Theoretical L-moments and TL-moments Using Combinatorial Identities and Finite Operators

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Moments have been traditionally used to characterize a probability distribution. Recently, L-moments and trimmed L-moments are appealing alternatives to the conventional moments. This paper focuses on the computation of theoretical L-moments and TL-moments and emphasizes the use of combinatorial identities. We are able to derive new closed-form formulas of L-moments and TL-moments for continuous probability distributions. Finally, closed-form formulas for the L-moments for the exponential distribution and the uniform distribution are also obtained.

Keywords L-moments; Trimmed L-moments; Combinatorial identities; Finite-element operators; Uniform distribution; Exponential distribution.

1 Introduction

Linear moments (L-moments) were first introduced by Hosking (1990) as new measures of the location, scale and shape of probability distributions. They are related to expected values of order statistics and analogous to the conventional moments (mean, variance, skewness, kurtosis and higher moments). Consider a random variable X and denote by $X_{j,m}$ the jth order statistic (that is the jth smallest variable of an i.i.d. sample of size m). The mth L-moment of X is defined as

$$\lambda_m = \frac{1}{m}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\mathbb{E}(X_{m-j,m}), \tag{1}$$

where $m \in \mathbb{N}^\star$ and $\binom{m-1}{j}$ denotes the usual binomial coefficient, see e.g. Olver et al. (2010). For ease of notation, the dependence of λ_m on the random variable X is not stressed. Throughout the paper, we assume that the expectation $\mathbb{E}(X_{m-j,m})$ exists, otherwise the corresponding L-moment is infinite and the corresponding algebraic manipulation is useless.

L-moments can be computed for a large variety of probability distributions but may not exist for heavy-tailed distributions such as the Cauchy distribution which has an infinite mean (see e.g. Kotz et al. (1994)). Trimmed L-moments (so-called TL-moment) are a natural extension of L-moments introduced by Elamir & Seheult (2003). The TL-moments are of two types: either symmetric $\lambda_m^{(t)}$ or asymmetric $\lambda_m^{(s,t)}$. They are defined as

$$\lambda_m^{(t)} = \frac{1}{m}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\mathbb{E}(X_{m+t-j,m+2t}), \tag{2}$$

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$$\lambda_m^{(s,t)} = \frac{1}{m}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\mathbb{E}(X_{m+s-j,m+s+t}). \quad (3)$$

The TL-moments have the advantage to exist for heavy-tailed distributions such as the generalized Pareto distribution, see e.g. Beirlant et al. (2004), which makes them particularly attractive for modeling heavy tails distributions. Fitting distributions by matching empirical and theoretical TL-moments are particularly adapted for location and scale parameters. Indeed, the quantile function is by construction an affine function of location and scale leading to immediate estimation. Furthermore, shape parameters may also be obtained as a root of the difference equation. Table 1 list the L/TL-moments for some common two-parameter distributions. As for centered moments, λ_3/λ_2 and $\lambda_3^{(1)}/\lambda_2^{(1)}$ are measures of skewness respectively L-skewness and TL-skewness, while λ_4/λ_2 and $\lambda_4^{(1)}/\lambda_2^{(1)}$ are measures of kurtosis respectively L-kurtosis and TL-kurtosis. The listed two-parameter distributions assume fixed values of such L/TL-skewness and L/TL-kurtosis. Therefore, looking for the distribution that best fit the data on such quantity is challenging. As stated above, TL-moments exist even if the corresponding L-moments do not, e.g. for the Cauchy distribution.

Distribution	Quantile $Q(p)$	λ_1	λ_2	λ_3/λ_2	λ_4/λ_2	$\lambda_1^{(1)}$	$\lambda_2^{(1)}$	$\lambda_3^{(1)}/\lambda_2^{(1)}$	$\lambda_4^{(1)}/\lambda_2^{(1)}$
Uniform	$a+(b-a)p$	$\frac{a+b}{2}$	$\frac{b-a}{6}$	0	0	$\frac{a+b}{2}$	$\frac{b-a}{10}$	0	0
Exponential	$-\lambda\log(1-p)$	λ	$\frac{\lambda}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5\lambda}{6}$	$\frac{\lambda}{4}$	$\frac{2}{9}$	$\frac{1}{12}$
Normal	$\mu+\sigma\Phi(p)$	μ	$\frac{\sigma}{\pi}$	0	0.1226	μ	0.297σ	0	0.062
Cauchy	$\mu+\sigma\tan(\pi p-\frac{\pi}{2})$	∞	∞			μ	0.698σ	0	0.343

Table 1: L/TL-moments of common distributions.

Typical application of fitting distribution by equalizing theoretical TL-moments to empirical ones include loss distribution in insurance, traffic volumes on computer networks, return of stock indexes, see e.g. Karvanen (2006), Hosking (2007) and the references therein.

This papers aims to provide a new point-of-view of the computation of theoretical TL-moments. Combinatorial methods are a powerful tool to simplify and to derive closed-form formulas or asymptotics of complex problems of finite or countable discrete structure, see e.g. Graham et al. (1994), Rosen et al. (2000), Sprugnoli (2006) and Flajolet & Sedgewick (2006). In particular, combinatorial methods provide techniques with combinatorial sums of type (1), (2), and (3). In this paper, we derive new formulas for the theoretical TL-moments as well as two examples of TL-moments for the uniform and exponential distributions. Firstly, we use combinatorial identities and manipulations of binomial coefficients to derive simplified formulas of (1), (2), (3) as well as a general expression for TL-moments of the uniform distribution. Secondly, we work with finite operators to derive TL-moments of the exponential distribution.

The paper is structured as follows: Section 2 presents necessary tools for studying combinatorial identities as well as the identities needed in the subsequent sections. In Section 3, we provide new formulation of TL-moments. Section 4 and 5 apply these new formulations for the computation of TL-moments of uniform distribution and exponential distribution, respectively. Finally, Section 6 concludes.

2 Preliminaries

In this section, we present tools and derive a serie of propositions that will be used in subsequent sections. In the following, the Kronecker delta is denoted by $\delta_{i,n}$, the rising factorial by $x^{\overline{n}} = x(x+1)\dots(x+n-1)$ and the falling factorial by $x^{\underline{n}} = x(x-1)\dots(x-n+1)$, see e.g. Olver et al. (2010). We start with identities based on combinatorial arguments and, then we follow up with identities based on finite operators.

2.1 Identities used for TL moment proved by combinatorial arguments

From Graham et al. (1994), we recall that the binomial coefficient is $\binom{r}{k} = r^{\underline{k}}/k!$, for $r \in \mathbb{R}, k \in \mathbb{N}$. Base manipulations are given in Table 2. The so-called useful identities (Equations (16), (17), (18), (19)) are given in Appendix A. Let us start with two identities based on the well-known Vandermonde identity (16).

symmetry	$\binom{n}{k} = \binom{n}{n-k}$	$n, k \in \mathbb{N}$
absorption	$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1} = \frac{r-k+1}{k}\binom{r}{k-1}$	$r \in \mathbb{R}, k \in \mathbb{N}$
sign change	$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$	$r \in \mathbb{R}, k \in \mathbb{N}$
trinomial revision	$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$	$r \in \mathbb{R}, k \in \mathbb{N}$

Table 2: Base manipulations of binomial coefficients.

Proposition 1. *Let $n \geq k \in \mathbb{N}$.*

$$\sum_{j=0}^{n-k} \binom{n}{j+k}\binom{n-k}{j} = \binom{2n-k}{n-k}.$$

Proof. Using the Vandermonde identity (16) with $r \leftarrow n-k$, $m \leftarrow n$, $n \leftarrow n-k$ and the symmetry rule (Table 2), we get

$$\sum_{j=0}^{n-k} \binom{n}{j+k}\binom{n-k}{j} = \sum_{j=0}^{n-k} \binom{n}{n-k-j}\binom{n-k}{j} = \sum_{l=0}^{n-k} \binom{n}{l}\binom{n-k}{n-k-l} = \binom{n+n-k}{n-k} = \binom{2n-k}{n-k}.$$

□

Proposition 2. *Let $k, n \in \mathbb{N}$.*

$$\sum_{j=k}^{n} \binom{j}{k}\binom{n}{j}^2 = \binom{n}{k}\binom{2n-k}{n-k}.$$

Proof. Using the trinomial revision (Table 2) combined with Proposition 1, we get

$$\sum_{j=k}^{n} \binom{j}{k}\binom{n}{j}^2 = \sum_{j=k}^{n} \binom{n}{k}\binom{n-k}{j-k}\binom{n}{j} = \binom{n}{k}\sum_{j=k}^{n} \binom{n-k}{j-k}\binom{n}{j} = \binom{n}{k}\sum_{l=0}^{n-k} \binom{n-k}{l}\binom{n}{l+k} = \binom{n}{k}\binom{2n-k}{n-k}.$$

□

We now focus on identities not linked to the Vandermonde identity. For the sake of completeness, we recall a result of (Graham et al. 1994, p. 185). The proof is also given in details as similar arguments will be used for the next propositions.

Proposition 3. *Let $k, n \in \mathbb{N}$.*

$$e_{n,k} = \sum_{l=0}^{n} (-1)^l \frac{1}{l+k}\binom{n+l}{2l}\binom{2l}{l} = \frac{(k-1)!n!}{(k+n)!}(-1)^n \binom{k-1}{n}.$$

In particular, $e_{n,k} = 0$ for $n > k-1$.

