Tiling the (musical) line with polynomials

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TILING THE (MUSICAL) LINE WITH POLYNOMIALS: SOME THEORETICAL AND IMPLEMENTATIONAL ASPECTS

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ABSTRACT

This paper aims at discussing the polynomial approach to the problem of tiling the (musical) time axis with translates of one tile. This mathematical construction naturally leads to a new family of rhythmic tiling canons having the property of being generated by cyclotomic polynomials (*tiling cyclotomic canons*).

1. INTRODUCTION

Tiling problems in music theory, analysis and composition have a relatively old history in mathematical music theory. Surprisingly, despite the well-known canonical equivalence (isomorphism) between a well-tempered division of the octave and the cyclic character of any periodic rhythm [19], the study of some tiling properties of the time-line by means of translates of a given rhythmic tile (or some usual transformations of it) is a relatively new research area inside mathematical music theory. Dan Tudor Vuza's algebraic model of tiling canon construction by the factorization of a cyclic group into a direct sum of two subsets [20] gave a strong impulse to the implementation of algebraic methods in music composition. [1] In this paper we focus on cyclotomic polynomials. Some preliminary definitions about cyclotomic polynomials, tiling of the line process and rhythmic tiling canon construction will be provided in Section 2. In Section 3 we show how this approach has been implemented in *OpenMusic* visual programming language and discuss some difficulties in directly applying the cyclotomic factorization to the canon construction. In Section 4 we discuss some interesting connections between Vuza's original model of Regular Complementary Canons of Maximal Category [20] and the polynomial approach by also showing how both approaches are intimately related to some mathematical conjectures.

[1] For a detailed presentation of the group factorization approach to the construction of tiling canons, together with the *OpenMusic* implementation, see [5]. For a combinatorial discussion of Vuza's model, also see [9] and [12]. For a different compositional approach to the problem tiling the line, see [13].

2. SOME PRELIMINARY DEFINITIONS

This section provides some definitions on cyclotomic polynomials and some general factorization theorems.

2.1. 0-1 polynomials and rhythmic tiling canons

A rhythmic tiling canon is a decomposition of a cyclic group Z_n into a direct sum of two subsets:

$$Z_n = A \oplus B$$

An enhancement of the ambient structure originates to [16]: put $A(x) = \sum_{a\in A} x^a$, then the above equation becomes a relation between 0-1 polynomials, that is to say polynomials with coefficients being either 0 or 1:

$$A(x) \times B(x) \equiv 1 + x + x^2 + \cdots + x^{n-1} \pmod{x^n - 1}$$

Factors of the polynomial $\Delta_n(x) = 1+x+x^2+\cdots+x^{n-1}$ are thus of paramount importance, especially those with 0-1 coefficients. We find a number of them by considering the cyclotomic approach.

2.2. Cyclotomic polynomials

Definition 1 *The n^{th} cyclotomic polynomial is*

$$\Phi_n(x) = \prod_{\gcd(k,n)=1} (x - e^{2i\,k\,\pi/n})$$

This the monic polynomial whose roots are the primitive units of order n, that is to say the $\xi \in C$ for which $z^n = 1$ though $z^r \neq 1$ for $1 \leq r < n$.

A classical result states that these polynomials have integer coefficients, i.e. $\Phi_n(x) \in Z[x]$.

Another classical result states that they are irreducible in the euclidean ring $Q[x]$, and hence in $Z[x]$. Another way to put it is that any polynomial in $Z[x]$ having a primitive unit root of order n has Φ_n as a factor.

Directly relevant to rhythmic canons is the fact that the product of a selection of cyclotomic polynomials can be expressed by the following equations:

$x^n - 1 = \prod_{d|n} \Phi_d(x)$
$\Delta_n(x) = \prod_{d|n, d\neq 1} \Phi_d(x)$ (1)

Formula (1) enables to compute efficiently all cyclotomic polynomials.

