Some nodal properties of the quantum harmonic oscillator and other Schrödinger operators in R^2

Pierre Bérard, Bernard Helffer

Some nodal properties of the quantum harmonic oscillator and other Schrödinger operators in $\mathbb{R}^2$

P. Bérard
Institut Fourier, Université Grenoble Alpes and CNRS, B.P.74,
F38402 Saint Martin d'Hères Cedex, France.
pierrehberard@gmail.com
and
B. Helffer
Laboratoire Jean Leray, Université de Nantes and CNRS,
F44332 Nantes Cedex France, and
Laboratoire de Mathématiques, Univ. Paris-Sud 11.
Bernard.Helffer@univ-nantes.fr

June 21, 2015, revised July 31, 2016

Abstract

For the spherical Laplacian on the sphere and for the Dirichlet Laplacian in the square, Antonie Stern claimed in her PhD thesis (1924) the existence of an infinite sequence of eigenvalues whose corresponding eigenspaces contain an eigenfunction with exactly two nodal domains. These results were given complete proofs respectively by Hans Lewy in 1977, and the authors in 2014 (see also Gauthier-Shalom–Przybytkowski, 2006). In this paper, we obtain similar results for the two dimensional isotropic quantum harmonic oscillator. In the opposite direction, we construct an infinite sequence of regular eigenfunctions with as many nodal domains as allowed by Courant's theorem, up to a factor $\frac{1}{4}$. A classical question for a 2-dimensional bounded domain is to estimate the length of the nodal set of a Dirichlet eigenfunction in terms of the square root of the energy. In the last section, we consider some Schrödinger operators $-\Delta + V$ in $\mathbb{R}^2$ and we provide bounds for the length of the nodal set of an eigenfunction with energy λ in the classically permitted region $\{V(x) < \lambda\}$.

Keywords: Quantum harmonic oscillator, Schrödinger operator, Nodal lines, Nodal domains, Courant nodal theorem.
MSC 2010: 35B05, 35Q40, 35P99, 58J50, 81Q05.

1 Introduction and main results

Given a finite interval $]a, b[$ and a continuous function $q : [a, b] \mapsto \mathbb{R}$, consider the one-dimensional self-adjoint eigenvalue problem

$$- y'' + qy = \lambda y \text{ in }]a, b[\,, \quad y(a) = y(b) = 0\,. \tag{1.1}$$

Arrange the eigenvalues in increasing order, $\lambda_1(q) < \lambda_2(q) < \cdots$. A classical theorem of C. Sturm [22] states that an eigenfunction u of (1.1) associated with $\lambda_k(q)$ has exactly $(k-1)$ zeros in $]a, b[$ or, equivalently, that the zeros of u divide $]a, b[$ into k sub-intervals.

In higher dimensions, one can consider the eigenvalue problem for the Laplace-Beltrami operator $-\Delta_g$ on a compact connected Riemannian manifold (M, g), with Dirichlet condition in case M has a boundary ∂M,

$$- \Delta u = \lambda u \text{ in } M, \quad u|_{\partial M} = 0. \tag{1.2}$$

Arrange the eigenvalues in non-decreasing order, with multiplicities,

$$\lambda_1(M, g) < \lambda_2(M, g) \leq \lambda_3(M, g) \leq \dots$$

Denote by M_0 the interior of M, $M_0 := M \setminus \partial M$.

Given an eigenfunction u of $-\Delta_g$, denote by

$$N(u) := \overline{\{x \in M_0 \mid u(x) = 0\}} \tag{1.3}$$

the *nodal set* of u, and by

$$\mu(u) := \#\{ \text{ connected components of } M_0 \setminus N(u)\} \tag{1.4}$$

the number of *nodal domains* of u i.e., the number of connected components of the complement of $N(u)$.

Courant's theorem [12] states that if $-\Delta_g u = \lambda_k(M, g)u$, then $\mu(u) \leq k$.

In this paper, we investigate three natural questions about Courant's theorem in the framework of the 2D isotropic quantum harmonic oscillator.

Question 1. In view of Sturm's theorem, it is natural to ask whether Courant's upper bound is sharp, and to look for lower bounds for the number of nodal domains, depending on the geometry of (M, g) and the eigenvalue. Note that for orthogonality reasons, for any $k \geq 2$ and any eigenfunction associated with $\lambda_k(M, g)$, we have $\mu(u) \geq 2$.

We shall say that $\lambda_k(M, g)$ is *Courant-sharp* if there exists an eigenfunction u, such that $-\Delta_g u = \lambda_k(M, g)u$ and $\mu(u) = k$. Clearly, $\lambda_1(M, g)$ and $\lambda_2(M, g)$ are always Courant-sharp eigenvalues. Note that if $\lambda_3(M, g) = \lambda_2(M, g)$, then $\lambda_3(M, g)$ is not Courant-sharp.

The first results concerning Question 1 were stated by Antonie Stern in her 1924 PhD thesis [36] written under the supervision of R. Courant.

Theorem 1.1. *[A. Stern, [36]]*

1. *For the square* $[0,\pi]\times[0,\pi]$ *with Dirichlet boundary condition, there is a sequence of eigenfunctions* $\{u_r, r\geq 1\}$ *such that*
$$-\Delta u_r = (1+4r^2)u_r\,,\ \textit{and}\ \mu(u_r)=2\,.$$

2. *For the sphere* $\mathbb{S}^2$, *there exists a sequence of eigenfunctions* $u_\ell, \ell\geq 1$ *such that*
$$-\Delta_{\mathbb{S}^2} u_\ell = \ell(\ell+1)u_\ell\,,\ \textit{and}\ \mu(u_\ell)=2\ \textit{or}\ 3\,,$$
depending on whether ℓ *is odd or even.*

Stern's arguments are not fully satisfactory. In 1977, H. Lewy [25] gave a complete independent proof for the case of the sphere, without any reference to [36] (see also [4]). More recently, the authors [2] gave a complete proof for the case of the square with Dirichlet boundary conditions (see also Gauthier-Shalom–Przybytkowski [17]).

The original motivation of this paper was to investigate the possibility to extend Stern's results to the case of the two-dimensional isotropic quantum harmonic oscillator $\widehat{H} := -\Delta + |x|^2$ acting on $L^2(\mathbb{R}^2,\mathbb{R})$ (we will say "harmonic oscillator" for short). After the publication of the first version of this paper [3], T. Hoffmann-Ostenhof informed us of the unpublished master degree thesis of J. Leydold [26].

An orthogonal basis of eigenfunctions of the harmonic oscillator $\widehat{H}$ is given by
$$\phi_{m,n}(x,y) = H_m(x)H_n(y)\exp(-\frac{x^2+y^2}{2})\,, \tag{1.5}$$
for $(m,n)\in\mathbb{N}^2$, where H_n denotes the Hermite polynomial of degree n. For Hermite polynomials, we use the definitions and notation of Szegö [37, §5.5].

The eigenfunction $\phi_{m,n}$ corresponds to the eigenvalue $2(m+n+1)$,
$$\widehat{H}\phi_{m,n} = 2(m+n+1)\,\phi_{m,n}\,. \tag{1.6}$$
It follows that the eigenspace $\mathcal{E}_n$ of $\widehat{H}$ associated with the eigenvalue $\hat{\lambda}(n)=2(n+1)$ has dimension $(n+1)$, and is generated by the eigenfunctions $\phi_{n,0}$, $\phi_{n-1,1}$, …, $\phi_{0,n}$.

