CELLS AND CACTI
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Abstract. — Let \((W, S)\) be a Coxeter system, let \(\varphi\) be a weight function on \(S\) and let \(\text{Cact}_W\) denote the associated \textit{cactus group}. Following an idea of I. Losev, we construct an action of \(\text{Cact}_W \times \text{Cact}_W\) on \(W\) which has nice properties with respect to the partition of \(W\) into left, right or two-sided cells (under some hypothesis, which hold for instance if \(\varphi\) is constant or if \(W\) is finite of rank \(\leq 4\)). It must be noticed that the action depends heavily on \(\varphi\).

Let \((W, S)\) be a Coxeter system with \(S\) finite and let \(\varphi\) be a positive \textit{weight function} on \(S\) as defined by Lusztig [Lu1]. We denote by \(\text{Cact}_W\) the \textit{Cactus group} associated with \(W\), as defined for instance in [Lo]. In [Lo], I. Losev has constructed, whenever \(W\) is a finite Weyl group and \(\varphi\) is constant, an action of \(\text{Cact}_W \times \text{Cact}_W\) on \(W\) which satisfies some good properties with respect to the partition of \(W\) into cells. His construction is realized as the combinatorial shadow of wall-crossing functors on the category \(\mathcal{O}\).

In [Lo, §5.1], I. Losev suggested that the construction of this action could be extended to other types of Coxeter groups and general weight function \(\varphi\), using some recent results of Lusztig [Lu2]. This is the aim of this paper to show that Losev’s idea works, by using extensively the results of [BoGe] and assuming that some of Lusztig’s Conjectures in [Lu1, §14.2] hold, as in [Lu2], as well as an hypothesis on a sign function (this will be made more precise in §3.B). Note that, if \(\varphi\) is constant, then these Conjectures hold (as well as the hypothesis on the sign function), so this provides at least an action in the equal parameter case: if moreover \(W\) is a Weyl group, this action coincides with the one constructed by Losev [Lo].

Let us state our main result (in this Theorem, if \(I \subset S\), we denote by \(W_I\) the subgroup generated by \(I\) and by \(\varphi_I\) the restriction of \(\varphi\) to \(I\)).

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**Theorem.**— Assume that the hypotheses of §3.B hold. There exists an action of $\text{Cact}_W \times \text{Cact}_W$ on the set $W$ such that, if we denote by $\tau_L^\varphi$ (respectively $\tau_R^\varphi$) the permutation of $W$ obtained through the action of $(\tau, 1) \in \text{Cact}_W \times \text{Cact}_W$ (respectively $(1, \tau) \in \text{Cact}_W \times \text{Cact}_W$), then:

(a) If $C$ is a left cell, then $\tau_L^\varphi(C)$ is a left cell and $\tau_L^\varphi$ induces an isomorphism of left $\mathcal{H}$-modules $\mathcal{H}^L[C] \sim \mathcal{H}^L[\tau_L^\varphi(C)]$.

(a') If $C$ is a right cell, then $\tau_R^\varphi(C)$ is a right cell and $\tau_R^\varphi$ induces an isomorphism of right $\mathcal{H}$-modules $\mathcal{H}^R[C] \sim \mathcal{H}^L[\tau_R^\varphi(C)]$.

(b) If $w \in W$, then $\tau_L^\varphi(w) \sim_R w$ and $\tau_R^\varphi(w) \sim_L w$.

In this Theorem, if $C$ is a left (respectively right) cell, then $\mathcal{H}^L[C]$ (respectively $\mathcal{H}^R[C]$) denotes the associated left (respectively right) $\mathcal{H}$-module (see §1.A).

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## 1. Notation

**Set-up.** We fix a Coxeter system $(W, S)$, whose length function is denoted by $\ell : W \to \mathbb{N}$. We also fix a totally ordered abelian $\mathcal{A}$ and we denote by $A$ the group algebra $\mathbb{Z}[\mathcal{A}]$. We use an exponential notation for $A$:

$$A = \bigoplus_{a \in \mathcal{A}} \mathbb{Z} v^a$$

where $v^a v^{a'} = v^{a+a'}$ for all $a, a' \in \mathcal{A}$.

If $a_0 \in \mathcal{A}$, we write $\mathcal{A}_{\leq a_0} = \{ a \in \mathcal{A} \mid a \leq a_0 \}$ and $A_{\leq a_0} = \bigoplus_{a \in \mathcal{A}_{\leq a_0}} \mathbb{Z} v^a$; we define similarly $A_{< a_0}, A_{\geq a_0}, A_{> a_0}, \ldots$. We denote by $\overline{\cdot} : A \to A$ the involutive automorphism such that $\overline{v^a} = v^{-a}$ for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is totally ordered, $A$ inherits two maps $\deg : A \to \mathcal{A} \cup \{-\infty\}$ and $\val : A \to \mathcal{A} \cup \{+\infty\}$ respectively called degree and valuation, and which are defined as usual.

