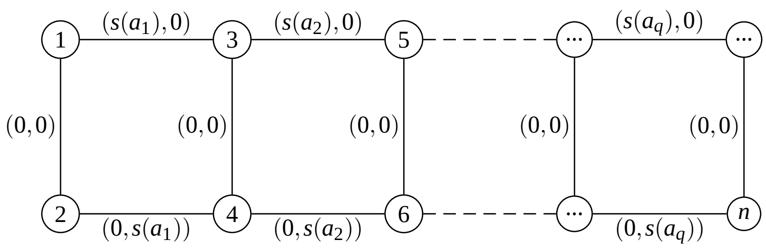

Fig. 7 Reduction from partition problem to robust spanning tree.

3.2 A general algorithmic approach

We show here how k best solutions algorithms can be used for robust discrete optimization. Ranking algorithms have been already used in various contexts in multicriteria combinatorial optimization:

– *Enumeration of P-efficient solutions in the bicriteria shortest path problem* (Climaco and Martins, 1982). The idea of the method is to start with a lexicographically optimal path (which is Pareto-optimal) and then construct the second best path for the first objective, the third, etc. until the nadir value on the first criterion is reached (the nadir point is characterized by the componentwise maximum of all P-efficient solutions).

– *Search for an optimal solution of the max-ordering multicriteria spanning tree problem* (Hamacher and Ruhe, 1994). In these problems, one looks for a solution minimizing the worst possible cost among criteria (i.e., a solution X minimizing $\max_{i=1,\dots,p} x_i$). The idea of the method is to enumerate the k best solutions with respect to a weighted sum of the criteria. The authors give a stopping condition in the enumeration of the k best solutions so that the optimal solution is included in the generated subset of solutions. Ehrgott and Skriver (2003) proposed recently a refinement in the bicriteria case. They suggest resorting to a two-phase method. The first phase allows to determine an adequate weighting vector for the ranking algorithm used in the second phase.

Similarly to this latter approach, our algorithm is based on the insertion of a stopping condition in the enumeration of the k best solutions. More precisely, it relies on the following result:

Proposition 4 *A cost-vector $(x_1,\dots,x_p)$ L-dominates any cost-vector $(y_1,\dots,y_p)$ such that*

$$\sum_{i=1}^{p} y_i > p.x_{(1)} \tag{1}$$

Proof By contradiction, assume that:

$$\sum_{i=1}^{p} y_{(i)} > p.x_{(1)} \quad (i)$$

and

$$\exists k \leq p \text{ such that } \sum_{i=1}^{k} y_{(i)} < \sum_{i=1}^{k} x_{(i)} \quad (ii)$$

Since $\sum_{i=1}^{k} x_{(i)} \leq k.x_{(1)}$, (ii) implies that:

$$\sum_{i=1}^{k} y_{(i)} < k.x_{(1)} \quad (iii)$$

Furthermore, we have:

$$\sum_{i=k+1}^{p} y_{(i)} \leq (p-k)y_{(k)} \quad (iv)$$

However, we know that $y_{(1)} > x_{(1)}$ by (i). Consequently, by (ii), $\exists j \in \{1,\dots,k\}$ such that $y_{(j)} < x_{(j)} \leq x_{(1)}$ and therefore $y_{(k)} < x_{(1)}$ (since $y_{(k)} \leq y_{(j)}$). Then, we deduce from (iv):

$$\sum_{i=k+1}^{p} y_{(i)} < (p-k)x_{(1)} \quad (v)$$

Finally, we get by (iii) and (v):

$$\sum_{i=1}^{p} y_{(i)} < p.x_{(1)}$$

which yields a contradiction with (i). ■

In the bi-scenario case, this result can be represented graphically (see Figure 8). In Figure 8, the hatching area corresponds to the set of vectors which have a total cost greater than $2 \times 7 = 14$. This approximated cone is as better as the considered cost-vector is well-balanced (near the bisecting line).

Thus, Proposition 4 suggests enumerating the k best solutions with respect to the sum of the criteria, and stopping the enumeration as soon as one finds a cost-vector y such that condition (1) holds. Hence, no L-efficient solution belongs to the set of remaining solutions. Furthermore, the following observation allows to speed up the search for L-efficient solutions among generated solutions:

Remark 1 A cost vector x cannot be L-dominated by a cost vector y such that:

$$\sum_{i=1}^{p} y_i > \sum_{i=1}^{p} x_i$$

Hence, we enumerate the list of solutions in increasing order of their total cost and once we know that all other solutions have a greater total cost, we only have to compare a solution with the previously generated ones to decide wether it is L-efficient or L-dominated.

In order to present more formally our algorithm, we introduce additional notations:

- X^k denotes the k^{th} best solution, and x^k is the associated cost-vector,
- b denotes the highest cost among generated solutions,
- LL denotes the set of L-efficient solutions currently found,
- YY denotes a buffer set of solutions which have all the same total cost.

