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Yohann Le Floch, Alvaro Pelayo

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Abstract. We introduce axiomatically a semiclassical quantization for non necessarily self-adjoint operators. Then we focus on unitary operators and prove that, in the semiclassical limit, the convex hull of the joint spectrum of a finite commuting family of semiclassical unitary operators converges to the convex hull of the joint image of the principal symbols, which can be shown to be a subset of a d-torus $(\mathbb{S}^1)^d$. This result covers in particular $\hbar$-pseudodifferential and Berezin-Toeplitz operators. Part of the paper is devoted to the definition of this notion of convex hull for subsets of tori. The proof of our result builds on recent results for semiclassical self-adjoint operators and involves the inverse Cayley transform for unitary operators.

1. Introduction

We introduce axiomatically a semiclassical quantization for non-necessarily self-adjoint operators. This extends the quantization given by the second author, Polterovich and Vũ Ngọc in [22] for self-adjoint operators, and includes the case of pseudodifferential and Berezin-Toeplitz operators. Then we focus on unitary operators. In this setting our goal is to generalize to semiclassical unitary operators the quantum mechanical principle that "the spectrum of a quantum mechanical system converges to the classical spectrum in the semiclassical limit". Unlike self-adjoint operators which have spectrum in $\mathbb{R}$, the spectrum of a unitary operator is a subset of $\mathbb{S}^1$, and this implies that making the above statement precise is more difficult.

One can show that semiclassical unitary operators must have circle valued principal symbols, and hence the image of the joint map of principal symbols of a family of commuting unitary operators is a subset $\mathcal{S}$ of the d-dimensional torus $\mathbb{T}^d = (\mathbb{S}^1)^d$. This paper shows that in the semiclassical limit, the convex hull of the joint spectrum of a quantum

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system converges to the convex hull of the joint image of its principal symbols (Theorem 4). This was known for self-adjoint Berezin-Toeplitz operators on compact symplectic manifolds and pseudodifferential operators on cotangent bundles (rigorous proofs of this appear in [22]), and this paper provides the first proof of the principle in a non necessarily self-adjoint case. A key ingredient of the proof is the inverse Cayley transform of a unitary operator.

In order to prove the convergence result first we must give the "right notion" of convex hull inside the torus $\mathbb{T}^d$, which is subtle to define as trying to use a lift to $\mathbb{R}^d$ is too naive. The first part of the paper is devoted to giving a notion of convex hull for subsets of d-dimensional tori, which is well suited for the study of spectral convergence.

The axioms of semiclassical quantization in this paper (Section 4.2) apply to pseudodifferential operators and Berezin-Toeplitz operators. The study of pseudodifferential operators is now a classical subject initiated about fifty years ago, and to which many have contributed, see for instance Hörmander's account of the subject in [17]. Berezin-Toeplitz operators, which fit inside the framework of the well known geometric quantization of Kostant [18] and Souriau [28], were introduced by Berezin [1], and their microlocal analysis initiated by Boutet de Monvel and Guillemin [5]. Non self-adjoint operators arise naturally in partial differential equations, for instance in problems concerning damped wave equations, scattering poles, and convection-diffusion.

After introducing the relevant notions and reviewing a result in [22] by the second author, Polterovich and Vũ Ngọc on which this paper builds (Theorem 3), we will state our main result in Section 3. The rest of the paper is devoted to its proof, generalizations, and an application to semiclassical Berezin-Toeplitz quantization of symplectic actions.

2. Joint spectrum and Hausdorff distance

We start by reviewing the case of self-adjoint operators from [22], but first let us recall the basic terminology used in the study of semiclassical operators. A finite number of normal operators $S_1, \ldots, S_d$ on a Hilbert space are said to be *mutually commuting* if their corresponding spectral measures $\mu_1, \ldots, \mu_d$ pairwise commute. In this case we may define the joint spectral measure $\mu := \mu_1 \otimes \cdots \otimes \mu_d$ on $\mathbb{C}^d$.

In this paper we are concerned with semiclassical operators, that is, the operator itself is given by a sequence of operators, labelled by the Planck constant $\hbar$. Let I be a subset of $(0,1]$ that accumulates at 0. Let

$$\mathcal{F} = \Big(T_1 := (T_1(\hbar))_{\hbar \in I}, \ldots, T_d := (T_d(\hbar))_{\hbar \in I} \Big)$$

be a collection of pairwise commuting semiclassical normal operators. These operators depend on the Planck constant $\hbar \in I$ and act on a Hilbert space $\mathcal{H}_\hbar$, $\hbar \in I$. We assume that at each $\hbar \in I$ the operators have a common dense domain $\mathcal{D}_\hbar \subset \mathcal{H}_\hbar$ such that the inclusion $T_j(\hbar)(\mathcal{D}_\hbar) \subset \mathcal{D}_\hbar$ holds for all $j = 1, \ldots, d$.

For a fixed value of $\hbar$, the *joint spectrum* of $(T_1(\hbar), \ldots, T_d(\hbar))$ is the support of their joint spectral measure. It is denoted by

$$\mathrm{JointSpec}(T_1(\hbar), \ldots, T_d(\hbar)).$$

For instance, if the Hilbert space $\mathcal{H}_\hbar$ is finite dimensional (eg. when the manifold M is closed, that is, compact and with no boundary), then $\mathrm{JointSpec}(T_1(\hbar), \ldots, T_d(\hbar))$ is the set

$$\Big\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d \mid \exists v \neq 0,\ T_j(\hbar)v = \lambda_j v, \forall j = 1, \ldots, d\Big\}.$$

Definition 1. The *joint spectrum* $\mathrm{JointSpec}(T_1, \ldots, T_d)$ of $(T_1, \ldots, T_d)$ is the collection of all joint spectra of $(T_1(\hbar), \ldots, T_d(\hbar)$, $\hbar \in I$.

Now suppose that (M, ω) is a connected quantizable manifold. Let $d \geqslant 1$ and let $(T_1, \ldots, T_d)$ be a family of pairwise commuting semiclassical operators on M. Following the physicists, we use the following definition, which is the classical analogue of the previous one in virtue of the classical-quantum correspondence.

Definition 2. The *classical spectrum* of $(T_1, \ldots, T_d)$ is the closure of the image $F(M) \subset \mathbb{R}^d$, where $F = (f_1, \ldots, f_d)$ is the map of principal symbols of $T_1, \ldots, T_d$.

The main result of this paper compares the Hausdorff distance between classical and quantum spectra, in the semiclassical limit, that is when $\hbar \to 0$. Let us recall the meaning of this distance.

The *Hausdorff distance* (see e.g.[6, Definition 7.3.1]) between two subsets $A \subset X$ and $B \subset X$ of a metric space (X, d) is the quantity $\mathrm{d}_H^X(A,\, B) := \inf\{\varepsilon > 0 \mid\ A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}$, where for any subset S of X, and any $\epsilon > 0$, the set S_ε is

$$S_\varepsilon := \bigcup_{s \in S} \{x \in X \mid d(s, x) \leqslant \varepsilon\}.$$

Recall that if $\mathrm{d}_H^X(A,\, B) = 0$ and A, B are closed sets, then $A = B$. When $X = \mathbb{R}^d$ with its Euclidean norm, we will simply use the notation d_H for the Hausdorff distance. Recently, Pelayo, Polterovich and Vũ Ngọc proved the following convergence result for semiclassical self-adjoint operators.

Theorem 3 ([22, Theorem 8])**.** *Let $(T_1, \dots, T_d)$ be a family of semiclassical self-adjoint operators on a quantizable manifold M. Assume that for every $j \in [\![1, d]\!]$, the principal symbol of T_j is bounded. Let $\mathcal{S} \subset \mathbb{R}^d$ be the classical spectrum of $(T_1, \dots, T_d)$. Then*

$$\text{Convex Hull}\,(\text{JointSpec}(T_1, \dots, T_d)) \underset{\hbar \to 0}{\longrightarrow} \text{Convex Hull}\,(\mathcal{S})$$

for the Hausdorff distance topology.

Given the background of this section, we are ready to state our main result, which is a similar statement for unitary operators, in the upcoming section.

3. Main theorem

Let (M, ω) be a connected quantizable manifold. Let $d \geqslant 1$ and let $\mathcal{F} := (U_1(\hbar), \dots, U_d(\hbar))$ be a family of pairwise commuting unitary semiclassical operators on M (necessarily their principal symbols are $\mathbb{S}^1$-valued, as we will prove in Lemma 13). Let $\mathcal{S} \subset \mathbb{T}^d$ be the classical spectrum of $\mathcal{F}$. Assume in addition that $f_1(M), \dots, f_d(M)$ are closed, where f_j is the principal symbol of $U_j(\hbar)$, that none of these principal symbols is onto, and that $F(M)$ is also closed, where $F = (f_1, \dots, f_d)$ is the joint principal symbol of the family. For a subset A of $\mathbb{T}^d$ we denote by $\text{Convex Hull}_{\mathbb{T}^d}(A)$ its "convex hull" in the torus (the construction of such convex hull is subtle and we carry it out in Section 5) For a complete version of the following statement, where the "genericity conditions" are spelled out, see Theorem 36.

Theorem 4. *From the family of joint spectra*

$$\Big\{\text{Convex Hull}_{\mathbb{T}^d}(\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))\Big\}_{\hbar \in I},$$

one can recover the convex hull of the classical spectrum $\mathcal{S} \subseteq \mathbb{T}^d$. Furthermore, under some genericity assumptions, this family of joint spectra converges in the semiclassical limit to the convex hull of the classical spectrum $\mathcal{S} \subseteq \mathbb{T}^d$, in the Hausdorff metric, in other words,

$$\lim_{\hbar \to 0} \text{Convex Hull}_{\mathbb{T}^d} \Big(\text{JointSpec}(\mathcal{F})\Big) \;=\; \text{Convex Hull}_{\mathbb{T}^d}\, \mathcal{S}$$

where the limit convergence is in the Hausdorff metric.

We also state the following conjecture (see Conjecture 39 for a detailed version).

Conjecture 5. *The same statement holds even when we remove the assumption that no principal symbol is onto.*

We provide with evidence in favor of the validity of this conjecture in Section 7.3.

Next we give an application of Theorem 4. Closed symplectic $2n$--dimensional manifolds (M, ω) that come endowed with an effective symplectic Hamiltonian torus action of an n-dimensional torus with momentum map $\mu\colon M \to \mathbb{R}^d$ are called *symplectic toric manifolds* (or *toric integrable systems*). Even though a non-Hamiltonian symplectic actions does not admit a momentum map with values in $\mathbb{R}^d$, by a theorem of McDuff ([20]) a symplectic manifold endowed with a symplectic but non Hamiltonian torus action always admits a torus valued momentum map $\mu\colon M \to \mathbb{T}^d$, and its natural semiclassical quantization is given by d semiclassical operators

$$U_1, \ldots, U_d,$$

whose principal symbols

$$\mu_1, \ldots, \mu_d$$

are precisely the components of μ. Hence the theorem above applies to this case. As an immediate consequence of Theorem 4 we obtain the following.

Corollary 6. *Let (M, ω) be a prequantizable closed connected symplectic manifold for which the cohomology class of ω is integral. Suppose that M comes endowed with a symplectic $\mathbb{T}^d$-action which is not Hamiltonian, and let $\mu := (\mu_1, \ldots, \mu_d)\colon M \to \mathbb{T}^d$ be the $\mathbb{S}^1$-valued momentum map with image $\mathcal{S}$. Suppose that $\mathcal{F} = (U_1, \ldots, U_d)$ is a family of pairwise commuting unitary Berezin-Toeplitz operators on M whose principal symbols are $\mu_1, \ldots, \mu_d$, and that none of the μ_j is onto. Then from the data of the family of joint spectra of $\mathcal{F}$, one can recover the convex hull of $\mathcal{S}$, and under some genericity assumptions,*

$$\lim_{\hbar \to 0} \text{Convex Hull}_{\mathbb{T}^d} \Big(\text{JointSpec}(\mathcal{F})\Big) \;=\; \text{Convex Hull}_{\mathbb{T}^d}\, \mathcal{S}$$

where the limit convergence is in the Hausdorff metric.

4. Semiclassical operators and semiclassical quantization

4.1. **Operators on Hilbert spaces.** Let $\mathcal{H}$ be a Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$; we use the notation $\| \cdot \|$ for the associated norm. We will need to work with possibly unbounded linear operators acting on $\mathcal{H}$, hence we introduce some standard terminology (for more details, we refer the reader to standard material, as [24, Chapter VIII] or [16, Appendix 3] for instance). A linear operator acting on $\mathcal{H}$ is the data of a linear subspace $\mathcal{D}(T) \subset \mathcal{H}$, called the domain of T, and a linear map

$$T : \mathcal{D}(T) \to \mathcal{H}.$$

Throughout the paper, $\mathcal{L}(\mathcal{H})$ will denote the set of densely defined (that is with dense domain) linear operators on $\mathcal{H}$. The range $\mathcal{R}(T)$ of a linear operator T is the set of all values Tu, $u \in \mathcal{D}(T)$.

We say that the operator T is *bounded* if there exists a constant $C \geqslant 0$ such that for every $u \in \mathcal{D}(T)$, $\|Tu\| \leqslant C\|u\|$. If this is the case, by a slight abuse of notation, we will write $\|T\|$ for its operator norm, defined as

$$\|T\| = \sup_{\substack{u \in \mathcal{D}(T) \\ u \neq 0}} \frac{\|Tu\|}{\|u\|}.$$

Let us recall that if T is a bounded operator, it admits a bounded extension with domain $\mathcal{H}$ (see [16, Proposition A.3.9] for example).

If T is a densely defined linear operator acting on $\mathcal{H}$, its adjoint is defined as follows: let $\mathcal{D}(T^*)$ be the set of $u \in \mathcal{H}$ such that there exists $v_u \in \mathcal{H}$ such that

$$\forall w \in \mathcal{D}(T) \qquad \langle Tw, u \rangle = \langle w, v_u \rangle .$$

Then for $u \in \mathcal{D}(T^*)$, this v_u is unique and we set $T^*u = v_u$. This defines a linear operator acting on $\mathcal{H}$, with domain $\mathcal{D}(T^*)$ not necessarily dense; T^* is called the adjoint of T. A densely defined closed operator is said to be *normal* when $TT^* = T^*T$ (this equality includes the fact that the domains of these operators agree). Normal operators are of particular interest because they satisfy the spectral theorem [11, Chapter X, Theorem 4.11] which associates to the operator a spectral measure and spectral projections. Two normal operators $A, B \in \mathcal{L}(\mathcal{H})$ are said to *commute* if and only if all their spectral projections commute (cf. for instance [26, Proposition 5.27]). A densely defined operator T is said to be *self-adjoint* when $T^* = T$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *positive*, in which case we will write $T \geqslant 0$, when $\langle Tu, u \rangle \geqslant 0$ for every $u \in \mathcal{D}(T)$; if there exists some constant $c \in \mathbb{R}$ such that $T - c \,\mathrm{Id} \geqslant 0$, then we write $T \geqslant c \,\mathrm{Id}$.