We summarize Leydold's main results in the following theorem.

Theorem 1.2. *[J. Leydold, [26]]*

1. *For* $n\geq 2$, *and for any nonzero* $u\in\mathcal{E}_n$,
$$\mu(u)\leq \mu_n^L := \frac{n^2}{2}+2\,. \tag{1.7}$$

2. *The lower bound on the number of nodal domains is given by*
$$\min\{\mu(u)\mid u\in\mathcal{E}_n, u\neq 0\} = \begin{cases} 1 & \textit{if}\ n=0\,,\\ 3 & \textit{if}\ n\equiv 0 \pmod 4\,, n\geq 4\,,\\ 2 & \textit{if}\ n\not\equiv 0 \pmod 4\,. \end{cases} \tag{1.8}$$

Remark. When $n \geq 3$, the estimate (1.7) is better than Courant's bound which is $\mu_n^C := \frac{n^2}{2} + \frac{n}{2} + 1$. The idea of the proof is to apply Courant's method separately to odd and to even eigenfunctions (with respect to the map $x \mapsto -x$). A consequence of (1.7) is that the only Courant-sharp eigenvalues of the harmonic oscillator are the first, second and fourth eigenvalues. The same ideas work for the sphere as well [26, 27]. For the analysis of Courant-sharp eigenvalues of the square with Dirichlet boundary conditions, see [30, 2].

Leydold's proof that there exist eigenfunctions of the harmonic oscillator satisfying (1.8) is quite involved. In this paper, we give a simple proof that, for any odd integer n, there exists a one-parameter family of eigenfunctions with exactly two nodal domains in $\mathcal{E}_n$. More precisely, for $\theta \in [0, \pi[$, we consider the following curve in $\mathcal{E}_n$,

$$\Phi_n^\theta := \cos\theta \, \phi_{n,0} + \sin\theta \, \phi_{0,n} \,, \tag{1.9}$$

$$\Phi_n^\theta(x,y) = (\cos\theta \, H_n(x) + \sin\theta \, H_n(y)) \, \exp(-\frac{x^2+y^2}{2}) \,.$$

We prove the following theorems (Sections 4 and 5).

Theorem 1.3. *Assume that n is odd. Then, there exists an open interval $I_{\frac{\pi}{4}}$ containing $\frac{\pi}{4}$, and an open interval $I_{\frac{3\pi}{4}}$, containing $\frac{3\pi}{4}$, such that for*

$$\theta \in I_{\frac{\pi}{4}} \cup I_{\frac{3\pi}{4}} \setminus \{\frac{\pi}{4}, \frac{3\pi}{4}\} \,,$$

the nodal set $N(\Phi_n^\theta)$ is a connected simple regular curve, and the eigenfunction Φ_n^θ has two nodal domains in $\mathbb{R}^2$.

Theorem 1.4. *Assume that n is odd. Then, there exists $\theta_c > 0$ such that, for $0 < \theta < \theta_c$, the nodal set $N(\Phi_n^\theta)$ is a connected simple regular curve, and the eigenfunction Φ_n^θ has two nodal domains in $\mathbb{R}^2$.*

Remark. The value θ_c and the intervals can be computed numerically. The proofs of the theorems actually show that $]0, \theta_c[\cap I_{\frac{\pi}{4}} = \emptyset$.

Question 2. How good/bad is Courant's upper bound on the number of nodal domains? Consider the eigenvalue problem (1.2). Given $k \geq 1$, define $\mu(k)$ to be the maximum value of $\mu(u)$ when u is an eigenfunction associated with the eigenvalue $\lambda_k(M, g)$. Then,

$$\limsup_{k \mapsto \infty} \frac{\mu(k)}{k} \leq \gamma(n) \tag{1.10}$$

where $\gamma(n)$ is a universal constant which only depends on the dimension n of M. Furthermore, for $n \geq 2$, $\gamma(n) < 1$. The idea of the proof, introduced by Pleijel in 1956, is to use a Faber-Krahn type isoperimetric inequality and Weyl's asymptotic law. Note that the constant $\gamma(n)$ is not sharp. For more details and references, see [30, 14].

As a corollary of the above result, the eigenvalue problem (1.2) has only finitely many Courant-sharp eigenvalues.

The above result gives a quantitative improvement of Courant's theorem in the case of the Dirichlet Laplacian in a bounded open set of $\mathbb{R}^2$. When trying to implement the strategy of Pleijel for the harmonic oscillator, we get into trouble because of the absence of a reasonable Faber-Krahn inequality. P. Charron [9, 10] has obtained the following theorem.

Theorem 1.5. *If (λ_n, u_n) is an infinite sequence of eigenpairs of $\widehat{H}$, then*

$$\limsup \frac{\mu(u_n)}{n} \leq \gamma(2) = \frac{4}{j_{0,1}^2}\,, \tag{1.11}$$

where $j_{0,1}$ is the first positive zero of the Bessel function of order 0.

This is in some sense surprising that the statement is exactly the same as in the case of the Dirichlet realization of the Laplacian in a bounded open set in $\mathbb{R}^2$. The proof does actually not use the isotropy of the harmonic potential, can be extended to the n-dimensional case, but strongly uses the explicit knowledge of the eigenfunctions. We refer to [11] for further results in this direction.

A related question concerning the estimate (1.10) is whether the order of magnitude is correct. In the case of the 2-sphere, using spherical coordinates, one can find decomposed spherical harmonics u_ℓ of degree ℓ, with associated eigenvalue $\ell(\ell+1)$, such that $\mu(u_\ell) \sim \frac{\ell^2}{2}$ when ℓ is large, whereas Courant's upper bound is equivalent to ℓ^2. These spherical harmonics have critical zeros and their nodal sets have self-intersections. In [16, Section 2], the authors construct spherical harmonics v_ℓ, of degree ℓ, without critical zeros i.e., whose nodal sets are disjoint closed regular curves, such that $\mu(v_\ell) \sim \frac{\ell^2}{4}$. These spherical harmonics v_ℓ have as many nodal domains as allowed by Courant's theorem, up to a factor $\frac{1}{4}$. Since the v_ℓ's are regular, this property is stable under small perturbations in the same eigenspace.

In Section 6, in a direction opposite to Theorems 1.3 and 1.4, we construct eigenfunctions of the harmonic oscillator $\widehat{H}$ with "many" nodal domains.

Theorem 1.6. *For the harmonic oscillator $\widehat{H}$ in $L^2(\mathbb{R}^2)$, there exists a sequence of eigenfunctions $\{u_k, k \geq 1\}$ such that $\widehat{H}u_k = \hat{\lambda}(k)u_k$, with $\hat{\lambda}(4k) = 2(4k+1)$, u_k as no critical zeros (i.e. has a regular nodal set), and*

$$\mu(u_k) \sim \frac{(4k)^2}{8}.$$

Remarks.
1. The above estimate is, up to a factor $\frac{1}{4}$, asymptotically the same as the upper bounds for the number of nodal domains given by Courant and Leydold.
2. A related question is to analyze the zero set when θ is a random variable. We refer to [18] for results in this direction. The above questions are related to the question of spectral minimal partitions [19]. In the case of the harmonic oscillator similar questions appear in the analysis of the properties of ultracold atoms (see for example [33]).