Remark that our method remains valid whatever the number of scenarios since we enumerate the list of solutions with respect to the total cost (and not with respect to a particular scenario as it is sometimes done to enumerate P-efficient solutions in bicriteria problems).

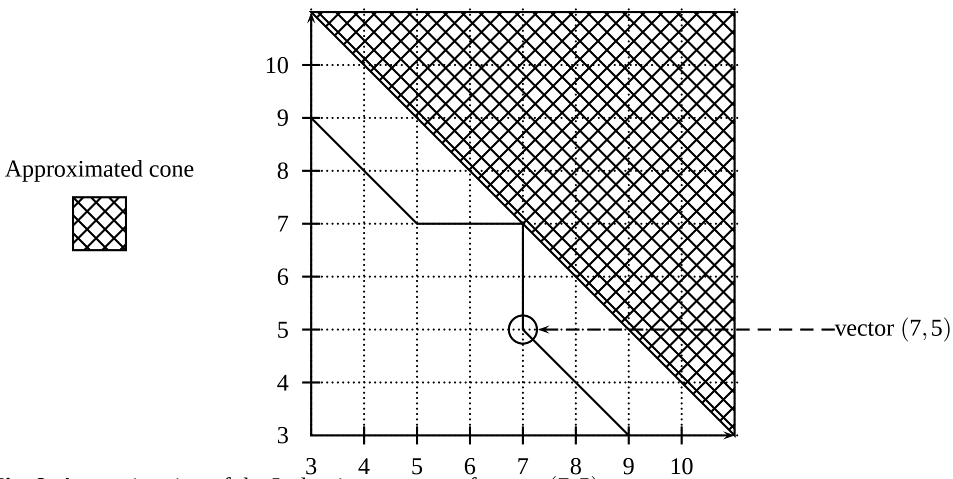

Fig. 8 Approximation of the L-dominance cone of vector $(7,5)$.

Algorithm 1 *Lorenz Optimization*

$X^1 \leftarrow \arg\min_{X \in \mathscr{F}} \sum_{i=1}^{p} x_i$;
Set $b \leftarrow \max_{i=1,\dots,p} x_i^1$; $LL \leftarrow \emptyset$; $YY \leftarrow \{X^1\}$; $k \leftarrow 2$;
While $\sum_{i=1}^{p} x_i^{k-1} \leq p \times b$
    Compute the k^{th} best solution X^k;
    If $\max_{i=1,\dots,p} x_i^k < b$ then $b \leftarrow \max_{i=1,\dots,p} x_i^k$;
    If $\sum_{i=1}^{p} x_i^k > \sum_{i=1}^{p} x_i^{k-1}$ then
        Complete the set LL with solutions in YY that are L-efficient in $LL \cup YY$;
        $YY \leftarrow \emptyset$;
    end
    $YY \leftarrow YY \cup \{X^k\}$;
    $k \leftarrow k+1$;
end
Output the set LL of L-efficient solutions;

end

We now illustrate the process of our algorithm on the instance of the robust shortest path problem given in Section 2 (see Figure 9). Note that we choose arbitrarily the order in which solutions with the same total cost are generated. First solution 10 is generated, and therefore the bound is set to $2 \times 13 = 26$. Then, solution 11 is generated, and the bound is updated to $2 \times 12 = 24$. Next solutions have a greater total cost, so we can already determine which solutions are L-efficient among solution 10 and solution 11. Solution 10 is L-dominated by solution 11, and therefore solution 11 is the unique L-efficient solution among both solutions. Afterwards, solutions 9, 3 and 8 are generated. The bound is updated to $2 \times 11 = 22$ (due to solution 3). Solutions 9 and 8 are discarded since they are L-dominated by solutions 3 and 11. Solution 3 is not L-dominated by solution 11, and therefore it is L-efficient. Finally, solutions 1, 2 and 7 are generated and the bound is updated to $2 \times 10 = 20$ (due to solution 1). Solution 1 is L-efficient and the other are L-dominated. The stopping condition is now satisfied since the other solutions have a total cost strictly greater than 20.

Due to the underlying use of k best solutions algorithms, our method can be applied to combinatorial problems for which efficient k best solutions algorithms are known. We focus here on the minimum spanning tree problem and the shortest path problem:

- for the generation of weighted spanning trees in order, we use an algorithm of Gabow (1977). Its complexity is $O(m\log(\beta(m,n))$ for a graph with n vertices and m edges, where $\beta(m,n) = \min\{i : \log^{(i)} n \leq m/n\}$ and $\log^{(i)} x$ denotes the log function iterated n times. It proceeds by successive edges exchanges.
- for the generation of weighted paths in order, we use an algorithm of Eppstein (1998). Its complexity is $O(m + n\log n + k)$ for a graph with n vertices and m edges. It proceeds by computing the shortest paths tree and constructing a new graph representing every possible deviations from the shortest path.