We say that $T \in \mathcal{L}(\mathcal{H})$ is *invertible* if it admits a bounded inverse, that is a bounded operator

$$T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$$

such that $TT^{-1} = \mathrm{Id}_{\mathcal{R}(T)}$ and $T^{-1}T = \mathrm{Id}_{\mathcal{D}(T)}$. In this case, T^{-1} is unique. A bounded operator U acting on $\mathcal{H}$ is said to be unitary if it is invertible and $U^{-1} = U^*$. Now, we define the spectrum $\mathrm{Sp}(T) \subset \mathbb{C}$

of a given $T \in \mathcal{L}(\mathcal{H})$ as follows:

$$\lambda \in \mathrm{Sp}(T) \iff \lambda \mathrm{Id} - T \text{ is not invertible.}$$

It can be proved that the spectrum of a self-adjoint (respectively unitary) operator is a subset of $\mathbb{R}$ (respectively the unit circle $\mathbb{S}^1$).

Finally, recall the following useful result about the norm of a self-adjoint operator. If A is self-adjoint, then

$$\sup_{\lambda \in \mathrm{Sp}(A)} |\lambda| = \sup_{\substack{u \in \mathcal{D}(A) \\ u \neq 0}} \frac{|\langle Au, u \rangle|}{\|u\|^2} = \sup_{\substack{u \in \mathcal{D}(A) \\ u \neq 0}} \frac{\|Au\|}{\|u\|} \leqslant +\infty. \tag{1}$$

This result is standard but very often stated for bounded operators only; a concise proof can be found in [22, Section 3].

4.2. **Semiclassical quantization.** Let M be a connected manifold. Let $\mathcal{A}_0$ be a subalgebra of $\mathscr{C}^\infty(M, \mathbb{C})$ containing the constants and the compactly supported functions, and stable by complex conjugation. Assume also that if $f \in \mathcal{A}_0$ never vanishes, then $1/f$ also belongs to $\mathcal{A}_0$. Let $I \subset (0, 1]$ be a set accumulating at zero. Given a bounded function $f \in \mathcal{A}_0$, its uniform norm will be denoted by $\|f\|_\infty$.

Definition 7. A *semiclassical quantization* of $(M, \mathcal{A}_0)$ consists of a family of complex Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar \in I}$ together with a family of $\mathbb{C}$-linear maps

$$\mathrm{Op}_\hbar : \mathcal{A}_0 \to \mathcal{L}(\mathcal{H}_\hbar)$$

satisfying the following properties (in the statement of which $f, g \in \mathcal{A}_0$):

(Q1) if f and g are bounded, then the composition $\mathrm{Op}_\hbar(f)\,\mathrm{Op}_\hbar(g)$ is well-defined and

$$\|\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(g) - \mathrm{Op}_\hbar(fg)\| = \mathcal{O}(\hbar)$$

(*composition*);

(Q2) for every $\hbar \in I$, $\mathrm{Op}_\hbar(f)^* = \mathrm{Op}_\hbar\left(\bar{f}\,\right)$ (*reality*);

(Q3) $\mathrm{Op}_\hbar(1) = \mathrm{Id}$ (*normalization*);

(Q4) if $f \geqslant 0$, then there exists a constant $C > 0$ such that for every $\hbar \in I$, $\mathrm{Op}_\hbar(f) \geqslant -C\hbar\,\mathrm{Id}$, (*quasi-positivity*);

(Q5) if $f \neq 0$ has compact support, then $\mathrm{Op}_\hbar(f)$ is bounded for every $\hbar \in I$ and

$$\liminf_{\hbar \to 0} \|\mathrm{Op}_\hbar(f)\| > 0$$

(*non-degeneracy*);

(Q6) if g has compact support, then for every $f \in \mathcal{A}_0$, $\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(g)$ is bounded and

$$\|\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(g) - \mathrm{Op}_\hbar(fg)\| = \mathcal{O}(\hbar)$$

(*product formula*);

If such a semiclassical quantization exists, we say that M is *quantizable*. Let us make a few comments regarding these axioms. Axioms (Q3), (Q4), (Q5) and (Q6) were introduced in [22] in order to work with self-adjoint semiclassical operators[1]. We introduce axioms (Q1) and (Q2) in order to extend this setting to include operators that are not necessarily self-adjoint. One can argue that there is some redundancy between axioms (Q1) and (Q6), but for the sake of clarity, we prefer to keep them both instead of stating some single axiom implying them both.

It was checked in [22] that the axioms (Q3), (Q4), (Q5) and (Q6) are satisfied by pseudodifferential and Berezin-Toeplitz operators. For pseudodifferential operators, axiom (Q1) is a consequence of the product formula for the Weyl quantization, which can be found in [12, Theorem 7.9] for example, while axiom (Q2) is given by Formula (7.3) in the same reference. The fact that axioms (Q1) and (Q2) hold for Berezin-Toeplitz quantization will be checked in Lemma 41.

Let us now derive a few consequences of these axioms. Firstly, note that axiom (Q2) implies in particular that $\mathrm{Op}_\hbar$ maps real-valued functions to self-adjoint operators. Similarly, axioms (Q1), (Q2) and (Q3) together imply that $\mathrm{Op}_\hbar$ maps $\mathbb{S}^1$-valued functions to "quasi-unitary" operators, that is to say operators $U_\hbar \in \mathcal{L}(\mathcal{H})$ such that

$$\|U_\hbar^* U_\hbar - \mathrm{Id}\| = \mathcal{O}(\hbar) \text{ and } \|U_\hbar U_\hbar^* - \mathrm{Id}\| = \mathcal{O}(\hbar).$$

Secondly, our axioms show the following.

Corollary 8. *If $f \in \mathcal{A}_0$ is bounded, then the operator $\mathrm{Op}_\hbar(f)$ is bounded and*

$$\|\mathrm{Op}_\hbar(f)\| \leqslant \|f\|_\infty + \mathcal{O}(\hbar). \tag{2}$$

Proof. Axioms (Q3) and (Q4) yield that

$$\mathrm{Op}_\hbar\left(|f|^2\right) \leqslant \|f\|_\infty^2 \,\mathrm{Id} + \mathcal{O}(\hbar).$$

[1] Actually, axiom (Q3) was stated in the weaker form $\|\mathrm{Op}_\hbar(1) - \mathrm{Id}\| = \mathcal{O}(\hbar)$, but our formulation does not seem too restrictive, since it is still true for both Berezin-Toeplitz and pseudodifferential quantizations.

Since $\mathrm{Op}_\hbar(|f|^2)$ is self-adjoint, this implies, by formula (1), that its norm satisfies

$$\left\|\mathrm{Op}_\hbar(|f|^2)\right\| \leqslant \|f\|_\infty^2 + \mathcal{O}(\hbar);$$

using axioms (Q1) and (Q2), this means that

$$\|\mathrm{Op}_\hbar(f)^*\mathrm{Op}_\hbar(f)\| \leqslant \|f\|_\infty^2 + \mathcal{O}(\hbar).$$

But this, in turn, yields the boundedness of $\mathrm{Op}_\hbar(f)$; indeed, if $u \in \mathcal{H}$ belongs to the domain of $\mathrm{Op}_\hbar(f)$, then we get by the Cauchy-Schwarz inequality that

$$\begin{aligned} |\langle \mathrm{Op}_\hbar(f)u, \mathrm{Op}_\hbar(f)u\rangle| &= |\langle \mathrm{Op}_\hbar(f)^*\mathrm{Op}_\hbar(f)u, u\rangle| \\ &\leqslant \|\mathrm{Op}_\hbar(f)^*\mathrm{Op}_\hbar(f)u\|\,\|u\|. \end{aligned}$$

Therefore, we obtain that

$$\|\mathrm{Op}_\hbar(f)u\| \leqslant \sqrt{\|\mathrm{Op}_\hbar(f)^*\mathrm{Op}_\hbar(f)\|}\,\|u\|,$$

which implies that $\mathrm{Op}_\hbar(f)$ is bounded and that its norm satisfies the inequality (2).

□

We state another useful corollary of our axioms regarding the invertibility of our operators.

Corollary 9. *Let $f \in \mathcal{A}_0$ be bounded. Then there exists $\hbar_0 \in I$ such that $\mathrm{Op}_\hbar(f)$ is invertible for every $\hbar \leqslant \hbar_0$ with inverse having norm uniformly bounded in $\hbar$ if and only if there exists $c > 0$ such that $|f| \geqslant c$.*

Proof. Note that since f is bounded, the previous corollary yields that $\mathrm{Op}_\hbar(f)$ is bounded with norm smaller than

$$\|f\|_\infty + \mathcal{O}(\hbar).$$

Assume that $\mathrm{Op}_\hbar(f)$ is invertible for $\hbar \leqslant \hbar_0$ with

$$\|\mathrm{Op}_\hbar(f)^{-1}\| \leqslant 1/c$$

for every $\hbar \leqslant \hbar_0$, for some constant $c > 0$. Then from the equality $\mathrm{Op}_\hbar(f)^{-1}\mathrm{Op}_\hbar(f) = \mathrm{Id}$ we derive the following:

$$\forall u \in \mathcal{H}_\hbar \qquad \|\mathrm{Op}_\hbar(f)u\| \geqslant \frac{\|u\|}{\|\mathrm{Op}_\hbar(f)^{-1}\|} \geqslant c\|u\|. \tag{3}$$

Let $m \in M$ and let $\chi \geqslant 0$ be a compactly supported smooth function identically 1 in a compact set K containing m. We claim that there exists $u_\hbar \in \mathcal{H}_\hbar$ of norm 1 such that

$$u_\hbar = \mathrm{Op}_\hbar(\chi)u_\hbar + \mathcal{O}(\hbar). \tag{4}$$

This claim is established in Step 3 of the proof of Lemma 11 in [22], but we present a sketch of its proof for the sake of completeness. Let η be a smooth, not identically vanishing function supported on K. By axiom (Q5), there exists $\gamma > 0$ such that

$$\|\mathrm{Op}_\hbar(\eta)\| \geqslant \gamma$$

for every $\hbar \leqslant \hbar_0$, so there exists some $v_\hbar \in \mathcal{H}_\hbar$ of norm 1 and such that

$$\|\mathrm{Op}_\hbar(\eta)v_\hbar\| > \gamma/2.$$

Choose $u_\hbar$ as follows:

$$u_\hbar = \frac{1}{\|\mathrm{Op}_\hbar(\eta)v_\hbar\|}\mathrm{Op}_\hbar(\eta)v_\hbar.$$

Thanks to axiom (Q6), we obtain

$$\mathrm{Op}_\hbar(\chi)u_\hbar = \frac{1}{\|\mathrm{Op}_\hbar(\eta)v_\hbar\|}\mathrm{Op}_\hbar(\chi\eta)v_\hbar + \mathcal{O}(\hbar)$$

which allows us to conclude that $u_\hbar$ satisfies formula (4), since $\chi\eta = \eta$.

We choose such a $u_\hbar$. By axiom (Q6), we get that

$$\|\mathrm{Op}_\hbar(\chi f)u_\hbar - \mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(\chi)u_\hbar\| = \mathcal{O}(\hbar).$$

Combining this estimate with the fact that $u_\hbar$ satisfies equation (4) yields

$$\|\mathrm{Op}_\hbar(\chi f)u_\hbar - \mathrm{Op}_\hbar(f)u_\hbar\| = \mathcal{O}(\hbar)$$

and using equations (2) and (3), this gives

$$\|\chi f\|_\infty + \mathcal{O}(\hbar) \geqslant \|\mathrm{Op}_\hbar(\chi f)u_\hbar\| \geqslant \|\mathrm{Op}_\hbar(f)u_\hbar\| + \mathcal{O}(\hbar) \geqslant c + \mathcal{O}(\hbar).$$

By choosing $\hbar$ sufficiently small, this yields

$$\|\chi f\|_\infty \geqslant c/2.$$

Since we can choose the support K of χ arbitrarily, the previous inequality implies that $|f(m)| \geqslant c/2$.

Conversely, assume that $|f| \geqslant c$ for some constant $c > 0$. Then $1/f$ is bounded, thus axiom (Q1) implies that

$$\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar\left(\frac{1}{f}\right) = \mathrm{Id} + R_\hbar$$

where $R_\hbar$ is bounded with norm $\mathcal{O}(\hbar)$. By a standard result (see for instance [16, Theorem A.3.30]), there exists $\hbar_1 \in I$ such that $\mathrm{Id} + R_\hbar$ is invertible whenever $\hbar \leqslant \hbar_1$, thus for such $\hbar$

$$\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar\left(\frac{1}{f}\right)(\mathrm{Id} + R_\hbar)^{-1} = \mathrm{Id},$$

therefore $\mathrm{Op}_\hbar(f)$ is surjective. Similarly, there exists a bounded operator $S_\hbar$ with norm $\mathcal{O}(\hbar)$ such that

$$\mathrm{Op}_\hbar\left(\frac{1}{f}\right)\mathrm{Op}_\hbar(f) = \mathrm{Id} + S_\hbar$$

and there exists $\hbar_2 \in I$ such that for every $\hbar \leqslant \hbar_2$, $\mathrm{Id} + S_\hbar$ is invertible, so

$$(\mathrm{Id} + S_\hbar)^{-1}\mathrm{Op}_\hbar\left(\frac{1}{f}\right)\mathrm{Op}_\hbar(f) = \mathrm{Id}$$

and hence $\mathrm{Op}_\hbar(f)$ is injective. Consequently, $\mathrm{Op}_\hbar(f)$ is bijective for every $\hbar \leqslant \hbar_0 := \min(\hbar_1, \hbar_2)$. Since $\mathrm{Op}_\hbar(f)$ is a bounded operator, the inverse mapping theorem [24, Theorem III.11] implies that it is invertible for every $\hbar \leqslant \hbar_0$. It remains to show that the norm of its inverse is uniformly bounded in $\hbar$. For this we notice that $\mathrm{Op}_\hbar(1/f)$ is bounded since $1/f$ is bounded, thus

$$\left\|\mathrm{Op}_\hbar(f)^{-1}\right\| \leqslant \|\mathrm{Id} + S_\hbar\|^{-1}\left\|\mathrm{Op}_\hbar\left(\frac{1}{f}\right)\right\| \leqslant \left\|\frac{1}{f}\right\|_\infty + \mathcal{O}(\hbar).$$

Taking a smaller $\hbar_0$ if necessary, this yields the result. $\square$

Remark 10. Note that as a byproduct of the proof of the second point of the corollary, we have that if f is bounded and $|f| \geqslant c$ for some $c > 0$, then

$$\left\|\mathrm{Op}_\hbar(f)^{-1} - \mathrm{Op}_\hbar\left(\frac{1}{f}\right)\right\| = \mathcal{O}(\hbar).$$

$\oslash$

4.3. **Semiclassical operators.** We now introduce an algebra $\mathcal{A}_I$ whose elements are families $f_I = (f_\hbar)_{\hbar \in I}$ of elements of $\mathcal{A}_0$ of the form

$$f_\hbar = f_0 + \hbar f_{1,\hbar}$$

with $f_0 \in \mathcal{A}_0$ and where the family $(f_{1,\hbar})_{\hbar \in I}$ is uniformly bounded in $\hbar$ and supported in a compact set $K \subset M$ independent of $\hbar$. If f_0 is also compactly supported, we say that f_I is compactly supported. We have a map

$$\mathrm{Op} : \mathcal{A}_I \to \prod_{\hbar \in I} \mathcal{L}(\mathcal{H}_\hbar), \quad f_I = (f_\hbar)_{\hbar \in I} \mapsto (\mathrm{Op}_\hbar(f_\hbar))_{\hbar \in I}.$$

Definition 11. A *semiclassical operator* is an element of the image $\Psi := \mathrm{Op}(\mathcal{A}_I)$ of this map.