Proof. We apply the trinomial revision (Table 2)

$$e_{n,k} = \sum_{l=0}^{n} (-1)^l \frac{1}{l+k}\binom{n}{l}\binom{n+l}{l} = \sum_{l=0}^{n} (-1)^l \frac{1}{l+1}\binom{n}{l}\binom{n+l}{n}\frac{l+1}{l+k}.$$

We cannot directly simplify this expression except when $k = 1$. Different manipulations are needed using (20),

$$\frac{(-l-2)+1}{(-l-2)+1-(k-1)} = \sum_{j \geq 0} \binom{k-1}{j} / \binom{-l-2}{j} \text{ and } \binom{-l-2}{j} = (-1)^j \binom{j+l+1}{j}.$$

Thus

$$\frac{l+1}{l+k} = \sum_{j\geq 0} \frac{\binom{k-1}{j}(-1)^j}{\binom{j+l+1}{j}} \Rightarrow \frac{l+1}{l+k}\frac{1}{l+1}\binom{n}{l} = \sum_{j\geq 0} \frac{\binom{k-1}{j}(-1)^j\binom{n+1}{l+1}}{\binom{j+l+1}{j}} = \sum_{j\geq 0} \frac{(k-1)!(n+1)!(-1)^j}{(k-1-j)!(j+l+1)!(n-l)!(n+1)}.$$

Using the sign change rule $\binom{n+l}{l}(-1)^l = \binom{-n-1}{l}$, we obtain

$$e_{n,k} = \frac{(k-1)!n!}{(k+n)!} \sum_{j\geq 0}\sum_{l=0}^{n}(-1)^j \binom{k+n}{n+1+j}\binom{n+1+j}{l+j+1}\binom{-n-1}{l}.$$

Using (17), we deduce that $\sum_{l=0}^{n}\binom{n+1+j}{l+j+1}\binom{-n-1}{l} = \binom{j}{n}$, leading by (18) to

$$e_{n,k} = \frac{(k-1)!n!}{(k+n)!}\sum_{j\geq 0}(-1)^j\binom{k+n}{n+1+j}\binom{j}{n} = -\frac{(k-1)!n!}{(k+n)!}\sum_{j<0}(-1)^j\binom{k+n}{n+1+j}\binom{j}{n}.$$

Letting $i = n+1+j$ we derive

$$\begin{aligned} e_{n,k} &= -\frac{(k-1)!n!}{(k+n)!}\sum_{i\leq n}(-1)^{i-n-1}\binom{k+n}{i}\binom{i-n-1}{n} = \frac{(k-1)!n!}{(k+n)!}\sum_{i\leq n}(-1)^i\binom{k+n}{i}\binom{i-n-1}{n}(-1)^n \\ &= \frac{(k-1)!n!}{(k+n)!}\sum_{i\leq n}(-1)^i\binom{k+n}{i}\binom{2n-i}{n} = \frac{(k-1)!n!}{(k+n)!}(-1)^n\binom{k-1}{n}, \end{aligned}$$

using the sign change rule and (19).

□

Similarly, we apply the same technique: trinomial revision, absorption and identities of Appendix A to derive a closed-form of the following sum.

Proposition 4. *Let $k, n \in \mathbb{N}$.*

$$\tilde{e}_{n,k} = \sum_{l=0}^{n}(-1)^l\frac{l!}{(l+k)!}\binom{n+l}{2l}\binom{2l}{l} = \frac{n!}{(n+k)!}\binom{k-1}{n}.$$

In particular, $\tilde{e}_{n,k} = 0$ for $n > k-1$.

Proof. Consider the sequence

$$\tilde{e}_{n,k} = \sum_{l=0}^{n}(-1)^l\frac{l!}{(l+k)!}\binom{2l}{l}\binom{n+l}{2l} = \sum_{l=0}^{n}(-1)^l\frac{l!}{(l+k)!}\binom{n}{l}\binom{n+l}{l},$$

by trinomial revision. Since

$$\frac{l!}{(l+k)!}\binom{n}{l} = \frac{n!}{(n-l)!(l+k)!} = \frac{n!}{(n+k)!}\binom{n+k}{l+k},$$

we deduce

$$\tilde{e}_{n,k} = \frac{n!}{(n+k)!}\sum_{ls=0}^{n}(-1)^l\binom{n+k}{l+k}\binom{n+l}{l} = \frac{n!}{(n+k)!}\sum_{l\geq 0}(-1)^l\binom{n+k}{k+l}\binom{n+l}{n}.$$

By (18), we obtain

$$\sum_{l\in\mathbb{Z}}(-1)^l\binom{n+k}{k+l}\binom{n+l}{n} = (-1)^{n+2k}\binom{n-k}{n-n-k} = (-1)^n\binom{n-k}{-k} = 0.$$

Thus using $i = k + l$,

$$\tilde{e}_{n,k} = \frac{n!}{(n+k)!}(-1)\sum_{l<0}(-1)^l\binom{n+k}{k+l}\binom{n+l}{n} = \frac{n!}{(n+k)!}(-1)\sum_{i\leq k-1}(-1)^{i-k}\binom{n+k}{i}\binom{n-k+i}{n}.$$

By the sign change rule, we have $\binom{n-(k-1-i)-1}{n}(-1)^n = \binom{k-1-i}{n}$. Hence by (19),

$$\tilde{e}_{n,k} = \frac{n!}{(n+k)!}(-1)^{k+n+1}\sum_{i\leq k-1}(-1)^i\binom{n+k}{i}\binom{k-1-i}{n} = \frac{n!}{(n+k)!}\binom{k-1}{n}.$$

□

Finally, we consider a sum related to Harmonic numbers. Let H_n be Harmonic numbers defined $H_n = \sum_{k=1}^n 1/k$ with the convention that $H_0 = 0$.

Proposition 5. *Let $n \in \mathbb{N}$.*

$$S_n = \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{(1+j)^2} = \frac{H_{n+1}}{n+1}.$$

In particular, $S_0 = 1$, $S_1 = 3/4$, $S_2 = 11/18$ and $S_3 = 25/48$.

Proof. Using the absorption rule, we get

$$S_n = \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{(1+j)^2} = \sum_{j=0}^{n}\binom{n+1}{j+1}\frac{(-1)^j}{(1+j)(n+1)} = \frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j}\frac{(-1)^{j-1}}{j}.$$

Indeed, $H_n = \sum_{j=1}^n \binom{n}{j}\frac{(-1)^{j-1}}{j}$ by Section 6.4 of Graham et al. (1994). Therefore $S_n = H_{n+1}/(n+1)$. □

2.2 Results proved by finite operators

Now, we present the finite operators E and Δ, first introduced by English mathematicians such as G. Boole, see e.g. Spiegel (1971). Forward operator E is defined as $Ef(x) = f(x+1)$ and difference operator Δ is $\Delta f(x) = f(x+1) - f(x)$. Base manipulations are given in Table 3.

linearity	$E[\alpha f(x) + \beta g(x)] = \alpha Ef(x) + \beta Eg(x)$	$\Delta[\alpha f(x) + \beta g(x)] = \alpha\Delta f(x) + \beta\Delta g(x)$
product	$E[f(x)g(x)] = Ef(x)Eg(x)$	$\Delta[f(x)g(x)] = \Delta f(x)Eg(x) + f(x)\Delta g(x)$
exponentiation	$E^n f(x) = f(x+n)$	$\Delta^n f(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(x+k)$

Table 3: Finite operators.

The two formulas for exponentiation can be used to derive closed-form of sums. Since we have

$$E^n = \sum_{k=0}^{n}\binom{n}{k}\Delta^k, \tag{4}$$

$$\Delta^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E^k. \tag{5}$$

If we are able to express Δ^n in a simpler way than Equation (5), then we can deduce $\sum_{k=0}^n\binom{n}{k}\Delta^k$ as E^n from Equation (4) by identifying E^k in Equation (5). Let us work on an illustrative example. From identities in Table 4, we have $\Delta^n[1/x] = (-1)^n n!/(x(x+1)\dots(x+n))$. Multiplying by x, we get

$$\Delta^n\left[\frac{1}{x}\right] = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{x}{x+k} = \frac{(-1)^n n!}{(x+1)\dots(x+n)}.$$

Therefore, identifying terms of (5) in the previous equation yields by (4) to

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k!}{(x+k)\dots(x+1)} = \frac{x}{x+n}.$$

This binomial sum looked particularly terrible at the first shot, but simplifies largely thanks to the exponentiation identities of Δ and E.

Now, we consider the following function $f(x) = \binom{p+x}{m} H_x$, that will be used in Section 5. We want to apply the same technique by simplifying the nth-order finite difference of f and inverting exponentation identities in order to compute the sum

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p+k}{m} H_k.$$

The computation of $\Delta^n f(x)$ requires some intermediate results which have been postponed in Appendix C. Firstly, we derive $\Delta^n f(x)$ for $n \le m$ in Proposition 6 using Lemmas 15 and 16. Secondly, Proposition 7 for $\Delta^n f(x)$ for $n > m$ is obtained using Lemmas 17 and 18.

In order to compute the Δ-differences of f, we use the function g defined below

$$g\left(x; {p, n \atop m}\right) = \binom{x+p}{m} \binom{x+p+n}{x+p} \binom{x+n}{x}^{-1}.$$

This helps in rewriting Δf

$$\Delta\left[\binom{p+x}{m} H_x\right] = \binom{p+x+1}{m} \frac{1}{x+1} - \left(\binom{p+x}{m} - \binom{p+x+1}{m}\right) H_x = \frac{1}{m} g\left(x; {p, 1 \atop m-1}\right) + \binom{p+x}{m-1} H_x.$$

Therefore, further finite order differences use the differences of g. By manual inspection of the first three finite difference $\Delta^1\left[\binom{p+x}{m}H_x\right]$, $\Delta^2\left[\binom{p+x}{m}H_x\right]$, $\Delta^3\left[\binom{p+x}{m}H_x\right]$, we guess Lemma 16 proved by recurrence. Choosing a specific x leads to the following proposition. This result is yet partial, since it is only valid for $n \le m$.

Proposition 6. *Let $m, p, n \in \mathbb{N}$ such that $n \le m$.*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p+k}{m} H_k = (-1)^n \binom{p}{m-n} \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} \binom{p+j}{j},$$

with the convention that $\sum_1^0$ cancels when $n = 0$.