Of course, our approach could also be applied to several other combinatorial problems for which efficient k best solutions algorithms are known, such as matching (e.g., Chegireddy and Hamacher, 1987), scheduling (e.g., Brucker and Hamacher, 1989), network flows (e.g., Hamacher, 1995).

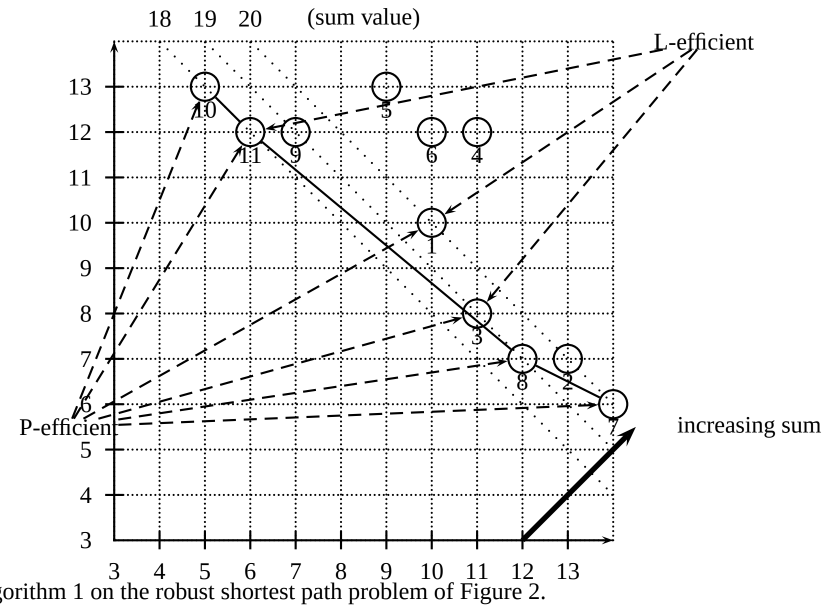

Fig. 9 Process of Algorithm 1 on the robust shortest path problem of Figure 2.

4 Discriminating between L-efficient solutions

As shown in the previous section, the set of L-efficient solutions is a subset of P-efficient solutions and, as such, it might contain an important number of elements. However, all these L-efficient elements are not equivalent for the decision maker. For example, cost vector $x = (10,10)$ might be seen as more robust than $y = (18,1)$, despite the fact that no dominance holds between $L(x) = (10,20)$ and $L(y) = (18,19)$. Besides, cost vector $z = (11,1)$ might be preferred to x despite the fact that no dominance holds between $L(z) = (11,12)$ and $L(x)$. Thus, L-dominance only provides a preliminary filter for robustness analysis, quite similarly to P-dominance in multi-objective optimization. To go further, we need a sharper preference model allowing better discrimination between solutions and possibly enabling different attitudes towards robustness to be captured. We propose below an axiomatic approach aiming at introducing a preference weak-order $\succsim$ on $X = \mathbb{R}^p_+$ consistent with L-dominance.

The first axiom requires that discrimination between solutions must be founded on Lorenz vectors. Hence, we do not want to discriminate between solutions having the same Lorenz vector, which writes:

Neutrality. For all x,y in $\mathbb{R}^p_+$, $L(x) = L(y) \Rightarrow x \sim y$.

We may define a preference relation $\succsim'$ among Lorenz vectors of $L(\mathbb{R}^p_+) = \{v \in \mathbb{R}^p_+ : \exists x \in \mathbb{R}^p_+, v = L(x)\}$ by setting:

$$\forall\, L,M \in L(\mathbb{R}^p_+),\ L \succsim' M \ \Leftrightarrow\ \exists x,y \in \mathbb{R}^p_+, \begin{cases} L(x) = L \text{ and } L(y) = M \\ x \succsim y \end{cases}$$

For the sake of convenience, we now use $\succsim$ instead of $\succsim'$ to denote the preference relation among Lorenz vectors. As we intend the preference relation to refine L-dominance, we need the following axiom:

Strict L-Monotonicity. $L(x) \succ_P L(y) \Rightarrow x \succ y$.

Then we introduce three axioms that can be seen as counterparts of von Neumann and Morgenstern (1947) axioms adapted for Lorenz vectors. As the former are used to characterized preferences representable by a utility function, we will use the latter to characterize a measure of robustness. The first of them is the weak-order assumption (in order to discriminate between solutions).

Complete weak-order. $\succsim$ is reflexive, transitive and complete.

We introduce now a continuity axiom for preferences over Lorenz vectors, using Jensen's classical formulation (Jensen, 1967):

Continuity. Let $L, M, N \in L(\mathbb{R}^p_+)$ such that $L \succ M \succ N$. There exists $\alpha, \beta \in]0,1[$ such that:

$$\alpha L + (1-\alpha)N \succ M \succ \beta L + (1-\beta)N$$

Continuity of preferences formalizes the intuitive notion that if two elements in $L(\mathbb{R}^p_+)$ are not very different, then their utilities should be closed together (Fishburn, 1970). More precisely, consider two vectors L and N such that $L \succ N$, and L_γ the vector resulting from the convex combination $\gamma L + (1-\gamma)N$. For any M such that $L \succ M \succ N$, when γ is closed to 1, L_γ is closed to L and therefore L_γ is preferred to M, provided continuity holds. Similarly, when γ is closed to 0, L_γ is closed to N and therefore M is preferred to L_γ.