We want to define a map $\sigma : \Psi \to \mathcal{A}_0$ which associates to $\mathrm{Op}_\hbar(f_I)$ the function $f_0 \in \mathcal{A}_0$. However, we need to check that the latter is unique.

Lemma 12. *The map σ is well-defined. Given $T = (T_\hbar)_{\hbar\in I} \in \Psi$, we call $\sigma(T)$ the* principal symbol *of T.*

Proof. This proof already appeared in [22, Section 4] but we recall it here for the sake of completeness. Let $f_I \in \mathcal{A}_I$ be such that $\mathrm{Op}(f_I) = 0$. Since all the functions $f_{1,\hbar}$ are supported in the same compact set, we deduce from Corollary 8 that

$$\|\mathrm{Op}_\hbar(f_\hbar) - \mathrm{Op}_\hbar(f_0)\| = \mathcal{O}(\hbar). \tag{5}$$

Let χ be any compactly supported smooth function. Using the previous estimate and axiom (Q6), we obtain that for every compactly supported $g_I \in \mathcal{A}_I$

$$\|\mathrm{Op}_\hbar(f_\hbar)\mathrm{Op}_\hbar(\chi) - \mathrm{Op}_\hbar(f_\hbar\chi))\| = \mathcal{O}(\hbar),$$

hence $\|\mathrm{Op}_\hbar(f_\hbar\chi))\| = \mathcal{O}(\hbar)$. Applying (5) to $f_\hbar\chi$ then yields

$$\|\mathrm{Op}_\hbar(f_0\chi))\| = \mathcal{O}(\hbar).$$

Therefore, by axiom (Q5), we conclude that $f_0\chi = 0$. Since χ was arbitrary, this means that $f_0 = 0$. $\square$

By axiom (Q3), the principal symbol of the identity is $\sigma(\mathrm{Id}) = 1$. Axiom (Q2) implies that the principal symbol of a self-adjoint semiclassical operator is real-valued. We can also draw conclusions about the principal symbol of a unitary operator.

Lemma 13. *The principal symbol of a unitary semiclassical operator is $\mathbb{S}^1$-valued.*

Proof. Let $U_\hbar$ be a unitary semiclassical operator. Since we are only interested in the principal symbol, we can assume that $U_\hbar = \mathrm{Op}_\hbar(f)$ for some $f \in \mathcal{A}_0$. Let $m \in M$ and let $\chi \geqslant 0$ be a smooth compactly supported function such that $\chi(m) = 1$. By axiom (Q6), we get

$$\left\|\mathrm{Op}_\hbar(\chi^2|f|^2) - \mathrm{Op}_\hbar(\chi\bar{f}\,)\mathrm{Op}_\hbar(\chi f)\right\| = \mathcal{O}(\hbar). \tag{6}$$

But, still because of axiom (Q6), we have that

$$\|\mathrm{Op}_\hbar(\chi f) - \mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(\chi)\| = \mathcal{O}(\hbar),$$

which yields thanks to Corollary 8 applied to $\chi\bar{f}$:

$$\left\|\mathrm{Op}_\hbar(\chi\bar{f}\,)\mathrm{Op}_\hbar(\chi f) - \mathrm{Op}_\hbar(\chi\bar{f}\,)\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(\chi)\right\| = \mathcal{O}(\hbar).$$

Therefore we obtain by using (6) and the triangle inequality:

$$\left\|\mathrm{Op}_\hbar(\chi^2|f|^2) - \mathrm{Op}_\hbar(\chi\bar{f}\,)\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(\chi)\right\| = \mathcal{O}(\hbar).$$

By iterating the same method, we eventually get

$$\left\|\mathrm{Op}_\hbar(\chi^2|f|^2) - \mathrm{Op}_\hbar(\chi)\mathrm{Op}_\hbar(\bar{f}\,)\mathrm{Op}_\hbar(f)\mathrm{Op}_\hbar(\chi)\right\| = \mathcal{O}(\hbar).$$

Now, using axiom (Q2) and the fact that $\mathrm{Op}_\hbar(f)$ is unitary, this yields

$$\left\|\mathrm{Op}_\hbar(\chi^2|f|^2) - \mathrm{Op}_\hbar(\chi)^2\right\| = \mathcal{O}(\hbar).$$

Finally, thanks to axiom (Q6) and the linearity of $\mathrm{Op}_\hbar$, we infer from this equality that

$$\left\|\mathrm{Op}_\hbar\left(\chi^2(|f|^2 - 1)\right)\right\| = \mathcal{O}(\hbar),$$

thus as a consequence of axiom (Q5) we have that $\chi^2(|f|^2 - 1) = 0$, hence $|f(m)|^2 = 1$. □

4.4. **Further assumptions.** Let $T_\hbar \in \Psi$ be a semiclassical operator with bounded principal symbol f, and such that $|f| \geqslant c$ for some $c > 0$. Then as a consequence of Corollary 9, $T_\hbar$ is invertible. Indeed, $\mathrm{Op}_\hbar(f)$ is invertible and

$$T_\hbar = \mathrm{Op}_\hbar(f) + \mathcal{O}(\hbar);$$

thus our claim comes from an application of Theorem A.3.30 in [16]. We now add one axiom for semiclassical operators, namely:

(Q7) if $S_\hbar, T_\hbar \in \Psi$ have bounded principal symbols, then $S_\hbar T_\hbar$ belongs to Ψ. Furthermore, if $|\sigma(T_\hbar)| \geqslant c$ for some $c > 0$, then $T_\hbar^{-1}$ belongs to Ψ. (*stability*)

Note that in the first situation, it follows from axiom (Q1) that the principal symbol of $S_\hbar T_\hbar$ is equal to the products of the principal symbols of $S_\hbar$ and $T_\hbar$. Moreover, in the second situation, as a consequence of Remark 10, we have that $\sigma(T_\hbar^{-1}) = 1/\sigma(T_\hbar)$.

Remark 14. We will need this axiom in our proof of Theorem 36. However, we will apply it to a very specific case, and we could have stated a weaker version of it. Nonetheless, we keep it in this form because it is a very natural property to require for a semiclassical quantization. In particular, it is satisfied by pseudodifferential operators (see for instance the lines below the proof of Proposition 8.4 in [12]) and by Berezin-Toeplitz operators (e.g. as a consequence of Theorem 1 in [7]–see also the references therein). ⊘

5. The convex hull of a subset of $\mathbb{T}^d$

5.1. **Preliminaries about $\mathbb{T}^d$.** We consider $\mathbb{T}^d = (\mathbb{S}^1)^d$ as the product of d copies of the unit circle. If z belongs to the unit circle, we will denote by $\arg(z)$ its argument in $(-\pi, \pi]$. We denote by

$$\arg : \mathbb{T}^d = (\mathbb{S}^1)^d \to (-\pi, \pi]^d$$

the function assigning its argument to each component of $z \in \mathbb{T}^d$:

$$\arg(z_1, \ldots, z_d) = (\arg(z_1), \ldots, \arg(z_d))\,.$$

Similarly, we will consider the function $\exp : \mathbb{C}^d \to \mathbb{C}^d$ given by

$$\exp(w) = (\exp(w_1), \ldots, \exp(w_d))$$

for $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$.

We endow $\mathbb{T}^d$ with the following distance: for $z, w \in \mathbb{T}^d$

$$d^{\mathbb{T}^d}(z, w) = \min_{\theta \in (2\pi\mathbb{Z})^d} \| \arg(z) - \arg(w) + \theta \|_{\mathbb{R}^d}.$$

The Hausdorff distance induced by this distance (cf Section 2) will be denoted by $d_H^{\mathbb{T}^d}$.

5.2. **Multiplication in $\mathbb{T}^d$.** Let $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d)$ be two points in $\mathbb{T}^d = (\mathbb{S}^1)^d$. Then we use the following notation for the product of a and b in $\mathbb{T}^d$: $a \cdot b = (a_1 b_1, \ldots, a_d b_d)$. Now, given a subset E of the torus $\mathbb{T}^d$ and a point $a \in \mathbb{T}^d$, we define the set $a.E$ as the set of all points of the form $a \cdot z$, $z \in E$. Moreover, we use the notation $a^{-1} \in \mathbb{T}^d$ to denote the point $(a_1^{-1}, \ldots, a_d^{-1})$. The following lemma is an easy consequence of our choice of distance on $\mathbb{T}^d$.

Lemma 15. *Let $E, F \subset \mathbb{T}^d$. Then*

$$d_H^{\mathbb{T}^d}(a \cdot E, a \cdot F) = d_H^{\mathbb{T}^d}(E, F)$$

for every $a \in \mathbb{T}^d$.

5.3. **Convex hulls for simple subsets of $\mathbb{T}^d$.** Let us define the notion of convex hull for subsets of $\mathbb{T}^d$. Because the topology of the torus is more complicated than the one of $\mathbb{R}^d$, we need to be careful. We begin by defining it in the case where we can lift everything to $\mathbb{R}^d$ without any trouble.

Definition 16. A subset $E \subset \mathbb{T}^d$ is called *very simple* if it has no point having a component equal to -1:

$$\forall z = (z_1, \ldots, z_d) \in E, \quad \forall j \in [\![1, d]\!], \qquad z_j \neq -1.$$

For $j \in [\![1, d]\!]$, let $p_j : \mathbb{T}^d \to \mathbb{S}^1$ be the natural projection on the j-th factor:

$$\forall z = (z_1, \ldots, z_d) \in \mathbb{T}^d, \qquad p_j(z) = z_j.$$

Definition 17. A subset $E \subset \mathbb{T}^d$ is called *simple* if none of the functions $p_{j|E}$ is onto.

Remark 18. Note that if E is simple, then there exists $a \in \mathbb{T}^d$ such that $a \cdot E$ is very simple. Note also that a set consisting of a finite number of points is always simple. $\oslash$

Recall that for a subset E of the torus $\mathbb{T}^d$ and a point $a \in \mathbb{T}^d$, we define the set $a \cdot E$ as the set of points of the form $a \cdot z$, $z \in E$.

Lemma 19. *Let $E \subset \mathbb{T}^d$ be simple and compact, with finitely many connected components $E_1, \ldots, E_n$. Let $b, c \in \mathbb{T}^d$ be such that $b \cdot E$ and $c \cdot E$ are very simple. Then for every $j \in [\![1, N]\!]$, there exists a constant $\theta_j^{(b,c)} \in (2\pi\mathbb{Z})^d$ such that*

$$\forall z \in E_j, \quad \arg(c \cdot z) = \arg(b \cdot z) + \arg(c \cdot b^{-1}) + \theta_j^{(b,c)}.$$

We call $\theta_j^{(b,c)}$ the phase shift *of E_j with respect to (b, c).*

Proof. Let $z \in E$; then $c \cdot z = (c \cdot b^{-1}) \cdot (b \cdot z)$, therefore

$$\arg(c \cdot z) = \arg(b \cdot z) + \arg(c \cdot b^{-1}) + \theta(z)$$

for some $\theta(z) \in (2\pi\mathbb{Z})^d$. But the function $z \mapsto \arg(c \cdot z) - \arg(b \cdot z)$ is continuous, since $b \cdot E$ and $c \cdot E$ are very simple; indeed, if a compact set $H \subset \mathbb{T}^d$ is very simple, then H is contained in some compact subset K of $(\mathbb{S}^1 \setminus \{-1\})^d$, and the function $\arg : K \to (-\pi, \pi)^d$ is continuous. Hence the same holds for $z \mapsto \theta(z)$, and thus $\theta(z) = \theta(w)$ whenever z and w belong to the same connected component of E. Consequently, for every $j \in [\![1, d]\!]$, there exists a constant $\theta_j^{(b,c)} \in (2\pi\mathbb{Z})^d$ such that for every $z \in E_j$, $\theta(z) = \theta_j^{(b,c)}$, which was to be proved. $\square$

Given a compact subset I of $\mathbb{R}$, we use the notation $\mathrm{diam}(I)$ for the diameter of I:

$$\mathrm{diam}(I) = \max \{|x - y|, \quad x, y \in I\},$$

with the convention that $\mathrm{diam}(\emptyset) = 0$. Furthermore, let η_j, $1 \leqslant j \leqslant d$ denote the natural projection $\mathbb{R}^d \to \mathbb{R}$, $(x_1, \ldots, x_d) \mapsto x_j$.

Lemma 20. *Let $E \subset \mathbb{T}^d$ be simple and compact, with finitely many connected components. Then there exists $b \in \mathbb{T}^d$ such that $b \cdot E$ is very simple and*

$$\forall j \in [\![1, d]\!], \quad \mathrm{diam}(\eta_j\left(\arg(b \cdot E)\right)) = \min \Lambda_j$$

where

$$\Lambda_j = \left\{\mathrm{diam}(\eta_j\left(\arg(c \cdot E)\right)),\ c \in \mathbb{T}^d, c \cdot E \text{ very simple}\right\}.$$

We say that such a point $b \in \mathbb{T}^d$ is admissible.

Proof. We start by proving that the sets Λ_j do admit minima. So fix $j \in [\![1, d]\!]$; since E is simple, Λ_j is not empty. We will prove that the set Λ_j consists of a finite number of values, which will yield the existence of its minimum. We now make the following simple observation: if $G = \arg(c \cdot E) \subset (-\pi, \pi]^d$ with $c \cdot E$ very simple is the image of $F = \arg(b \cdot E) \subset (-\pi, \pi]^d$, $b \cdot E$ very simple, by a translation, then

$$\mathrm{diam}(\eta_j(F)) = \mathrm{diam}(\eta_j(G)).$$

Let $\theta_1^{(b,c)}, \ldots \theta_N^{(b,c)}$ be the phase shifts of $E_1, \ldots, E_N$ with respect to (b, c) as introduced in Lemma 19. If G is not the image of F by a translation, then necessarily there exists $i \neq k \in [\![1, N]\!]$ such that $\theta_i^{(b,c)} \neq \theta_k^{(b,c)}$. But each $\theta_i^{(b,c)}$ is an element of $(2\pi\mathbb{Z})^d \cap [-2\pi, 2\pi]^d$, hence we can only get a finite number of different values for $\theta_i^{(b,c)}$ by changing b and c. Consequently, there is only a finite number of ways to make G not be the image of F by a translation, hence Λ_j is finite.

