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Pierre Gaillard, Mickael Gastineau. Tenth order solutions to the NLS equation with eighteen parameters. 2015. hal-01165324

HAL Id: hal-01165324
https://hal.science/hal-01165324

Preprint submitted on 18 Jun 2015

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Tenth order solutions to the NLS equation with eighteen parameters.

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April 29, 2015

Abstract

We present here new solutions of the focusing one dimensional non linear Schrödinger equation which appear as deformations of the Peregrine breather of order 10 with 18 real parameters. With this method, we obtain new families of quasi-rational solutions of the NLS equation, and we obtain explicit quotients of polynomial of degree 110 in x and t by a product of an exponential depending on t. We construct new patterns of different types of rogue waves and recover the triangular configurations as well as rings and concentric as found for the lower orders.

PACS : 35Q55, 37K10.
Keywords : NLS equation, wronskians, Peregrine breather, rogue waves.

1 Introduction

In 1972 Zakharov and Shabat solved the nonlinear Schrodinger equation (NLS) [18, 19] using the inverse scattering method [1, 2]. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [3]. The first quasi-rational solutions of NLS equation were constructed in 1983 by Peregrine [4], nowadays called worldwide Peregrine breathers. In 1986 Akhmediev, Eleonski and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather [5, 6]. Other families of higher order 3 and 4 were constructed in a series of articles by Akhmediev et al. [7, 8], using Darboux transformations.
In 2011, solutions of the NLS equation have been presented as a quotient of two

wronskians of order $2N$ in [11, 12].
In 2012, Guo, Ling and Liu have constructed an other representation of the solutions of the NLS equation, as a quotient of two determinants [**?**] using generalized Darboux transform.
A new approach has been presented in [**?**] where solutions of the NLS equation have been expressed by means of a determinant, obtained from Hirota bilinear method.
Then it was found in [13] for the order N (for determinants of order 2N), solutions depending on $2N-2$ real parameters.
With this method, we construct new solutions of the focusing one dimensional nonlinear Schrödinger equation which appear as deformations of the (analogue) Peregrine breather of order 10 with 18 real parameters : when all the parameters are equal to 0, we recover the famous P_{10} breather. These solutions are completely expressed as a quotient of two polynomials of degree 110 in x and t by an exponential depending on t. We dont have the space to present them here; we only present plots in the $(x;t)$ plane to analyse the evolution of the solutions in function of the different parameters.

2 Solutions of NLS equation in terms of determinant

2.1 Quasi-rational limit solutions of the NLS equation

We consider the one dimensional focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

We recall briefly the results obtained in [11, 12, 13]. We consider $2N$ parameters λ_ν, $\nu = 1, \ldots, 2N$ satisfying the relations

$$0 < \lambda_j < 1,\ \lambda_{N+j} = -\lambda_j,\ 1 \leq j \leq N. \tag{2}$$

We define the terms κ_ν, δ_ν, γ_ν by the following equations,

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2},\ \delta_\nu = \kappa_\nu \lambda_\nu,\ \gamma_\nu = \sqrt{\tfrac{1-\lambda_\nu}{1+\lambda_\nu}}, \tag{3}$$

and

$$\begin{array}{l} \kappa_{N+j} = \kappa_j,\ \delta_{N+j} = -\delta_j,\ \gamma_{N+j} = 1/\gamma_j, \\ 1 \leq j \leq N. \end{array} \tag{4}$$

The terms $x_{r,\nu}$ ($r = 3,\ 1$) are defined by

$$x_{r,\nu} = (r-1)\ln \tfrac{\gamma_\nu - i}{\gamma_\nu + i},\ 1 \leq \nu \leq 2N. \tag{5}$$

The parameters e_ν are defined by

$$\begin{array}{l} e_j = ia_j - b_j,\ e_{N+j} = ia_j + b_j, \\ 1 \leq j \leq N, \end{array} \tag{6}$$

where a_j and b_j, for $1 \leq j \leq N$ are arbitrary real numbers.
The terms ϵ_ν are defined by :

$$\begin{aligned} \epsilon_\nu &= 0, \quad 1 \leq \nu \leq N \\ \epsilon_\nu &= 1, \quad N+1 \leq \nu \leq 2N. \end{aligned}$$