Question 3. Consider the eigenvalue problem (1.2), and assume for simplicity that M is a bounded domain in $\mathbb{R}^2$. Fix any small number r, and a point $x \in M$ such that $B(x,r) \subset M$. Let u be a Dirichlet eigenfunction associated with the eigenvalue λ and assume that $\lambda \geq \frac{\pi j_{0,1}^2}{r^2}$. Then $N(u) \cap B(x,r) \neq \emptyset$. This fact follows from the monotonicity of the Dirichlet eigenvalues, and indicates that the length of the nodal set should tend to infinity as the eigenvalue tends to infinity. The first results in this direction are due to Brüning and Gromes [6, 7] who show that the length of the nodal set $N(u)$ is bounded from below by a constant times $\sqrt{\lambda}$. For further results in this direction (Yau's conjecture), we refer to [15, 29, 28, 34, 35]. In Section 7, we investigate this question for the harmonic oscillator.

Theorem 1.7. *Let $\delta \in]0,1[$ be given. Then, there exists a positive constant C_δ such that for λ large enough, and for any nonzero eigenfunction of the isotropic 2D quantum harmonic oscillator,*

$$\widehat{H} := -\Delta + |x|^2\,, \quad \widehat{H}u = \lambda u\,, \tag{1.12}$$

the length of $N(u) \cap B\left(\sqrt{\delta\lambda}\right)$ is bounded from below by $C_\delta\, \lambda^{\frac{3}{2}}$.

As a matter of fact, we prove a lower bound for more general Schrödinger operators in $\mathbb{R}^2$ (Propositions 7.2 and 7.8), shedding some light on the exponent $\frac{3}{2}$ in the above estimate. In Section 7.3, we investigate upper and lower bounds for the length of the nodal sets, using the method of Long Jin [29].

Acknowledgements. The authors would like to thank P. Charron for providing an earlier copy of [9], T. Hoffmann-Ostenhof for pointing out [26], J. Leydold for providing a copy of his master degree thesis [26], Long Jin for enlightening discussions concerning [29] and Section 7, I. Polterovich for pointing out [9], as well as D. Jakobson and M. Persson-Sundqvist for useful comments. During the preparation of this paper, B. H. was Simons foundation visiting fellow at the Isaac Newton Institute in Cambridge. Finally, the authors would like to thank the referee for his remarks which helped improve the paper.

2 A reminder on Hermite polynomials

We use the definition, normalization, and notation of Szegö's book [37]. With these choices, the Hermite polynomial H_n has the following properties, [37, § 5.5 and Theorem 6.32].

1. H_n satisfies the differential equation

$$y''(t) - 2t\, y'(t) + 2n\, y(t) = 0\,.$$

2. H_n is a polynomial of degree n which is even (*resp.* odd) for n even (*resp.* odd).

3. $H_n(t) = 2t\, H_{n-1}(t) - 2(n-1)\, H_{n-2}(t)\,,\ n \geq 2\,,\quad H_0(t) = 1\,,\quad H_1(t) = 2t\,.$

4. H_n has n simple zeros $t_{n,1} < t_{n,2} < \cdots < t_{n,n}$.

5. $$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t)\,.$$

6. $$H'_n(t) = 2nH_{n-1}(t)\,. \tag{2.1}$$

7. The coefficient of t^n in H_n is 2^n.

8. $$\int_{-\infty}^{+\infty} e^{-t^2}|H_n(t)|^2\,dt = \pi^{\frac{1}{2}}\,2^n\,n!\,.$$

9. The first zero $t_{n,1}$ of H_n satisfies
$$t_{n,1} = (2n+1)^{\frac{1}{2}} - 6^{-\frac{1}{2}}(2n+1)^{-\frac{1}{6}}(i_1 + \epsilon_n)\,, \tag{2.2}$$
where i_1 is the first positive real zero of the Airy function, and $\lim_{n\to+\infty}\epsilon_n = 0$.

The following result (Theorem 7.6.1 in Szegö's book [37]) will also be useful.

Lemma 2.1. *The successive relative maxima of* $t \mapsto |H_n(t)|$ *form an increasing sequence for* $t \geq 0$ *.*

Proof.
It is enough to observe that the function
$$\Theta_n(t) := 2nH_n(t)^2 + H'_n(t)^2$$
satisfies
$$\Theta'_n(t) = 4t\,(H'_n(t))^2\,.$$
□

3 Stern-like constructions for the harmonic oscillator in the case n-odd

3.1 The case of the square

Consider the square $[0,\pi]^2$, with Dirichlet boundary conditions, and the following families of eigenfunctions associated with the eigenvalues $\hat{\lambda}(1,2r) := 1+4r^2$, where r is a positive integer, and $\theta \in [0,\pi/4]$,
$$(x,y) \mapsto \cos\theta\,\sin x\,\sin(2ry) + \sin\theta\,\sin(2rx)\,\sin y\,.$$

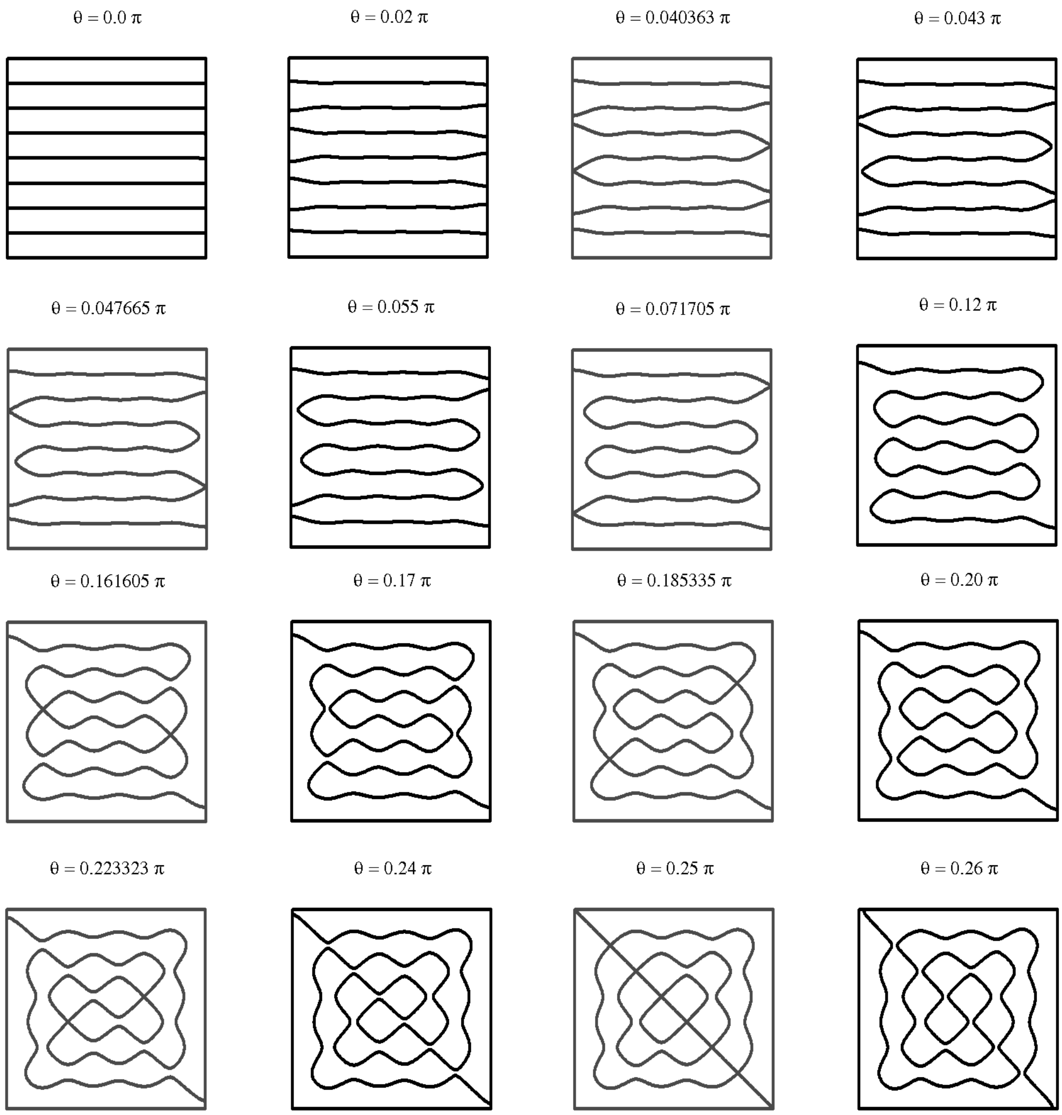

Figure 1: Nodal sets for the Dirichlet eigenvalue $\hat{\lambda}(1,8)$ of the square.