It remains to prove that there exists a common $b \in \mathbb{T}^d$ minimizing all the Λ_j, $1 \leqslant j \leqslant d$. If $d = 1$, this is obvious, thus let us assume that $d \geqslant 2$. Obviously we can pick some $b \in \mathbb{T}^d$ which is a minimizer for Λ_1. Now let $j \in [\![1, d-1]\!]$ and assume that we have found $b^j \in \mathbb{T}^d$ such that

$$\forall i \in [\![1, j]\!], \quad \mathrm{diam}(\eta_i\left(\arg(b^j \cdot E)\right)) = \min \Lambda_i.$$

Consider the set $C_j \subset \mathbb{T}^d$ of points of the form $(1, \tilde{c}) \cdot b_j$, $\tilde{c} \in \mathbb{T}^{d-j}$. Then clearly, for every $c \in C_j$ with $c \cdot E$ very simple,

$$\forall i \in [\![1, j]\!], \quad \mathrm{diam}(\eta_i\left(\arg(c \cdot E)\right)) = \min \Lambda_i.$$

Now, let $\Xi_{j+1} = \{\mathrm{diam}(\eta_{j+1}\left(\arg(c \cdot E)\right)), c \in C_j, c \cdot E \text{ very simple}\}$. We want to prove that $\min \Xi_{j+1} = \min \Lambda_{j+1}$; this follows from the fact that the map

$$\varphi_j : \mathbb{T}^j \times C_j \to \mathbb{T}^d, \quad (a, (1, \tilde{c}) \cdot b_j) \mapsto (a, \tilde{c}) \cdot b_j$$

is a bijection satisfying $\eta_{j+1}(\arg(\varphi_j(a, c) \cdot z)) = \eta_{j+1}(\arg(c \cdot z))$ for every $(a, c) \in \mathbb{T}^j \times C_j$ and $z \in E$. We conclude by (finite) induction. □

We would now like to define the convex hull of $E \subset \mathbb{T}^d$ simple, compact, with finitely many connected components, as the set

$$b^{-1} \cdot \exp\left(i \text{Convex Hull}(\arg(b \cdot E))\right)$$

where $b \in \mathbb{T}^d$ is given by Lemma 20 and for any set $F \subset \mathbb{R}^d$

$$\exp\left(iF\right) = \left\{\exp(i\theta),\ \theta \in F\right\}.$$

The problem is that the point $b \in \mathbb{T}^d$ is, in general, far from being unique. Hence, in order to use this definition, we would need this set to not depend on the choice of b. But it can depend on this choice if E displays some symmetries; this is why we use the following definition.

Definition 21. Let $E \subset \mathbb{T}^d$ be simple, compact, and with finitely many connected components $E_1, \ldots, E_N$. We define the convex hull of E as follows:

(1) if for every admissible $b, c \in \mathbb{T}^d$, all the phase shifts $\theta_1^{(b,c)}, \ldots, \theta_N^{(b,c)}$ are equal, then

$$\text{Convex Hull}_{\mathbb{T}^d}(E) := b^{-1} \cdot \exp\left(i \text{ Convex Hull}(\arg(b \cdot E))\right);$$

(2) otherwise, $\text{Convex Hull}_{\mathbb{T}^d}(E) := \mathbb{T}^d$.

Remark 22. In view of this definition, we see that if $E \subset \mathbb{T}^d$ is simple, compact and connected, its convex hull is simply defined as

$$\text{Convex Hull}_{\mathbb{T}^d}(E) := b^{-1} \cdot \exp\left(i \text{ Convex Hull}(\arg(b \cdot E))\right)$$

for any $b \in \mathbb{T}^d$ such that $b \cdot E$ is very simple. ⊘

Definition 21 makes sense because of the following lemma.

Lemma 23. *Let $b, c \in \mathbb{T}^d$ be two admissible points such that the equality $\theta_1^{(b,c)} = \ldots = \theta_N^{(b,c)}$ holds. Then*

$$c^{-1} \cdot \exp\left(i \text{ Convex Hull}(\arg(c \cdot E))\right) = b^{-1} \cdot \exp\left(i \text{ Convex Hull}(\arg(b \cdot E))\right).$$

Proof. Because of the assumption, we have that for every $z \in E$

$$\arg(c \cdot z) = \arg(b \cdot z) + \arg(c \cdot b^{-1}) + \theta$$

where θ is the common value of the $\theta_j^{(b,c)}$. Hence

$$\text{Convex Hull}(\arg(c \cdot E)) = \arg(c \cdot b^{-1}) + \theta + \text{Convex Hull}(\arg(b \cdot E))$$

which implies, since θ belongs to $(2\pi\mathbb{Z})^d$, that

$$\exp\left(i \text{ Convex Hull}(\arg(c \cdot E))\right) = c \cdot b^{-1} \cdot \exp\left(i \text{ Convex Hull}(\arg(b \cdot E))\right)$$

and the result follows. □

A simple compact subset $E \subset \mathbb{T}^d$ with finitely many connected components and satisfying the first condition in the above definition will be called *generic*. This terminology makes sense because such sets are, indeed, generic in the following sense.

Lemma 24. *Let $E \subset \mathbb{T}^d$ be simple, compact, with finitely many connected components, and such that there exists $b, c \in \mathbb{T}^d$ admissible such that not all the phase shifts $\theta_j^{(b,c)}$ are equal. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \leqslant \varepsilon_0$, there exists a compact simple subset $E_\varepsilon \subset \mathbb{T}^d$ such that E_ε is generic and $d_H^{\mathbb{T}^d}(E, E_\varepsilon) \leqslant \varepsilon$.*

This result, of which we will give a proof later, is a corollary of the next two lemmas.

Lemma 25. *Let $E = \{z_1, \ldots, z_N\} \subset \mathbb{T}^d$ be such that there exists $b, c \in \mathbb{T}^d$ admissible such that not all the phase shifts $\theta_j^{(b,c)}$ are equal. Then*

- *either there exist $p, q, r, s \in [\![1, N]\!]$, $j \in [\![1, d]\!]$ and $\theta \in 2\pi\mathbb{Z}$ such that $\{p, q\} \neq \{r, s\}$ and*

$$|\eta_j\left(\arg(b \cdot z_r)\right) - \eta_j\left(\arg(b \cdot z_s)\right) + \theta| = |\eta_j\left(\arg(b \cdot z_p)\right) - \eta_j\left(\arg(b \cdot z_q)\right)|,$$

- *or there exist $p, q \in [\![1, N]\!]$ and $j \in [\![1, d]\!]$ such that*

$$|\eta_j\left(\arg(b \cdot z_p)\right) - \eta_j\left(\arg(b \cdot z_q)\right)| = \pi.$$

It would be interesting to give a simpler characterization of non generic sets. An example of non generic set when $d = 1$ is E consisting of a finite number of points uniformly distributed on $\mathbb{S}^1$, but there are also sets with weaker symmetries which are not generic, for example

$$E = \{\exp(i\phi_1), \exp(i\phi_2), \exp(i\phi_3)\} \subset \mathbb{S}^1$$

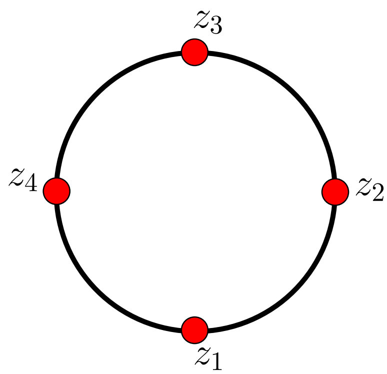
where $\phi_3 = \pi + (\phi_1 + \phi_2)/2$ (see Figure 1).

Proof. Firstly, note that the existence of the pair (b, c) satisfying the assumptions of the lemma implies that $N > 1$. Moreover, replacing E by $b \cdot E$ and c by $c \cdot b^{-1}$ if necessary, we can assume that $b = 1$. To simplify the notation, we will set $\theta_\ell := \theta_\ell^{(1,c)}$, $1 \leqslant \ell \leqslant N$.

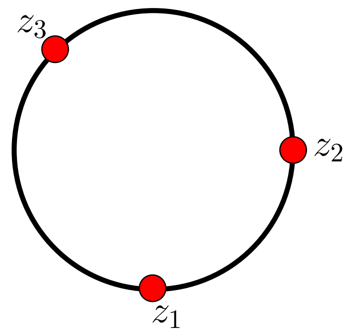
Let us start with some considerations for fixed $j \in [\![1, d]\!]$. Let $p, q \in [\![1, N]\!]$ be such that

$$\operatorname{diam}\left(\eta_j(\arg(E))\right) = |\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))|.$$

If there exist $r, s \in [\![1, N]\!]$ with $\{p, q\} \neq \{r, s\}$ such that this diameter is also equal to $|\eta_j(\arg(z_r)) - \eta_j(\arg(z_s))|$, then we are done. So from now



(a) $\{\exp\left(\frac{i\ell\pi}{2}\right)\}, 0 \leqslant \ell \leqslant 3$



(b) $\{\exp\left(\frac{-i\pi}{2}\right), 1, \exp\left(\frac{3i\pi}{4}\right)\}$

FIGURE 1. Two examples of non generic subsets of $\mathbb{S}^1$.

on we assume that it is not the case. We choose indices $r, s \in [\![1, N]\!]$ such that

$$\mathrm{diam}\left(\eta_j(\arg(c \cdot E))\right) = \left|\eta_j(\arg(c \cdot z_r)) - \eta_j(\arg(c \cdot z_s))\right|.$$

Since 1 and c are admissible, the equality

$$|\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))| = |\eta_j(\arg(c \cdot z_r)) - \eta_j(\arg(c \cdot z_s))|$$

holds; it can be rewritten as

$$|\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))| = |\eta_j(\arg(z_r)) - \eta_j(\arg(z_s)) + \eta_j(\theta_r) - \eta_j(\theta_s)|.$$

If $\{p, q\} \neq \{r, s\}$, then we are done. If $\{p, q\} = \{r, s\}$ and $\eta_j(\theta_p) \neq \eta_j(\theta_q)$, then we are also done. Indeed, this means that

$$|\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))| = |\eta_j(\arg(z_p)) - \eta_j(\arg(z_q)) + \theta|,$$

where $\theta = \pm 2\pi$. Assuming for instance that $\eta_j(\arg(z_p)) > \eta_j(\arg(z_q))$, this yields

$$2(\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))) = \pm 2\pi.$$

Therefore, let us consider the case where $\{p, q\} = \{r, s\}$ and $\eta_j(\theta_p) = \eta_j(\theta_q) = \mu$; we will call this case the *exceptional case*. Exchanging the roles of p and q if necessary, we can assume that $\eta_j(\arg(z_p)) > \eta_j(\arg(z_q))$. Then for every $\ell \notin \{p, q\}$, we have that

$$\eta_j(\arg(z_q)) < \eta_j(\arg(z_\ell)) < \eta_j(\arg(z_p)). \tag{7}$$

But we also know that $\eta_j(\arg(c \cdot z_p)) > \eta_j(\arg(c \cdot z_q))$, because

$$\eta_j(\arg(c \cdot z_p)) = \eta_j(\arg(z_p)) + \eta_j(\arg(c)) + \mu$$

and

$$\eta_j(\arg(c \cdot z_q)) = \eta_j(\arg(z_q)) + \eta_j(\arg(c)) + \mu.$$

Therefore, we also have that for every $\ell \notin \{p, q\}$,

$$\eta_j(\arg(c \cdot z_q)) < \eta_j(\arg(c \cdot z_\ell)) < \eta_j(\arg(c \cdot z_p)),$$

which implies that

$$\eta_j(\arg(z_q)) + \mu < \eta_j(\arg(z_\ell)) + \eta_j(\theta_\ell) < \eta_j(\arg(c \cdot z_p)) + \mu.$$

Combining this with inequality (7), we get that for every $\ell \in [\![1, N]\!]$, $\eta_j(\theta_\ell) = \mu$.

Let us sum up the situation. If for some $j \in [\![1, d]\!]$, we are not in the exceptional case, then we are done. But there must exist such a j, because otherwise we would have that

$$\eta_j(\theta_1) = \ldots = \eta_j(\theta_N)$$

for every j, that is to say $\theta_1 = \ldots = \theta_N$.

□

Lemma 26. *Let E be a compact simple subset of $\mathbb{T}^d$, with finitely many connected components $E_1, \ldots, E_N$, and such that for every $j \in [\![1, d]\!]$ and $p \in [\![1, N]\!]$, there exists a unique point $z^-_{j,p}$ (respectively $z^+_{j,p}$) such that for every b such that $b \cdot E$ is very simple*

$$\min_{z \in E_p} \eta_j(\arg(b \cdot z)) = \eta_j(\arg(b \cdot z^-_{j,p}))$$

(respectively $\max_{z \in E_p} \eta_j(\arg(b \cdot z)) = \eta_j(\arg(b \cdot z^+_{j,p}))$). If E satisfies the assumption of Lemma 24, then the set

$$F = \{z^-_{1,1}, z^+_{1,1}, \ldots, z^-_{d,N}, z^+_{d,N}\}$$

satisfies the assumption of Lemma 25.