We also fix a weight function $\varphi : S \to \mathcal{A}_{>0}$ (that is, $\varphi(s) = \varphi(t)$ or all $s, t \in S$ which are conjugate in $W$) and, if $I \subset S$, we denote by $\varphi_I : I \to \mathcal{A}_{>0}$ the restriction of $\varphi$.

**Commentary.**— In [Lu1, §14.2], Lusztig states several Conjectures relating the so-called Lusztig’s $a$-function and the partition of $W$ into cells. Throughout this paper, if $1 \leq i \leq 15$, the expression Lusztig’s Conjecture $Pi$ will refer to [Lu1, §14.2, Conjecture $Pi$].
Most of the results of this paper only hold if Lusztig’s Conjectures P1, P4, P8 and P9 hold for all finite standard parabolic subgroups of $W$. In particular, they hold if $\varphi$ is constant [Lu1, §15].

1.A. Cells. — Let $\mathcal{H} = \mathcal{H}(W, S, \varphi)$ denote the Iwahori-Hecke algebra associated with the triple $(W, S, \varphi)$. This $A$-algebra is free as an $A$-module, with a standard basis denoted by $(T_w)_{w \in W}$. The multiplication is completely determined by the following two rules:

\[
\begin{align*}
T_w T_{w'} &= T_{w w'} & \text{if } \ell(w w') = \ell(w) + \ell(w'), \\
(T_s - u^{\varphi(s)})(T_s + u^{-\varphi(s)}) &= 0 & \text{if } s \in S.
\end{align*}
\]

The involution $\ast$ on $A$ can be extended to an $A$-semilinear involutive automorphism $\ast : \mathcal{H} \to \mathcal{H}$ by setting $\overline{T_w} = T^{-1}_w$. Let $\mathcal{H}_{<0} = \bigoplus_{w \leq W} A_{<0} T_w$.

If $w \in W$, there exists [Lu1] a unique $C_w \in \mathcal{H}$ such that

\[
\begin{align*}
\sum_{w \in W} C_w &= C_w, \\
C_w &\equiv T_w \mod \mathcal{H}_{<0}.
\end{align*}
\]

It is well-known [Lu1] that $(C_w)_{w \in W}$ is an $A$-basis of $\mathcal{H}$ (called the Kazhdan-Lusztig basis) and we will denote by $h_{x,y,z}$ the structure constants, defined by

\[
C_x C_y = \sum_{z \in W} h_{x,y,z} C_z.
\]

We also set

\[
C_y = \sum_{x \in W} p_{x,y}^* T_x,
\]

and recall that $p_{x,y}^* = 1$ and $p_{x,y}^* \in A_{<0}$ if $x \neq y$.

We will denote by $\leq_L$, $\leq_R$, $\leq_{LR}$, $\leq_{LR}$, $\sim_L$, $\sim_R$ and $\sim_{LR}$ the relations defined in [Lu1] and associated with the triple $(W, S, \varphi)$. Also, we will call left, right and two-sided cells the equivalence classes for the relations $\sim_L$, $\sim_R$ and $\sim_{LR}$ respectively. If $C$ is a left cell, we set

\[
\mathcal{H}^{<L,C} = \bigoplus_{w \leq L} A C_w, \quad \mathcal{H}^{<L,C} = \bigoplus_{w \leq L} A C_w \quad \text{and} \quad \mathcal{H}^L[C] = \mathcal{H}^{<L,C} / \mathcal{H}^{<L,C}.
\]

These are left $\mathcal{H}$-modules. If $w \in C$, we denote by $c_w^L$ the image of $C_w$ in the quotient $\mathcal{H}^L[C]$ and $\mathcal{H}^{<L,C}$ and $\mathcal{H}^{<L,C}$ might be also denoted by $\mathcal{H}^{<L,w}$ and $\mathcal{H}^{<L,w}$ respectively: it is clear that $(c_w^L)_{w \in C}$ is an $A$-basis of $\mathcal{H}^L[C]$. If $C$ is a right (respectively two-sided) cell, we define similarly $\mathcal{H}^{<R,C}$, $\mathcal{H}^{<R,C}$ and $\mathcal{H}^R[C]$ (respectively $\mathcal{H}^{<LR,C}$, $\mathcal{H}^{<LR,C}$ and $\mathcal{H}^{LR}[C]$), as well as $c_w^R$ (respectively $c_w^{LR}$).
1.B. Parabolic subgroups. — We denote by $\mathcal{P}(S)$ the set of subsets of $S$. If $I \subset S$, we denote by $W_I$ the standard parabolic subgroup generated by $I$ and by $X_I$ the set of elements $x \in W$ which have minimal length in $xW_I$. We also define $\text{pr}^L_I : W \to W_I$ and $\text{pr}^R_I : W \to W_I$ by the following formulas:

$$\forall x \in X_I, \forall w \in W_I, \quad \text{pr}^L_I(xw) = w \quad \text{and} \quad \text{pr}^R_I(wx^{-1}) = w.$$  

If $\delta : W_I \to W_I$ is any map, we denote by $\delta_L^L : W \to W$ and $\delta_R^R : W \to W$ the maps defined by

$$\delta^L(xw) = x\delta(w) \quad \text{and} \quad \delta^R(wx^{-1}) = \delta(w)x^{-1}$$

for all $x \in X_I$ and $w \in W_I$ (see [BoGe, §6]). Note that

$$(1.1) \quad \text{pr}^L_I \circ \delta^L = \delta \circ \text{pr}^L_I \quad \text{and} \quad \text{pr}^R_I \circ \delta^R = \delta \circ \text{pr}^R_I.$$  

We also denote by $\delta^{op}_I : W_I \to W_I$ the map defined by

$$\delta^{op}(w) = \delta(w^{-1})^{-1}$$

for all $w \in W$. Note that $\delta^R = ((\delta^{op})^L)^L$. If $\sigma : W \to W$ is any automorphism such that $\sigma(S) = S$, then

$$(1.2) \quad \sigma \circ \text{pr}^L_I = \text{pr}^L_{\sigma(I)} \circ \sigma \quad \text{and} \quad \sigma \circ \text{pr}^R_I = \text{pr}^R_{\sigma(I)} \circ \sigma.$$  

The Hecke algebra $\mathcal{H}(W_I, I, \varphi_I)$ will be denoted by $\mathcal{H}_I$ and will be viewed as a subalgebra of $\mathcal{H}$ in the natural way. Since $\mathcal{H}$ is free as a left or as a right $\mathcal{H}_I$-module, we will identify, for $\mathfrak{H}$ a left (respectively right) ideal of $\mathcal{H}_I$, the left (respectively right) ideal $\mathcal{H}_I \mathfrak{H}$ (respectively $\mathfrak{H} \mathcal{H}_I$) with $\mathcal{H} \otimes_{\mathcal{H}_I} \mathfrak{H}$ (respectively $\mathfrak{H} \otimes_{\mathcal{H}_I} \mathcal{H}$).

Let $\mathcal{P}(S)$ (respectively $\mathcal{P}_{itf}(S)$) denote the set of subsets $I$ of $S$ such that $W_I$ is finite (respectively such that $W_I$ is finite and the Coxeter graph of $(W_I, I)$ is connected). If $I \in \mathcal{P}(S)$, we denote by $w_I$ the longest element of $W_I$ and we set

$$\omega_I : W_I \to W_I \quad w \mapsto w_I w w_I.$$  

It is an automorphism of $W_I$ and it satisfies $\omega_I(I) = I$. If $W$ is finite, then $w_s$ will be denoted by $w_0$, according to the tradition. Also, $\omega_s$ will be denoted by $\omega_0$.

If $I \in \mathcal{P}(S)$, we denote by $a_I : W \to \mathcal{A}$ the Lusztig’s $a$-function defined by

$$a_I(z) = \max_{x, y \in W_I} \deg(h_{x,y,z})$$

for all $z \in W_I$. If $W$ itself is finite, then $a_s$ will be simply denoted by $a$. 


1.C. Cactus group. — We define the cactus group associated with $W$, and we denote by $\text{Cact}_W$, the group with the following presentation:

- **Generators:** $(\tau_I)_{I \in \mathcal{P}_{\text{ir},f}(S)}$
- **Relations:** for all $I, J \in \mathcal{P}_{\text{ir},f}(S)$, we have:

\[
\begin{align*}
(C1) & \quad \tau_I^2 = 1, \\
(C2) & \quad [\tau_I, \tau_J] = 1 \quad \text{if } I \cup J \text{ is disconnected}, \\
(C3) & \quad \tau_I \tau_J = \tau_J \tau_{\omega(I)} \quad \text{if } I \subset J.
\end{align*}
\]

By construction, the map $\tau_I \mapsto w_I$ extends to a surjective morphism $\text{Cact}_W \longrightarrow W$ which will not be used in this paper.