Independence. Let L, M, N belong to $L(\mathbb{R}^p_+)$. Then, for all $\alpha \in]0,1[$:

$$L \succ M \Longrightarrow \alpha L + (1-\alpha)N \succ \alpha M + (1-\alpha)N$$

This axiom requires that the preference between two Lorenz vectors does not depend on their common components. It is important to observe that this independence axiom is a weakening of the usual independence axiom on $\mathbb{R}^p_+$, obtained by restriction to comonotonic vectors. Recall that x and y in $\mathbb{R}^p_+$ are said to be *comonotonic* if $x_i > x_j$ and $y_i < y_j$ for no $i, j \in \{1,\dots,p\}$ (see Yaari, 1987). Indeed, for any pair x, y of comonotonic vectors, there exists a permutation π of $\{1,\dots,p\}$ such that $x_{\pi(1)} \geq x_{\pi(2)} \geq \dots \geq x_{\pi(p)}$ and $y_{\pi(1)} \geq y_{\pi(2)} \geq \dots \geq y_{\pi(p)}$. Consequently, $L(\alpha x + (1-\alpha)y) = \alpha L(x) + (1-\alpha)L(y)$. Hence, for all comonotonic vectors $x, y, z \in \mathbb{R}^p_+$, if $x \succ y \Longrightarrow \alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$ then $L(x) \succ L(y) \Longrightarrow \alpha L(x) + (1-\alpha)L(z) \succ \alpha L(y) + (1-\alpha)L(z)$. Observing that for any triple L, M, N of Lorenz vectors, there exists x, y, z, three comonotonic vectors in $\mathbb{R}^p_+$ such that $L = L(x), M = L(y)$ and $N = L(z)$, we deduce that usual independence on $\mathbb{R}^p_+$ implies independence on $L(\mathbb{R}^p_+)$.

Note that weakening the usual independence axiom is necessary in our framework due to its incompatibility with the Strict L-monotonicity axiom, as shown by the following:

Example 3 Let us consider $x = (24,24)$, $y = (22,26)$ and $z = (26,22)$. Due to Strict L-monotonicity, $x \succ y$. Hence, usual independence would imply $(25,23) = \frac{1}{2}x + \frac{1}{2}z \succ \frac{1}{2}y + \frac{1}{2}z = (24,24)$ which is in contradiction with $(24,24) \succ_L (25,23)$.

The conflict here can be explained as follows: on the one hand, the cost-dispersion of vector $(25,23)$ resulting from the combination of x and z is greater than that of $x = (24,24)$; on the other hand, the cost dispersion of vector $(24,24)$ resulting from the combination of y and z is smaller than that of

$y = (22,26)$. This situation cannot occur when x, y and z are pairwise comonotonic, which explains the very idea of our independence axiom. Indeed, assuming that x, y and z are pairwise comonotonic, this axiom states that if an individual prefers a cost-vector x to a cost vector y, then he should also prefer a $\alpha/(1-\alpha)$ chance of getting x or y to a $\alpha/(1-\alpha)$ chance of getting x or z.

Actually, a similar idea was already present in Dual Choice Theory under Risk (see Yaari, 1987) in the form of the *Dual independance axiom*. The link with Yaari's theory under Risk is natural here since Lorenz vectors can be seen as counterparts of cumulative distribution functions in decision under risk.

Before introducing our representation theorem, we need to show that $L(\mathbb{R}^p_+)$ with the usual convex combination in vector spaces is a *mixture set* (Herstein and Milnor, 1953):

Definition 5 *A set $\mathscr{M}$ is said to be a mixture set if for any $x,y \in \mathscr{M}$ and for any α we can associate another element, which we write as $\alpha x + (1-\alpha)y$, which is again in $\mathscr{M}$, and where:*

M1. $1x + 0y = x$,
M2. $\alpha x + (1-\alpha)y = (1-\alpha)y + \alpha x$,
M3. $\alpha[\beta x + (1-\beta)y] + (1-\alpha)y = (\alpha\beta)x + (1-\alpha\beta)y$,

for all x,y in $\mathscr{M}$ and all α,β in $[0,1]$.