Proof. The result can be deduced from the following observation: for every $j \in [\![1, d]\!]$ and $p \in [\![1, N]\!]$, and for every $b \in \mathbb{T}^d$ such that $b \cdot E$ is very simple,

$$\operatorname{diam}(\eta_j(\arg(b \cdot E))) = \operatorname{diam}(\eta_j(\arg(b \cdot F))).$$

□

Proof of Lemma 24. Firstly, we can slightly modify the connected components of E in order to get an ε-close set $\tilde{E}_\varepsilon$ satisfying the assumption in the previous lemma (see Figure 2), because if this assumption is true for one b such that $b \cdot E$ is very simple, it is true for all such b. To this new set $\tilde{E}_\varepsilon$, we then associate a set

$$F_\varepsilon = \{z^-_{1,1}(\varepsilon), z^+_{1,1}(\varepsilon), \ldots, z^-_{d,N}(\varepsilon), z^+_{d,N}(\varepsilon)\}$$

as in this lemma. Then equalities as in Lemma 25 occur for F_ε for a certain number of couples (b, c) of admissible points. But recall that there is only a finite number of different values of

$$\left(\theta_1^{(b,c)}, \ldots, \theta_N^{(b,c)}\right)$$

that we can obtain by changing (b, c). Hence, given $\varepsilon > 0$ small enough, by performing small perturbations of the connected components of $\tilde{E}_\varepsilon$ around the points $z_1(\varepsilon), \ldots, z_M(\varepsilon)$, we can construct a set E_ε which is ε-close to $\tilde{E}_\varepsilon$ with respect to the Hausdorff distance and such that no equality as in Lemma 25 ever occurs, which means that E_ε is generic. $\square$

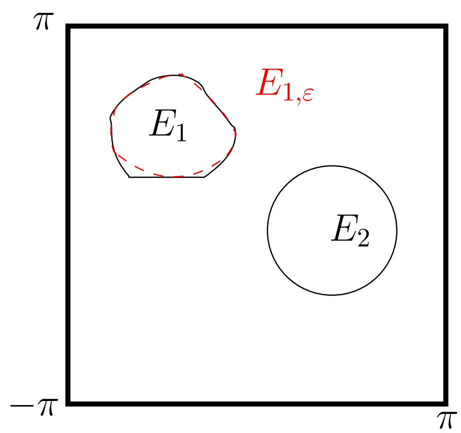


FIGURE 2. Approximating E by a set satisfying the assumptions of Lemma 26.

Before concluding this section, note that the convex hull does not depend on the position of E inside the torus: if E is as in Definition 21, then

$$\text{Convex Hull}_{\mathbb{T}^d}(b \cdot E) = \text{Convex Hull}_{\mathbb{T}^d}(E)$$

for every $b \in \mathbb{T}^d$.

Remark 27. We will not give a definition of the convex hull of a general subset of the torus, since we will always keep these assumptions of compactness and finite number of connected components. However, our definition allows us to handle, in particular, compact connected subsets and sets consisting of a finite number of points. Back to our initial problem, the former corresponds to the closed image of a joint principal symbol, while the latter corresponds to the joint spectrum of a family of pairwise commuting operators acting on finite-dimensional spaces. Moreover, computing the convex hull of a finite number of points on tori seems to be of interest in computational geometry [13]. ⊘

5.4. Convex hull for compact, connected subsets of $\mathbb{T}^d$.

We finally turn to the definition of the convex hull for non necessarily simple compact connected subsets of $\mathbb{T}^d$, which is more involved. We start by proving the following result on approximation by simple subsets.

Lemma 28. *Let E be a compact connected subset of $\mathbb{T}^d$. Assume that there exists a sequence $(E_n)_{n\geqslant 1}$ of compact connected very simple subsets such that*

(1) $E_n \underset{n\to\infty}{\longrightarrow} E$ with respect to the Hausdorff distance,
(2) $E_n \subset E_{n+1}$,
(3) $d_H^{\mathbb{T}^d}(E_n, E_{n+1}) \leqslant \frac{1}{2^n}\min\left(1, d\left(\arg(E_n), \partial\left([-\pi,\pi]^d\right)\right)\right)$.

We call such a sequence a very simple approximation *of E. Then there exists a compact subset $C \subset \mathbb{T}^d$ such that the sequence $(C_n)_{n\geqslant 1}$ of subsets of $\mathbb{T}^d$ defined by*

$$C_n = \text{Convex Hull}_{\mathbb{T}^d}(E_n)$$

converges to C for the Hausdorff distance topology.

Before proving this result, we state the following useful lemma. It is a standard exercise to show that in $\mathbb{R}^d$, taking the convex hull is a 1-Lipschitz operation for the Hausdorff distance; it turns out that the same does not hold in general for simple subsets of $\mathbb{T}^d$, see Figure 3 for a counterexample. However, the following weaker version of this property holds.

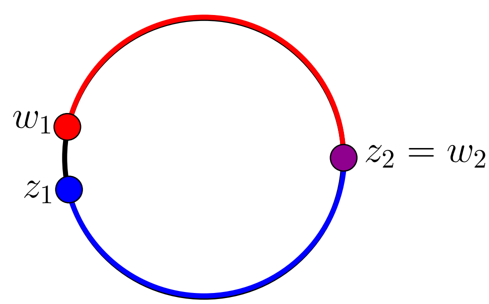


FIGURE 3. Two subsets $E = \{z_1, z_2\}$ and $F = \{w_1, w_2\}$ of $\mathbb{S}^1$ which convex hulls (in blue, the convex hull of E, in red, the convex hull of F) are at Hausdorff distance greater than the Hausdorff distance between E and F.

Lemma 29. *Let $E, F \subset \mathbb{T}^d$ be compact connected very simple subsets such that*
$$d_H^{\mathbb{T}^d}(E, F) \leqslant \delta$$
where
$$\delta = \frac{1}{2} d\left(\arg(F), \partial\left([-\pi, \pi]^d\right)\right).$$
Then
$$d_H^{\mathbb{T}^d}\left(\text{Convex Hull}_{\mathbb{T}^d}(E), \text{Convex Hull}_{\mathbb{T}^d}(F)\right) \leqslant d_H^{\mathbb{T}^d}(E, F).$$

Proof. Before starting the proof, we recall that, because of Definition 21 and the remark following it, we have that
$$\text{Convex Hull}_{\mathbb{T}^d}(E) = \exp\left(i\left(\text{Convex Hull}_{\mathbb{T}^d}(\arg(E))\right)\right)$$
and similarly for F.

Let $z \in \text{Convex Hull}_{\mathbb{T}^d}(F)$; there exists $\theta \in \text{Convex Hull}(\arg(F))$ such that $z = \exp(i\theta)$. Thus θ can be written as a finite linear combination of elements of $\arg(F)$:
$$\theta = \sum_{\ell=1}^{m} \alpha_\ell \theta^\ell, \quad \alpha_1, \ldots, \alpha_m \in [0,1], \sum_{\ell=1}^{m} \alpha_\ell = 1, \quad \theta^1, \ldots, \theta^d \in \arg(F)$$
(here we use superscripts to avoid confusion with the components of elements of $\mathbb{R}^d$). For $1 \leqslant \ell \leqslant m$, let
$$z^\ell = \exp(i\theta^\ell) \in F.$$
Fix $1 < \gamma < 2$; there exists $w^1, \ldots, w^m \in E$ such that for $1 \leqslant \ell \leqslant m$
$$d^{\mathbb{T}^d}(z^\ell, w^\ell) \leqslant \gamma\, d_H^{\mathbb{T}^d}(E, F) \leqslant \gamma\delta. \tag{8}$$
Consider the element
$$w = \exp\left(i \sum_{\ell=1}^{m} \alpha_\ell \arg(w^\ell)\right)$$
of $\text{Convex Hull}_{\mathbb{T}^d}(E)$. For $\ell \in [\![1, m]\!]$, choose a non-zero $\phi^\ell \in (2\pi\mathbb{Z})^d$; then $\arg(w^\ell) - \phi^\ell$ does not belong to $[-\pi, \pi]^d$, thus
$$\left\|\arg(z^\ell) - \arg(w^\ell) + \phi^\ell\right\|_{\mathbb{R}^d} \geqslant d\left(\arg(F), \partial\left([-\pi, \pi]^d\right)\right),$$
which implies, by the choice of γ, that
$$\left\|\arg(z^\ell) - \arg(w^\ell) + \phi^\ell\right\|_{\mathbb{R}^d} \geqslant \gamma\delta \geqslant \left\|\arg(z^\ell) - \arg(w^\ell)\right\|_{\mathbb{R}^d}.$$
Therefore
$$d^{\mathbb{T}^d}(z^\ell, w^\ell) = \left\|\theta^\ell - \arg(w^\ell)\right\|_{\mathbb{R}^d}.$$

Combining this equality with the fact that

$$d^{\mathbb{T}^d}(z,w) \leqslant \left\| \sum_{\ell=1}^{m} \alpha_\ell \left(\arg(w^\ell) - \theta^\ell \right) \right\|_{\mathbb{R}^d} \leqslant \sum_{\ell=1}^{m} \alpha_\ell \left\| \arg(w^\ell) - \theta^\ell \right\|_{\mathbb{R}^d},$$

and equation (8), we obtain that

$$d^{\mathbb{T}^d}(z,w) \leqslant \gamma \sum_{\ell=1}^{m} \alpha_\ell \, d_H^{\mathbb{T}^d}(E,F) = \gamma d_H^{\mathbb{T}^d}(E,F).$$

Hence

$$d^{\mathbb{T}^d}(z, \text{Convex Hull}_{\mathbb{T}^d}(E)) \leqslant \gamma d_H^{\mathbb{T}^d}(E,F).$$

Exchanging the roles of E and F, we obtain that for every $w \in$ Convex $\text{Hull}_{\mathbb{T}^d}(E)$, the inequality

$$d^{\mathbb{T}^d}(w, \text{Convex Hull}_{\mathbb{T}^d}(F)) \leqslant \gamma d_H^{\mathbb{T}^d}(E,F)$$

holds. Thus, from the following characterization of the Hausdorff distance (see for instance [6, Exercise 7.3.2]):

(9)
$$d_H^{\mathbb{T}^d}(A,B) \leqslant r \Leftrightarrow \Big(\forall a \in A,\ d^{\mathbb{T}^d}(a,B) \leqslant r \text{ and } \forall b \in B,\ d^{\mathbb{T}^d}(b,A) \leqslant r \Big),$$

we deduce that

$$d_H^{\mathbb{T}^d}(\text{Convex Hull}_{\mathbb{T}^d}(E), \text{Convex Hull}_{\mathbb{T}^d}(F)) \leqslant \gamma d_H^{\mathbb{T}^d}(E,F).$$

Since $\gamma > 1$ was arbitrary, this concludes the proof. □

Proof of Lemma 28. Since $(E_n)_{n \geqslant 1}$ converges, it is a Cauchy sequence, and furthermore, thanks to the previous lemma, we have that for every $n \geqslant 1$

$$d_H^{\mathbb{T}^d}(C_n, C_{n+1}) \leqslant d_H^{\mathbb{T}^d}(E_n, E_{n+1}).$$

Therefore, the triangle inequality yields, for every $n, p \geqslant 1$,

$$d_H^{\mathbb{T}^d}(C_n, C_{n+p}) \leqslant \sum_{\ell=n}^{n+p-1} d_H^{\mathbb{T}^d}(E_\ell, E_{\ell+1}).$$

But the series

$$\sum_{\ell \geqslant 1} d_H^{\mathbb{T}^d}(E_\ell, E_{\ell+1})$$

converges; this implies that the sequence $(C_n)_{n \geqslant 1}$ is a Cauchy sequence as well. But it is a well-known fact that the set of compact subsets of a complete metric space, endowed with the Hausdorff distance, is complete [6, Proposition 7.3.7]. Applying this to our context, we get that the sequence $(C_n)_{n \geqslant 1}$ converges to some compact subset $C \subset \mathbb{T}^d$. □

Lemma 30. *If E is a compact connected subset of $\mathbb{T}^d$ and $(E_n)_{n\geqslant 1}$, $(F_n)_{n\geqslant 1}$ are two very simple approximations of E (see Lemma 28 for terminology), then*

$$\lim_{n\to+\infty} \text{Convex Hull}_{\mathbb{T}^d}(E_n) = \lim_{n\to+\infty} \text{Convex Hull}_{\mathbb{T}^d}(F_n),$$

where, as usual, the limit is with respect to the Hausdorff distance.

Proof. For $n \geqslant 1$, put

$$C_n = \text{Convex Hull}_{\mathbb{T}^d}(E_n)$$

and

$$D_n = \text{Convex Hull}_{\mathbb{T}^d}(F_n).$$

Denote by C (respectively D) the limit of the sequence $(C_n)_{n\geqslant 1}$ (respectively $(D_n)_{n\geqslant 1}$). By the triangle inequality, we have that

$$d_H^{\mathbb{T}^d}(C,D) \leqslant d_H^{\mathbb{T}^d}(C,C_n) + d_H^{\mathbb{T}^d}(C_n,D_n) + d_H^{\mathbb{T}^d}(D_n,D). \tag{10}$$

The first and third terms on the right hand side of this inequality go to zero as n goes to infinity. Let us estimate the second one. For every $p \geqslant n$, we have, using the triangle inequality again:

$$\begin{aligned} d_H^{\mathbb{T}^d}(E_n,F_n) \leqslant \frac{1}{2^n}\Big(&\min\left(1, d\left(\arg(E_n), \partial\left([-\pi,\pi]^d\right)\right)\right) \\ &+ \min\left(1, d\left(\arg(F_n), \partial\left([-\pi,\pi]^d\right)\right)\right)\Big) + d_H^{\mathbb{T}^d}(E_{n+1},F_{n+1}). \end{aligned}$$

Iterating and using the fact that

$$d\left(\arg(E_{n+1}), \partial\left([-\pi,\pi]^d\right)\right) \leqslant d\left(\arg(E_n), \partial\left([-\pi,\pi]^d\right)\right),$$

we obtain that for every $p \geqslant 1$

$$\begin{aligned} d_H^{\mathbb{T}^d}(E_n,F_n) \leqslant \frac{1}{2^n}\left(\sum_{\ell=0}^{p}\frac{1}{2^\ell}\right)\Big(&\min\left(1, d\left(\arg(E_n), \partial\left([-\pi,\pi]^d\right)\right)\right) \\ &+ \min\left(1, d\left(\arg(F_n), \partial\left([-\pi,\pi]^d\right)\right)\right)\Big) + d_H^{\mathbb{T}^d}(E_{n+p},F_{n+p}). \end{aligned}$$

Letting p go to infinity, we deduce that

$$d_H^{\mathbb{T}^d}(E_n,F_n) \leqslant \frac{1}{2^{n-1}} \max\left(d\left(\arg(E_n), \partial\left([-\pi,\pi]^d\right)\right), d\left(\arg(F_n), \partial\left([-\pi,\pi]^d\right)\right)\right).$$

Therefore, thanks to Lemma 29, the second term in the right hand side of equation (10) satisfies

$$d_H^{\mathbb{T}^d}(C_n,D_n) \leqslant d_H^{\mathbb{T}^d}(E_n,F_n)$$

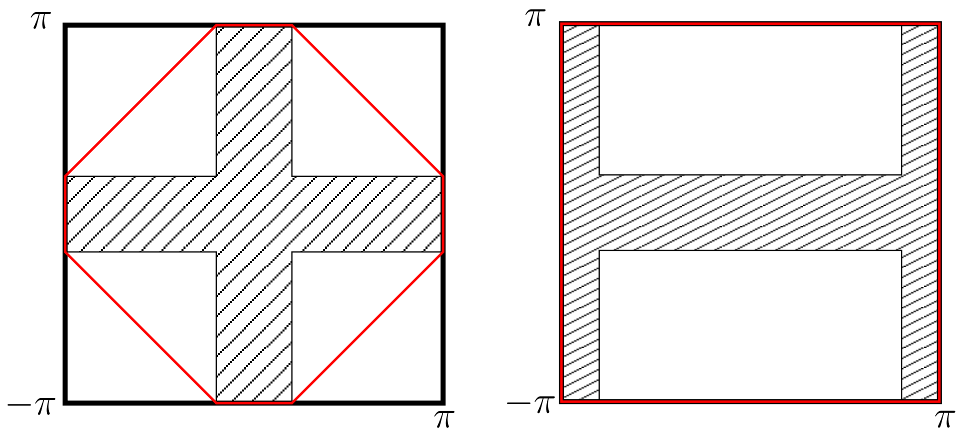
and converges to zero as well. Hence, letting n go to infinity in equation (10), we obtain that $d_H^{\mathbb{T}^d}(C,D) = 0$, which yields $C = D$ because C and D are closed.