Proof. By Lemma 16, we deduce that for $x \in \mathbb{R}$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{p+x+k}{m} H_{x+k} = \binom{p+x}{m-n} H_x + \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} g\left(x; {p, j \atop m-n}\right).$$

In particular for $x = 0$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{p+k}{m} H_k = \binom{p}{m-n} H_0 + \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} g\left(x; {p, j \atop m-n}\right),$$

where

$$g\left(0; {p, j \atop m-n}\right) = \binom{p}{m-n} \binom{p+j}{j} \binom{j}{j}^{-1} = \binom{p}{m-n} \binom{p+j}{j}.$$

That is

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{p+k}{m} H_k = \binom{p}{m-n} \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} \binom{p+j}{j}$$

$$\Leftrightarrow \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p+k}{m} H_k = (-1)^n \binom{p}{m-n} \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} \binom{p+j}{j}.$$

□

So far, we get half the job since we only have $\Delta^n f$ (and the corresponding sum) for $n \leq m$. Let us study the Δ-difference for $n = m$. Using Proposition 6 for $n = m$ yields to

$$\Delta^m \left[\binom{p+x}{m} H_x \right] = H_x + \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j} \binom{m}{j} g\left(x; \begin{matrix} p, j \\ 0 \end{matrix}\right).$$

Finite difference of the first term is obtained by Sprugnoli (2006)

$$\Delta^i H_x = \frac{(-1)^{i-1}}{i} \binom{x+i}{i}^{-1}. \tag{6}$$

Thus, the computation of $\Delta^{m+i} f$ simply relies on the computation of $\Delta^i g$ (second term). For ease of notation, we introduce a new function h of x with parameters a, b, c, d defined as

$$h\left(x; \begin{matrix} a, b \\ c, d \end{matrix}\right) = \frac{(x+a)^{\underline{b}}}{(x+c)^{\underline{d}}}.$$

By convention, when b or d equals 0, the corresponding term equals 1, and when $c = d$, d is omitted. This h function is directly linked to g by

$$g\left(x; \begin{matrix} p, n \\ 0 \end{matrix}\right) = \binom{x+p+n}{n} \binom{x+n}{n}^{-1} = h\left(x; \begin{matrix} p+n, n \\ n \end{matrix}\right).$$

Therefore, finite difference $\Delta^{m+i} f$ uses the differences of h. Again by manual inspection of the first three finite difference $\Delta^{m+1}\left[\binom{p+x}{m} H_x\right]$, $\Delta^{m+2}\left[\binom{p+x}{m} H_x\right]$, $\Delta^{m+3}\left[\binom{p+x}{m} H_x\right]$, we guess Lemma 17 that we prove by recurrence. Using Lemma 17 and (6), we get the Δ-difference of f in Lemma 18. At last, we are able to derive a new formulation of the sum.

Proposition 7. *Let $i, m, p \in \mathbb{N}$ such that $i, m \geq 1$. We have*

$$\sum_{k=0}^{m+i} \binom{m+i}{k} (-1)^k \binom{p+k}{m} H_k = (-1)^{m+i} \frac{(-1)^{i-1}}{i} + (-1)^{m+i} \sum_{j=1}^{i} \sum_{l=0}^{j-1} \frac{(-1)^j (l+1)}{j(i+l+1)} \binom{m}{j} \binom{j}{l+1} \binom{p}{l}$$
$$+ (-1)^{m+i} \frac{(i-1)! p!}{(p+i)!} \sum_{j=i+1}^{m} \sum_{l=1}^{i} (-1)^j \binom{m}{j} \binom{i}{l} \binom{j-1+l}{j} \binom{p+j}{j+l}.$$

By convention $\sum_{j=i+1}^{m}$ cancels when $i \geq m$.

Proof. Choosing $x = 0$ in Lemma 18, we get

$$h\left(0; \begin{matrix} p+j, j-i\wedge j \\ j+i-k \end{matrix}\right) = \frac{(p+j)^{\underline{j-i\wedge j}}}{(j+i-k)^{\underline{j+i-k}}}, \quad \binom{0+i}{i}^{-1} = 1,$$

and

$$\Delta^{m+i} \left[\binom{p+x}{m} H_x \right]_{x=0} = \frac{(-1)^{i-1}}{i} + \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j} \binom{m}{j} (-1)^i \sum_{k=0}^{i\wedge j-1} \binom{i\wedge j}{i\wedge j-k}$$
$$\times \frac{(i\wedge j-1)! j^{\overline{i\wedge j-k}} p^{\underline{i\wedge j-k}}}{(i\wedge j-k-1)!} \frac{(j+i-k-1)!}{(j+i\wedge j-k-1)!} \frac{(p+j)^{\underline{j-i\wedge j}}}{(j+i-k)^{\underline{j+i-k}}}.$$

Since for $x = 0$

$$\Delta^{m+i} \left[\binom{x+p}{m} H_x \right]_{x=0} = \sum_{k=0}^{m+i} \binom{m+i}{k} (-1)^{m+i-k} \binom{p+k}{m} H_k,$$

we obtain by simplifying $(-1)^i$

$$\sum_{k=0}^{m+i} \binom{m+i}{k} (-1)^k \binom{p+k}{m} H_k = \frac{(-1)^{m+1}}{i} + (-1)^{m+1} \sum_{j=1}^{m} \frac{(-1)^j}{j} \binom{m}{j} \sum_{k=0}^{i\wedge j-1} \binom{i\wedge j}{i\wedge j-k}$$
$$\times \frac{(i\wedge j-1)! j^{\overline{i\wedge j-k}} p^{\underline{i\wedge j-k}}}{(i\wedge j-k-1)!} \frac{(j+i-k-1)!}{(j+i\wedge j-k-1)!} \frac{(p+j)^{\underline{j-i\wedge j}}}{(j+i-k)^{\underline{j+i-k}}}.$$

When $i < m$, we can split the outer sum, say S_j.

$$\begin{aligned}
S_{j,i<j} &= \sum_{k=0}^{i\wedge j-1} \binom{i\wedge j}{i\wedge j-k} \frac{(i\wedge j-1)!\, j^{\overline{i\wedge j-k}} p^{\underline{i\wedge j-k}}}{(i\wedge j-k-1)!} \frac{(j+i-k-1)!}{(j+i\wedge j-k-1)!} \frac{(p+j)^{\underline{j-i\wedge j}}}{(j+i-k)^{\underline{j+i-k}}} \\
&= \sum_{k=0}^{i-1} \binom{i}{i-k} \frac{(i-1)!\, j^{\overline{i-k}} p^{\underline{i-k}}}{(i-k-1)!} \frac{(j+i-k-1)!}{(j+i-k-1)!} \frac{(p+j)^{\underline{j-i}}}{(j+i-k)^{\underline{j+i-k}}} \\
&= \sum_{k=0}^{i-1} \binom{i}{i-k} (i-1)! \frac{(j+i-k-1)!}{(j-1)!(i-k-1)!} \frac{p!}{(p+k-i)!(j+i-k)!} \frac{(p+j)!}{(p+i)!} \\
&= \frac{j(i-1)!p!}{(p+i)!} \sum_{k=0}^{i-1} \binom{i}{k} \binom{j+i-k-1}{j} \binom{p+j}{p+k-i} = \frac{j(i-1)!p!}{(p+i)!} \sum_{l=1}^{i} \binom{i}{l} \binom{j-1+l}{j} \binom{p+j}{j+l}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
S_{j,i\geq j} &= \sum_{k=0}^{i\wedge j-1} \binom{i\wedge j}{i\wedge j-k} \frac{(i\wedge j-1)!\, j^{\overline{i\wedge j-k}} p^{\underline{i\wedge j-k}}}{(i\wedge j-k-1)!} \frac{(j+i-k-1)!}{(j+i\wedge j-k-1)!} \frac{(p+j)^{\underline{j-i\wedge j}}}{(j+i-k)^{\underline{j+i-k}}} \\
&= \sum_{k=0}^{j-1} \binom{j}{j-k} \frac{(j-1)!\, j^{\overline{j-k}} p^{\underline{j-k}}}{(j-k-1)!} \frac{(j+i-k-1)!}{(j+j-k-1)!} \frac{(p+j)^{\underline{j-j}}}{(j+i-k)^{\underline{j+i-k}}} \\
&= \sum_{k=0}^{j-1} \binom{j}{j-k} \frac{(2j-k-1)!p!}{(j-k-1)!(p-j+k)!} \frac{(j+i-k-1)!}{(2j-k-1)!} \frac{1}{(j+i-k)!} \\
&= \sum_{k=0}^{j-1} \binom{j}{j-k} \binom{p}{j-k-1} \frac{j-k}{(j+i-k)} = \sum_{l=0}^{j-1} \binom{j}{l+1} \binom{p}{l} \frac{l+1}{i+l+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^{m} \frac{(-1)^j}{j} \binom{m}{j} S_j = \sum_{j=i+1}^{m} \frac{(-1)^j}{j} \binom{m}{j} S_{j,i<j} + \sum_{j=1}^{i} \frac{(-1)^j}{j} \binom{m}{j} S_{j,i\geq j} \\
&= \sum_{j=i+1}^{m} \frac{(-1)^j}{j} \binom{m}{j} \frac{j(i-1)!p!}{(p+i)!} \sum_{l=1}^{i} \binom{i}{l} \binom{j-1+l}{j} \binom{p+j}{j+l} + \sum_{j=1}^{i} \frac{(-1)^j}{j} \binom{m}{j} p \sum_{l=0}^{j-1} \binom{j}{l+1} \binom{p-1}{l} \frac{1}{i+l+1} \\
&= \frac{(i-1)!p!}{(p+i)!} \sum_{j=i+1}^{m} \sum_{l=1}^{i} (-1)^j \binom{m}{j} \binom{i}{l} \binom{j-1+l}{j} \binom{p+j}{j+l} + \sum_{j=1}^{i} \sum_{l=0}^{j-1} \frac{(-1)^j (l+1)}{j(i+l+1)} \binom{m}{j} \binom{j}{l+1} \binom{p}{l}.
\end{aligned}$$

Otherwise when $i \geq m$, we always have $i > j$. Finally, we obtain

$$\begin{aligned}
\sum_{k=0}^{m+i} \binom{m+i}{k} (-1)^k \binom{p+k}{m} H_k \quad &= \quad (-1)^{m+i} \frac{(-1)^{i-1}}{i} + (-1)^{m+i} \sum_{j=1}^{i} \sum_{l=0}^{j-1} \frac{(-1)^j (l+1)}{j(i+l+1)} \binom{m}{j} \binom{j}{l+1} \binom{p}{l} \\
&\quad + (-1)^{m+i} \frac{(i-1)!p!}{(p+i)!} \sum_{j=i+1}^{m} \sum_{l=1}^{i} (-1)^j \binom{m}{j} \binom{i}{l} \binom{j-1+l}{j} \binom{p+j}{j+l},
\end{aligned}$$

where $\sum_{j=i+1}^{m}$ cancels for $i \geq m$.