Usually Φ_n have integer coefficients which are often $0, 1$ or -1. In particular the rhythm of a metronome is easily expressed as a product of cyclotomic polynomials:

$$1+x^k+x^{2k}+\dots x^{(m-1)k} = \frac{x^{mk}-1}{x^k-1} = \prod_{d|mk, d\nmid k} \Phi_d(x)$$

Note that some of these cyclotomic factors are NOT 0-1 polynomials, though their product is. For instance:

$$\Phi_6 \times \Phi_3 = (1-x+x^2)(1+x+x^2) = 1+x^2+x^4$$

2.3. CYCLOTOMIC POLYNOMIALS AND THE RHYTHMIC TILING CANON CONSTRUCTION

The importance of these particular polynomials lies in the following lemme:

Lemme 1 *If $A \oplus B = Z_n$, then for all $d \mid n$ ($d \neq 1$) Φ_d is a divisor of either $A(x)$ or $B(x)$.*

Thus, cyclotomic polynomials occur in all rhythmic canons. Conversely, in the *OpenMusic* implementation we have tried to use these polynomials to build up some rhythmic canons. As we will see, some other rhythmic canons are left out of this schema, but nevertheless it gives an interesting degree of control over the canons construction.

Except in special cases([3]), it is not known whether a given rhythmic motif enables to make a canon unless one is able to exhibit such a canon; but looking for the outer rhythm knowing only the inner rhythm is impractical as far as computing time is concerned.

Since 1998, there is a useful sufficient condition (also necessary when the number of notes in the rhythmic pattern has at most two prime factors) dealing with the cyclotomic factors. Let A be an inner rhythm (i.e. a rhythmic pattern that tiles, see [3]) and $A(x)$ the associated 0-1 polynomial. Put $R_A = \{d, \Phi_d \text{ divides } A(x)\}$ and $S_A = \{p^\alpha \in R_A\}$ the subset of prime powers. It is proved in [7] that if both following conditions (henceforth the Coven-Meyerowitz conditions) are true, then A enables to build a rhythmic canon :

$(T_1) : A(1) = \prod_{p^\alpha \in S_A} p$

$(T_2) : p^\alpha, q^\beta \dots \in S_A \Rightarrow (p^\alpha \times q^\beta \dots) \in R_A$

The condition (T_1) is also always necessary. We will look again at these conditions in connection with the spectral conjecture and its relationship with Vuza canons in the final section. The next section deals with construction of rhythmic canons in *OpenMusic* using these conditions.

3. THE OPENMUSIC IMPLEMENTATION

3.1. The list of cyclotomic products having the T_1 and T_2 properties

For periods $n > 10$ we need to test the two conditions of Coven and Meyerowitz in order to be sure that the associated rhythmic canon tiles the time axis. The algorithm thus tests all sublists of the list of divisors of n for conditions T_1, T_2 and computes the corresponding products of cyclotomic polynomials. We will discuss in detail the cases $n = 12$ and $n = 16$.

3.1.1. *The case n=12*

Since 12 has five divisors $(2, 3, 4, 6, 12)$, the cyclotomic factors may be combined in several ways in order to tile the line. Apart from $\Phi_6 = 1-x+x^2$ and $\Phi_{12} = 1-x^2+x^4$ (which are not 0-1 polynomials), each cyclotomic polynomial of index $k \in \{2, 3, 4\}$ tiles the line by itself.

We lose somehow the symmetric distribution of the simpler cases. For instance $A = \Phi_2\Phi_4\Phi_6$ should tile, as conditions T_1, T_2 are fulfilled: $S_A = \{2, 4\}$ and there is only (T_1) to check, namely $A(1) = 2 \times 2$.

But the simple trick of multiplying the remaining cyclotomic factors of $1+x+x^2+\dots x^{11}$ does not work this time, as

$$\Phi_3\Phi_{12} = 1+x-x^3+x^5+x^6$$

is not a 0-1 polynomial.

The outer rhythm may still be produced with cyclotomic polynomials, following the proof of the Coven and Meyerowitz theorem, but the formula is more complicated:

$$B(x) = \Phi_3(x^4) = 1+x^4+x^8 = \Phi_3\Phi_6\Phi_{12}$$

(see figure 1)