We have:

Lemma 1 *$L(\mathbb{R}^p_+)$ is a mixture set with respect to the usual convex combination in vector spaces.*

Proof Let $L,M \in L(\mathbb{R}^p_+)$. We first establish that $\alpha L + (1-\alpha)M$ belongs to $L(\mathbb{R}^p_+)$. Since L and M are Lorenz vectors, there exists x and y in $\mathbb{R}^p_+$ such that $L(x) = L$ and $L(y) = M$. Consider now $\overline{x} = (x_{(1)},\ldots,x_{(p)})$ and $\overline{y} = (y_{(1)},\ldots,y_{(p)})$. Remark that $L(\overline{x}) = L(x) = L$ and $L(\overline{y}) = L(y) = M$. It is easy to check that $\alpha L + (1-\alpha)M = \alpha L(\overline{x}) + (1-\alpha)L(\overline{y}) = L(\alpha\overline{x} + (1-\alpha)\overline{y})$ since $\overline{x}$ and $\overline{y}$ are comonotonic by construction. Therefore $\alpha L + (1-\alpha)M \in L(\mathbb{R}^p_+)$. Then, M1 and M2 being straightforward, we only prove M3: $\alpha[\beta L + (1-\beta)M] + (1-\alpha)M = \alpha\beta L + \alpha M - \alpha\beta M + M - \alpha M = \alpha\beta L + (1-\alpha\beta)M$. ∎

A linear function on a mixture set is defined as follows:

Definition 6 *$\varphi : \mathscr{M} \to \mathbb{R}$ is linear if $\varphi(\alpha x + (1-\alpha)y) = \alpha\varphi(x) + (1-\alpha)\varphi(y)$ for all $\alpha \in [0,1]$ and $x,y \in \mathscr{M}$.*

Note that here, since the mixture operation coincides with the usual convex combination in vector spaces, φ is automatically p-linear:

$$\varphi(\sum_{i=1}^{p} \alpha_i x_i) = \sum_{i=1}^{p} \alpha_i \varphi(x_i)$$

with $\sum_{i=1}^{p} \alpha_i = 1$ and $\alpha_i \in [0,1]$ for all i (proof by induction).

Moreover, vectors $\ell_i = (1,2,\ldots,i-1,i,\ldots,i)$ for $i = 1,\ldots,p$, form a basis of $L(\mathbb{R}^p_+)$; in particular, every element of $L(\mathbb{R}^p_+)$ can be seen as a linear combination of those vectors. Indeed, by setting $\ell_0 = (0,\ldots,0)$ and $\ell_{p+1} = \ell_p$, we can write $e_i = 2\ell_i - \ell_{i-1} - \ell_{i+1}$ for all i in $\{1,\ldots,p\}$, where e_i is the vector whose i^{th} component equals 1, all others being null. Consequently, every vector L of $L(\mathbb{R}^p_+)$ can be written:

$$\begin{aligned} L = \textstyle\sum_{i=1}^{p} L_i e_i &= \textstyle\sum_{i=1}^{p} L_i(2\ell_i - \ell_{i-1} - \ell_{i+1}) \\ &= \textstyle\sum_{i=1}^{p} (2L_i - L_{i-1} - L_{i+1})\ell_i \end{aligned} \tag{2}$$

with the convention $L_0 = 0$ and $L_{p+1} = L_p$. We can now establish our representation theorem:

Theorem 2 *A preference relation $\succsim$ satisfies Neutrality, Strict L-monotonicity, Complete weak-order, Continuity and Independence if and only if there exists a linear function φ on $L(\mathbb{R}^p_+)$ such that:*

$$x \succsim y \iff \varphi(L(x)) \leq \varphi(L(y))$$

$$\begin{aligned} \textit{where } & \varphi(L(x)) = \textstyle\sum_{i=1}^{p}(2\varphi(\ell_i) - \varphi(\ell_{i-1}) - \varphi(\ell_{i+1}))L_i(x) \\ \textit{and } & \varphi(\ell_i) - \varphi(\ell_{i-1}) > \varphi(\ell_{i+1}) - \varphi(\ell_i) > 0 \ \textit{ for all } i \end{aligned}$$

Proof By Neutrality, $x \succsim y$ iff $L(x) \succsim L(y)$ and therefore assuming a complete weak-order on $\mathbb{R}^p_+$ amounts to assuming a complete weak-order on $L(\mathbb{R}^p_+)$. Consequently, Complete weak-order, Continuity and Independence hold. Herstein and Milnor (1953) have shown that, given $\mathscr{M}$ a mixture set and $\succsim$ a preference relation on $\mathscr{M}$, the following two statements are equivalent:

– Complete weak-order, Continuity and Independence hold;
– there exists a linear function φ on $\mathscr{M}$ such that, for all $x, y \in \mathscr{M}$, $x \succ y \iff \varphi(x) < \varphi(y)$.

By Lemma 1, $L(\mathbb{R}^p_+)$ is a mixture set. Hence, there exists a linear function φ on $L(\mathbb{R}^p_+)$ such that, for all $L, M \in L(\mathbb{R}^p_+)$, $L \succ M \iff \varphi(L) < \varphi(M)$.