According to [36], for any given $r \geq 1$, the typical evolution of the nodal sets when θ varies is similar to the case $r = 4$ shown in Figure 1 [2, Figure 6.9]. Generally speaking, the nodal sets deform continuously, except for *finitely many* values of θ, for which self-intersections of the nodal set appear or disappear or, equivalently, for which critical zeros of the eigenfunction appear/disappear.

We would like to get similar results for the isotropic quantum harmonic oscillator.

3.2 Symmetries

Recall the notation,

$$\Phi_n^\theta(x,y) := \cos\theta\,\phi_{n,0}(x,y) + \sin\theta\,\phi_{0,n}(x,y)\,. \tag{3.1}$$

More simply,

$$\Phi_n^\theta(x,y) = \exp\left(-\frac{x^2+y^2}{2}\right)\,(\cos\theta\,H_n(x) + \sin\theta\,H_n(y))\ .$$

Since $\Phi_n^{\theta+\pi} = -\Phi_n^\theta$, it suffices to vary the parameter θ in the interval $[0,\pi[$.

Assuming n is odd, we have the following symmetries.

$$\begin{cases} \Phi_n^\theta(-x,y) &= \Phi_n^{\pi-\theta}(x,y)\,, \\ \Phi_n^\theta(x,-y) &= -\Phi_n^{\pi-\theta}(x,y)\,, \\ \Phi_n^\theta(y,x) &= \Phi_n^{\frac{\pi}{2}-\theta}(x,y)\,. \end{cases} \tag{3.2}$$

When n is odd, it therefore suffices to vary the parameter θ in the interval $[0,\frac{\pi}{4}]$. The case $\theta = 0$ is particular, so that we shall mainly consider $\theta \in]0,\frac{\pi}{4}]$.

3.3 Critical zeros

A *critical zero* of Φ_n^θ is a point $(x,y) \in \mathbb{R}^2$ such that both Φ_n^θ and its differential $d\Phi_n^\theta$ vanish at (x,y). The critical zeros of Φ_n^θ satisfy the following equations.

$$\begin{cases} \cos\theta\,H_n(x) + \sin\theta\,H_n(y) &= 0\,, \\ \cos\theta\,H_n'(x) &= 0\,, \\ \sin\theta\,H_n'(y) &= 0\,. \end{cases} \tag{3.3}$$

Equivalently, using the properties of the Hermite polynomials, a point (x,y) is a critical zero of Φ_n^θ if and only if

$$\begin{cases} \cos\theta\,H_n(x) + \sin\theta\,H_n(y) &= 0\,, \\ \cos\theta\,H_{n-1}(x) &= 0\,, \\ \sin\theta\,H_{n-1}(y) &= 0\,. \end{cases} \tag{3.4}$$

The only *possible* critical zeros of the eigenfunction Φ_n^θ are the points $(t_{n-1,i}\,,t_{n-1,j})$ for $1 \le i,j \le (n-1)$, where the coordinates are the zeros of the Hermite polynomial H_{n-1}. The point $(t_{n-1,i}\,,t_{n-1,j})$ is a critical zero of Φ_n^θ if and only if $\theta = \theta(i,j)$, where $\theta(i,j) \in]0,\pi[$ is uniquely determined by the equation,

$$\cos(\theta(i,j))\;H_n(t_{n-1,i}) + \sin(\theta(i,j))\;H_n(t_{n-1,j}) = 0\,. \tag{3.5}$$

The values $\theta(i,j)$ will be called *critical values* of the parameter θ, the other values *regular values*. Here we have used the fact that H_n and H_n' have no common zeros. We have proved the following lemma.

Lemma 3.1. *For $\theta \in [0, \pi[$, the eigenfunction Φ_n^θ has no critical zero, unless θ is one of the critical values $\theta(i,j)$ defined by equation (3.5). In particular Φ_n^θ has no critical zero, except for finitely many values of the parameter $\theta \in [0, \pi[$. Given a pair $(i_0, j_0) \in \{1, \ldots, n-1\}$, let $\theta_0 = \theta(i_0, j_0)$, be defined by (3.5) for the pair $(t_{n-1,i_0}, t_{n-1,j_0})$. Then, the function $\Phi_n^{\theta_0}$ has finitely many critical zeros, namely the points $(t_{n-1,i}, t_{n-1,j})$ which satisfy*

$$\cos\theta_0 \, H_n(t_{n-1,i}) + \sin\theta_0 \, H_n(t_{n-1,j}) = 0 \,, \tag{3.6}$$

among them the point $(t_{n-1,i_0}, t_{n-1,j_0})$.

Remarks.
From the general properties of nodal lines [2, Properties 5.2], we derive the following facts.

1. When $\theta \notin \{\theta(i,j) \mid 1 \le i,j \le n-1\}$, the nodal set $N(\Phi_n^\theta)$ of the eigenfunction Φ_n^θ, is a smooth 1-dimensional submanifold of $\mathbb{R}^2$ (a collection of pairwise distinct connected simple regular curves).

2. When $\theta \in \{\theta(i,j) \mid 1 \le i,j \le n-1\}$, the nodal set $N(\Phi_n^\theta)$ has finitely many singularities which are double crossings[1]. Indeed, the Hessian of the function Φ_n^θ at a *critical zero* $(t_{n-1,i}, t_{n-1,j})$ is given by

$$\mathrm{Hess}_{(t_{n-1,i},t_{n-1,j})}\Phi_n^\theta = \exp\left(-\frac{t_{n-1,i}^2 + t_{n-1,j}^2}{2}\right) \begin{pmatrix} \cos\theta \, H_n''(t_{n-1,i}) & 0 \\ 0 & \sin\theta \, H_n''(t_{n-1,j}) \end{pmatrix} ,$$

and the assertion follows from the fact that H_{n-1} has simple zeros.

3.4 General properties of the nodal set $N(\Phi_n^\theta)$

Denote by $\mathcal{L}$ the finite lattice

$$\mathcal{L} := \{(t_{n,i}, t_{n,j}) \mid 1 \le i,j \le n\} \subset \mathbb{R}^2 \,, \tag{3.7}$$

consisting of points whose coordinates are the zeros of the Hermite polynomial H_n.