□

3 Theoretical L and TL-moments

Computing theoretical L-moments (1) and TL-moments (2), (3) needs the expectation of the order statistic $\mathbb{E}(X_{j,n})$. Let us recall a well known result of probability theory, see e.g. Johnson et al. (1994). Consider a sample $X_1, \ldots, X_n$ of independent and identically distributed random variables $X_i \sim X$. We assume that the random variable X has a density function f and a cumulative distribution function F defined on $(a,b) \mapsto \mathbb{R}$. Possibly a and/or b can be infinite, e.g. $a = -\infty$ and $b = +\infty$ for a normal distribution. The jth order statistic $X_{j,n}$ has the following density

$$f_{X_{j,n}}(x) = \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} \left(1 - F(x)\right)^{n-j} f(x). \tag{7}$$

By simple manipulations, the expectation can be written as

$$\mathbb{E}(X_{j,n}) = j\binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^{n-j-k} I_F(n-k-1), \tag{8}$$

where I_F is defined as

$$I_F(k) = \int_a^b x F(x)^k f(x) dx = \int_0^1 Q(p) p^k dp, \tag{9}$$

with $Q = F^{-1}$ the quantile function. Note that $I_F(0) = \mathbb{E}(X)$. We now present the central result of this paper simplifying in a sense the computation of TL-moments.

Proposition 8. *Let $m, s, t \in \mathbb{N}$. The TL-moment of X can be expressed as*

$$\lambda_m^{(s,t)} = \frac{m+t+s}{m} \sum_{l=0}^{m+t-1} (-1)^{m+l+1} I_F(s+l) \binom{m+s+t-1}{s+l} \binom{m+s+l-1}{l}. \tag{10}$$

Proof. Using (3), (8), the absorption and the symmetry rules, we have

$$\lambda_m^{(s,t)} = \frac{1}{m} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \mathbb{E}(X_{m+s-j,m+s+t}) \tag{11}$$

$$= \frac{1}{m} \sum_{k=0}^{m+t-1} (-1)^{t-k} I_F(s+m+t-1-k) \sum_{j=k-t}^{m-1} \binom{m-1}{j} \binom{m+s+t}{m+s-j} \binom{t+j}{k} (m+s-j) \tag{12}$$

$$= \frac{m+s+t}{m} \sum_{l=0}^{m+t-1} (-1)^{l+m+1} I_F(s+l) \sum_{j=m-1-l}^{m-1} \binom{m-1}{j} \binom{m+s+t-1}{t+j} \binom{t+j}{m+t-1-l}. \tag{13}$$

Then using the trinomial revision, we get

$$\lambda_m^{(s,t)} = \frac{m+s+t}{m} \sum_{l=0}^{m+t-1} (-1)^{m+l+1} I_F(s+l) \binom{m+t+s-1}{s+l} \sum_{j=m-1-l}^{m-1} \binom{m-1}{j} \binom{s+l}{l-m+j+1}.$$

Let

$$S_{m,l} = \sum_{j=m-1-l}^{m-1} \binom{m-1}{j} \binom{s+l}{l-m+j+1}.$$

Changing the summation index and using the Vandermonde identity (16), we have

$$S_{m,l} = \sum_{i=0}^{l} \binom{m-1}{m-1-i} \binom{s+l}{l-m+(m-1-i)+1} = \sum_{i=0}^{l} \binom{m-1}{i} \binom{s+l}{l-i} = \binom{m+s+l-1}{l}.$$

Therefore,

$$\lambda_m^{(s,t)} = \frac{m+s+t}{m} \sum_{l=0}^{m+t-1} (-1)^{m+l+1} I_F(s+l) \binom{m+t+s-1}{s+l} \binom{m+s+l-1}{l}.$$

□

Hosking (2007) established links with shifted Legendre polynomials as

$$\lambda_m^{(s,t)} = \frac{(m-1)!(m+s+t)!}{(m+s-1)!(m+t-1)!m} \int_0^1 u^s(1-u)^t P_{m-1}^{\star(t,s)}(u) Q(u) du,$$

where $P_m^{\star(t,s)}$ is the mth Legendre polynomial, i.e.

$$P_m^{\star(t,s)}(u) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m+t}{j} \binom{m+s}{s+j} u^j (1-u)^{m-j}.$$

The result (10) is in line with this representation. Indeed,

$$\begin{aligned}
u^s(1-u)^t P_m^{\star(t,s)}(u) &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m+t}{j} \binom{m+s}{s+j} u^{s+j} (1-u)^{m+t-j} \\
&= \sum_{j=0}^{m} (-1)^{m-j} \binom{m+t}{j} \binom{m+s}{s+j} u^{s+j} \sum_{k=0}^{m+t-j} \binom{m+t-j}{k} (-u)^{m+t-j-k} \\
&= \sum_{j=0}^{m} \sum_{k=0}^{m+t-j} (-1)^{t-k} \binom{m+t}{j} \binom{m+s}{s+j} \binom{m+t-j}{k} u^{m+t+s-k}.
\end{aligned}$$

Hence inverting indexes j and k,

$$\begin{aligned}
&\int_0^1 u^s(1-u)^t P_{m-1}^{\star(t,s)}(u) Q(u) du \\
&= \sum_{j=0}^{m-1} \sum_{k=0}^{m+t-1-j} (-1)^{t-k} \binom{m+t-1}{j} \binom{m+s-1}{s+j} \binom{m+t-1-j}{k} \int_0^1 u^{m+t+s-k-1} Q(u) du. \\
&= \sum_{k=0}^{m+t-1} \sum_{j=0}^{m+t-1-k} (-1)^{t-k} \binom{m+t-1}{j} \binom{m+s-1}{s+j} \binom{m+t-1-j}{k} I_F(m+t+s-k-1) \\
&= \sum_{l=0}^{m+t-1} \sum_{j=0}^{l} (-1)^{l-m+1} \binom{m+t-1}{j} \binom{m+s-1}{s+j} \binom{m+t-1-j}{m+t-1-l} I_F(s+l) \\
&= \sum_{l=0}^{m+t-1} I_F(s+l)(-1)^{l-m+1} \sum_{j=0}^{l} \binom{m+t-1}{j} \binom{m+s-1}{s+j} \binom{m+t-1-j}{l-j}.
\end{aligned}$$

Since

$$\binom{m+t-1}{j} \binom{m+s-1}{s+j} \binom{m+t-1-j}{l-j} = \frac{(m+t-1)!(m+s-1)!}{(m-1)!(s+l)!(m+t-1-l)!} \binom{m-1}{j} \binom{s+l}{l-j},$$

and by (16)

$$\sum_{j=0}^{l} \binom{m-1}{j} \binom{s+l}{l-j} = \binom{m+s+l-1}{l},$$

we get

$$\begin{aligned}
&\int_0^1 u^s(1-u)^t P_{m-1}^{\star(t,s)}(u) Q(u) du \\
&= \sum_{l=0}^{m+t-1} I_F(s+l)(-1)^{l-m+1} \frac{(m+t-1)!(m+s-1)!}{(m-1)!(s+l)!(m+t-1-l)!} \binom{m+s+l-1}{l} \\
&= \frac{(m+t-1)!(m+s-1)!}{(m-1)!(m+s+t-1)!} \sum_{l=0}^{m+t-1} I_F(s+l)(-1)^{l-m+1} \binom{m+s+t-1}{s+l} \binom{m+s+l-1}{l}.
\end{aligned}$$

Multiplying by $\frac{(m-1)!(m+s+t)!}{(m+s-1)!(m+t-1)!m}$ leads to Equation (10).

Now, we turn our attention to the symmetric case when $s = t$. We can easily derive a new formulation of $\lambda_m^{(t)}$.

Proposition 9. *Let $m, t \in \mathbb{N}$. The TL-moment of X can be expressed as*

$$\lambda_m^{(t)} = \frac{m+2t}{m} \sum_{l=0}^{m+t-1} (-1)^{m+l+1} I_F(t+l) \binom{m+2t-1}{t+l} \binom{m+t+l-1}{l}. \tag{14}$$

Proof. Using (10) with $t = s$ yields to

$$\begin{aligned}
\lambda_m^{(t)} &= \lambda_m^{(t,t)} = \frac{m+t+t}{m} \sum_{l=0}^{m+t-1} (-1)^{m+l+1} I_F(t+l) \binom{m+t+t-1}{t+l} \binom{m+t+l-1}{l} \\
&= \frac{m+2t}{m} \sum_{l=0}^{m+t-1} (-1)^{m+l+1} I_F(t+l) \binom{m+2t-1}{t+l} \binom{m+t+l-1}{l}.
\end{aligned}$$

□

This result is in line with Equation (7) of Elamir & Seheult (2003)

$$\lambda_m^{(t)} = \frac{m+2t}{m} \sum_{j=0}^{m-1} (-1)^j I_Q(m,t,j) \binom{m+2t-1}{t+j} \binom{m-1}{j},$$

where $I_Q(m,t,j) = \int_0^1 Q(u) u^{m+t-j-1}(1-u)^{t+j} du$. In fact, we have

$$\begin{aligned}
\lambda_m^{(t)} &= \frac{m+2t}{m} \sum_{j=0}^{m-1} (-1)^j \binom{m+2t-1}{t+j} \binom{m-1}{j} \int_0^1 Q(u) u^{m+t-j-1} (1-u)^{t+j} du \\
&= \frac{m+2t}{m} \sum_{j=0}^{m-1} (-1)^j \binom{m+2t-1}{t+j} \binom{m-1}{j} \int_0^1 Q(u) u^{m+t-j-1} \sum_{k=0}^{t+j} \binom{t+j}{k} 1^k (-u)^{t+j-k} du \\
&= \frac{m+2t}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{t+j} (-1)^{t-k} \binom{m+2t-1}{t+j} \binom{m-1}{j} \binom{t+j}{k} \int_0^1 Q(u) u^{m+2t-k-1} du \\
&= \frac{m+2t}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{t+j} (-1)^{t-k} \binom{m+2t-1}{t+j} \binom{m-1}{j} \binom{t+j}{k} I_F(m+2t-k-1) \\
&= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{t+j} (-1)^{t-k} (m+2t-j) \binom{m+2t}{t+j} \binom{m-1}{j} \binom{t+j}{k} I_F(m+2t-k-1),
\end{aligned}$$

using $\binom{m+2t}{t+j} = \frac{m+2t}{m+2t-j} \binom{m+2t-1}{t+j}$. The last equation is exactly Equation (12) with $s = t$.

This result is also in line with Hosking (2007) who establishes links with shifted Legendre polynomials as $\lambda_m^{(s,t)}$ is valid for shifted Legendre polynomials. That is

$$\lambda_m^{(t)} = \frac{(m-1)!(m+2t)!}{((m+t-1)!)^2 m} \int_0^1 \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m+t-1}{j} \binom{m+t-1}{t+j} u^{t+j} (1-u)^{m+t-1-j} Q(u) du.$$

Finally, we get a new formulation for the L-moment λ_m.