In other respects, for every vector $L(x)$ of $L(\mathbb{R}^p_+) \setminus \{\ell_0\}$ we have:

$$2L_i(x) - L_{i-1}(x) - L_{i+1}(x) = x_{(i)} - x_{(i+1)} \geq 0$$

for $i = 1, \ldots, p$ with the convention $x_{(p+1)} = 0$. Moreover we have:

$$\sum_{i=1}^{p}(2L_i(x) - L_{i-1}(x) - L_{i+1}(x)) = \sum_{i=1}^{p} x_{(i)} - \sum_{i=1}^{p} x_{(i+1)} = x_{(1)}$$

Hence the coefficients $(2L_i(x) - L_{i-1}(x) - L_{i+1}(x))/x_{(1)}$ are positive and add-up to 1. By the p-linearity of φ, $\varphi(\ell_0) = 0$ and for every vector $L(x)$ of $L(\mathbb{R}^p_+) \setminus \{\ell_0\}$ we get from Equation 2:

$$\begin{aligned} \varphi(L(x)/x_{(1)}) &= \varphi(\sum_{i=1}^{p}[(2L_i(x) - L_{i-1}(x) - L_{i+1}(x))/x_{(1)}]\ell_i) \\ &= \sum_{i=1}^{p}[(2L_i(x) - L_{i-1}(x) - L_{i+1}(x))/x_{(1)}]\varphi(\ell_i) \\ &= \frac{1}{x_{(1)}}\sum_{i=1}^{p}(2L_i(x) - L_{i-1}(x) - L_{i+1}(x))\varphi(\ell_i) \end{aligned}$$

Then multiplication by $x_{(1)}$ and linearity yield:

$$\begin{aligned} \varphi(L(x)) &= \sum_{i=1}^{p}(2L_i(x) - L_{i-1}(x) - L_{i+1}(x))\varphi(\ell_i) \\ &= \sum_{i=1}^{p}(2\varphi(\ell_i) - \varphi(\ell_{i-1}) - \varphi(\ell_{i+1}))L_i(x) \end{aligned}$$

Moreover, Strict L-monotonicity implies, for all $i = 1, \dots, p$:

$$2\varphi(\ell_i) > \varphi(\ell_{i+1}) + \varphi(\ell_{i-1}) \text{ since } \ell_{i+1} + \ell_{i-1} \succ_P 2\ell_i$$
$$\text{and } \varphi(\ell_{i+1}) > \varphi(\ell_i) \text{ since } \ell_i \succ_P \ell_{i+1}$$

Conversely, if $\varphi(\ell_i) - \varphi(\ell_{i-1}) > \varphi(\ell_{i+1}) - \varphi(\ell_i) > 0$ for all $i \in \{1, \dots, p\}$, then Strict L-Monotonicity clearly holds. This concludes the proof. ■

In order to get a better interpretation of φ, let us formulate the corresponding function ψ on $\mathbb{R}^p_+$ (such that $x \succsim y \iff \psi(x) \leq \psi(y)$) using the definition of components $L_i(x)$. We get:

$$\psi(x) = \sum_{i=1}^{p} (\varphi(\ell_i) - \varphi(\ell_{i-1}))x_{(i)} \tag{3}$$

We recognize an Ordered Weighted Average (OWA, Yager, 1988) with strictly decreasing and strictly positive weights $w_i = \varphi(\ell_i) - \varphi(\ell_{i-1})$. Using these weights, Equation 3 writes:

$$\psi_w(x) = \sum_{i=1}^{p} w_i x_{(i)}$$

This is consistent with a result obtained by Ogryczak (2000) showing that any solution minimizing an ordered weighted average with strictly decreasing and strictly positive weights is L-efficient. This can be also linked to the characterization of Gini indices by Weymark (1981). Indeed, when $\psi_w(x) < \psi_w(y)$, x is preferred to y in terms of robustness. Hence, the function $\psi_w(x)$ can be seen as a measure of robustness. The use of "rank-dependent" weights in ψ_w can be used to express various attitude towards robustness. For example, let us mention the following particular cases:

- *Max criterion:* by setting $w_1 = 1, w_2 = 0, \dots, w_p = 0$, our measure reduces to the classical minimax criterion, used by Kouvelis and Yu (1997) to define absolute robustness.
- *Leximax criterion:* assuming a big-stepped distribution of weights (i.e., $w_1 \gg w_2 \gg \dots \gg w_p$) yields a leximax comparison rule (see e.g., Dubois and Fortemps, 2004): two cost vectors x and y are compared on the basis of their worst component; in case of tie, comparison involves the second worst components of each vector and so on until breaking the tie, if possible. Such an approach to robustness based on worse cases analysis is convenient for prudent decision makers.
- *Average:* choosing $w_i = 1/p$, vectors are ranked according to the average of costs. Here, when evaluating the robustness of a cost vector x by a scalar $\psi_w(x)$, the existence of a bad scenario for x can be fully compensated by a collection of more favorable scenarios.