2. Action of the longest element

In [Lu2, Theorem 2.3], Lusztig proves the following result (which generalizes to the unequal parameter case a result of Mathas [Ma]):

**Theorem 2.1 (Mathas, Lusztig).** — Assume that $W$ is finite and that Lusztig’s Conjectures P1, P4, P8 and P9 hold for $(W, S, \varphi)$. Then there exists a unique involution $\rho$ of the set $W$ such that, for all $w \in W$,

\[
\nu^{\varphi(w_0, w) - \varphi(w)} T_{w_0} C_w \equiv \eta^R_w C_{\rho(w)} \mod \mathcal{H}^{<L,R < L}. \tag{1}
\]

for some $\eta^R_w \in \{1, -1\}$. Similarly, there exists a unique involution $\lambda$ of the set $W$ such that, for all $w \in W$,

\[
\nu^{\varphi(w_0, w) - \varphi(w)} C_w T_{w_0} \equiv \eta^L_w C_{\lambda(w)} \mod \mathcal{H}^{<L,R < L}. \tag{2}
\]

Note that $\lambda = \rho^\text{op}$ and $\rho = \lambda \circ \omega_0$, and that $\eta^L_w = \eta^R_{w^{-1}}$.

In the equal parameter case, Mathas proved moreover that the map $w \mapsto \eta^R_w$ is constant on two-sided cells.

**Remark 2.2.** — We will explain here why we only need to assume that Lusztig’s Conjectures P1, P4, P8 and P9 hold for the above Theorem to hold (in [Lu2, Theorem 2.5], Lusztig assumed that P1, P2, ..., P14 and P15 hold). This will be a consequence of a simplification of the proof of [Lu2, Lemma 1.13], based on the ideas of [Bo]. In particular, we avoid the use of the difficult Lusztig’s Conjecture P15 and the construction/properties of the asymptotic algebra.
So assume that Lusztig’s Conjectures P1, P4, P8 and P9 hold. Let us write

\[ T_{w_0} C_y = \sum_{x \leq_L y} \lambda_{x,y} C_x, \]

with \( \lambda_{x,y} \in A \). Note that

\[ T_{w_0}^{-1} C_y = \sum_{x \leq_L y} \overline{\lambda}_{x,y} C_x. \]

By \([\text{Bo}, \text{Proposition 1.4(a)}]\),

\[ \deg(\lambda_{x,y}) \leq a(x) - a(w_0 x) \text{ with equality only if } x \sim_L y. \]

By \([\text{Bo}, \text{Proposition 1.4(b)}]\),

\[ \deg(\overline{\lambda}_{x,y}) \leq a(w_0 y) - a(y) \text{ with equality only if } x \sim_L y. \]

Assume now that \( x \sim_L y \). Then \( a(x) = a(y) \) and \( a(w_0 x) = a(w_0 y) \) by P4 and \([\text{Lu1}, \text{Corollary 11.7}]\), so

\[ \deg(\lambda_{x,y}) \leq a(y) - a(w_0 y) \leq \text{val}(\lambda_{x,y}). \]

So

\[ \text{if } x \sim_L y, \text{ then } v^{a(w_0 y) - a(y)} \lambda_{x,y} \in \mathbb{Z}, \]

Thanks to P9, this is exactly the statement in \([\text{Lu2}, \text{Lemma 1.13(a)}]\). Note also that \([\text{Lu2}, \text{Lemma 1.13(b)}]\) is already proved in \([\text{Bo}, \text{Proposition 1.4(c)}]\).

One can then check that, once \([\text{Lu2}, \text{Lemma 1.13}]\) is proved, the argument developed in \([\text{Lu2}, \text{Proof of Theorem 2.3}]\) to obtain Theorem 2.1 does not make use any more of Lusztig’s Conjectures. ■

**Notation.**— If \( h, h' \in \mathcal{H} \) and \( \mathfrak{H} \) is an \( A \)-submodule of \( \mathcal{H} \), we will write

\[ h \equiv h' \mod \mathfrak{H} \]

if there exists \( a \in A^\times \) such that

\[ h \equiv a h' \mod \mathfrak{H}. \]

For instance, Theorem 2.1 can be rewritten

\[ T_{w_0} C_w \equiv C_{\rho'(w)} \mod \mathcal{H}^<L^w \]

and this property determines the map \( \rho \). ■
3. Action of the cactus group

3.A. Cellular maps. — We recall [BoGe, Definition 4.1]:

**Definition 3.1.** — A map $\delta : W \to W$ is called **left cellular** if the following conditions are satisfied for every left cell $C$ of $W$:

(LC1) $\delta(C)$ is also a left cell.
(LC2) The $A$-linear map $\mathcal{H}^l[C] \to \mathcal{H}^l[\delta(C)]$, $c^l_w \mapsto c^l_{\delta(w)}$ is an isomorphism of left $\mathcal{H}$-modules.