We use the following notations :
$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \leq \nu \leq 2N.$
We consider the functions

$$\begin{aligned} \phi_{r,\nu}(y) &= \sin\Theta_{r,\nu},\ 1 \leq \nu \leq N, \\ \phi_{r,\nu}(y) &= \cos\Theta_{r,\nu},\ N+1 \leq \nu \leq 2N. \end{aligned}$$

$W_r(y) = W(\phi_{r,1}, \ldots, \phi_{r,2N})$ is the wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu\in[1,\ldots,2N]}].$$

Then we get the following statement [12]

Theorem 2.1 *The function v defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)}\exp(2it - i\varphi). \tag{7}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

To obtain quasi-rational solutions of the NLS equation, we take the limit when the parameters $\lambda_j \to 1$ for $1 \leq j \leq N$ and $\lambda_j \to -1$ for $N+1 \leq j \leq 2N$.
For that, we consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2j^2\epsilon^2,\ 1 \leq j \leq N. \tag{8}$$

When ϵ goes to 0, we obtain quasi-rational solutions of the NLS equation given by :

Theorem 2.2 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi)\lim_{\epsilon\to 0}\frac{W_3(0)}{W_1(0)}, \tag{9}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

2.2 Expression of solutions of NLS equation in terms of a quotient of two determinants

We construct the solutions of the NLS equation expressed as a quotient of two determinants which does not involve a passage to the limit.
We use the following notations :

$$A_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,$$

$$B_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (3), (4) and (5).
The parameters e_ν are defined by (6).
Here, the parameters a_j and b_j, for $1 \leq N$ are chosen in the form

$$\begin{array}{l} a_j = \sum_{k=1}^{N-1} \tilde{a_k} \epsilon^{2k+1} j^{2k+1}, \\ b_j = \sum_{k=1}^{N-1} \tilde{b_k} \epsilon^{2k+1} j^{2k+1}, \\ 1 \leq j \leq N. \end{array} \tag{10}$$

We consider the following functions :

$$\begin{array}{l} f_{4j+1,k} = \gamma_k^{4j-1} \sin A_k, \\ f_{4j+2,k} = \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} = -\gamma_k^{4j+1} \sin A_k, \\ f_{4j+4,k} = -\gamma_k^{4j+2} \cos A_k, \end{array} \tag{11}$$

for $1 \leq k \leq N$, and

$$\begin{array}{l} f_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos A_{N+k}, \\ f_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin A_{N+k}, \\ f_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos A_{N+k}, \\ f_{4j+4,k} = \gamma_k^{2N-4j-5} \sin A_{N+k}, \end{array} \tag{12}$$

for $1 \leq k \leq N$.
We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, we replace only the term A_k by B_k.

$$\begin{array}{l} g_{4j+1,k} = \gamma_k^{4j-1} \sin B_k, \\ g_{4j+2,k} = \gamma_k^{4j} \cos B_k, \\ g_{4j+3,k} = -\gamma_k^{4j+1} \sin B_k, \\ g_{4j+4,k} = -\gamma_k^{4j+2} \cos B_k, \end{array} \tag{13}$$

for $1 \leq k \leq N$, and

$$\begin{array}{l} g_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos B_{N+k}, \\ g_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin B_{N+k}, \\ g_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos B_{N+k}, \\ g_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin B_{N+k}, \end{array}$$

for $1 \leq k \leq N$.
Then we get the following result :

Theorem 2.3 *The function v defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{(2it-i\varphi)}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$\begin{array}{l} n_{j1} = f_{j,1}(x,t,0), \\ n_{jk} = \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = f_{j,N+1}(x,t,0), \\ n_{jN+k} = \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ d_{j1} = g_{j,1}(x,t,0), \\ d_{jk} = \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = g_{j,N+1}(x,t,0), \\ d_{jN+k} = \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ 2 \leq k \leq N,\ 1 \leq j \leq 2N \end{array}$$

The functions f and g are defined in (11),(12), (13), (14).