Between these two extreme cases, various attitudes towards robustness can be defined, depending on the way the weights w_i spread over components, and allowing more or less compensation between scenarios. For example, coming back to set of paths corresponding to Figure 2, criterion ψ_w can be used to evaluate the relative robustness of any L-efficient path and to rank them by decreasing order of preference:

Example 4 Assume that $\varphi(\ell_0) = 0$, $\varphi(\ell_1) = 0.9$ and $\varphi(\ell_2) = 1$, so that the weights are $w_1 = 0.9$ and $w_2 = 0.1$, adding up to 1. Table 1 provides the evaluation of solutions. The first part of the table provides the ranking of L-efficient solutions (here paths 1, 3 and 11) and the second part of the table provides the ranking of L-dominated solutions. The selection proposed in the first part of the table can be seen as a pessimistic view since focused on the worst case. Note that decreasing the strictly positive difference $w_1 - w_2 = 2\varphi(\ell_1) - \varphi(\ell_2)$ reflects a less pessimistic view and favors other Lorenz optima. For example, choosing $\varphi(\ell_0) = 0$, $\varphi(\ell_1) = 0.51$ and $\varphi(\ell_2) = 1$ so that the weights $w_1 = 0.51$ and $w_2 = 0.49$, we get Table 2 (with the same presentation as in Table 1).

Path	*Vertices*	*Costs*
1	(a,b,e,g)	10.0
3	(a,b,c,f,g)	10.7
11	(a,d,f,g)	11.4
8	(a,c,f,g)	11.5
9	(a,d,c,e,g)	11.5
6	(a,b,d,f,g)	11.8
4	(a,b,d,c,e,g)	11.9
10	(a,d,c,f,g)	12.2
2	(a,b,c,e,g)	12.4
5	(a,b,d,c,f,g)	12.6
7	(a,c,e,g)	13.2

Table 1 The paths and OWA values (0.9,0.1).

Path	*Vertices*	*Costs*
11	(a,d,f,g)	9.06
3	(a,b,c,f,g)	9.53
1	(a,b,e,g)	10.0
10	(a,d,c,f,g)	9.08
8	(a,c,f,g)	9.55
9	(a,d,c,e,g)	9.55
2	(a,b,c,e,g)	10.06
7	(a,c,e,g)	10.08
6	(a,b,d,f,g)	11.02
5	(a,b,d,c,f,g)	11.04
4	(a,b,d,c,e,g)	11.51

Table 2 The paths and OWA values (0.51,0.49).

Note that, in this case, the subset of L-efficient paths does not form the top of the ranking. Indeed, paths 10 which is L-dominated by 11 received a better evaluation than paths 3 and 1. For this reason, it is not recommended to use the OWA criterion directly on the entire set of paths. We recommend to use the following procedure:

PRESENTATION OF ROBUST SOLUTIONS

1. Determine the L-efficient solutions.
2. Choose a weighting vector w and rank the above list by decreasing order of preference (i.e., by increasing order of $\psi_w(x)$).

Hence we see that function ψ_w can be used to discriminate between L-efficient solutions, with the possibility of handling various attitudes towards robustness depending on the values of coefficients $\varphi(\ell_i), i = 1,\dots,p$. Due to strict-L-monotonicity, we know that any solution minimizing function ψ_w over the set of feasible solution is L-efficient. Conversely, one may wonder if any L-efficient solution can be obtained by minimizing function $\psi_w(x) = \sum_{i=1}^{p} w_i x_{(i)}$ over a set $X \subseteq \mathbb{R}_+^p$ with an appropriate choice of the weighting vector w. In the general case, the answer is negative, as shown by the following example:

Example 5 Consider a simple problem with 2 scenarios and 3 feasible solutions x,y,z such that $x = (50,50)$, $y = (80,10)$ and $z = (65,30)$. The corresponding Lorenz vectors are $L(x) = (50,100)$, $L(y) = (80,90)$ and $L(z) = (65,95)$. Remark that if $X = \{x,y,z\}$ no element is L-dominated by another. Assume the Decision Maker prefers solution z to the two others.

Such a preference cannot be described with an OWA. Indeed,

$$z \succ x \Rightarrow w_1 \times 65 + w_2 \times 30 < w_1 \times 50 + w_2 \times 50$$
$$z \succ y \Rightarrow w_1 \times 65 + w_2 \times 30 < w_1 \times 80 + w_2 \times 10$$

Hence we get:

$$15 \times w_1 < 20 \times w_2 \Longrightarrow \frac{w_1}{w_2} < \frac{4}{3}$$
$$15 \times w_1 > 20 \times w_2 \Longrightarrow \frac{w_1}{w_2} > \frac{4}{3}$$