□

Now, let E be any compact connected subset of $\mathbb{T}^d$, and let $\mathrm{App}(E) \subset \mathbb{T}^d$ be the set of $a \in \mathbb{T}^d$ such that $a \cdot E$ admits a very simple approximation (see Lemma 28). We define a map C_E from $\mathrm{App}(E)$ to the set of compact subsets of $\mathbb{T}^d$ as follows: for $a \in \mathrm{App}(E)$, $C_E(a)$ is the limit of the convex hull of any very simple approximation of $a \cdot E$.

Definition 31. We say that E is *hullizable* if $\mathrm{App}(E) \neq \emptyset$ and the map C_E is constant.

Figure 4 displays an example of non-hullizable set.



(a) $C_E(a) \neq \mathbb{T}^2$. (b) $C_E(a) = \mathbb{T}^2$.

FIGURE 4. An example of non hullizable set $E \subset \mathbb{T}^2$. The figure displays $\arg(a \cdot E)$ for two different values of $a \in \mathbb{T}^2$, and the corresponding set $\arg(C_E(a))$ (the boundary of which is represented by a red line).

Definition 32. Let E be a hullizable compact connected subset of $\mathbb{T}^d$. We define the convex hull of E as

$$\text{Convex Hull}_{\mathbb{T}^d}(E) := a^{-1} \cdot C_E(a)$$

for any $a \in \mathrm{App}(E)$.

This definition agrees with Definition 21 when E is simple. Indeed, if $a \in \mathbb{T}^d$ is such that $a \cdot E$ is very simple, then $(F_n = a \cdot E)_{n \geqslant 1}$ is a very simple approximation of $a \cdot E$, hence

$$C_E(a) = \text{Convex Hull}_{\mathbb{T}^d}(a \cdot E) = \exp\left(i \ \text{Convex Hull}(\arg(a \cdot E))\right)$$

Consequently, E is hullizable and its convex hull is computed as in Definition 21.

6. Proof of the main theorem

In this section, we prove our main result (Theorem 36), using the results proved in [22] for the self-adjoint case. We also state a conjecture about more general cases not satisfying our assumptions. Before doing so, we introduce the main ingredient of the proof.

6.1. The inverse Cayley transform for unitary operators. Let us recall the definition of the inverse Cayley transform of a unitary operator [25, Definition 3.17]. Let U be a unitary operator such that $-1 \notin \mathrm{Sp}(U)$. We define the inverse Cayley transform of U as

$$\mathcal{C}(U) = i(\mathrm{Id} - U)(\mathrm{Id} + U)^{-1}.$$

Then $\mathcal{C}(U)$ is a self-adjoint operator.

Lemma 33. *Let U, V be commuting unitary operators acting on a Hilbert space $\mathcal{H}$, none of them having -1 in its spectrum. Then $\mathcal{C}(U)$ and $\mathcal{C}(V)$ commute.*

Proof. This is a consequence of the following fact: if A is a normal operator acting on a Hilbert space $\mathcal{H}$, with spectral measure E_A, S is a Borel set and $f : \mathbb{C} \to \mathbb{C}$ is a measurable function, then

$$E_{f(A)}(S) = E_A(f^{-1}(S)).$$

Therefore, if B is another normal operator which commutes with A and g is another measurable function, the spectral projections $E_{f(A)}(S)$ and $E_{g(B)}(T)$ commute for every Borel sets S, T. Hence $f(A)$ and $g(B)$ commute. □

Consequently, it makes sense to talk about the joint spectrum of the family $\mathcal{C}(U_1), \ldots, \mathcal{C}(U_d)$. We recall that the joint spectrum of a finite family of pairwise commuting normal operators is defined as the support of its joint spectral measure.

Lemma 34. *Let $U_1, \ldots, U_d$ be commuting unitary operators acting on a Hilbert space $\mathcal{H}$. Then*

$$\mathrm{JointSpec}(\mathcal{C}(U_1), \ldots, \mathcal{C}(U_d)) = \overline{\left\{ \frac{1}{2}\arg(\lambda),\ \lambda \in \mathrm{JointSpec}(U_1, \ldots, U_d) \right\}}.$$

Proof. We mimic the reasoning of the proof of Proposition 5.25 in [26] (which deals with the spectrum of one single operator). For every $j \in [\![1, d]\!]$, we have that

$$\mathcal{C}(U_j) = \phi(U_j)$$

with

$$\phi : \mathbb{C} \setminus \{-1\} \to \mathbb{C}, \quad z \mapsto i\frac{1-z}{1+z}.$$

Let

$$\mu = E_{U_1} \otimes \ldots \otimes E_{U_d}$$

be the joint spectral measure of $U_1, \ldots, U_d$, and let

$$\nu = E_{\mathcal{C}(U_1)} \otimes \ldots \otimes E_{\mathcal{C}(U_d)}$$

be the joint spectral measure of $\mathcal{C}(U_1), \ldots, \mathcal{C}(U_d)$; we need to prove that

$$\operatorname{supp}(\nu) = \overline{\{(\phi(\lambda_1), \ldots, \phi(\lambda_d)),\ \lambda \in \operatorname{supp}(\mu)\}} =: S.$$

Indeed, a straightforward computation shows that for $z \in \mathbb{S}^1 \setminus \{-1\}$, $\phi(z) = \frac{1}{2} \arg z$. Firstly, let $\zeta = (\zeta_1, \ldots, \zeta_d) \in S$, and let $\varepsilon_1, \ldots, \varepsilon_d > 0$ small enough; there exists $\lambda = (\lambda_1, \ldots, \lambda_d) \in \operatorname{supp}(\mu)$ such that for every $j \in [\![1, d]\!]$, the inequality

$$|\zeta_j - \lambda_j| < \varepsilon_j$$

holds. Since ϕ is continuous in a neighbourhood of $\operatorname{Sp}(U_j)$ in $\mathbb{S}^1$ (because $\operatorname{Sp}(U_j)$ is closed and does not contain -1), there exists $\delta_j > 0$ such that

$$D(\lambda_j, \delta_j) \subset \{z \in \mathbb{C},\ |\phi(z) - \phi(\lambda_j)| < \varepsilon_j\} \subset \phi^{-1}\left(D(\zeta_j, 2\varepsilon_j)\right)$$

where $D(z, r)$ stands for the open disk of radius r centered at z in $\mathbb{C}$. We deduce from this inclusion that

$$E_{U_j}\left(\phi^{-1}\left(D(\zeta_j, 2\varepsilon_j)\right)\right) \geqslant E_{U_j}\left(D(\lambda_j, \delta_j)\right) > 0,$$

where the last inequality comes from the fact that λ belongs to the support of E_{U_j}. Consequently, if

$$D := D(\zeta_1, 2\varepsilon_1) \times \ldots \times D(\zeta_d, 2\varepsilon_d),$$

we have that

$$\nu(D) = \prod_{j=1}^{d} E_{\phi(U_j)}(D(\zeta_j, 2\varepsilon_j)) = \prod_{j=1}^{d} E_{U_j}(\phi^{-1}\left(D(\zeta_j, 2\varepsilon_j)\right)) > 0,$$

which means that ζ belongs to the support of ν.

Conversely, if $\zeta \notin S$, there exists $j \in [\![1, d]\!]$ such that $\phi^{-1}\left(D(\zeta_j, \varepsilon_j)\right)$ is empty for every $\varepsilon_j > 0$ small enough, and we conclude with similar computations that $\zeta \notin \operatorname{supp}(\nu)$.

$\square$

6.2. **When none of the principal symbols is onto.** In this section, we consider pairwise commuting unitary semiclassical operators $U_1(\hbar), \dots, U_d(\hbar)$ with joint principal symbol $F = (f_0^1, \dots, f_0^d)$. We assume that for every $j \in [\![1, d]\!]$, $f_0^j(M)$ is closed, and that the same holds for $F(M)$. Assume moreover that none of the principal symbols

$$f_0^j : M \to \mathbb{S}^1,$$

$j \in [\![1, d]\!]$, is onto; using the terminology introduced earlier, this means that $F(M)$ is a simple compact subset of $\mathbb{T}^d$. Note that this set is connected since it is the image of M, connected, by a continuous function.

Let us introduce an additional assumption in the case where the joint spectrum of $(U_1(\hbar), \dots, U_d(\hbar))$ is generic (see Lemma 24):

(A1) There exists $\hbar_0 \in I$ and a point $b \in \mathbb{T}^d$ which is admissible (see Lemma 20 for the terminology) for all the joint spectra $\mathrm{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))$, $\hbar \leqslant \hbar_0$, and such that $b \cdot F(M)$ is very simple.

Remark 35. This assumption might seem strange but will be crucial for a part of our analysis. Indeed, it may not hold if the joint spectrum is too sparse (see Figure 5). In this situation, given the data of the joint spectrum only, its convex hull computed thanks to our definition will be far from the convex hull of $F(M)$. However, this assumption is reasonable, because it holds for Berezin-Toeplitz and pseudodifferential operators, as a corollary of the Bohr-Sommerfeld rules which imply that the joint spectrum is "dense" (when $\hbar \to 0$) in the set of regular values of F (see [15] for pseudodifferential operators and [8] for Berezin-Toeplitz operators). Nevertheless, our assumption is much weaker than the Bohr-Sommerfeld rules. $\oslash$

Now, we no longer assume that the joint spectrum is generic. Our goal is to prove the following result.

Theorem 36. *We have that, for every $b \in \mathbb{T}^d$ such that $b \cdot F(M)$ is very simple,*

$$b^{-1} \cdot \exp\left(i \text{ Convex Hull}(\arg(b \cdot \mathrm{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))))\right)$$

converges, when $\hbar \to 0$, to

$$\overline{\text{Convex Hull}_{\mathbb{T}^d}(F(M))}$$

with respect to the Hausdorff distance on $\mathbb{T}^d$. In particular, if the joint spectrum is generic and assumption (A1) holds, then

$$\text{Convex Hull}_{\mathbb{T}^d}(\mathrm{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))) \underset{\hbar \to 0}{\longrightarrow} \overline{\text{Convex Hull}_{\mathbb{T}^d}(F(M))}. \tag{11}$$

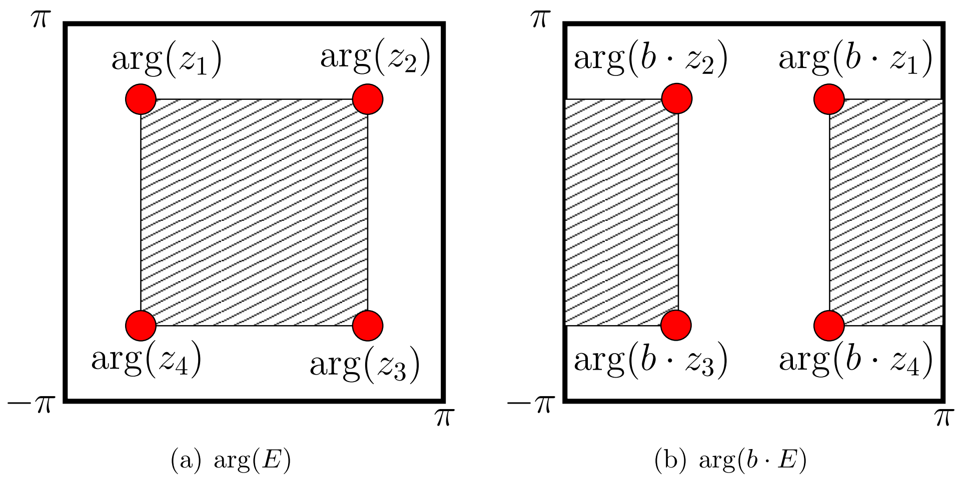


FIGURE 5. An example for which assumption (A1) does not hold.

In this statement, we use the fact that $b\cdot\mathrm{JointSpec}(U_1(\hbar),\ldots,U_d(\hbar)))$ is very simple, for $\hbar \in I$ small enough, whenever $b\cdot F(M)$ is very simple. This is a consequence of the following lemma, which will also be useful in the proof of the theorem.

Lemma 37. *Let j in $[\![1,d]\!]$, and let $a \in \mathbb{S}^1 \setminus f_0^j(M)$. Then there exists $\hbar_0 \in I$ such that for every $\hbar \leqslant \hbar_0$ in I, $a \notin \mathrm{Sp}(U_j(\hbar))$.*

Proof. This is a consequence of Corollary 9 (more precisely, of its consequence stated right before axiom (Q7)). Indeed, since $f_0^j(M)$ is closed, there exists a small open neighbourhood of a in $\mathbb{S}^1$ not intersecting it. Thus there exists $c > 0$ such that

$$|f_0^j - a| \geqslant c.$$

Hence $U_j(\hbar) - a\mathrm{Id}$ is invertible, thus a does not belong to the spectrum of $U_j(\hbar)$. □

Before proving Theorem 36, we state one last technical lemma.

Lemma 38. *Let E be a compact subset of $(-\pi,\pi)^d$ and let $(E_\varepsilon)_{\varepsilon>0}$ be a family of compact subsets of $(-\pi,\pi)^d$ such that*

$$d_H\left(E, E_\varepsilon\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Then $d_H^{\mathbb{T}^d}\left(\exp(iE), \exp(iE_\varepsilon)\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$

Proof. Let

$$\delta_0 = d\left(E, \partial([-\pi,\pi]^d)\right)$$

be the distance between E and the boundary of $[-\pi,\pi]^d$ in $\mathbb{R}^d$. Choose a positive number $\delta \leqslant \frac{1}{2}\delta_0$; there exists $\varepsilon > 0$ such that $d_H(E, E_\varepsilon) \leqslant \delta$. Let γ be such that $1 < \gamma < 2$. Let $u \in E$; by definition of the Hausdorff distance, there exists $v \in E_\varepsilon$ such that

$$\|u - v\|_{\mathbb{R}^d} \leqslant \gamma d_H(E, E_\varepsilon).$$

Now, let $\theta \in (2\pi\mathbb{Z})^d$ be non-zero; then $v-\theta$ does not belong to $[-\pi,\pi]^d$, thus

$$\|u - v + \theta\|_{\mathbb{R}^d} \geqslant \delta_0 \geqslant \gamma\delta \geqslant \|u - v\|_{\mathbb{R}^d}.$$

Consequently, we have that

$$d^{\mathbb{T}^d}(\exp(iu), \exp(iv)) = \|u - v\|_{\mathbb{R}^d} \leqslant \gamma d_H(E, E_\varepsilon) \leqslant \gamma\delta.$$

Therefore $d^{\mathbb{T}^d}(\exp(iu), \exp(iE_\varepsilon)) \leqslant \gamma\delta$. Exchanging the roles of E and E_ε, we also get that for every v in E_ε,

$$d^{\mathbb{T}^d}(\exp(iv), \exp(iE)) \leqslant \gamma\delta.$$

This implies that $d_H^{\mathbb{T}^d}(\exp(iE), \exp(iE_\varepsilon)) \leqslant \gamma\delta$, because of characterization (9). □

We are finally ready to give a proof of the main result of this section.