The solutions of the NLS equation can also be written in the form :

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

where $Q(x,t)$ is defined by :

$$\frac{\begin{vmatrix} f_{1,1}[0] & \dots & f_{1,1}[N-1] & f_{1,N+1}[0] & \dots & f_{1,N+1}[N-1] \\ f_{2,1}[0] & \dots & f_{2,1}[N-1] & f_{2,N+1}[0] & \dots & f_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}[0] & \dots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \dots & f_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} g_{1,1}[0] & \dots & g_{1,1}[N-1] & g_{1,N+1}[0] & \dots & g_{1,N+1}[N-1] \\ g_{2,1}[0] & \dots & g_{2,1}[N-1] & g_{2,N+1}[0] & \dots & g_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{2N,1}[0] & \dots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \dots & g_{2N,N+1}[N-1] \end{vmatrix}}$$

3 Quasi-rational solutions of order 10 with eighteen parameters

Wa have constructed in [11] and in other papers on the archive hal, solutions for the cases from $N = 1$ until $N = 9$ with $2N - 2$ parameters.

We do not give the analytic expression of the solution of NLS equation of order 10 with eighteen parameters because of the length of the expression. The computations were done using the computer algebra systems Maple and TRIP [29]. The number of terms of the polynomials of the numerator $d3$ and denominator $d1$ of the solutions are shown in the table above (3) when other a_i and b_i are set to 0.

N=10	number of terms
$W3(a_1, b_1, x, t)$	397478
$W1(a_1, b_1, x, t)$	202314
$W3(a_2, b_2, x, t)$	153368
$W1(a_2, b_2, x, t)$	78087
$W3(a_3, b_3, x, t)$	82200
$W1(a_3, b_3, x, t)$	41832
$W3(a_4, b_4, x, t)$	53991
$W1(a_4, b_4, x, t)$	27488
$W3(a_5, b_5, x, t)$	38526
$W1(a_5, b_5, x, t)$	19616

Table 1: Number of terms for the polynomials $d3$ and $d1$ of the solutions of the NLS equation.

We construct figures to show deformations of the tenth Peregrine breather. We get different types of symmetries in the plots in the (x, t) plane. We give some examples of this fact in the following

3.1 Peregrine breather of order 10

If we choose $\tilde{a}_i = \tilde{b}_i = 0$ for $1 \leq i \leq 8$, we obtain the classical Peregrine breather

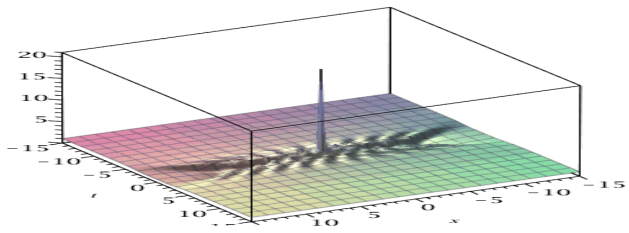


Figure 1: Solution of NLS, N=10, all parameters equal to 0, Peregrine breather P_{10}.

3.2 Variation of parameters

With other choices of parameters, we obtain all types of configurations : triangles and multiple concentric rings configurations with a maximum of 45 peaks.

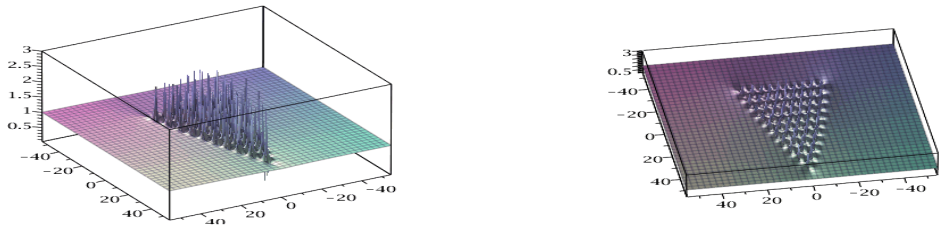


Figure 2: Solution of NLS, N=10, $\tilde{a}_1 = 10^3$: triangle with 55 peaks; on the right, sight of top.