This yields a contradiction. Therefore, there is no weighting vector $w = (w_1,\dots,w_p)$ such that $z \in \operatorname{Arg\,max}_{x\in X} \psi_w(x)$. In such a case, we say that x is not an *admissible OWA minimizer* on X, the set of admissible OWA minimizer in X being defined by:

$$OWA(X) = \bigcup_{w \in W} \operatorname{Arg}\max_{x \in X} \psi_w(x)$$

where W is the set of admissible weights defined by:

$$W = \{w \in \mathbb{R}_+^p \ / \ w_1 > w_2 > \ldots > w_p\}$$

This impossibility to obtain z by optimizing an admissible OWA function can easily be explained by the violation of the independence axiom. Indeed, assuming we have:

$$\begin{array}{l}(65,95) \succ (80,90)\\ (65,95) \succ (50,100)\end{array}$$

the independence axiom implies:

$$\begin{array}{l}\frac{1}{2}(65,95) + \frac{1}{2}(65,95) \succ \frac{1}{2}(80,90) + \frac{1}{2}(50,100)\\ \text{which yields } (65,95) \succ (65,95)\end{array}$$

Hence we get a contradiction. This shows that preferring z to the two other solutions is not compatible with the independence axiom. In such a case, there is no way to obtain z by optimizing function ψ over X. Hence, given a set X of cost-vectors associated to feasible solutions, and denoting $PE(X)$, $LE(X)$, $OWA(X)$ the subsets of P-efficient elements, L-efficient elements, and admissible OWA minimizers respectively, we have:

$$OWA(X) \subseteq LE(X) \subseteq PE(X)$$

but any of these inclusions can be strict. However, note that a heuristic search algorithm specially designed to determine an OWA minimizer has been provided in Perny and Spanjaard (2003). This algorithm is based on a refinement of a multicriteria search algorithm named MOA* (Stewart and White III, 1991). It could be easily adapted for the shortest path problem.

5 Numerical experiments

In this section we present some numerical experiments in order to evaluate the performance of the method described in Section 3.2. Both algorithms (for the robust spanning trees problem and for the robust shortest paths problem) have been implemented in C++ and all the tests have been carried out on a computer equipped with a PENTIUM IV 2.6Ghz and 1Gb of memory.

The algorithms are applied on graphs with randomly generated costs between 0 and 1000, for 5 scenarios. We evaluate the efficiency of our algorithms with respect to the size of the input graph. For the robust spanning tree problem, we consider complete graphs the number of vertices of which are between 10 and 34 (twenty instances for each value). For the robust shortest path problem, we consider graphs of density 0.5 (an arc is included between two vertices with a probability of 0.5) the number of vertices of which are between 1000 and 3000 (twenty instances for each value). The corresponding numerical results are indicated on Table 3 and Table 4. For each set of instances, we record the average number of L-efficient solutions (LE), the average number of generated solutions (GS) and the average execution time.

Although the number of L-efficient solutions can be huge on pathological instances, it remains quite low on a large sample of randomly generated instances (less than 20 on average for trees, less than 4 on

size	*LE*	*GS*	*time*
10	2	11	0
14	4	623	0.0117
20	7	1168	0.0328
24	10	25147	0.9681
26	10	15604	0.65235
28	11	43515	1.93675
30	17	51145	2.50955
32	16	47139	2.42965
34	19	75013	4.13535

Table 3 Numerical results for trees.

size	*LE*	*GS*	*time*
1000	1	237	0.2492
1250	2	285	0.4469
1500	2	320	0.7047
1750	3	383	1.06105
2000	2	466	1.29385
2250	4	501	1.85115
2500	3	553	2.22375
2750	2	599	2.90835
3000	2	662	3.21185

Table 4 Numerical results for paths.

average for paths). The number of paths generated during the search remains quite low too. However, the number of generated trees may significantly increases with the number of vertices. In the robust spanning trees problem, the critical ressource for running our algorithm is therefore the memory space and not the processing time. Note that experimentations have been carried out for 5 scenarios. This gives a good idea of the potential efficiency of our algorithms in practice since most decision problems involve a few number of scenarios or criteria (less than 10).

6 Conclusion

We have introduced a new formal framework to define robustness in combinatorial problems. This framework can be seen as a special case of multicriteria combinatorial optimization, where all scales are commensurate. Taking advantage of this specific feature, we have justified the use of Lorenz-dominance as a useful refinement of Pareto-dominance to compare solutions according to multiple scenarios. Then, we have proposed a general algorithmic approach to seek for Lorenz-efficient solutions for robust shortest path and robust spanning tree problems. This approach relies on a k best solutions algorithm on a monovalued graph, combined with a stopping condition which reveals efficient in practice. This stopping condition prunes the search while guaranteeing that all Lorenz-efficient solutions have been found.

We have then refined the notion of Lorenz-dominance by introducing the ordered weighted average as an axiomatically founded measure of robustness. This operator enables to handle various behavior patterns towards robustness, depending on the choice of the weights. The elicitation of these weights to capture the attitude of a given decision maker is not discussed in the paper but can clearly be derived from classical methods used for assessing utility functions. Another important issue might be to investigate the extension of our work when additional information about the likelihood of scenarios is present.

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