Proposition 10. *Let $m \in \mathbb{N}$. The L-moment of X can be expressed as*

$$\lambda_m = \sum_{l=0}^{m-1} (-1)^{m+l+1} I_F(l) \binom{m-1}{l} \binom{m+l-1}{l}. \tag{15}$$

Proof. Using (10) with $t = s = 0$ yields to

$$\begin{aligned}
\lambda_m &= \lambda_m^{(0,0)} = \frac{m+0+0}{m} \sum_{l=0}^{m+0-1} (-1)^{m+l+1} I_F(0+l) \binom{m+0+0-1}{0+l} \binom{m+0+l-1}{l} \\
&= \sum_{l=0}^{m-1} (-1)^{m+l+1} I_F(l) \binom{m-1}{l} \binom{m+l-1}{l}.
\end{aligned}$$

A direct proof is also possible. Let $m > 1$, the L-moment is given by

$$\lambda_m = \frac{1}{m} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} E(X_{m-j,m}) = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{j}{k} \binom{m-1}{j} \binom{m}{j} (-1)^k I_F(m-k-1)(m-j).$$

Since $\binom{m}{j} = \frac{m}{m-j} \binom{m-1}{j}$, we get

$$\lambda_m = \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{j}{k} \binom{m-1}{j} \binom{m-1}{j} (-1)^k I_F(m-k-1) = \sum_{k=0}^{m-1} (-1)^k I_F(m-k-1) \sum_{j=k}^{m-1} \binom{m-1}{j}^2 \binom{j}{k}.$$

Using Proposition (2), this yields to

$$\lambda_m = \sum_{k=0}^{m-1} (-1)^k I_F(m-k-1) \binom{m-1}{k} \binom{2m-2-k}{m-1-k} = \sum_{l=0}^{m-1} (-1)^{m-1-l} I_F(l) \binom{m-1}{l} \binom{m+l-1}{l}.$$

□

Logically, this formula is still in line with Hosking (2007) who establishes links with Legendre polynomials.

4 Application to the uniform distribution

Let us consider the uniform distribution $\mathcal{U}(a, b)$ whose quantile and distribution functions are given by

$$Q(p) = a + (b-a)p, \quad F(x) = \frac{x-a}{b-a} \mathbb{1}_{[a,b]}(x) + \mathbb{1}_{]b,\infty[}(x),$$

for $p \in [0, 1]$ and $x \in \mathbb{R}$. Therefore, the integral function I_F defined in (9) is

$$I_F(k) = \int_0^1 (a + (b-a)u) u^k du = \left[\frac{au^{k+1}}{k+1} + (b-a) \frac{u^{k+2}}{k+2} \right]_0^1 = \frac{a}{k+1} + \frac{b-a}{k+2}.$$

4.1 L-moments

Using preliminary calculus of Section 1, we get a result generalizing Table 1 of Hosking (1990).

Proposition 11. *For uniform distribution $\mathcal{U}(a,b)$, the L-moments are given by*

$$\lambda_m = \delta_{m,1}\frac{a+b}{2} + \delta_{m,2}\frac{b-a}{6}.$$

In particular, $\lambda_m = 0$ for $m > 2$.

Proof. Using Equation (15), we get

$$\begin{aligned}\lambda_m &= \sum_{l=0}^{m-1}(-1)^{m+l+1}\left(\frac{a}{l+1}+\frac{b-a}{l+2}\right)\binom{m-1}{l}\binom{m+l-1}{l}\\ &= a(-1)^{m+1}\sum_{l=0}^{m-1}(-1)^l\frac{1}{l+1}\binom{m-1}{l}\binom{m+l-1}{l} + (b-a)(-1)^{m+1}\sum_{l=0}^{m-1}(-1)^l\frac{1}{l+2}\binom{m-1}{l}\binom{m+l-1}{l}.\end{aligned}$$

Using

$$\binom{m-1}{l}\binom{m+l-1}{l} = \binom{m+l-1}{2l}\binom{2l}{l},$$

we obtain

$$\lambda_m = a(-1)^{m+1}e_{m-1,1} + (b-a)(-1)^{m+1}e_{m-1,2}.$$

Therefore by Proposition 3

$$\lambda_m = a\frac{1}{m}\binom{0}{m-1} + (b-a)\frac{1}{m(m+1)}\binom{1}{m-1} = a\delta_{m,1} + (b-a)\frac{1}{2}\delta_{m,1} + (b-a)\frac{1}{6}\delta_{m,2} = \delta_{m,1}\frac{a+b}{2} + \delta_{m,2}\frac{b-a}{6}.$$

□

4.2 Symmetric TL moments

Using preliminary calculus of Section 1, we generalized the previous proposition.

Proposition 12. *For uniform distribution $\mathcal{U}(a,b)$, the symmetric TL-moments are given by*

$$\lambda_m^{(t)} = \frac{(a+b)(t+1)}{2t+2}\delta_{m,1} + \frac{(b-a)}{2(2t+3)}\delta_{m,2}.$$

In particular, $\lambda_m^{(t)} = 0$ for $m > 2$, and with $t = 0$, we get back to Proposition 11.

Proof. Using Equation (14), we get

$$\begin{aligned}\lambda_m^{(t)} &= \frac{m+2t}{m}\sum_{l=0}^{m+t-1}(-1)^{m+l+1}\left(\frac{a}{t+l+1}+\frac{b-a}{t+l+2}\right)\binom{m+2t-1}{t+l}\binom{m+t+l-1}{l}\\ &= \frac{a(m+2t)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\frac{(-1)^l}{t+l+1}\binom{m+2t-1}{t+l}\binom{m+t+l-1}{l}\\ &\quad + \frac{(b-a)(m+2t)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\frac{(-1)^l}{t+l+2}\binom{m+2t-1}{t+l}\binom{m+t+l-1}{l}.\end{aligned}$$

Since

$$\binom{m+2t-1}{t+l}\binom{m+t+l-1}{l} = \binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(m+2t-1)!}{(m+t-1)!}\frac{l!}{(t+l)!},$$

we get

$$\begin{aligned}
\lambda_m^{(t)} &= \frac{(m+2t-1)!}{(m+t-1)!}\frac{a(m+2t)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\frac{(-1)^l}{t+l+1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{l!}{(t+l)!} \\
&\quad+\frac{(m+2t-1)!}{(m+t-1)!}\frac{(b-a)(m+2t)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\frac{(-1)^l}{t+l+2}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{l!}{(t+l)!} \\
&= \frac{(m+2t)!}{(m+t-1)!}\frac{a}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!}{(t+l+1)!} \\
&\quad+\frac{(m+2t)!}{(m+t-1)!}\frac{(b-a)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!(t+l+1)}{(t+l+2)!} \\
&= \frac{(m+2t)!}{(m+t-1)!}\frac{a}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!}{(t+l+1)!} \\
&\quad+\frac{(m+2t)!}{(m+t-1)!}\frac{(b-a)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!}{(t+l+1)!} \\
&\quad-\frac{(m+2t)!}{(m+t-1)!}\frac{(b-a)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!}{(t+l+2)!} \\
&= \frac{(m+2t)!}{(m+t-1)!}\frac{b}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!}{(t+l+1)!} \\
&\quad-\frac{(m+2t)!}{(m+t-1)!}\frac{(b-a)}{m}(-1)^{m+1}\sum_{l=0}^{m+t-1}\binom{m+t-1+l}{2l}\binom{2l}{l}\frac{(-1)^l l!}{(t+l+2)!},
\end{aligned}$$

using $\frac{l!(t+l+1)}{(t+l+2)!} = \frac{l!(t+l+2)}{(t+l+2)!} - \frac{l!}{(t+l+2)!}$.

Using Proposition 4, we get

$$\begin{aligned}
\lambda_m^{(t)} &= \frac{(m+2t)!}{(m+t-1)!}\frac{b}{m}(-1)^{m+1}\tilde{e}_{m+t-1,t+1} - \frac{(m+2t)!}{(m+t-1)!}\frac{(b-a)}{m}(-1)^{m+1}\tilde{e}_{m+t-1,t+2} \\
&= \frac{b}{m}(-1)^{m+1}\binom{t}{m+t-1} - \frac{b-a}{m(m+2t+1)}(-1)^{m+1}\binom{t+1}{m+t-1}.
\end{aligned}$$

Furthermore, $\binom{t}{m+t-1} = t\delta_{m,0} + \delta_{m,1}$, and $\binom{t+1}{m+t-1} = t(t+1)\delta_{m,0} + (t+1)\delta_{m,1} + \delta_{m,2}$. So

$$\begin{aligned}
\lambda_m^{(t)} &= \frac{b}{m}(-1)^{m+1}\delta_{m,1} - \frac{(b-a)}{m(m+2t+1)}(-1)^{m+1}((t+1)\delta_{m,1} + \delta_{m,2}) \\
&= b\delta_{m,1} - \frac{(b-a)}{2t+2}(t+1)\delta_{m,1} - \frac{(b-a)}{2(2t+3)}(-1)\delta_{m,2} \\
&= \delta_{m,1}\frac{1}{2t+2}\left(b(2t+2) - (b-a)(t+1)\right) + \frac{b-a)}{2(2t+3)}\delta_{m,2} = \frac{(a+b)(t+1)}{2t+2}\delta_{m,1} + \frac{(b-a)}{2(2t+3)}\delta_{m,2}.
\end{aligned}$$

□

5 Application to the exponential distribution

Let us consider the exponential distribution $\mathcal{E}(\lambda)$ whose quantile and distribution functions are

$$Q(p) = -\log(1-p)/\lambda, \quad F(x) = (1-e^{-\lambda x})\mathbb{1}_{\mathbb{R}_+}(x).$$

This enables to compute I_F by Proposition 5

$$\begin{aligned} I_F(k) &= \int_0^\infty x(1-e^{-\lambda x})^k \lambda e^{-\lambda x} dx = \sum_{j=0}^k \binom{k}{j} (-1)^j \int_0^\infty \lambda x e^{-\lambda x(1+j)} dx \\ &= \sum_{j=0}^k \binom{k}{j} (-1)^j \int_0^\infty \frac{t}{1+j} e^{-t} \frac{dt}{\lambda(1+j)} = \frac{1}{\lambda} \sum_{j=0}^k \binom{k}{j} \frac{(-1)^j}{(1+j)^2} = \frac{H_{k+1}}{\lambda(k+1)}. \end{aligned}$$

5.1 L moments

Proposition 13. *For an exponential distribution $\mathcal{E}(\lambda)$, the L-moments are given by*

$$\lambda_m = \delta_{m1} \frac{1}{\lambda} + (1-\delta_{m1}) \frac{1}{m(m-1)\lambda},$$

for $m \in \mathbb{N}^\star$.