The horizontal and vertical lines $\{y = t_{n,i}\}$ and $\{x = t_{n,j}\}$, $1 \le i,j \le n$, form a checkerboard like pattern in $\mathbb{R}^2$ which can be colored according to the sign of the function $H_n(x)\,H_n(y)$ (grey where the function is positive, white where it is negative). We will refer to the following properties as *the checkerboard argument*, compare with [36, 2].

For symmetry reasons, we can assume that $\theta \in]0, \frac{\pi}{4}]$.

(i) We have the following inclusions for the nodal sets $N(\Phi_n^\theta)$,

$$\mathcal{L} \subset N(\Phi_n^\theta) \subset \mathcal{L} \cup \left\{(x,y) \in \mathbb{R}^2 \mid H_n(x)\,H_n(y) < 0\right\} . \tag{3.8}$$

(ii) The nodal set $N(\Phi_n^\theta)$ does not meet the vertical lines $\{x = t_{n,i}\}$, or the horizontal lines $\{y = t_{n,i}\}$ away from the set $\mathcal{L}$.

[1] This result is actually general for any eigenfunction of the harmonic oscillator, as stated in [26], on the basis of Euler's formula and Courant's theorem.

(iii) The lattice point $(t_{n,i}, t_{n,j})$ is not a critical zero of Φ_n^θ (because H_n and H_n' have no common zero). As a matter of fact, near a lattice point, the nodal set $N(\Phi_n^\theta)$ is a single arc through the lattice point, with a tangent which is neither horizontal, nor vertical.

Figure 2 shows the evolution of the nodal set of Φ_n^θ, for $n = 7$, when θ varies in the interval $]0, \frac{\pi}{4}]$. The pictures in the first column correspond to regular values of θ whereas the pictures in the second column correspond to critical values of θ. The form of the nodal set is stable in the open interval between two consecutive critical values of the parameter θ. In the figures, the thick curves represent the nodal sets $N(\Phi_7^\theta)$, the thin lines correspond to the zeros of H_7, and the grey lines to the zeros of H_7', i.e. to the zeros of H_6.

We now describe the nodal set $N(\Phi_n^\theta)$ outside a large enough square which contains the lattice $\mathcal{L}$. For this purpose, we give the following two *barrier lemmas* which describe the intersections of the nodal set with horizontal and vertical lines.

Lemma 3.2. *Assume that $\theta \in]0, \frac{\pi}{4}]$. For n odd, define $t_{n-1,0}$ to be the unique point in $]-\infty, t_{n,1}[$ such that $H_n(t_{n-1,0}) = -H_n(t_{n-1,1})$. Then,*

1. *$\forall t \le t_{n,1}$, the function $y \mapsto \Phi_n^\theta(t,y)$ has exactly one zero in the interval $[t_{n,n}, +\infty[$;*

2. *$\forall t < t_{n-1,0}$, the function $y \mapsto \Phi_n^\theta(t,y)$ has exactly one zero in the interval $]-\infty, +\infty[$.*

Using the symmetry with respect to the vertical line $\{x = 0\}$, one has similar statements for $t \ge t_{n,n}$ and for $t > -t_{n-1,0}$.

Proof.
Let $v(y) := \exp(\frac{t^2+y^2}{2})\, \Phi_n^\theta(t,y)$. In $]t_{n,n}, +\infty[$, $v'(y)$ is positive, and $v(t_{n,n}) \le 0$. The first assertion follows. The local extrema of v occur at the points $t_{n-1,j}$, for $1 \le j \le (n-1)$. The second assertion follows from the definition of $t_{n-1,0}$, and from the inequalities,

$$\begin{aligned}\cos\theta\, H_n(t) + \sin\theta\, H_n(t_{n-1,j}) &\le \frac{1}{\sqrt{2}}\Big(H_n(t) + |H_n|(t_{n-1,j})\Big)\\ &< -\frac{1}{\sqrt{2}}\Big(H_n(t_{n-1,1}) - |H_n|(t_{n-1,j})\Big) \le 0\,,\end{aligned}$$

for $t < t_{n-1,0}$, where we have used Lemma 2.1. □

Lemma 3.3. *Let $\theta \in]0, \frac{\pi}{4}]$. Define $t_{n-1,n}^\theta \in]t_{n,n}, \infty[$ to be the unique point such that $\tan\theta\, H_n(t_{n-1,n}^\theta) = H_n(t_{n-1,1})$. Then,*

1. *$\forall t \ge t_{n,n}$, the function $x \mapsto \Phi_n^\theta(x,t)$ has exactly one zero in the interval $]-\infty, t_{n,1}]$;*

2. *$\forall t > t_{n-1,n}^\theta$, the function $x \mapsto \Phi_n^\theta(x,t)$ has exactly one zero in the interval $]-\infty, \infty[$.*

3. *For $\theta_2 > \theta_1$, we have $t_{n-1,n}^{\theta_2} < t_{n-1,n}^{\theta_1}$.*

Using the symmetry with respect to the horizontal line $\{y = 0\}$, one has similar statements for $t \le t_{n,1}$ and for $t < -t_{n-1,n}^\theta$.

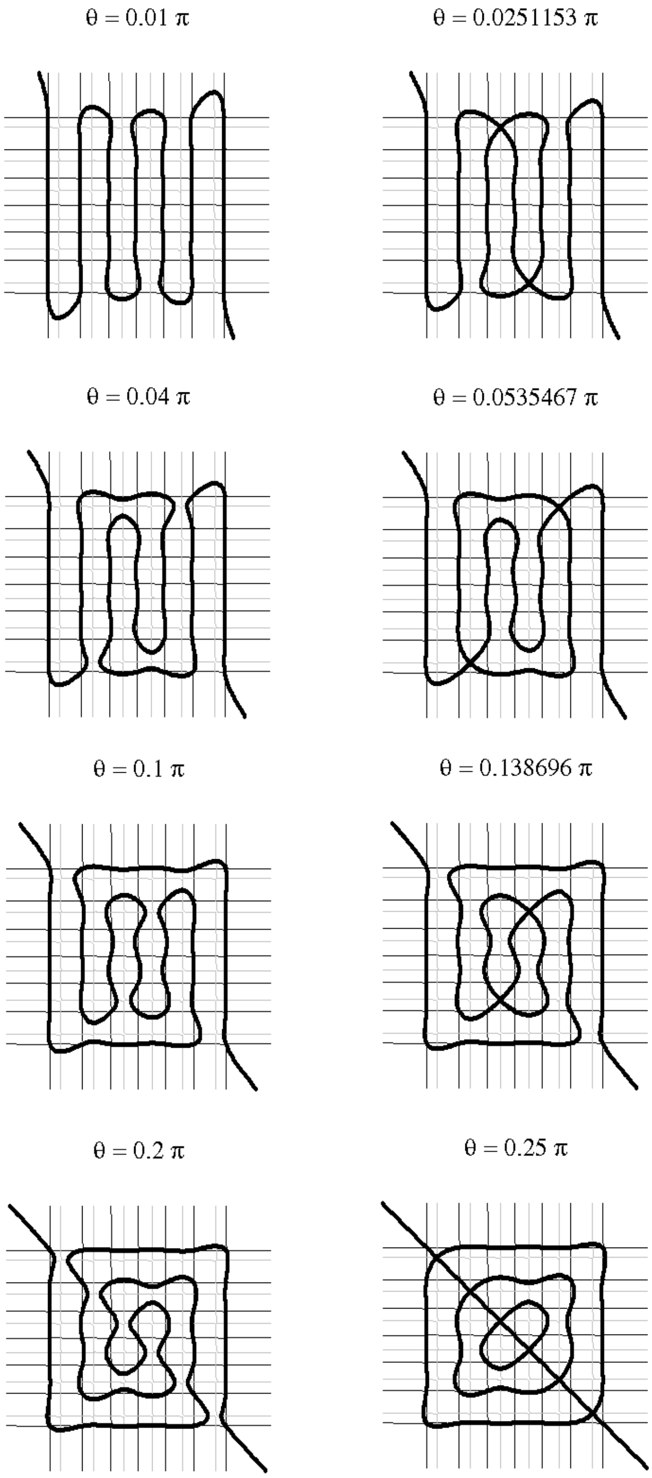

Figure 2: Evolution of the nodal set $N(\Phi_n^\theta)$, for $n = 7$ and $\theta \in]0, \frac{\pi}{4}]$.