Proof of Theorem 36. Let $b = (b_1, \ldots, b_d) \in \mathbb{T}^d$ be such that $b \cdot F(M)$ is very simple . For every $j \in [\![1, d]\!]$, consider the operator

$$V_j(\hbar) = b_j U_j(\hbar),$$

which is still a semiclassical unitary operator, with principal symbol

$$g_0^j = b_j f_0^j.$$

By Lemma 37, there exists $\hbar_j \in I$ such that $-1 \notin \mathrm{Sp}(U_j(\hbar))$ whenever $\hbar \leqslant \hbar_j$. Let

$$\hbar_0 = \min_{1 \leqslant j \leqslant d} \hbar_j;$$

in the rest of the proof we will assume that $\hbar \leqslant \hbar_0$. We can therefore consider the self-adjoint operators

$$T_j(\hbar) = 2\,\mathcal{C}(V_j(\hbar)), \quad 1 \leqslant j \leqslant d$$

where we recall that $\mathcal{C}$ stands for the inverse Cayley transform (see Section 6.1). Thanks to axiom (Q7), $T_j(\hbar)$ is a semiclassical operator. We deduce from the comments following this axiom that its principal symbol is equal to

$$a_0^j = 2\,\phi \circ g_0^j$$

where we recall that ϕ is defined as

$$\phi : \mathbb{C} \setminus \{-1\} \to \mathbb{C}, \quad z \mapsto i\frac{1-z}{1+z}.$$

We also recall that for $z \in \mathbb{S}^1 \setminus \{-1\}$, $\phi(z) = \frac{1}{2} \arg z$, and thus

$$a_0^j = \arg g_0^j = \arg(b_j f_0^j).$$

Furthermore, by Lemma 33, $T_j(\hbar)$ and $T_m(\hbar)$ commute for every $j, m \in [\![1, d]\!]$. Let $A = (a_0^1, \ldots, a_0^d)$; by Theorem 3, we have that

$$\text{Convex Hull}(\text{JointSpec}(T_1(\hbar), \ldots, T_d(\hbar))) \underset{\hbar \to 0}{\longrightarrow} \overline{\text{Convex Hull}(A(M))} \tag{12}$$

with respect to the Hausdorff distance on $\mathbb{R}^d$. But on the one hand, we have that

$$\text{Convex Hull}(A(M)) = \text{Convex Hull}(\arg(b \cdot F(M)))$$

On the other hand, Lemma 34 yields

$$\text{JointSpec}(T_1(\hbar), \ldots, T_d(\hbar))) = \overline{\arg(\text{JointSpec}(V_1(\hbar), \ldots, V_d(\hbar)))}.$$

Using these results in equation (12), we finally obtain that

$$\text{Convex Hull}(b \cdot \arg(\text{JointSpec}(U_1(\hbar), \ldots, U_d(\hbar))))$$

converges, when $\hbar$ goes to zero, to

$$\overline{\text{Convex Hull}(b \cdot \arg(F(M)))}$$

with respect to the Hausdorff distance on $\mathbb{R}^d$. By Lemma 38, this implies that

$$\exp\left(i \text{ Convex Hull}(b \cdot \arg(\text{JointSpec}(U_1(\hbar), \ldots, U_d(\hbar))))\right)$$

converges to

$$\exp\left(i \ \overline{\text{Convex Hull}(b \cdot \arg(F(M)))}\right)$$

for the Hausdorff distance on $\mathbb{T}^d$ when $\hbar$ goes to zero. Using the continuity of exp and of the restriction of arg to $(-\pi, \pi)^d$, the latter coincides with the set

$$\overline{\exp\left(i \text{ Convex Hull}(b \cdot \arg(F(M)))\right)}.$$

Finally, using that $z \in \mathbb{T}^d \mapsto b^{-1} \cdot z$ is continuous and preserves the Hausdorff distance (Lemma 15), this yields the first part of the Theorem.

For the second statement of the Theorem, just apply the first part with a point $b = (b_1, \ldots, b_d) \in \mathbb{T}^d$ admissible for all the joint spectra $\text{JointSpec}(U_1(\hbar), \ldots, U_d(\hbar))$, $\hbar \leqslant \hbar_0$, and such that the set $b \cdot F(M)$ is very simple, remembering Definition 21.

∎

6.3. **A conjecture in the general case.** We would like to get rid of the assumption on the surjectivity of the principals symbols. We consider pairwise commuting unitary semiclassical operators

$$U_1(\hbar), \dots, U_d(\hbar)$$

with joint principal symbol

$$F = (f_0^1, \dots, f_0^d).$$

We still assume that $F(M)$ is closed.

Conjecture 39. *Assume that $F(M)$ is hullizable (see Definition 31). From the behaviour of the joint spectrum* $\mathrm{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))$ *when $\hbar$ goes to zero, one can recover the convex hull of $F(M)$.*

We give evidence for this conjecture in Section 7.3, but first let us make a few comments about it. Firstly, "recover" can have several meanings, but it would be appreciable to obtain a statement similar to Theorem 36 involving the convex hull of the joint spectrum; however, the latter may no longer be simple, so we would need to give a meaning to its convex hull. Secondly, in order to prove this conjecture, using axioms (Q1) to (Q7) only might not be enough, thus a natural problem would be to look for the minimal set of additional axioms needed for this proof. Finally, it could be interesting to derive a sufficient condition for $F(M)$ to be hullizable.

7. An application: quantization of circle-valued momentum maps

7.1. **Construction of the circle valued momentum map.** We identify throughout the circle $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$ and denote by $\pi : \mathbb{R} \ni t \mapsto [t] \in \mathbb{R}/\mathbb{Z}$ the natural projection map. The length form $\lambda \in \Omega^1(\mathbb{R}/\mathbb{Z})$ is given by the expression $\lambda([t])\,(T_t\pi(r)) := r$.

Let (M, ω) be a connected symplectic manifold, that is, M is a smooth manifold and ω is a smooth 2-form on M which is non-degenerate and closed.

Let $\Phi : (\mathbb{R}/\mathbb{Z}) \times M \to M$ be a smooth symplectic action, that is a smooth action by diffeomorphisms $\Phi_{[t]} : M \to M$ that preserves the symplectic form ω (these are called *symplectomorphisms*).

For $r \in \mathbb{R}$ denote by $r_M \in \mathfrak{X}(M)$ the action infinitesimal generator determined by r whose value at $x \in M$ is

$$r_M(x) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_{[r\varepsilon]}(x).$$

Definition 40. The $\mathbb{R}/\mathbb{Z}$-action on (M,ω) is *Hamiltonian* if there is a smooth map $\mu\colon M \to \mathbb{R}$ such that $\mathbf{i}_{1_M}\omega := \omega(1_M,\cdot) = \mathrm{d}\mu$. The map μ is called the *momentum map* of the action.

Note that the existence of μ is equivalent to the one-form $\mathbf{i}_{1_M}\omega$ being exact, and therefore if the first cohomology group $H^1(M;\mathbb{R})$ vanishes then every symplectic $\mathbb{R}/\mathbb{Z}$-action on M is in fact Hamiltonian.

If the $\mathbb{R}/\mathbb{Z}$-action does not have a momentum map in the sense above, then the action must be non trivial. Hence, if the action is not Hamiltonian, then $\mathbf{i}_{1_M}\omega$ is not exact. These type of actions also admit an analogue of the momentum map, called the *circle valued momentum map*, and which now takes values in $\mathbb{R}/\mathbb{Z}$. A *circle valued momentum map* $\mu : M \to \mathbb{R}/\mathbb{Z}$ is determined by the equation

$$\mu^*\lambda = \mathbf{i}_{1_M}\omega.$$

Such a map μ always exists, for either ω itself, or a very close perturbation of it. To be more precise, suppose that $\mathbb{R}/\mathbb{Z}$ acts symplectically on the closed symplectic manifold (M,ω), but not Hamiltonianly. Whenever the symplectic form ω is integral (that is, $[\omega] \in H^2(M;\mathbb{Z})$), then the action admits a circle valued momentum map $\mu : M \to \mathbb{R}/\mathbb{Z}$ for ω (this result is due to McDuff, see [20], and is valid for some symplectic form even when the integral cohomology assumption is invalid).

For the sake of completeness and because it is a very simple construction, we review it here. It follows from [23, Lemma 7] that

$$[\mathbf{i}_{1_M}\omega] \in H^1(M;\mathbb{Z}).$$

Fix $m_0 \in M$ and let γ_m be an arbitrary smooth path in M, from m_0 to m, and define $\mu : M \to \mathbb{R}/\mathbb{Z}$ by

$$\mu(m) := \left[\int_{\gamma_m} \mathbf{i}_{1_M}\omega \right]. \tag{13}$$

It is immediate that the definition of μ is independent of paths, so it is is well defined. Also, μ is clearly smooth, and for every $v_m \in T_mM$, we have

$$T_m\mu(v_m) = T_{\int_{\gamma_m} \mathbf{i}_{1_M}\omega}\pi\big(\mathbf{i}_{1_M}\omega(m)(v_m)\big),$$

and consequently

$$(\mu^*\lambda)(m)(v_m) = \lambda(\mu(m))\,(T_m\mu(v_m)) = (\mathbf{i}_{1_M}\omega)\,(m)(v_m),$$

as desired. Of course the map μ is defined up to the addition of constants (because of the freedom in the choice of m_0).

7.2. **Circle action and Berezin-Toeplitz quantization.** It turns out that there exists a natural way to derive a semiclassical quantization of this circle-valued moment map when M is compact and ω is integral (in fact, integral up a factor 2π); this semiclassical quantization is called Berezin-Toeplitz quantization. It builds on geometric quantization, due to Kostant [18] and Souriau [28]. Berezin-Toeplitz operators were introduced by Berezin [1], their microlocal analysis was initiated by Boutet de Monvel and Guillemin [5], and they have been studied by many authors since [2, 3, 7, 19, 29].

Assume that (M, ω) is a compact, connected, Kähler manifold, which means that it is endowed with an almost complex structure which is compatible with ω and integrable. We recall that an almost complex structure j on M is a smooth section of the bundle $\mathrm{End}(TM) \to M$ such that $j^2 = -\mathrm{id}_{TM}$, and j being integrable means that it induces on M a structure of complex manifold. Compatibility between ω and j means that $\omega(\cdot, j\cdot)$ is a Riemannian metric on M.

Assumer moreover that the cohomology class $[\omega/2\pi]$ lies in $H^2(M, \mathbb{Z})$. Then there exists a prequantum line bundle $L \to M$, that is a holomorphic, Hermitian complex line bundle whose Chern connection (the unique connection compatible with both the holomorphic and Hermitian structures) has curvature form equal to $-i\omega$. Then for any integer $k \geqslant 1$, the space

$$\mathcal{H}_k = H^0\left(M, L^{\otimes k}\right)$$

of holomorphic sections of the line bundle $L^{\otimes k} \to M$, endowed with the Hermitian product

$$\phi, \psi \in \mathcal{H}_k \mapsto \langle \phi, \psi \rangle_k = \int_M h_k(\phi, \psi) \mu_M$$

where μ_M is the Liouville measure associated with ω and h_k is the Hermitian form on $L^{\otimes k}$ inherited from the one of L, is a finite dimensional Hilbert space.

Now, the quantization map

$$\mathrm{Op}_k : \mathscr{C}^\infty(M, \mathbb{C}) \to \mathcal{L}(\mathcal{H}_k)$$

is defined as follows: let $L^2(M, L^{\otimes k})$ be the space of square integrable sections of $L^{\otimes k} \to M$, that is the completion of $\mathscr{C}^\infty(M, L^{\otimes k})$ with respect to $\langle \cdot, \cdot \rangle_k$, and let Π_k be the orthogonal projector from $L^2(M, L^{\otimes k})$ to $\mathcal{H}_k$. Then, given $f \in \mathscr{C}^\infty(M, \mathbb{C})$, let

$$\mathrm{Op}_k(f) = \Pi_k f$$

where, by a slight abuse of notation, f stands for the operator of multiplication by f in $L^2(M, L^{\otimes k})$. Here the integer parameter k plays the part of the inverse of $\hbar$, therefore the semiclassical limit corresponds to $k \to +\infty$ instead of $\hbar \to 0$.

Lemma 41. *The Berezin-Toeplitz quantization is a semiclassical quantization in the sense of Section 4.2.*

Proof. This work was done in [22] for axioms (Q3) to (Q6). The fact that axiom (Q1) is satisfied comes, for instance, from [2]. For axiom (Q2), take $\phi, \psi \in \mathcal{H}_k$, and compute

$$\langle \Pi_k f\phi, \psi \rangle_k = \langle f\phi, \psi \rangle_k = \left\langle \phi, \bar{f}\psi \right\rangle_k = \left\langle \phi, \Pi_k \bar{f}\psi \right\rangle_k,$$

where, to derive the first equality, we used the fact that Π_k is self-adjoint and $\Pi_k\phi = \phi$, and we used a similar reasoning for the last equality. The second equality follows from the definition of $\langle \cdot, \cdot \rangle_k$ and the sesquilinearity of h_k. □

Remark 42. We have assumed that M is Kähler for convenience, but there exist ways to construct a Berezin-Toeplitz quantization on a compact symplectic, not necessarily Kähler, manifold (M, ω) with $[\omega/(2\pi)]$ integral [4, 5, 9, 14, 19, 27]. ⊘

Assume now that M is endowed with a smooth symplectic, but not Hamiltonian, action of $\mathbb{S}^1$. We now identify $\mathbb{R}/\mathbb{Z}$ with the unit circle $\mathbb{S}^1$ in $\mathbb{C}$ by means of the map

$$\mathbb{R}/\mathbb{Z} \to \mathbb{S}^1, \quad [t] \mapsto \exp(2i\pi t).$$

Since the symplectic form $\tilde{\omega} = \omega/2\pi$ is integral, there exists a circle valued momentum map $\tilde{\mu}$ with respect to $\tilde{\omega}$ for the action, whose value at $m \in M$ is given by the formula

$$\tilde{\mu}(m) = \left[\int_{\gamma_m} \mathbf{i}_{1_M} \tilde{\omega} \right],$$

where γ_m is a smooth path connecting a fixed point $m_0 \in M$ to m. Hence we get a function $\mu \in \mathscr{C}^\infty(M, \mathbb{S}^1)$ defined as

$$\mu(m) = \exp(2i\pi\tilde{\mu}(m)) = \exp\left(\int_{\gamma_m} \mathbf{i}_{1_M} \omega \right).$$

We associate to this function a unitary Berezin-Toeplitz operator as follows. Set $V(k) = \mathrm{Op}_k(\mu)$; then $V(k)$ is a Berezin-Toeplitz operator with principal symbol μ but may not be unitary. However, the operator

$$U(k) := V(k) \left(V(k)^* V(k) \right)^{-1/2}$$

is well-defined, clearly unitary, and it follows from the stability of Berezin-Toeplitz operators with respect to smooth functional calculus

[7, Proposition 12] that it is a Berezin-Toeplitz operator with principal symbol μ.