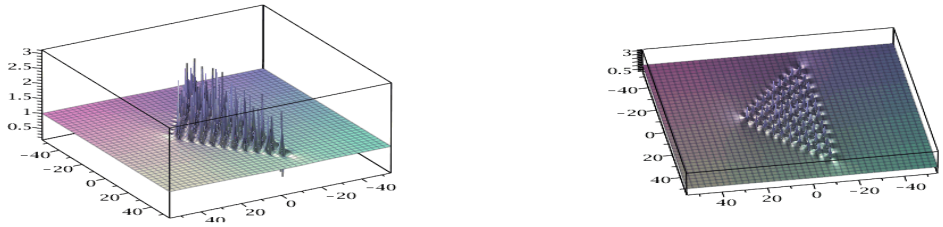


Figure 3: Solution of NLS, N=10, $\tilde{b}_1 = 10^3$: triangle with 55 peaks; on the right, sight of top.

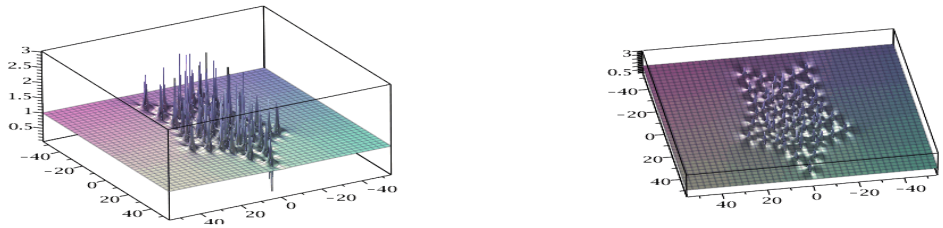


Figure 4: Solution of NLS, N=10, $\tilde{a}_2 = 10^5$: 8 rings without a peak in the center; on the right, sight of top.

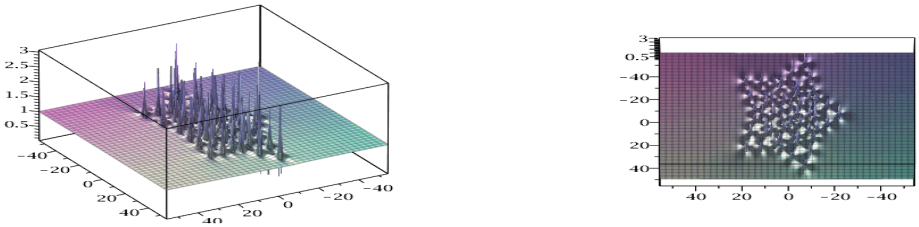


Figure 5: Solution of NLS, N=10, $\tilde{b}_2 = 10^5$: 8 rings without a peak in the center; on the right, sight of top.

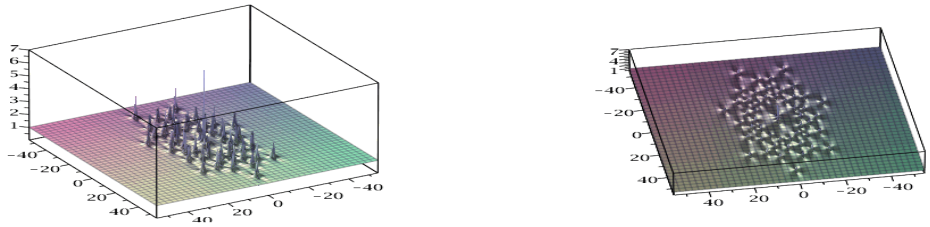


Figure 6: Solution of NLS, N=10, $\tilde{a}_3 = 10^7$: 4 rings with in the center P_3; on the right, sight of top.

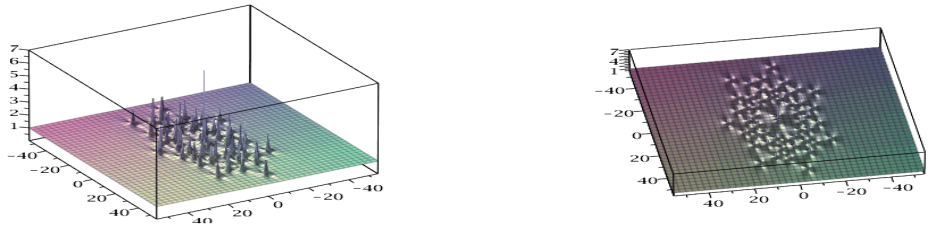


Figure 7: Solution of NLS, N=10, $\tilde{b}_3 = 10^7$: 4 rings with in the center P_3; on the right, sight of top.