Proof. Let $h(x) := \exp(\frac{x^2+t^2}{2})\, \Phi_n^\theta(x,t)$. In the interval $]-\infty, t_{n,1}]$, the derivative $h'(x)$ is positive, $h(t_{n,1}) > 0$, and $\lim_{x\to -\infty} h(x) = -\infty$, since n is odd. The first assertion follows. The local extrema of h are achieved at the points $t_{n-1,j}$. Using Lemma 2.1, for $t \geq t^{\theta}_{n-1,n}$, we have the inequalities,

$$\begin{aligned} H_n(t_{n-1,j}) + \tan\theta\, H_n(t) &\geq \tan\theta\, H_n(t^{\theta}_{n-1,n}) - |H_n(t_{n-1,j})| \\ &= H_n(t_{n-1,1}) - |H_n(t_{n-1,j})| \geq 0\,. \end{aligned}$$

□

As a consequence of the above lemmas, we have the following description of the nodal set far enough from $(0,0)$.

Proposition 3.4. *Let $\theta \in]0, \frac{\pi}{4}]$. In the set $\mathbb{R}^2 \backslash]-t^{\theta}_{n-1,n}, t^{\theta}_{n-1,n}[\times]t_{n-1,0}, |t_{n-1,0}|[$, the nodal set $N(\Phi^{\theta}_n)$ consists of two regular arcs. The first arc is a graph $y(x)$ over the interval $]-\infty, t_{n,1}]$, starting from the point $(t_{n,1}, t_{n,n})$ and escaping to infinity with,*

$$\lim_{x \to -\infty} \frac{y(x)}{x} = -\sqrt[n]{\cot\theta}\,.$$

The second arc is the image of the first one under the symmetry with respect to $(0,0)$ in $\mathbb{R}^2$.

3.5 Local nodal patterns

As in the case of the Dirichlet eigenvalues for the square, we study the possible local nodal patterns taking into account the fact that the nodal set contains the lattice points $\mathcal{L}$, can only visit the connected components of the set $\{H_n(x)\, H_n(y) < 0\}$ (colored white), and consists of a simple arc at the lattice points. The following figure summarized the possible nodal patterns in the interior of the square [2, Figure 6.4],

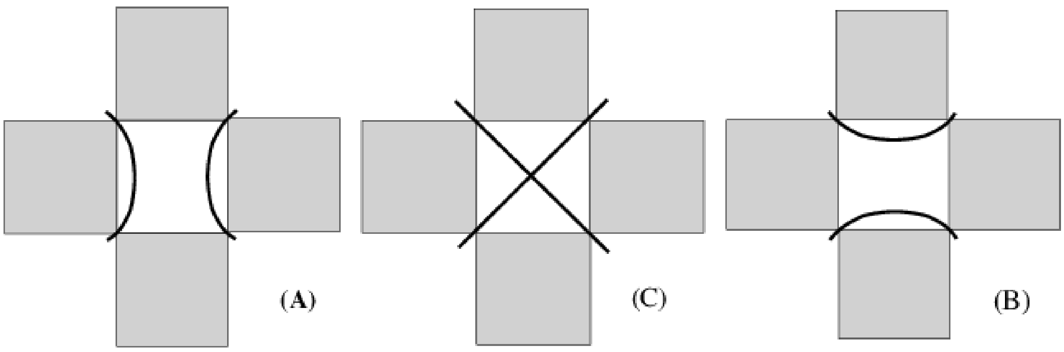

Figure 3: Local nodal patterns for Dirichlet eigenfunctions of the square.

Except for nodal arcs which escape to infinity, the local nodal patterns for the quantum harmonic oscillator are similar (note that in the present case, the connected components of the set $\{H_n(x)\, H_n(y) < 0\}$ are rectangles, no longer equal squares). The checkerboard argument and the location of the possible critical zeros determine the possible local patterns: (A), (B) or (C). Case (C) occurs near a critical zero. Following the same ideas as in the case of the square, in order to decide between cases (A) and (B), we use the barrier lemmas, Lemma 3.2 or 3.3, the vertical lines $\{x = t_{n-1,j}\}$, or the horizontal lines $\{y = t_{n-1,j}\}$.

4 Proof of Theorem 1.3

Note that

$$\phi_{n,0}(x,y) - \phi_{0,n}(x,y) = -\sqrt{2}\,\Phi_n^{\frac{3\pi}{4}}(x,y) = -\sqrt{2}\Phi_n^{\frac{\pi}{4}}(x,-y)\,.$$

Hence, up to symmetry, it is the same to work with $\theta = \frac{\pi}{4}$ and the anti-diagonal, or to work with $\theta = \frac{3\pi}{4}$ and the diagonal. For notational convenience, we work with $\frac{3\pi}{4}$.

4.1 The nodal set of $\Phi_n^{\frac{3\pi}{4}}$

The purpose of this section is to prove the following result which is the starting point for the proof of Theorem 1.3.

Proposition 4.1. *Let $\{t_{n-1,i}\,, 1 \leq i \leq n-1\}$ denote the zeroes of H_{n-1}. For n odd, the nodal set of $\phi_{n,0} - \phi_{0,n}$ consists of the diagonal $x = y$, and of $\frac{n-1}{2}$ disjoint simple closed curves crossing the diagonal at the $(n-1)$ points $(t_{n-1,i}\,, t_{n-1,i})$, and the anti-diagonal at the $(n-1)$ points $(t_{n,i}\,, -t_{n,i})$.*

To prove Proposition 4.1, we first observe that it is enough to analyze the zero set of

$$(x,y) \mapsto \Psi_n(x,y) := H_n(x) - H_n(y)\,.$$

4.1.1 Critical zeros

The only possible critical zeros of Ψ_n are determined by

$$H_n'(x) = 0\,,\; H_n'(y) = 0\,.$$

Hence, they consist of the $(n-1)^2$ points $(t_{n-1,i}\,, t_{n-1,j})$, for $1 \leq i, j \leq (n-1)$, where $t_{n-1,i}$ is the i-th zero of the polynomial H_{n-1} .

The zero set of Ψ_n contains the diagonal $\{x = y\}$. Since n is odd, there are only n points belonging to the zero set on the anti-diagonal $\{x + y = 0\}$.

On the diagonal, there are $(n-1)$ critical points. We claim that there are no critical zeros outside the diagonal. Indeed, let $(t_{n-1,i}\,, t_{n-1,j})$ be a critical zero. Then, $H_n(t_{n-1,i}) = H_n(t_{n-1,j})$. Using Lemma 2.1 and the parity properties of Hermite polynomials, we see that $|H_n(t_{n-1,i})| = |H_n(t_{n-1,j})|$ occurs if and only if $t_{n-1,i} = \pm t_{n-1,j}$. Since n is odd, we can conclude that $H_n(t_{n-1,i}) = H_n(t_{n-1,j})$ occurs if and only if $t_{n-1,i} = t_{n-1,j}$.