It is called **strongly left cellular** if it is left cellular and if moreover

(LC3) $\delta(w) \sim_R w$ for all $w \in W$.

We define similarly the notions of **right cellular** and **strongly right cellular** maps.

Of course, $\delta : W \to W$ is left cellular (respectively strongly left cellular) if and only if $\delta^\text{op}$ is right cellular (respectively strongly right cellular).

3.B. Longest element of finite parabolic subgroups. — We will denote by (H) the following property:

(H) The sign map $w \mapsto \eta^R_w$ defined in Theorem 2.1 is constant on right cells.

We will now work under the following hypothesis:

**Hypothesis.** From now on, and until the end of this paper, we assume that Lusztig’s Conjectures P1, P4, P8 and P9 and Hypothesis (H) hold for all triples $(W_I, I, \varphi_I)$, where $I \in \mathcal{P}(S)$.

**Remark.** — If $\varphi$ is constant, then Lusztig’s Conjectures P1, P2, . . . , P15 hold [Lu1, §15], as well as Hypothesis (H): in fact, the map $w \mapsto \eta^R_w$ is constant on two-sided cells in this case [Ma].

Let $I \in \mathcal{P}_{\text{ir,f}}(S)$. We denote by $\lambda_I$ (respectively $\rho_I$) the map $W_I \to W_I$ denoted by $\lambda$ (respectively $\rho$) in Theorem 2.1 in the case where $I = S$.

**Proposition 3.2.** — The map $\lambda_I : W_I \to W_I$ (respectively $\rho_I : W_I \to W_I$) is strongly left (respectively right) cellular.
Proof. — We may, and we will, assume that $W$ is finite and $I = S$, and we set $\rho = \rho_I$ and $\lambda = \lambda_I$. It is sufficient to prove that $\lambda$ is strongly left cellular. By Theorem 2.1,

$$\eta^L w \cdot v^{\langle w, w \rangle} C_w T_{w_0} \equiv C_{\lambda(w)} \bmod \mathcal{H}^{< L R},$$

so $\lambda(w) \leq_{L} w$. Similarly, $w = \lambda(\lambda(w)) \leq_{R} \lambda(w)$, so (LC3) holds.

Let $x$ and $y$ be two elements of $W$ such that $x \sim_L y$. Let $\Gamma$ (respectively $C$) denote the two-sided (respectively left) cell containing $x$ and $y$. For simplifying the notation, we set $\mathbf{a}_\Gamma = \mathbf{a}(\Gamma) - \mathbf{a}(w_0 \Gamma)$. Then there exists $x = x_0, x_1, \ldots, x_m = y = y_0, y_1, \ldots, y_n = x$ in $W$ and elements $h_1, \ldots, h_m, h'_1, \ldots, h'_n$ of $\mathcal{H}$ such that $C_{x_i}$ (respectively $C_{y_j}$) appears with a non-zero coefficient in the expression of $h_i C_{x_{i-1}}$ (respectively $h'_j C_{y_{j-1}}$) in the Kazhdan-Lusztig basis for $1 \leq i \leq m$ (respectively $1 \leq j \leq n$). By Lusztig’s Conjectures P4 and P9, $x_i, y_j \in C$. So, if we write

$$h_i C_{x_{i-1}} \equiv \sum_{u \in \Gamma} a_{i, u} C_u \bmod \mathcal{H}^{< L R},$$

then $a_{i, x_i} \neq 0$ and

$$h_i C_{x_{i-1}} T_{w_0} \equiv \sum_{u \in \Gamma} a_{i, u} C_u T_{w_0} \bmod \mathcal{H}^{< L R},$$

Therefore, by Theorem 2.1 and Hypothesis (H),

$$h_i v^{a_{i, u}} C_{\lambda(x_{i-1})} \equiv \sum_{u \in \Gamma} a_{i, u} v^{a_{i, u}} C_{\lambda(u)} \bmod \mathcal{H}^{< L R},$$

and so $\lambda(x_i) \leq_{L} \lambda(x_{i-1})$. This shows that $\lambda(y) \leq_{L} \lambda(x)$ and we can prove similarly that $\lambda(x) \leq_{L} \lambda(y)$. Therefore, $\lambda(C)$ is contained in a unique left cell $C'$. But, similarly, $\lambda(C')$ is contained in a unique left cell, and contains $C$. So $\lambda(C) = C'$ is a left cell. This shows (LC1).