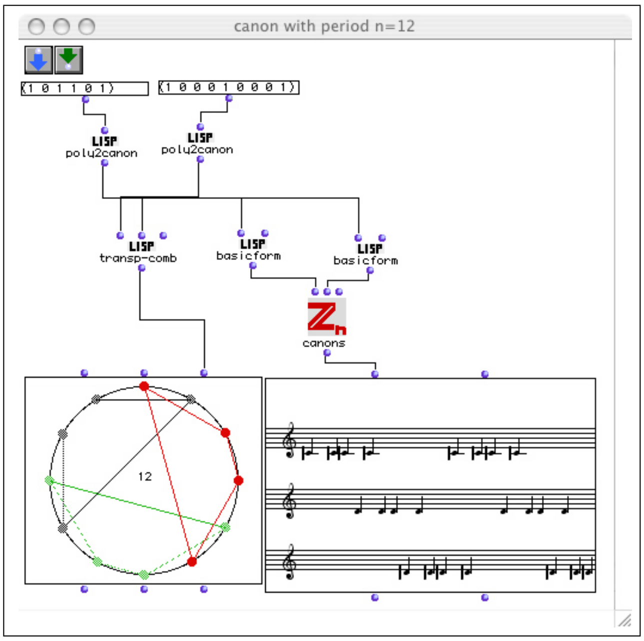

Figure 1. A 3 voices tiling rhythmic canon associated with the two polynomials $A(x)$ and $B(x)$.