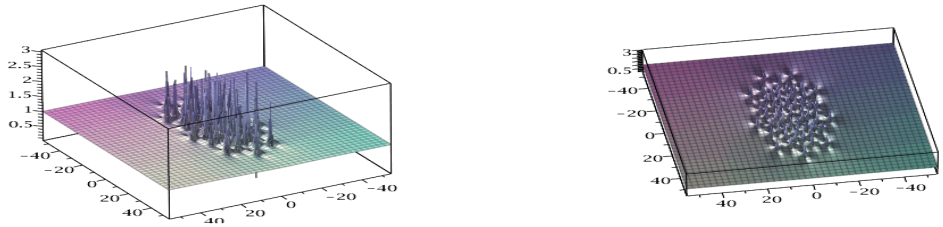


Figure 8: Solution of NLS, N=10, $\tilde{a}_4 = 10^8$: 5 rings with a central peak; on the right, sight of top.

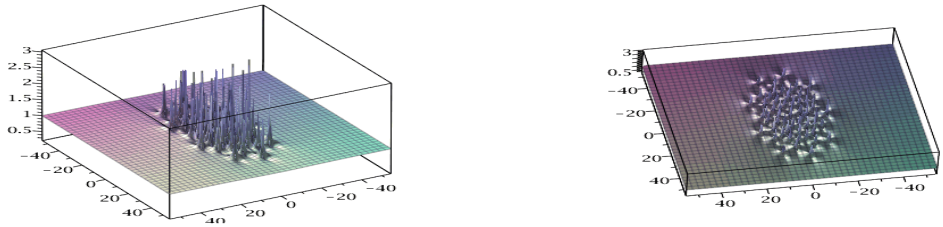


Figure 9: Solution of NLS, N=10, $\tilde{b}_4 = 10^8$: 5 rings with a central peak; on the right, sight of top.

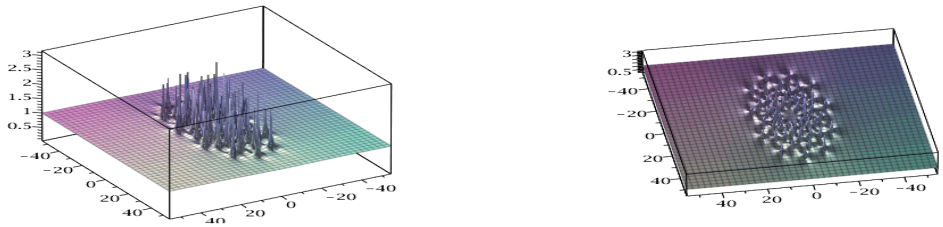


Figure 10: Solution of NLS, N=10, $\tilde{a}_5 = 10^{10}$: 5 rings without a central one peak; on the right, sight of top.

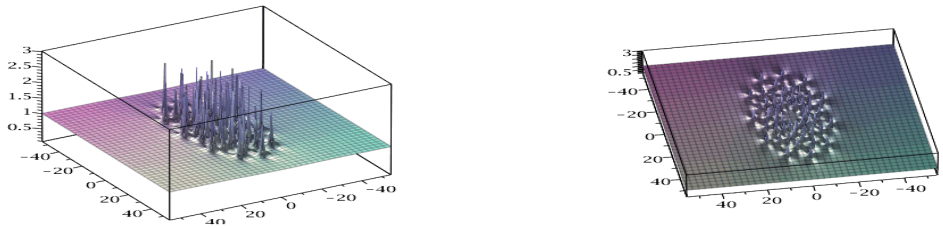


Figure 11: Solution of NLS, N=10, $\tilde{b}_5 = 10^{10}$: 5 rings without a central one peak; on the right, sight of top.

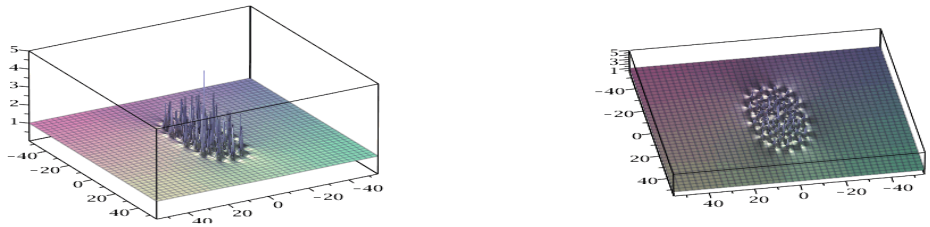


Figure 12: Solution of NLS, N=10, $\tilde{a}_6 = 10^{10}$: 4 rings with in the center the Peregrine breather of order 2; on the right, sight of top.