Finally the map $\lambda$ is obtained through the right multiplication by $\eta v^{\mathbf{a}_\Gamma} T_{w_0}$. Since this right multiplication commutes with the left action of $\mathcal{H}$, this implies (LC2). \(\Box\)

**Corollary 3.3.** — The map $\lambda^L_I : W \to W$ (respectively $\rho^R_I : W \to W$) is strongly left (respectively right) cellular.

Proof. — This follows from Proposition 3.2 and [BoGe, Theorem 6.2]. \(\Box\)

It must be noticed that the maps $\lambda^L_I$ and $\rho^R_I$ depend on the weight function $\varphi$, even if it is not clear from the notation. The canonicity of their construction shows that, if $\sigma : W \to W$ is an automorphism such that $\sigma(S) = S$ and $\varphi \circ \sigma = \varphi$, then

$$\sigma \circ \lambda^L_I = \lambda^L_{\sigma(I)} \circ \sigma \quad \text{and} \quad \sigma \circ \rho^R_I = \rho^R_{\sigma(I)} \circ \sigma.$$
For instance, if $W$ is finite, then $\omega_0 : W \to W$ satisfies the above properties and so
\begin{equation}
\omega_0 \circ \lambda^L = \lambda^L_{\omega_0(i)} \circ \omega_0 \quad \text{and} \quad \omega_0 \circ \rho^R = \rho^R_{\omega_0(i)} \circ \omega_0.
\end{equation}

**Corollary 3.6.** — Let $w \in W$. Then
\begin{equation}
\nu^{a_i(w_1,pr^R_i(w))}_{a_i[pr^R_i(w)]}) T_{w_1} C_w \equiv \eta^R_{I,w} C_{\rho^R_{i}(w)} \mod \mathcal{H}^{<R}_{I,w} \text{pr}^R_i(w)
\end{equation}
where $\eta^R_{I,w}$ is the sign associated with the two-sided cell of $pr^R_i(w)$ in $W_i$ through the Theorem 2.1 for $W_i$. Similarly,
\begin{equation}
\nu^{a_i(w_1,pr^R_i(w))}_{a_i[pr^R_i(w)]}) C_w T_{w_1} \equiv \eta^L_{I,w} C_{\lambda^L_{i}(w)} \mod \mathcal{H}^{<L}_{I,w} \text{pr}^R_i(w),
\end{equation}
where $\eta^L_{I,w} = \eta^R_{I,w^{-1}}$.

**Proof.** — It is sufficient to prove the first congruence. Write $w = w'x$, with $w' \in W_i$ and $x \in X_i^{-1}$. By [Ge1], there exists a family $(p^I_{w',x,u,a})_{w',x \in W_i, u,a \in X_i^{-1}}$ such that
\begin{equation}
C_w = C_{w'} \cdot T_x + \sum_{(u,a) \in W_i \times X_i^{-1} \text{ such that } a < x \text{ and } u \sim_R w'} p^I_{w',x,u,a} C_u T_a.
\end{equation}
Therefore,
\begin{equation}
C_w \equiv C_{w'} \cdot T_x + \sum_{(u,a) \in W_i \times X_i^{-1} \text{ such that } a < x \text{ and } u \sim_R w'} p^I_{w',x,u,a} C_u T_a \mod \mathcal{H}^{<R}_{I,w'}.
\end{equation}
If $u \sim_R w'$, then $a_i(u) = a_i(w')$ and $a_i(w_1 u) = a_i(w_1 w')$ by P4, so it follows from Theorem 2.1 and Hypothesis (H) for $W_i$ that
\begin{equation}
\eta^R_{I,w} \nu^{a_i(w_1,pr^R_i(w))}_{a_i[pr^R_i(w)]}) T_{w_1} C_w \equiv C_{\rho^R_{i}(w)} \cdot T_a + \sum_{(u,a) \in W_i \times X_i^{-1} \text{ such that } a < x \text{ and } u \sim_R w'} p^I_{w',x,u,a} C_{\rho^R_{i}(u)} T_a \mod \mathcal{H}^{<R}_{I,w'}.
\end{equation}
But, by [Ge2, Lemma 3.8], $p^I_{w',x,u,a} = p^I_{w',x,\rho^R_{i}(u),a}$ so
\begin{equation}
\eta^R_{I,w} \nu^{a_i(w_1,pr^R_i(w))}_{a_i[pr^R_i(w)]}) T_{w_1} C_w \equiv C_{\rho^R_{i}(w')} \mod \mathcal{H}^{<R}_{I,w'}
\end{equation}
as desired. \qed
3.C. Cactus group. — The main result of this section is the following:

**Theorem 3.7.** — Let $I, J \in \mathcal{A}_{u, \lambda}(S)$. Then:

(a) $[\lambda_I^L, \rho^R_I] = \text{Id}_W$.