Proof. Let us consider the simplest case ($m = 1$).

$$\lambda_1 = (-1)^{1+0+1} \frac{H_1}{\lambda} \binom{0}{0} \binom{0}{0} = \frac{1}{\lambda}.$$

In the following, we assume $m \geq 2$. Using Equation (15) and the absorption rule, we get

$$\begin{aligned} \lambda_m &= \sum_{l=0}^{m-1} (-1)^{m+l+1} \frac{H_{l+1}}{\lambda(l+1)} \binom{m-1}{l} \binom{m+l-1}{l} = \frac{(-1)^m}{m\lambda} \sum_{l=0}^{m-1} (-1)^{l+1} H_{l+1} \binom{m}{l+1} \binom{m+l-1}{m-1} \\ &= \frac{(-1)^m}{m\lambda} \sum_{l=1}^{m} (-1)^l H_l \binom{m}{l} \binom{m+l-2}{m-1} = \frac{(-1)^m}{m\lambda} \sum_{l=0}^{m} \binom{m}{l} (-1)^l H_l \binom{m-2+l}{m-1}, \end{aligned}$$

since both $H_0 = \binom{m-2}{m-1} = 0$. Using Proposition 7 with $m \leftarrow m-1$, $p \leftarrow m-2$, $i \leftarrow 1$, $i \wedge j = 1$,

$$\begin{aligned} \sum_{l=0}^{m} \binom{m}{l} (-1)^l H_l \binom{m-2+l}{m-1} &= \frac{(-1)^{m-1+1}}{1} + (-1)^{m-1+1} \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \binom{m-1}{j} \\ &\quad \times \sum_{k=0}^{1-1} \binom{1}{1-k} \frac{(1-1)! j^{\overline{1-k}} (m-2)^{\underline{1-k}}}{(1-k-1)!} \frac{(j+1-k-1)!}{(j+1-k-1)!} \frac{(m-2+j)^{\underline{j-1}}}{(j+1-k)^{\underline{j+1-k}}} \\ &= (-1)^m + (-1)^m \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \binom{m-1}{j} j(m-2) \frac{(m-2+j)^{\underline{j-1}}}{(j+1)!} \\ &= (-1)^m + (-1)^m \frac{1}{m-1} \sum_{j=1}^{m-1} (-1)^j \binom{m-1}{j} \frac{(m-2+j)\dots(m-2)}{(j+1)!} \\ &= (-1)^m + (-1)^m \frac{1}{m-1} \sum_{j=1}^{m-1} (-1)^j \binom{m-1}{j} \binom{m-2+j}{m-3}. \end{aligned}$$

Hence,

$$\lambda_m = \frac{1}{m\lambda} + \frac{1}{m(m-1)\lambda} \sum_{j=1}^{m-1} (-1)^j \binom{m-1}{j} \binom{m-2+j}{m-3}.$$

Using (19) with $l \leftarrow m-1$, $m \leftarrow 0$, $s \leftarrow m-2$, $n \leftarrow m-3$, we have for $m > 1$

$$\begin{aligned} \sum_{j=1}^{m-1} (-1)^j \binom{m-1}{j} \binom{m-2+j}{m-3} &= \sum_{j=0}^{+\infty} (-1)^j \binom{m-1}{j} \binom{m-2+j}{m-3} - (-1)^0 \binom{m-1}{0} \binom{m-2+0}{m-3} \\ &= (-1)^{m+1} \binom{m-2}{-2} - (m-2) = 0 + 2 - m. \end{aligned}$$

Thus,

$$\lambda_m = \delta_{m1}\frac{1}{\lambda} + (1-\delta_{m1})\left(\frac{1}{m\lambda} + \frac{2-m}{m(m-1)\lambda}\right) = \delta_{m1}\frac{1}{\lambda} + (1-\delta_{m1})\frac{1}{m(m-1)\lambda}.$$

□

This result is in line with Hosking & Wallis (1997), where the first four L-moments are given

$$\lambda_1 = 1/\lambda,\ \lambda_2 = 1/(2\lambda),\ \lambda_3 = 1/(6\lambda),\ \lambda_4 = 1/(12\lambda).$$

5.2 symmetric TL-moments

Proposition 14. *For an exponential distribution $\mathcal{E}(\lambda)$, the L-moments are given by*

$$\begin{aligned}\lambda_m^{(t)} &= \frac{1}{\lambda m(t+1)} + \frac{(-1)^t}{\lambda m}\frac{t!p!}{(m+t-1)!}\sum_{j=t+2}^{m+t-1}\sum_{l=1}^{t+1}(-1)^j\binom{m+t-1}{j}\binom{t+1}{l}\binom{j-1+l}{j}\binom{m-2+j}{j+l}\\ &+\frac{(-1)^t}{\lambda m}\sum_{j=1}^{t+1}\sum_{l=0}^{j-1}\frac{(-1)^j(l+1)}{j(i+l+1)}\binom{m+t-1}{j}\binom{j}{l+1}\binom{m-2}{l},\end{aligned}$$

for $m \in \mathbb{N}^\star$.

Proof. Using Equation (14), we get

$$\begin{aligned}\lambda_m^{(t)} &= \frac{m+2t}{m}\sum_{l=0}^{m+t-1}(-1)^{m+l+1}\frac{H_{t+l+1}}{\lambda(t+l+1)}\binom{m+2t-1}{t+l}\binom{m+t+l-1}{l}\\ &= \frac{(-1)^m}{\lambda m}\sum_{l=0}^{m+t-1}(-1)^{l+1}H_{t+l+1}\binom{m+2t}{t+l+1}\binom{m+t+l-1}{l}\\ &= \frac{(-1)^m}{\lambda m}\sum_{j=t+1}^{m+2t}(-1)^{j-t}H_j\binom{m+2t}{j}\binom{m-2+j}{m+t-1}\\ &= \frac{(-1)^{m+t}}{\lambda m}\sum_{j=0}^{m+2t}\binom{m+2t}{j}(-1)^jH_j\binom{m-2+j}{m+t-1}.\end{aligned}$$

since $\binom{m-2+j}{m+t-1} = 0$ for $j < t+1$. We now use Proposition 7 with $m \leftarrow m+t-1$, $i \leftarrow t+1$, $p \leftarrow m-2$. We get for $m \geq 2$

$$\begin{aligned}S_{m+2t} &= \sum_{k=0}^{m+2t}\binom{m+2t}{k}(-1)^k\binom{m-2+k}{m+t-1}H_k\\ &= (-1)^{m+2t}\frac{(-1)^t}{t+1} + (-1)^{m+2t}\sum_{j=1}^{t+1}\sum_{l=0}^{j-1}\frac{(-1)^j(l+1)}{j(i+l+1)}\binom{m+t-1}{j}\binom{j}{l+1}\binom{m-2}{l}\\ &+(-1)^{m+2t}\frac{t!p!}{(m+t-1)!}\sum_{j=t+2}^{m+t-1}\sum_{l=1}^{t+1}(-1)^j\binom{m+t-1}{j}\binom{t+1}{l}\binom{j-1+l}{j}\binom{m-2+j}{j+l}.\end{aligned}$$

Multiplying by $(-1)^{m+t}/(\lambda m)$ leads to the desired result. □

6 Conclusion

This paper proposes a new direction to compute closed-form formulas of theoretical TL-moments. Proposition 8 provides a general formulation of TL-moments for any continuous probability distribution. This formulation has been applied on two particular distributions: the uniform and the exponential distributions (Propositions 11, 12, 13 and 14). Other distributions such as Normal or Cauchy could be studied in this way. In future research, we plan to study two directions: the computation of sample TL-moments that used a double combinatorial sum, and the computation of TL-moments of other distributions possibly multivariate distributions.

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A Known combinatorial identities

We give below known identities, see e.g (Graham et al. 1994, Chap. 5). The Vandermonde Identity is

$$\sum_{l=0}^{r} \binom{m}{l}\binom{n}{r-l} = \binom{m+n}{r}. \tag{16}$$

Let $s \in \mathbb{R}$, $m, n \in \mathbb{Z}$ and $l \in \mathbb{N}$

$$\sum_{k\in\mathbb{Z}} \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}. \tag{17}$$

Let $s \in \mathbb{R}$, $m, n \in \mathbb{Z}$ and $l \in \mathbb{N}$

$$\sum_{k\in\mathbb{Z}} \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}. \tag{18}$$

Let $s \in \mathbb{R}$, $m, n \in \mathbb{Z}$ and $l \in \mathbb{N}$

$$\sum_{k\in\mathbb{Z}} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}. \tag{19}$$

Let $m, n \in \mathbb{Z}$ such that $n \neq m-1$

$$\frac{n+1}{n+1-m} = \sum_{j\geq 0} \binom{m}{j} \Big/ \binom{n}{j}. \tag{20}$$

B Finite calculus

Let us study the powerful tool of finite element analysis : E, Δ.

$Ec = c$	$\Delta c = 0$
$Ex^m = (x+1)^m$	$\Delta x^m = \sum_{k=0}^{m-1} \binom{m}{k} x^k$
$Ex^{\underline{m}} = (x+1)^{\underline{m}}$	$\Delta x^{\underline{m}} = mx^{\underline{m-1}}$
$E\frac{1}{x+m} = \frac{1}{x+m+1}$	$\Delta \frac{1}{x+m} = \frac{1}{(x+m)(x+m+1)}$
$E^n \frac{1}{x+m} = \frac{1}{x+m+n}$	$\Delta^n \frac{1}{x+m} = \frac{(-1)^n n!}{(x+m)\ldots(x+m+n)}$
$E\binom{x}{m} = \binom{x+1}{m}$	$\Delta\binom{x}{m} = \binom{x}{m-1}$
$E^n\binom{x}{m} = \binom{x+n}{m}$	$\Delta^n\binom{x}{m} = \binom{x}{m-n}$
$E\binom{x}{m}^{-1} = \binom{x+1}{m}^{-1}$	$\Delta\binom{x}{m}^{-1} = -\frac{m}{m+1}\binom{x+1}{m+1}^{-1}$
$E^n\binom{x}{m}^{-1} = \binom{x+n}{m}^{-1}$	$\Delta^n\binom{x}{m}^{-1} = (-1)^n \frac{m}{m+n}\binom{x+n}{m+n}^{-1}$

Table 4: Known finite operations.