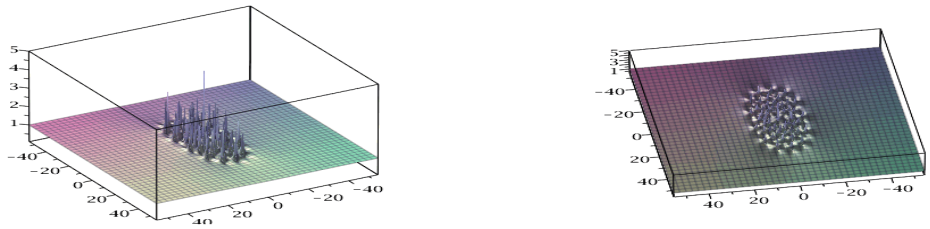


Figure 13: Solution of NLS, N=10, $\tilde{b}_6 = 10^{10}$: 4 rings with in the center the Peregrine breather of order 2; on the right, sight of top.

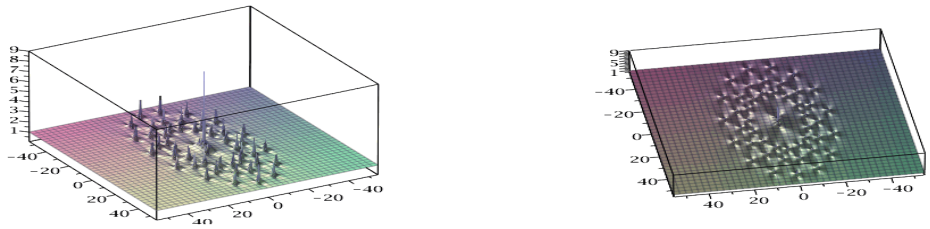


Figure 14: Solution of NLS, N=10, $\tilde{a}_7 = 10^{16}$: 3 rings with in the center the Peregrine breather of order 4; on the right, sight of top.

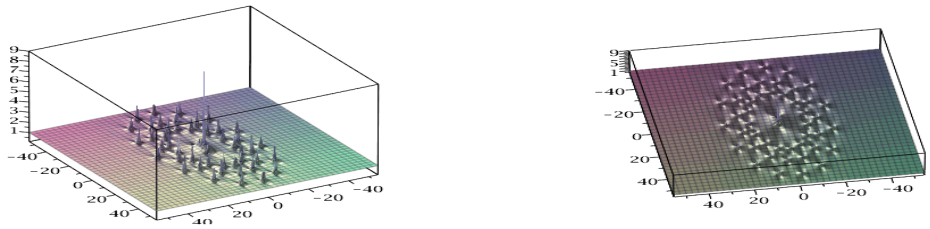


Figure 15: Solution of NLS, N=9, $\tilde{b}_7 = 10^{16}$: 3 rings with in the center the Peregrine breather of order 4; on the right, sight of top.

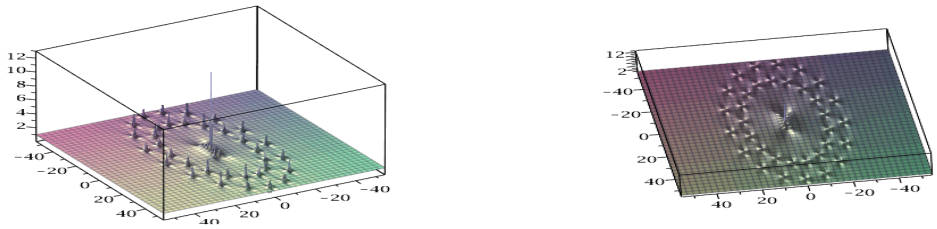


Figure 16: Solution of NLS, N=10, $\tilde{a}_8 = 10^{19}$: 2 ring with in the center the Peregrine breather of order 6; on the right, sight of top.