(b) $(\lambda_I^L)^2 = (\rho^R_I)^2 = \text{Id}_W$.

(c) If $I \cup J$ is disconnected, then $[\lambda_I^L, \lambda_J^L] = [\rho_I^R, \rho_J^R] = \text{Id}_W$.

(d) If $I \subset J$, then $\lambda_I^L \lambda_J^L = \lambda_I^L \omega_{\omega_I^L(I)}$ and $\rho_I^R \rho_J^R = \rho_I^R \rho_{\omega_I^L(I)}$.

**Proof.** — (a) It follows from Corollary 3.6 that

$$T_{w_I} C_{w_I} T_{w_I} \cong \eta^R_{I,w} C_{\lambda_I^L(w)} T_{w_I} \mod \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \mathcal{H}$$

and so

$$T_{w_I} C_{w_I} T_{w_I} \cong \eta^L_{I,\rho^R_I(w)} \eta^R_{I,w} C_{\lambda_I^L(\rho^R_I(w))} \mod \left( \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \mathcal{H} + \mathcal{H} \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \right).$$

It then follows from Corollary 3.3 that $\rho^R_I(w) \sim_L w$, and so $\mathcal{P}^r_{I}(w) \sim_L \mathcal{P}^r_{I}(w)$ by [Ge1]. Therefore, $\eta^L_{I,\rho^R_I(w)} = \eta^L_{I,w}$ and

$$T_{w_I} C_{w_I} T_{w_I} \cong \eta^L_{I,w} \eta^R_{I,w} C_{\lambda_I^L(\rho^R_I(w))} \mod \left( \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \mathcal{H} + \mathcal{H} \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \right).$$

Symmetrically,

$$T_{w_I} C_{w_I} T_{w_I} \cong \eta^L_{I,w} \eta^R_{I,w} C_{\lambda_I^L(\rho^R_I(w))} \mod \left( \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \mathcal{H} + \mathcal{H} \mathcal{H}^\sim_{\leq 2} \mathcal{P}^r_{I}(w) \right).$$

Let $\mathcal{E}$ be the set

$$\mathcal{E} = \{ u \in W \mid \mathcal{P}^r_I(u) <_L \mathcal{P}^r_I(w) \} \cup \{ u \in W \mid \mathcal{P}^r_I(u) <_R \mathcal{P}^r_J(w) \}.$$

By [Ge1], the previous congruences imply

$$C_{\lambda_I^L(\rho^R_I(w))} \cong C_{\rho^R_I(\lambda_I^L(w))} \mod \bigoplus_{u \in \mathcal{E}} A_{C_{w}}.$$

So it is sufficient to prove that $\lambda_I^L(\rho^R_I(w)) \notin \mathcal{E}$.

- Let us first assume that $\mathcal{P}^r_I(\lambda_I^L(\rho^R_I(w))) <_L \mathcal{P}^r_I(w)$. Since, by (1.1), $\mathcal{P}^r_I \circ \lambda_I^L = \lambda_I \circ \mathcal{P}^r_I$, we have $\lambda_I(\mathcal{P}^r_I(\rho^R_I(w))) <_L \mathcal{P}^r_I(w)$. But again, $\rho^R_I(w) \sim_L w$ by Corollary 3.3 and, by [Ge1], $\mathcal{P}^r_I(\rho^R_I(w)) \sim_L \mathcal{P}^r_I(w)$. This shows that $\lambda_I(\mathcal{P}^r_I(w)) <_L \mathcal{P}^r_I(w)$, which contradicts (thanks to P9) the fact that $\lambda_I(u) \sim_{LR} u$ for all $u \in W_I$.

- Let us now assume that $\mathcal{P}^r_I(\lambda_I^L(\rho^R_I(w))) <_R \mathcal{P}^r_I(w)$. Still by Corollary 3.3, we have $\lambda_I^L(\rho^R_I(w)) <_R \rho^R_I$ and so, by [Ge1], $\mathcal{P}^r_I(\rho^R_I(w)) <_R \mathcal{P}^r_I(w)$. Using now (1.1), we get $\mathcal{P}^r_I(\rho^R_I(w)) = \mathcal{P}^r_I(\mathcal{P}^r_I(w))$, which contradicts (thanks to P9) the fact that $\rho_I(u) \sim_{LR} u$ for all $u \in W_I$. 
The proof of (a) is now complete.