By elementary manipulations, we have

$$\Delta(f(x)g(x)h(x)) = \Delta(f(x))E(g(x))E(h(x)) + f(x)\Delta(g(x))E(h(x)) + f(x)g(x)\Delta(h(x)). \tag{21}$$

C Lemmas for Section 2

We introduce two auxiliary functions g and h in order to deal with the computation of Δf.

Lemma 15. *Let us define the auxiliary function g of x with parameters p, m, n as*

$$g\left(x; \begin{matrix} p, n \\ m \end{matrix}\right) = \binom{x+p}{m} \frac{(x+p+1)\dots(x+p+n)}{(x+1)\dots(x+n)} = \binom{x+p}{m}\binom{x+p+n}{x+p}\binom{x+n}{x}^{-1},$$

for $m \in \mathbb{N}$. We have for all $m \geq 1$

$$\Delta g\left(x; \begin{matrix} p, n \\ m \end{matrix}\right) = g\left(x; \begin{matrix} p, n \\ m-1 \end{matrix}\right)\frac{n+m}{m} - g\left(x; \begin{matrix} p, n+1 \\ m-1 \end{matrix}\right)\frac{n}{m}.$$

Proof. We have to compute the Δ-difference of $\binom{p+x}{m}$ and $\binom{n+x}{n}^{-1}$ in order to use (21). From Table 4, we have

$$\Delta\binom{x+p}{m} = \binom{x+p}{m-1},\ \Delta\binom{x+p+n}{n} = \binom{x+p+n}{n-1},\ \Delta\binom{x+n}{n}^{-1} = -\frac{n}{n+1}\binom{x+n+1}{n+1}^{-1}.$$

Using (21), we get

$$\begin{aligned}\Delta g\left(x; \begin{matrix} p, n \\ m \end{matrix}\right) &= \binom{x+p}{m-1}\binom{x+p+n+1}{n}\binom{x+n+1}{n}^{-1} + \binom{x+p}{m}\binom{x+p+n}{n-1}\binom{x+n+1}{n}^{-1} \\ &\quad -\binom{x+p}{m}\binom{x+p+n}{n}\frac{n}{n+1}\binom{x+n+1}{n+1}^{-1}.\end{aligned}$$

Using the absorption rule (Table 2), we have $\binom{x+p+n+1}{n} = \frac{x+p+n+1}{n}\binom{x+p+n}{n-1}$, $\binom{x+p}{m} = \frac{x+p+1-m}{m}\binom{x+p}{m-1}$, $\binom{x+n+1}{n+1}^{-1} = \frac{n+1}{x+n+1}\binom{x+n}{n}^{-1}$. Thus

$$\begin{aligned}&\Delta g\left(x; \begin{matrix} p, n \\ m \end{matrix}\right) \\ =\ &\binom{x+p}{m-1}\binom{x+p+n}{n-1}\binom{x+n+1}{n}^{-1}\frac{x+p+n+1}{n} + \binom{x+p}{m-1}\binom{x+p+n}{n-1}\binom{x+n+1}{n}^{-1}\frac{x+p+1-m}{m} \\ &-\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n+1}{n+1}^{-1}\frac{x+p+1-m}{m}\frac{n}{n+1} \\ =\ &\binom{x+p}{m-1}\binom{x+p+n}{n-1}\binom{x+n+1}{n}^{-1}\frac{(x+p+1)(n+m)}{nm} \\ &-\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n+1}{n+1}^{-1}\frac{x+p+1-m}{m}\frac{n}{n+1} \\ =\ &\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n}{n}^{-1}\frac{n+m}{m}\frac{x+1}{x+n+1} \\ &-\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n+1}{n+1}^{-1}\frac{x+p+n+1-m-n}{m}\frac{n}{n+1} \\ =\ &\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n}{n}^{-1}\frac{n+m}{m}\frac{x+1}{x+n+1} \\ &+\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n}{n}^{-1}\frac{n+m}{m}\frac{n}{n+1}\frac{n+1}{x+n+1} - \binom{x+p}{m-1}\binom{x+p+n+1}{n+1}\binom{x+n+1}{n+1}^{-1}\frac{n}{m} \\ =\ &\binom{x+p}{m-1}\binom{x+p+n}{n}\binom{x+n}{n}^{-1}\frac{n+m}{m} - \binom{x+p}{m-1}\binom{x+p+n+1}{n+1}\binom{x+n+1}{n+1}^{-1}\frac{n}{m}.\end{aligned}$$

Replacing the corresponding term by $g(x;)$ leads to the result. ☐

Lemma 16. *Let $p, n, m \in \mathbb{N}$ such that $0 \le n \le m$, we have*

$$\Delta^n \left[\binom{p+x}{m} H_x \right] = \binom{p+x}{m-n} H_x + \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} g\left(x; \begin{matrix} p, j \\ m-n \end{matrix}\right), \tag{22}$$

with the convention that the sum $\sum_1^0$ cancels. The coefficients $\binom{n}{j}(j-1)!$ are related to the number of permutations of the symmetric group that are pure j-cycles, see e.g. https://oeis.org/A111492.

Proof. Let us prove (22) by recurrence. For $n = 1$, the proposition is verified. Indeed,

$$\Delta \left[\binom{p+x}{m} H_x \right] = \binom{p+x}{m-1} H_x + \frac{(-1)^0 (0)!}{(m-1+1)^{\overline{1}}} \binom{1}{1} g\left(x; \begin{matrix} p, 1 \\ m-1 \end{matrix}\right).$$

Assume the formula (22) valid for $n \in \mathbb{N}$.

$$\Delta^{n+1} \left[\binom{p+x}{m} H_x \right] = \Delta \left[\binom{p+x}{m-n} H_x \right] + \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} \Delta g\left(x; \begin{matrix} p, j \\ m-n \end{matrix}\right).$$

The first term gives

$$\Delta \left[\binom{p+x}{m-n} H_x \right] = \binom{p+x}{m-n-1} H_x + \frac{1}{m-n} g\left(x; \begin{matrix} p, 1 \\ m-n-1 \end{matrix}\right).$$

In the sum, we use

$$\Delta g\left(x; \begin{matrix} p, j \\ m-n \end{matrix}\right) = g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right) \frac{j+m-n}{m-n} - g\left(x; \begin{matrix} p, j+1 \\ m-n-1 \end{matrix}\right) \frac{j}{m-n}.$$

By splitting the sum, this yields to

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n+1)^{\overline{j}}} \binom{n}{j} \Delta g\left(x; \begin{matrix} p, j \\ m-n \end{matrix}\right) \\
= \;& \sum_{j=1}^{n} \frac{(-1)^{j} j!}{(m-n)^{\overline{j+1}}} \binom{n}{j} g\left(x; \begin{matrix} p, j+1 \\ m-n-1 \end{matrix}\right) + \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n)^{\overline{j}}} \binom{n}{j} g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right) \\
= \;& \sum_{j=2}^{n+1} \frac{(-1)^{j-1}(j-1)!}{(m-n)^{\overline{j}}} \binom{n}{j-1} g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right) + \sum_{j=1}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n)^{\overline{j}}} \binom{n}{j} g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right) \\
= \;& \sum_{j=2}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n)^{\overline{j}}} \left(\binom{n}{j-1} + \binom{n}{j} \right) g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right) + \frac{(-1)^n n!}{(m-n)^{\overline{n+1}}} \binom{n}{n} g\left(x; \begin{matrix} p, n+1 \\ m-n-1 \end{matrix}\right) \\
& + \frac{(-1)^0 0!}{(m-n)^{\overline{1}}} \binom{n}{1} g\left(x; \begin{matrix} p, 1 \\ m-n-1 \end{matrix}\right) \\
= \;& \sum_{j=2}^{n} \frac{(-1)^{j-1}(j-1)!}{(m-n)^{\overline{j}}} \binom{n+1}{j} g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right) + \frac{(-1)^n n!}{(m-n)^{\overline{n+1}}} \binom{n+1}{n+1} g\left(x; \begin{matrix} p, n+1 \\ m-n-1 \end{matrix}\right) \\
& + \frac{n}{m-n} g\left(x; \begin{matrix} p, 1 \\ m-n-1 \end{matrix}\right).
\end{aligned}$$

Regrouping $g\left(x; \begin{matrix} p, 1 \\ m-n-1 \end{matrix}\right)$ leads to

$$\frac{n+1}{m-n} g\left(x; \begin{matrix} p, 1 \\ m-n-1 \end{matrix}\right) = \frac{(-1)^0 0!}{(m-n)^{\overline{1}}} \binom{n+1}{1} g\left(x; \begin{matrix} p, 1 \\ m-n-1 \end{matrix}\right).$$

Hence,

$$\Delta^{n+1} \left[\binom{p+x}{m} H_x \right] = \binom{p+x}{m-n-1} H_x + \sum_{j=1}^{n+1} \frac{(-1)^{j-1}(j-1)!}{(m-n)^{\overline{j}}} \binom{n+1}{j} g\left(x; \begin{matrix} p, j \\ m-n-1 \end{matrix}\right).$$

Thus, the recurrence (22) is valid for $n \in \mathbb{N}^\star$. □

Lemma 17. *Let $p, n, i \in \mathbb{N}$ such that $1 \le i$, we have*

$$\Delta^i h\left(x; \begin{matrix} p+n, n \\ n \end{matrix}\right) = (-1)^i \sum_{k=0}^{i\wedge n-1} \binom{i\wedge n}{i\wedge n-k} \frac{(i\wedge n-1)! n^{\overline{i\wedge n-k}} p^{\underline{i\wedge n-k}}}{(i\wedge n-k-1)!} \frac{(n+i-k-1)!}{(n+i\wedge n-k-1)!} h\left(x; \begin{matrix} p+n, n-i\wedge n \\ n+i-k \end{matrix}\right),$$

with $i \wedge n = \min(i, n)$. The coefficients $\binom{i}{i-k}\frac{(i-1)!}{(i-k-1)!}$ are related to triangular arrays, see e.g. https://oeis.org/A089231.