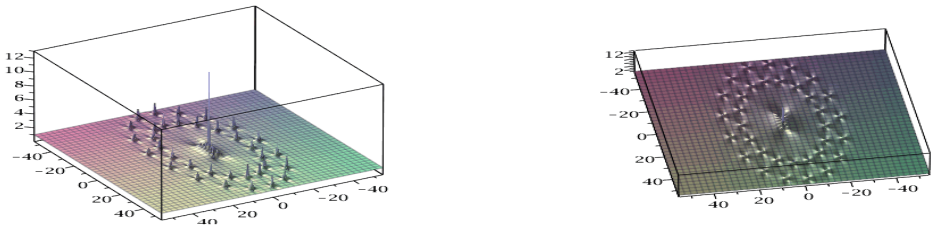


Figure 17: Solution of NLS, N=10, $\tilde{b}_8 = 10^{19}$: 2 ring with in the center the Peregrine breather of order 6; on the right, sight of top.

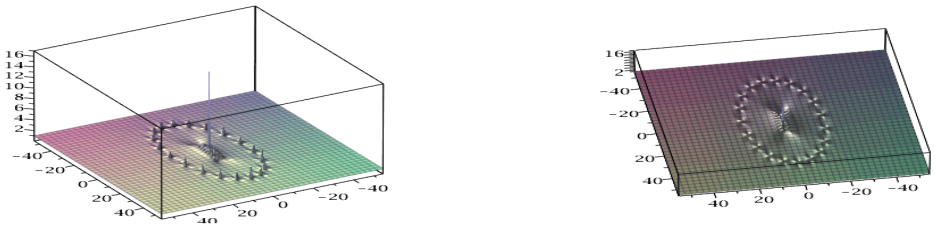


Figure 18: Solution of NLS, N=10, $\tilde{a}_9 = 10^{19}$: one ring with in the center the Peregrine breather of order 8; on the right, sight of top.

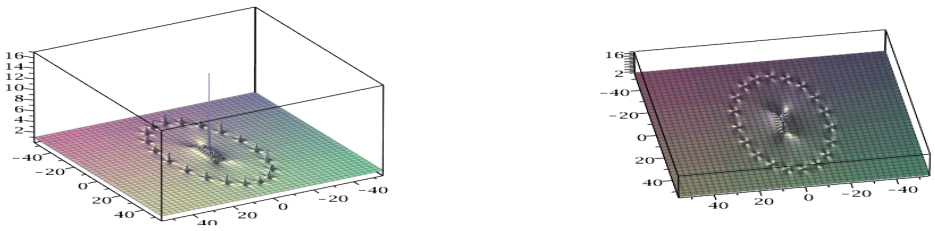


Figure 19: Solution of NLS, N=10, $\tilde{b}_9 = 10^{19}$: one ring with in the center the Peregrine breather of order 8; on the right, sight of top.

4 Conclusion

We have constructed explicitly solutions of the NLS equation of order 10 with 18 real parameters. The explicit representation in terms of polynomials of degree 110 in x and t is too large to be published in this text. The method described here provides a powerful tool to get explicitly solutions of the NLS equation. As my knowledge, it is the first time that the Peregrine breather of order ten with its deformations with eighteen parameters is presented. It confirms the conjecture about the shape of the breather in the (x,t) coordinates, the maximum of amplitude equal to $2N+1$ and the degree of polynomials in x and t here equal to $N(N+1)$.
By different choices of these parameters, we obtained new patterns in the $(x;t)$ plane; we recognized ring shape as already observed in the case of deformations depending on two parameters [11, 13]. We get news triangular shapes and multiple concentric rings.
It is important to underline the symmetrical role played by the parameters a_j and b_j; the configurations obtained for one of these two parameters $a_j \neq 0$ or $b_j \neq 0$ for the index j are the same ones.
We can mention important applications for example in the fields of nonlinear optics and hydrodynamics; we can cite in particular the works of Chabchoub et al [30] or Kibler et al. [31].
This study gives a classification of the solutions of order 10 of the NLS equation as it was begun recently, but differently for lower orders in [32] and try to bring a better understanding of the phenomenon of the rogue waves. It would be important in the future, to try to classify the solutions of NLS equation in the general case of the order N.

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