3.1.2. *The case n=16*

All possible solutions for period $n = 16$ are given in Figure 2.

```
;============PERIOD 16============
'(16
(((1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1) nil (2 4 8 16)))
(((1 1) (1 0 1 0 1 0 1 0 1 0 1 0 1 0 1) (2)))
(((1 0 1) (1 1 0 0 1 1 0 0 1 1 0 0 1 1) (4)))
(((1 0 0 0 1) (1 1 1 1 0 0 0 0 1 1 1 1) (8)))
(((1 0 0 0 0 0 0 0 1) (1 1 1 1 1 1 1 1) (16)))
(((1 1 1 1) (1 0 0 0 1 0 0 0 1 0 0 0 1) (2 4)))
(((1 1 0 0 1 1) (1 0 1 0 0 0 0 0 1 0 1) (2 8)))
(((1 1 0 0 0 0 0 0 1 1) (1 0 1 0 1 0 1) (2 16)))
(((1 0 1 0 1 0 1) (1 1 0 0 0 0 0 0 1 1) (4 8)))
(((1 0 1 0 0 0 0 0 1 0 1) (1 1 0 0 1 1) (4 16)))
(((1 0 0 0 1 0 0 0 1 0 0 0 1) (1 1 1 1) (8 16)))
(((1 1 1 1 1 1 1 1) (1 0 0 0 0 0 0 0 1) (2 4 8)))
(((1 1 1 1 0 0 0 0 1 1 1 1) (1 0 0 0 1) (2 4 16)))
(((1 1 0 0 1 1 0 0 1 1 0 0 1 1) (1 0 1) (2 8 16)))
(((1 0 1 0 1 0 1 0 1 0 1 0 1 0 1) (1 1) (4 8 16)))
)
.
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Figure 2. Catalogue of tiling rhythmic canons of period n=16 that are directly given by simple cyclotomic factors (or any given product of them).

Notice that there is a symmetry principle in the distribution of solutions for the "inner" and "outer" rhythm. Take for example the solution given by the cyclotomic polynomial Φ_4. The "outer rhythm" is provided by the product $\Phi_2 \times \Phi_8 \times \Phi_{16}$ (see figure 3).

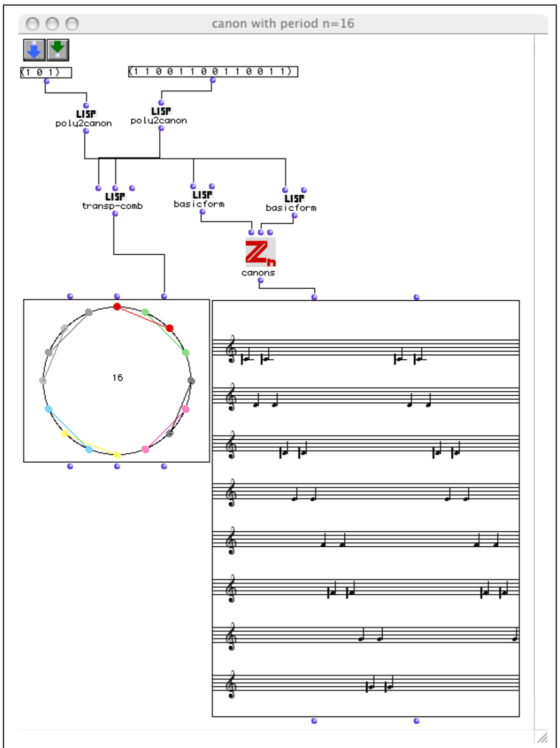

Figure 3. A 8 voices tiling rhythmic canon associated with the two polynomial Φ_4 and $\Phi_2 \times \Phi_8 \times \Phi_{16}$

4. RHYTHMIC TILING CANONS AND SOME MATHEMATICAL CONJECTURES

As we mentioned, Vuza's original model of canons focused on a very special family of tiling rhythmic canons. These were group-theoretically formalized as the solution of factorizing a given cyclic group into the direct sum of two non-periodic subsets. This theory has been established by Vuza independently of any consideration of geometric tiling conjectures. Nevertheless, as we have already shown, it is possible to directly link Vuza's model of rhythmic canons to Minkowski's original conjecture of the tiling of the n-dimensional space by unit cubes [5]. Enumeration techniques inspired by Polya and Burnside combinatorial algebra enabled H. Fripertinger to list all Vuza canons for periods 72 and 108; it is thus now known that Vuza canons are less than one out of a million. We now show that Vuza canons are related to a second famous conjecture, Fuglede's (or spectral) conjecture.

4.1. Vuza canons and the spectral conjecture

Flugede's conjecture[10], still unsolved in dimension 1, asserts that a measurable set A of R^n tiles R^n by translations if and only if A is spectral i.e. the Lebesgue space $L^2(A)$ has an orthogonal basis $\{e^{2i\pi\lambda.x}\}_{\lambda\in\Lambda}$. For the integers, the conjecture asserts that :

Fuglede's Conjecture (1974). The finite set $A \subset N$ tiles the integers Z by translations if and only if A is spectral, i.e. the set $\Lambda = \{\lambda_0, \lambda_1, ..., \lambda_{n-1}\}$ where λ_j are distinct, $\lambda_0 = 0$, and $n = A(1)$ verifies

$$A(e^{2i\pi(\lambda_j-\lambda_k)}) = 0$$

for all $0 \le j, k \le n-1$, and $j \ne k$.

This conjecture, not unlike the conjectures by Minkowski and Hajos [18], links geometry to harmonic analysis. It was noticed shortly after the discovery by Coven-Meyerowitz discovery that the two conditions $(T_1), (T_2)$ are strongly linked to the spectral condition for a tiling: if a motif is spectral then (T_1) is true, if furthermore (T_2) is true then the motif is spectral [14].

But a canon that is NOT a Vuza canon may be decomposed into an union of smaller canons. For instance, the rhythmic pattern $(0, 1, 4, 5)$ tiling with period 8 is two copies of the rhythm $(0, 1)$ which tiles Z_4.

It was proved [2] that this operation preserves condition (T_2). Hence if a rhythmic canon tiles without being spectral, it shall be reduced to a Vuza canon with the same property. In other words:

If the spectral conjecture is false then it is false for some Vuza canon. If all Vuza canons are spectral then so are all canons of any kind.

The scarcity of Vuza canons is a further indication that the spectral conjecture might be true in dimension 1. Furthermore, all the algorithms currently used for producing Vuza canons are guaranteed to produce canons verifying (T_2) (from other kinds of reduction techniques). But there is as yet no certainty as to the general Vuza canon, for there is no known way to produce or descript the cyclotomic structure of all of them.

5. CONCLUSIONS

The *OM* implementation of the cyclotomic approach is a necessary step in our general search of all possible solutions of tiling rhythmic canons of a given period. We have shown some connections between the classical approach based on the factorization of a cyclic group into two subsets and this new approach that makes use of the mathematical theory of tiling the line by translates of a cyclotomic polynomial (or of a product of cyclotomic polynomials). Coven-Meyerowitz conditions have been the starting point for implementing compositional algorhitms enabling to tile the musical line via cyclotomic polynomials. Although the two conditions are also necessary in some special cases, the problem of establishing necessary and sufficient conditions for the tiling of the line process remains open. We have shown some difficulties in trying to recover Vuza's original theory by the cyclotomic decomposition (and vice-versa). As in the case of the classical approach of rhythmic tiling canons construction by factorization of cyclic groups, the polynomial approach naturally leads to some still open mathematical conjectures, as the spectral (or Fuglede) conjecture. We suggest that the special family of *Regular Complementary Canons of Maximal Category*, originally conceived by Vuza in terms of factorizations of cyclic groups into non-periodic subsets, could play a major role in the musical interpretation (and mathematical solution) of the spectral conjecture.

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