Proof. By standard calculations, we have

$$\Delta h\left(x; \begin{matrix} a, b \\ c, d \end{matrix}\right) = b \times h\left(x; \begin{matrix} a, b-1 \\ c, d \end{matrix}\right) - d \times h\left(x; \begin{matrix} a+1, b \\ c+1, d+1 \end{matrix}\right). \tag{23}$$

Hence by (23) with $a \leftarrow p+n$, $b \leftarrow n-d$, $c \leftarrow n+d$ and $d \leftarrow n+d$, we obtain

$$\Delta h\left(x; \begin{matrix} p+n, n-d \\ n+d \end{matrix}\right) = -(n+d)(p-d) h\left(x; \begin{matrix} p+n, n-d-1 \\ n+d+1 \end{matrix}\right) - 2dh\left(x; \begin{matrix} p+n, n-d-1 \\ n+d \end{matrix}\right). \tag{24}$$

For $i = 1$, we obtain

$$\begin{aligned} \Delta^1 h\left(x; \begin{matrix} p+n, n \\ n \end{matrix}\right) &= (-1)^1 \sum_{k=0}^{0} \binom{1}{1-k} \frac{(1-1)! n^{\overline{1-k}} p^{\underline{1-k}}}{(1-k-1)!} \frac{(n+1-k-1)!}{(n+1-k-1)!} h\left(x; \begin{matrix} p+n, n-1 \\ n+1-k \end{matrix}\right) \\ &= -\frac{n^{\overline{1}} p^{\underline{1}}}{0!} \frac{n!}{n!} h\left(x; \begin{matrix} p+n, n-1 \\ n \end{matrix}\right) = -nph\left(x; \begin{matrix} p+n, n-1 \\ n \end{matrix}\right). \end{aligned}$$

The last expression is valid by (24). Now, assume the property correct for $i \ge 1$. Let us compute $\Delta^{i+1} h(x; .)$

$$\Delta^{i+1} h\left(x; \begin{matrix} p+n, n \\ n \end{matrix}\right) = (-1)^i \sum_{k=0}^{i\wedge n-1} \binom{i\wedge n}{i\wedge n-k} \frac{(i\wedge n-1)! n^{\overline{i\wedge n-k}} p^{\underline{i\wedge n-k}}}{(i\wedge n-k-1)!} \frac{(n+i-k-1)!}{(n+i\wedge n-k-1)!} \Delta h\left(x; \begin{matrix} p+n, n-i\wedge n \\ n+i-k \end{matrix}\right).$$

By (23), we have

$$\Delta h\left(x; \begin{matrix} p+n, n-i\wedge n \\ n+i-k \end{matrix}\right) = h\left(x; \begin{matrix} p+n, n-i\wedge n-1 \\ n+i-k+1 \end{matrix}\right) ((n-i\wedge n)(x+n+i-k+1) - (n+i-k)(x+p+n+1)).$$

Consider first the case $i < n$. This simplifies to

$$(n-i\wedge n)(x+n+i-k+1) - (n+i-k)(x+p+n+1) = (x+n+i+1-k)(k-2i) - (n+i-k)(p-i+k),$$

we get for $i < n$,

$$\Delta h\left(x; \begin{matrix} p+n, n-i\wedge n \\ n+i-k \end{matrix}\right) = -(n+i-k)(p-i+k) h\left(x; \begin{matrix} p+n, n-i-1 \\ n+i-k+1 \end{matrix}\right) - (2i-k) h\left(x; \begin{matrix} p+n, n-i-1 \\ n+i-k \end{matrix}\right),$$

and

$$\begin{aligned} \Delta^{i+1} h\left(x; \begin{matrix} p+n, n \\ n \end{matrix}\right) &= (-1)^i \sum_{k=0}^{i-1} \binom{i}{i-k} \frac{(i-1)! n^{\overline{i-k}} p^{\underline{i-k}}}{(i-k-1)!} \left[-(n+i-k)(p-i+k) h\left(x; \begin{matrix} p+n, n-i-1 \\ n+i-k+1 \end{matrix}\right) \right. \\ &\quad \left. -(2i-k) h\left(x; \begin{matrix} p+n, n-i-1 \\ n+i-k \end{matrix}\right) \right]. \end{aligned}$$

So for $i < n$, splitting the sum and removing respectively the first term ($k = 1$) and the last term ($k = i-1$)

of the two sums gives

$$\begin{aligned}
& (-1)^{i+1}\sum_{k=1}^{i-1}\binom{i}{i-k}\frac{(i-1)!n^{\overline{i-k}}p^{\underline{i-k}}}{(i-k-1)!}(n+i-k)(p-i+k)h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right)\\
& +(-1)^{i+1}\sum_{k=0}^{i-2}\binom{i}{i-k}\frac{(i-1)!n^{\overline{i-k}}p^{\underline{i-k}}}{(i-k-1)!}(2i-k)h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k\end{matrix}\right)\\
= \;& (-1)^{i+1}\sum_{k=1}^{i-1}\binom{i}{i-k}\frac{(i-1)!n^{\overline{i+1-k}}p^{\underline{i+1-k}}}{(i-k-1)!}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right)\\
& +(-1)^{i+1}\sum_{k=1}^{i-1}\binom{i}{i-k+1}\frac{(i-1)!n^{\overline{i-k+1}}p^{\underline{i-k+1}}}{(i-k)!}(2i-k+1)h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right)\\
= \;& (-1)^{i+1}\sum_{k=1}^{i-1}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right)n^{\overline{i+1-k}}p^{\underline{i+1-k}}\\
& \times\left(\binom{i}{i-k}\frac{(i-1)!}{(i-k-1)!}+\binom{i}{i-k+1}\frac{(i-1)!}{(i-k)!}(2i-k+1)\right).
\end{aligned}$$

Since the summand simplifies to

$$\binom{i}{i-k}\frac{(i-1)!}{(i-k-1)!}+\binom{i}{i-k+1}\frac{(i-1)!}{(i-k)!}(2i-k+1)=\binom{i+1}{i+1-k}\frac{(i+1-1)!}{(i+1-k-1)!},$$

we get

$$(-1)^{i+1}\sum_{k=1}^{i-1}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right)n^{\overline{i+1-k}}p^{\underline{i+1-k}}\binom{i+1}{i+1-k}\frac{(i+1-1)!}{(i+1-k-1)!}.$$

The terms removed when splitting the sum are respectively (up to $(-1)^{i+1}$)

$$n^{\overline{i+1}}p^{\underline{i+1}}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right)=\binom{i+1}{i+1}\frac{(i+1-1)!n^{\overline{i+1}}p^{\underline{i+1}}}{(i+1-1)!}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i-k+1\end{matrix}\right),$$

and

$$i(i-1)!n^{\overline{1}}p^{\underline{1}}(i+1)h\left(x;\begin{matrix}p+n,n-i-1\\ n+1\end{matrix}\right)=\binom{i+1}{1}\frac{(i+1-1)!n^{\overline{1}}p^{\underline{1}}}{0!}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i+1-i\end{matrix}\right).$$

Hence for $i<n$

$$\Delta^{i+1}h\left(x;\begin{matrix}p+n,n\\ n\end{matrix}\right)=(-1)^{i+1}\sum_{k=0}^{i}h\left(x;\begin{matrix}p+n,n-i-1\\ n+i+1-k\end{matrix}\right)n^{\overline{i+1-k}}p^{\underline{i+1-k}}\binom{i+1}{i+1-k}\frac{(i+1-1)!}{(i+1-k-1)!}.$$

Consider now the case $i\geq n$. By Table 4, we get

$$\begin{aligned}
\Delta h\left(x;\begin{matrix}p+n,n-i\wedge n\\ n+i-k\end{matrix}\right) &= \Delta h\left(x;\begin{matrix}p+n,0\\ n+i-k\end{matrix}\right)=\Delta\frac{1}{(n+i-k)!}\binom{x+n+i-k}{n+i-k}^{-1}\\
&= \frac{1}{(n+i-k)!}\frac{-(n+i-k)}{n+i-k+1}\binom{x+n+i+1-k}{n+i+1-k}^{-1}\\
&= -(n+i-k)h\left(x;\begin{matrix}p+n,0\\ n+i+1-k\end{matrix}\right).
\end{aligned}$$

So for $i\geq n$,

$$\begin{aligned}
\Delta^{i+1}h\left(x;\begin{matrix}p+n,n\\ n\end{matrix}\right) &= (-1)^{i+1}\sum_{k=0}^{n-1}\binom{n}{n-k}\frac{(n-1)!n^{\overline{n-k}}p^{\underline{n-k}}}{(n-k-1)!}\\
&\quad\times\frac{(n+i+1-k-1)!}{(n+n-k-1)!}h\left(x;\begin{matrix}p+n,0\\ n+i+1-k\end{matrix}\right).
\end{aligned}$$

Therefore, the property is valid for all $i\in\mathbb{N}^{\star}$. ☐

Lemma 18. *Let $p, m, i \in \mathbb{N}$ such that $1 \leq i$, we have*

$$\Delta^{m+i}\left[\binom{p+x}{m}H_x\right] = \frac{(-1)^{i-1}}{i}\binom{x+i}{i}^{-1} + \sum_{j=1}^{m}\frac{(-1)^{j-1}}{j}\binom{m}{j}(-1)^i\sum_{k=0}^{i\wedge j-1}\binom{i\wedge j}{i\wedge j-k} \times\frac{(i\wedge j-1)!j^{\overline{i\wedge j-k}}p^{\underline{i\wedge j-k}}}{(i\wedge j-k-1)!}\frac{(j+i-k-1)!}{(j+i\wedge j-k-1)!}h\left(x;\begin{matrix}p+j, j-i\wedge j\\ j+i-k\end{matrix}\right).$$

Proof. We have for $i \geq 1$

$$\Delta^{m+i}\left[\binom{p+x}{m}H_x\right] = \Delta^i H_x + \sum_{j=1}^{m}\frac{(-1)^{j-1}}{j}\binom{m}{j}\Delta^i h\left(x;\begin{matrix}p+j, j\\ j\end{matrix}\right).$$

Using Lemma 17,

$$\Delta^i h\left(x;\begin{matrix}p+j, j\\ j\end{matrix}\right) = (-1)^i\sum_{k=0}^{i\wedge j-1}\binom{i\wedge j}{i\wedge j-k}\frac{(i\wedge j-1)!j^{\overline{i\wedge j-k}}p^{\underline{i\wedge j-k}}}{(i\wedge j-k-1)!}\frac{(j+i-k-1)!}{(j+i\wedge j-k-1)!}h\left(x;\begin{matrix}p+j, j-i\wedge j\\ j+i-k\end{matrix}\right).$$

Using (6) leads to the desired result. □