4.1.2 Existence of disjoint simple closed curves in the nodal set of $\Phi_n^{\frac{3\pi}{4}}$

The second part in the proof of the proposition follows closely the proof in the case of the Dirichlet Laplacian for the square (see Section 5 in [2]). Essentially, the Chebyshev polynomials are replaced by the Hermite polynomials. Note however that the checkerboard is no more with equal squares, and that the square $[0,\pi]^2$ has to be replaced in the argument by the rectangle $[t_{n-1,0}, -t_{n-1,0}] \times [-t^{\theta}_{n-1,n}, t^{\theta}_{n-1,n}]$, for some $\theta \in]0, \frac{3\pi}{4}[$, see Lemmas 3.2 and 3.3 .

The checkerboard argument holds, see (3.8) and the properties at the beginning of Section 3.4.

The separation lemmas of our previous paper [2] must be substituted by Lemmas 3.2 and 3.3 , and similar statements with the lines $\{x = t_{n-1,j}\}$ and $\{y = t_{n-1,j}\}$, for $1 \leq j \leq (n-1)$.

One needs to control what is going on at infinity. As a matter of fact, outside a specific rectangle centered at the origin, the zero set is the diagonal $\{x = y\}$, see Proposition 3.4 .

Hence in this way (like for the square), we obtain that the nodal set of Ψ_n consists of the diagonal and $\frac{n-1}{2}$ disjoint simple closed curves turning around the origin. The set $\mathcal{L}$ is contained in the union of these closed curves.

4.1.3 No other closed curve in the nodal set of $\Phi_n^{\frac{3\pi}{4}}$

It remains to show that there are no other closed curves which do not cross the diagonal. The "energy" considerations of our previous papers [2, 4] work here as well. Here is a simple alternative argument.

We look at the line $y = \alpha\, x$ for some $\alpha \neq 1$. The intersection of the zero set with this line corresponds to the zeroes of the polynomial $x \mapsto H_n(x) - H_n(\alpha\, x)$ which has at most n zeroes. But in our previous construction, we get at least n zeroes. So the presence of extra curves would lead to a contradiction for some α. This argument solves the problem at infinity as well. □

4.2 Perturbation argument

Figure 4 shows the desingularization of the nodal set $N(\Phi_n^{\frac{3\pi}{4}})$, from below and from above. The picture is the same as in the case of the square (see Figure 1), all the critical points disappear at the same time and in the same manner, i.e. all the double crossings open up horizontally or vertically depending whether θ is less than or bigger than $\frac{3\pi}{4}$.

As in the case of the square, in order to show that the nodal set can be desingularized under small perturbation, we look at the signs of the eigenfunction $\Phi_n^{\frac{3\pi}{4}}$ near the critical zeros. We use the cases (I) and (II) which appear in Figure 5 below (see also [2, Figure 6.7]).

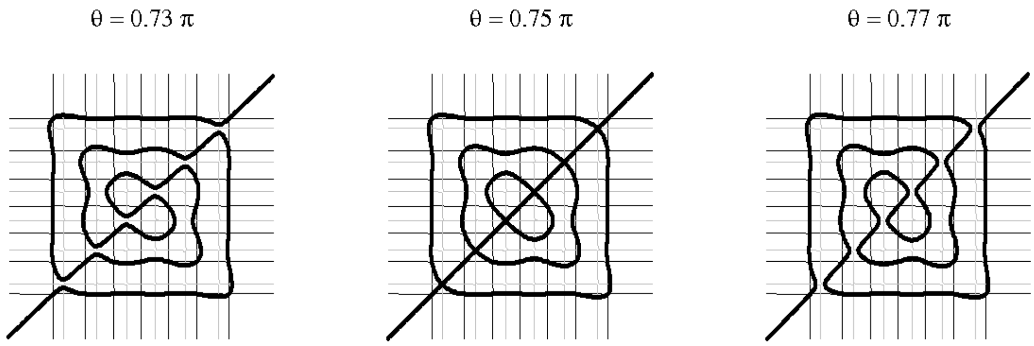

Figure 4: The nodal set of $N(\Phi_n^\theta)$ near $\frac{3\pi}{4}$ (here $n = 7$).

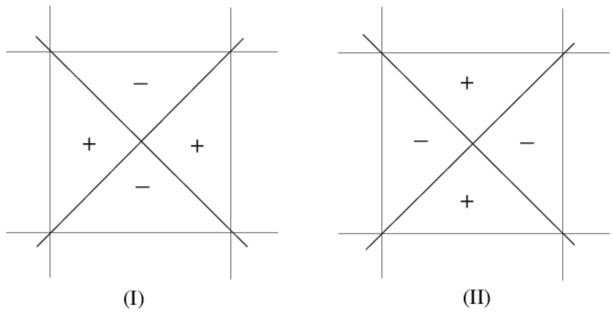

Figure 5: Signs near a critical zero.

The sign configuration for $\phi_{n,0}(x, y) - \phi_{0,n}(x, y)$ near the critical zero $(t_{n-1,i}, t_{n-1,i})$ is given by Figure 5

$$\begin{cases} \text{case (I), if } i \text{ is even,} \\ \text{case (II), if } i \text{ is odd.} \end{cases}$$

Looking at the intersection of the nodal set with the vertical line $\{y = t_{n-1,i}\}$, we have that

$$(-1)^i \left(H_n(t) - H_n(t_{n-1,i})\right) \geq 0\,, \text{ for } t \in]t_{n,i}, t_{n,i+1}[\,.$$

For positive ϵ small, we write

$$(-1)^i \left(H_n(t) - (1+\epsilon)H_n(t_{n-1,i})\right) = (-1)^i \left(H_n(t) - H_n(t_{n-1,i})\right) + \epsilon(-1)^{i+1}\, H_n(t_{n-1,i})\,,$$

so that

$$(-1)^i \left(H_n(t) - (1+\epsilon)H_n(t_{n-1,i})\right) \geq 0\,, \text{ for } t \in]t_{n,i}, t_{n,i+1}[\,.$$

A similar statement can be written for horizontal line $\{x = t_{n-1,i}\}$ and $-\epsilon$, with $\epsilon > 0$, small enough. These inequalities describe how the crossings all open up at the same time, and in the same manner, vertically (case I) or horizontally (case II), see Figure 6, as in the case of the square [2, Figure 6.8].

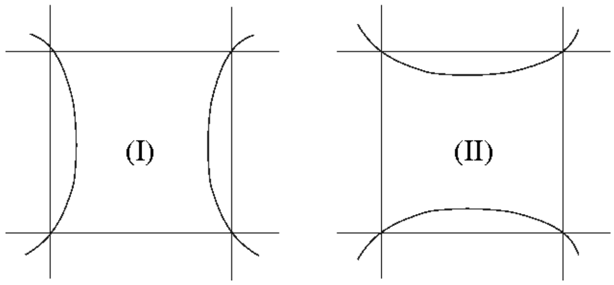

Figure 6: Desingularization at a critical zero.

We can then conclude as in the case of the square, using the local nodal patterns, Section 3.5.

Remark. Because the local nodal patterns can only change when θ passes through one of the values $\theta(i,j)$ defined in (3.5), the above arguments work for $\theta \in J \setminus \{\frac{3\pi}{4}\}$, for any interval J containing $\frac{3\pi}{4}$ and no other critical value $\theta(i,j)$.