7.3. **A family of examples.** Following these constructions, we introduce a family of examples for manifolds $M = \mathbb{T}^{2d}$. We start with the case $d = 1$.

An example when $d = 1$. A famous example of symplectic but non Hamiltonian circle action is the action of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the formula:

$$[t] \cdot ([q, p]) = ([t + q, p]) .$$

Here the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is endowed with the symplectic form coming from the standard one on $\mathbb{R}^2$, that is:

$$\omega = dp \wedge dq.$$

The action is clearly symplectic, and is not Hamiltonian, for instance because it has no fixed point.

Lemma 43. *The circle-valued momentum map associated with this action is*

$$\tilde{\mu}([q, p]) = [p]$$

up to the addition of a constant.

Proof. Using the notation of the previous section, we have that

$$\Phi_{[t]}([q, p]) = [t + q, p],$$

hence

$$1_M([q, p]) = \frac{\partial}{\partial q},$$

therefore $\mathbf{i}_{1_M}\omega = dp$. Take $m_0 = [0, 0] \in \mathbb{T}^2$ and let $m = [q, p]$ be any point in $\mathbb{T}^2$. Then

$$\gamma_m : [0, 1] \to \mathbb{T}^2, \quad t \mapsto [tq, tp]$$

is a smooth path connecting m_0 to m. Thus

$$\tilde{\mu}(m) = \int_{\gamma_m} dp = \int_0^1 p\ dt = p.$$

□

As in the previous part, this map gives rise to a map

$$\mu \in \mathscr{C}^\infty\left(\mathbb{T}^2, \mathbb{S}^1\right), \quad \mu([q, p]) = \exp(2i\pi p).$$

We have a natural semiclassical operator associated with this momentum map, in the setting of Berezin-Toeplitz quantization. Firstly, let us briefly describe the geometric quantization of the torus, although it is now quite standard (see [21, Chapter I.3] for instance). Let

$$L_{\mathbb{R}^2} \to \mathbb{R}^2$$

be the trivial line bundle with standard Hermitian form and connection $d - i\alpha$, where α is the 1-form defined as

$$\alpha_u(v) = \frac{1}{2}\omega(u, v),$$

equipped with the unique holomorphic structure compatible with the Hermitian structure and the connection. Consider a lattice $\Lambda \subset \mathbb{R}^2$ of symplectic volume 4π. The Heisenberg group

$$H = \mathbb{R}^2 \times U(1)$$

with product

$$(x, u) \star (y, v) = \left(x + y, uv \exp\left(\frac{i}{2}\omega_0(x, y)\right)\right)$$

acts on $L_{\mathbb{R}^2}$, with action given by the same formula. This action preserves all the relevant structures, and the lattice Λ injects into H; therefore, by taking the quotient, we obtain a prequantum line bundle L over $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$. Furthermore, the action extends to the line bundle $L_{\mathbb{R}^2}^{\otimes k}$ by

$$(x, u).(y, v) = \left(x + y, u^k v \exp\left(\frac{ik}{2}\omega_0(x, y)\right)\right).$$

We thus get an action

$$T^* : \Lambda \to \operatorname{End}\left(\mathscr{C}^\infty\left(\mathbb{R}^2, L_{\mathbb{R}^2}^{\otimes k}\right)\right), \quad u \mapsto T_u^*.$$

The Hilbert space $\mathcal{H}_k = H^0(\mathbb{T}^2, L^{\otimes k})$ can naturally be identified with the space $\mathcal{H}_{\Lambda,k}$ of holomorphic sections of $L_{\mathbb{R}^2}^{\otimes k} \to \mathbb{R}^2$ which are invariant under the action of Λ, endowed with the Hermitian product

$$\langle \phi, \psi \rangle_k = \int_D \phi \overline{\psi} \ |\omega|$$

where D is the fundamental domain of the lattice. Furthermore, $\Lambda/2k$ acts on $\mathcal{H}_{\Lambda,k}$. Let e and f be generators of Λ satisfying $\omega(e, f) = 4\pi$; one can show that there exists an orthonormal basis $(\psi_\ell)_{\ell \in \mathbb{Z}/2k\mathbb{Z}}$ of $\mathcal{H}_{\Lambda,k}$ such that

$$\forall \ell \in \mathbb{Z}/2k\mathbb{Z} \qquad \begin{cases} T_{e/2k}^* \psi_\ell = w^\ell \psi_\ell \\ \\ T_{f/2k}^* \psi_\ell = \psi_{\ell+1} \end{cases}$$

with $w = \exp\left(\frac{i\pi}{k}\right)$. The basis sections ψ_ℓ can be computed using Theta functions.

Now, set

$$U(k) = T^*_{e/2k} : \mathcal{H}_k \to \mathcal{H}_k;$$

of course, $U(k)$ is unitary. Let (q,p) be coordinates on $\mathbb{R}^2$ associated with the basis (e,f) and $[q,p]$ be the equivalence class of (q,p). It is known [10, Theorem 3.1] that $U(k)$ is a Berezin-Toeplitz operator with principal symbol

$$[q,p] \mapsto \exp(2i\pi p),$$

which is precisely μ. Trivially, the spectrum of $U(k)$ is

$$\mathrm{Sp}(U(k)) = \left\{ \exp\left(\frac{i\pi\ell}{k}\right), \quad 0 \leqslant \ell \leqslant 2k-1 \right\}$$

which is dense in $\mu(\mathbb{T}^2) = \mathbb{S}^1$ when k goes to infinity. Thus, this example is interesting because the assumptions of Theorem 36 are not satisfied, since μ is onto, yet we can recover $\mu(M)$ from the spectrum of $U(k)$ when $k \to +\infty$. This provides with evidence for Conjecture 39 in the $d = 1$ case; we will now explain how to do the same in higher dimension.

The higher dimensional case. More generally, we can consider d symplectic but non Hamiltonian circle actions on $M = \mathbb{T}^{2d} = (\mathbb{T}^2)^d$, endowed with the symplectic form coming from

$$\omega = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \ldots + \mathrm{d}p_d \wedge \mathrm{d}q_d$$

as follows: for $j \in [\![1,d]\!]$, the j-th action is the action of $\mathbb{S}^1$ described above applied to the j-th copy of $\mathbb{T}^2$:

$$[t].[q_1,p_1,\ldots,q_d,p_d] = [q_1,p_1,\ldots,q_{j-1},p_{j-1},t+q_j,p_j,q_{j+1},p_{j+1},\ldots,q_d,p_d].$$

This action admits the circle valued moment map

$$\mu_j \in \mathscr{C}^\infty\left(\mathbb{T}^{2d},\mathbb{S}^1\right), \quad \mu_j([q_1,p_1,\ldots,q_d,p_d]) = \exp(2i\pi p_j).$$

Now, we recall the following useful property of Berezin-Toeplitz quantization with respect to direct products: if M_1, M_2 are two compact connected Kähler manifolds endowed with prequantum line bundles L_1 and L_2 respectively, the line bundle

$$L = L_1 \boxtimes L_2 := \pi_1^* L_1 \otimes \pi_2^* L_2 \to M = M_1 \times M_2$$

is a prequantum line bundle (here $\pi_j : M \to M_j$ is the natural projection). Moreover, the quantum Hilbert spaces satisfy

$$H^0\left(M, L^{\otimes k}\right) = H^0\left(M_1, L_1^{\otimes k}\right) \otimes H^0\left(M_2, L_2^{\otimes k}\right)$$

and, if $f_j \in \mathscr{C}^\infty(M,\mathbb{C})$, $j = 1,2$, then

$$\mathrm{Op}_k(f) = \mathrm{Op}_k(f_1) \otimes \mathrm{Op}_k(f_2)$$

for $f(m_1, m_2) = f(m_1)f(m_2)$. Coming back to our example where $M = \mathbb{T}^2 \times \ldots \times \mathbb{T}^2$, we quantize $\mathbb{T}^2$ as explained in the previous section and we obtain a family of quantum spaces

$$\mathcal{H}_k = H^0\left(\mathbb{T}^2, L^{\otimes k}\right)^{\otimes d}$$

with orthonormal basis

$$(\psi_{\ell_1} \otimes \ldots \otimes \psi_{\ell_d})_{\ell_1,\ldots,\ell_d \in \mathbb{Z}/2k\mathbb{Z}}.$$

Let $U(k)$ be the same operator as in the previous section, and introduce the operator

$$V_j(k) := \mathrm{Id} \otimes \ldots \otimes \mathrm{Id} \otimes \underbrace{U(k)}_{j-\text{th position}} \otimes \ldots \otimes \mathrm{Id}$$

for every $j \in [\![1, d]\!]$. Then $(V_1(k), \ldots, V_d(k))$ is a family of pairwise commuting unitary Berezin-Toeplitz operator acting on $\mathcal{H}_k$, with joint principal symbol $\mu = (\mu_1, \ldots, \mu_d)$. Its joint spectrum is equal to

$$\left\{ \left(\exp\left(\frac{i\pi\ell_1}{k}\right), \ldots, \exp\left(\frac{i\pi\ell_d}{k}\right) \right),\ \ell_1, \ldots, \ell_d \in \mathbb{Z}/2k\mathbb{Z} \right\}$$

and again, from this we recover $\mu(M) = \mathbb{T}^d$ when k goes to infinity.

8. Concluding remarks

A certain number of questions of interest remain. In what follows we gather some of them.

Convex hulls on tori. Our definition of the convex hull of a simple subset of $\mathbb{T}^d$ is quite involved, yet it does not apply to every kind of subset. Is there a way to get a more flexible definition? Can one find a way to handle subsets of $\mathbb{T}^d$ with infinitely many connected components?

More general non-self-adjoint operators. The notion of semiclassical quantization introduced in Section 4.2 allows us to handle general non-self-adjoint semiclassical operators. It would be interesting to see if from the family of joint spectra of a family of pairwise commuting normal operators (not necessarily self-adjoint or unitary) one can recover some information about the classical spectrum $\mathcal{S}$. One can maybe try to recover the convex hull of $\mathcal{S} \subset \mathbb{C}^d \simeq \mathbb{R}^{2d}$, but the analysis seems quite complicated. Furthermore, we would lose information for unitary operators because then the convex hull of $\mathcal{S} \subset \mathbb{C}^d$ would be larger than its convex hull computed as a subset of $\mathbb{T}^d$.

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References

[1] F. A. Berezin. General concept of quantization. *Comm. Math. Phys.*, 40:153–174, 1975.

[2] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. Toeplitz quantization of Kähler manifolds and $\mathrm{gl}(N)$, $N \to \infty$ limits. *Comm. Math. Phys.*, 165(2):281–296, 1994.

[3] D. Borthwick, T. Paul, and A. Uribe. Semiclassical spectral estimates for Toeplitz operators. *Ann. Inst. Fourier (Grenoble)*, 48(4):1189–1229, 1998.

[4] D. Borthwick and A. Uribe. Almost complex structures and geometric quantization. *Math. Res. Lett.*, 3(6):845–861, 1996.

[5] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1981.

[6] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[7] L. Charles. Berezin-Toeplitz operators, a semi-classical approach. *Comm. Math. Phys.*, 239(1-2):1–28, 2003.

[8] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. *Comm. Partial Differential Equations*, 28(9-10):1527–1566, 2003.

[9] L. Charles. Quantization of compact symplectic manifolds. Preprint, http://arxiv.org/abs/1409.8507, 2014.

[10] L. Charles and J. Marché. Knot state asymptotics I: AJ conjecture and Abelian representations. *Publications mathématiques de l'IHÉS*, 121(1):279–322, 2015.

[11] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[12] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[13] C. I. Grima and A. Márquez. *Computational geometry on surfaces. Performing computational geometry on the cylinder, the sphere, the torus, and the cone.* Dordrecht: Kluwer Academic Publishers. xv, 2001.

[14] V. Guillemin. Star products on compact pre-quantizable symplectic manifolds. *Lett. Math. Phys.*, 35(1):85–89, 1995.

[15] B. Helffer and D. Robert. Puits de potentiel généralisés et asymptotique semi-classique. *Ann. Inst. H. Poincaré Phys. Théor.*, 41(3):291–331, 1984.

[16] P. D. Hislop and I. M. Sigal. *Introduction to spectral theory*, volume 113 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. With applications to Schrödinger operators.

[17] L. Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[18] B. Kostant. Quantization and unitary representations. *Uspehi Mat. Nauk*, 28(1(169)):163–225, 1973. Translated from the English (Lectures in Modern Analysis and Applications, III, pp. 87–208, Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970) by A. A. Kirillov.

[19] X. Ma and G. Marinescu. Toeplitz operators on symplectic manifolds. *J. Geom. Anal.*, 18(2):565–611, 2008.

[20] D. McDuff. The moment map for circle actions on symplectic manifolds. *J. Geom. Phys.*, 5(2):149–160, 1988.

[21] D. Mumford. *Tata lectures on theta. I*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition.

[22] Á. Pelayo, L. Polterovich, and S. Vũ Ngọc. Semiclassical quantization and spectral limits of $\hbar$-pseudodifferential and Berezin-Toeplitz operators. *Proc. Lond. Math. Soc. (3)*, 109(3):676–696, 2014.

[23] Á. Pelayo and T. S. Ratiu. Circle-valued momentum maps for symplectic periodic flows. *Enseign. Math. (2)*, 58(1-2):205–219, 2012.

[24] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

[25] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[26] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2012.

[27] B. Shiffman and S. Zelditch. Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds. *J. Reine Angew. Math.*, 544:181–222, 2002.

[28] J.-M. Souriau. Quantification géométrique. *Comm. Math. Phys.*, 1:374–398, 1966.

[29] S. Zelditch. Szegő kernels and a theorem of Tian. *Internat. Math. Res. Notices*, (6):317–331, 1998.

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