(b) is obvious.

(c) Assume that \( I \cup J \) is disconnected. We only need to prove that \([\lambda^1_I, \lambda^1_J] = \text{Id}_W\), the proof of the other equality is similar. Let \( w \in W \) and write \( w = x w' \), with \( x \in X_{I \cup J} \) and \( w' \in W_{I \cup J} \). Since \( I \cup J \) is disconnected, we have \( W_{I \cup J} = W_I \times W_J \) and so there exists \( w_1 \in W_I \) and \( w_2 \in W_J \) such that \( w' = w_1 w_2 = w_2 w_1 \). Note also that \( x w_1 \in X_J, x \lambda_I(w_1) \in X_J, x w_2 \in X_I \) and \( x \lambda_J(w_2) \in X_I \). Therefore,

\[
\lambda^1_I(\lambda^1_J(w)) = \lambda^1_I(x w_1 \lambda_J(w)) = \lambda^1_I(x \lambda_I(w_2) \lambda_J(w_1))
\]

and, similarly,

\[
\lambda^1_J(\lambda^1_I(w)) = x \lambda_J(w_1) \lambda_I(w_2).
\]

So \([\lambda^1_I, \lambda^1_J] = \text{Id}_W\), as desired.

(d) Assume here that \( I \subset J \). It is easily checked that we may assume that \( W \) is finite and \( J = S \). Let \( w \in W \). Then

\[
\lambda^L_S(\lambda^L_I(w)) = \rho_S^L(\omega_0(\lambda^L_I(w))) = \rho_S\lambda^L_{\omega_0(I)}(\omega_0(w)) = \lambda^L_{\omega_0(I)}(\rho_S(\omega_0(w))) = \lambda^L_{\omega_0(I)}(\lambda^L_S(w))
\]

by Theorem 2.1.

This proves the first equality and the second follows from a similar argument. □

Let \( \mathcal{S}_W \) denote the symmetric group on the set \( W \). The statements (b), (c) and (d) of the previous Theorem 3.7 show that there exists a unique morphism of groups

\[
\begin{align*}
\text{Cact}_W & \rightarrow \mathcal{S}_W \\
\tau & \mapsto \tau^L_\varphi
\end{align*}
\]

such that

\[
\tau^L_{I, \varphi} = \lambda^L_I
\]

for all \( I \in \mathcal{P}_{ir,f}(S) \). Note that we have here emphasized the fact that the map depends on \( \varphi \). The same statements also show that there exists a unique morphism of groups

\[
\begin{align*}
\text{Cact}_W & \rightarrow \mathcal{S}_W \\
\tau & \mapsto \tau^R_\varphi
\end{align*}
\]

such that

\[
\tau^R_{I, \varphi} = \rho^R_I
\]
for all $I \in \mathcal{P}_{k,f}(S)$. Moreover, Theorem 3.7(a) shows that both actions commute or, in other words, that the map

$$\text{Cact}_W \times \text{Cact}_W \rightarrow \mathcal{S}_W \quad (\tau_1, \tau_2) \mapsto \tau_{L, \varphi}^1 \tau_{R, \varphi}^2$$

is a morphism of groups. Let us summarize the properties of this morphism which are proved in this paper:

**Theorem 3.9.** — Assume that the hypotheses of §3.B hold. Let $\tau \in \text{Cact}_W$. Then:

(a) If $C$ is a left cell, then $\tau_{L, \varphi}(C)$ is a left cell and the $A$-linear map $\mathcal{H}^L[C] \rightarrow \mathcal{H}^L[\tau_{L, \varphi}(C)]$, $c_w^L \mapsto c_{\tau_{L, \varphi}(w)}^L$ is an isomorphism of left $\mathcal{H}$-modules.

(a') If $C$ is a right cell, then $\tau_{R, \varphi}(C)$ is a right cell and the $A$-linear map $\mathcal{H}^R[C] \rightarrow \mathcal{H}^R[\tau_{R, \varphi}(C)]$, $c_w^R \mapsto c_{\tau_{R, \varphi}(w)}^R$ is an isomorphism of right $\mathcal{H}$-modules.

(b) If $w \in W$, then $\tau_{L, \varphi}(w) \sim_r w$ and $\tau_{R, \varphi}(w) \sim_l w$.

(c) If $\tau' \in \text{Cact}_W$, then $[\tau_{L, \varphi}, \tau_{R, \varphi}^R] = \text{Id}_W$.

**References**


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