5 Proof of Theorem 1.4

Proposition 5.1. *The conclusion of Theorem 1.4 holds with*

$$\theta_c := \inf \{\theta(i,j) \mid 1 \le i,j \le n-1\} \,, \tag{5.1}$$

where the critical values $\theta(i,j)$ are defined by (3.5)*.*

Proof. The proof consists in the following steps. For simplicity, we call N the nodal set $N(\Phi_n^\theta)$.

- Step 1. By Proposition 3.4, the structure of the nodal set N is known outside a large coordinate rectangle centered at $(0,0)$ whose sides are defined by the ad hoc numbers in Lemmas 3.2 and 3.3. Notice that the sides of the rectangle serve as barriers for the arguments using the local nodal patterns as in our paper for the square.

- Step 2. For $1 \le j \le n-1$, the line $\{x = t_{n-1,j}\}$ intersects the set N at exactly one point $(t_{n-1,j}, y_j)$, with $y_j > t_{n,n}$ when j is odd, *resp.* with $y_j < t_{n,1}$ when j is even. The proof is given below, and is similar to the proofs of Lemmas 3.2 or 3.3 .

- Step 3. Any connected component of N has at least one point in common with the set $\mathcal{L}$. This follows from the argument with $y = \alpha x$ or from the energy argument (see Subsection 4.1.3).

- Step 4. Follow the nodal set from the point $(t_{n,1}, t_{n,n})$ to the point $(t_{n,n}, t_{n,1})$, using the analysis of the local nodal patterns as in the case of the square.

Proof of Step 2. For $1 \leq j \leq (n-1)$, define the function v_j by

$$v_j(y) := \cos\theta\, H_n(t_{n-1,j}) + \sin\theta\, H_n(y)\,.$$

The local extrema of v_j are achieved at the points $t_{n-1,i}$, for $1 \leq i \leq (n-1)$, and we have

$$v_j(t_{n-1,i}) = \cos\theta\, H_n(t_{n-1,j}) + \sin\theta\, H_n(t_{n-1,i})\,,$$

which can be rewritten, using (3.5), as

$$v_j(t_{n-1,i}) = \frac{H_n(t_{n-1,j})}{\sin\theta(j,i)}\, \sin\left(\theta(j,i) - \theta\right).$$

The first term in the right-hand side has the sign of $(-1)^{j+1}$ and the second term is positive provided that $0 < \theta < \theta_c$. Under this last assumption, we have

$$(-1)^{j+1}\, v_j(t_{n-1,i}) > 0\,,\ \forall i\,,\ 1 \leq i \leq (n-1)\,. \tag{5.2}$$

The assertion follows. $\square$

6 Eigenfunctions with "many" nodal domains, proof of Theorem 1.6

This section is devoted to the proof of Theorem 1.6 i.e., to the constructions of eigenfunctions of $\widehat{H}$ with regular nodal sets (no self-intersections) and "many" nodal domains. We work in polar coordinates. An orthogonal basis of $\mathcal{E}_\ell$ is given by the functions $\Omega^{\pm}_{\ell,n}$,

$$\Omega^{\pm}_{\ell,n}(r,\varphi) = \exp(-\frac{r^2}{2})\, r^{\ell-2n}\, L_n^{(\ell-2n)}(r^2)\, \exp\left(\pm i(\ell-2n)\varphi\right)\,, \tag{6.1}$$

with $0 \leq n \leq \left[\frac{\ell}{2}\right]$, see [26, Section 2.1]. In this formula $L_n^{(\alpha)}$ is the generalized Laguerre polynomial of degree n and parameter α, see [37, Chapter 5]. Recall that the Laguerre polynomial L_n is the polynomial $L_n^{(0)}$.

Assumption 6.1. *From now on, we assume that $\ell = 4k$, with k even.*

Since ℓ is even, we have a rotation invariant eigenfunction $\exp(-\frac{r^2}{2})\, L_{2k}(r^2)$ which has $(2k+1)$ nodal domains. We also look at the eigenfunctions $\omega_{\ell,n}$,

$$\omega_{\ell,n}(r,\varphi) = \exp(-\frac{r^2}{2})\, r^{\ell-2n}\, L_n^{(\ell-2n)}(r^2)\, \sin\left((\ell-2n)\varphi\right)\,, \tag{6.2}$$

with $0 \le n < \left[\frac{\ell}{2}\right]$.
The number of nodal domains of these eigenfunctions is $\mu(\omega_{\ell,n}) = 2(n+1)(\ell-2n)$, because the Laguerre polynomial of degree n has n simple positive roots. When $\ell = 4k$, the largest of these numbers is

$$\mu_\ell := 4k(k+1)\,, \tag{6.3}$$

and this is achieved for $n = k$.

When k tends to infinity, we have $\mu_\ell \sim \frac{\ell^2}{4}$, the same order of magnitude as Leydold's upper bound $\mu_\ell^L \sim \frac{\ell^2}{2}$.

We want now to construct eigenfunctions $u_k \in \mathcal{E}_\ell\,, \ell = 4k$ with *regular* nodal sets, and "many" nodal domains (or equivalently, "many" nodal connected components), more precisely with $\mu(u_k) \sim \frac{\ell^2}{8}$.

The construction consists of the following steps.

1. Choose $A \in \mathcal{E}_\ell$ such that $\mu(A) = \mu_\ell$.
2. Choose $B \in \mathcal{E}_\ell$ such that for a small enough, the perturbed eigenfunction $F_a := A + a\,B$ has no critical zero except the origin, and a nodal set with many components. Fix such an a.
3. Choose $C \in \mathcal{E}_\ell$ such that for b small enough (and a fixed), $G_{a,b} := A + a\,B + b\,C$ has no critical zero.

From now on, we fix some ϵ, $0 < \epsilon < 1$. We assume that a is positive (to be chosen small enough later on), and that b is non zero (to be chosen small enough, either positive or negative later on).

In the remaining part of this section, we skip the exponential factor in the eigenfunctions since it is irrelevant to study the nodal sets.

Under Assumption 6.1, define

$$A(r,\varphi) := r^{2k}\, L_k^{(2k)}(r^2)\, \sin(2k\varphi)\,,\ B(r,\varphi) := r^{4k}\, \sin(4k\varphi - \epsilon\pi)\,,\ C(r,\varphi) := L_{2k}(r^2)\,. \tag{6.4}$$

We consider the deformations $F_a = A + a\,B$ and $G_{a,b} = A + a\,B + b\,C$. Both functions are invariant under the rotation of angle $\frac{\pi}{k}$, so that we can restrict to $\varphi \in [0, \frac{\pi}{k}]$.

For later purposes, we introduce the angles $\varphi_j = \frac{j\pi}{k}$, for $0 \le j \le 4k-1$, and $\psi_m = \frac{(m+\epsilon)\pi}{4k}$, for $0 \le m \le 8k-1$. We denote by $t_i, 1 \le i \le k$ the zeros of $L_k^{(2k)}$, listed in increasing order. They are simple and positive, so that the numbers $r_i = \sqrt{t_i}$ are well defined. For notational convenience, we denote by $\dot{L}_k^{(2k)}$ the derivative of the polynomial $L_k^{(2k)}$. This polynomial has $(k-1)$ simple zeros, which we denote by $t_i', 1 \le i \le k-1$, with $t_i < t_i' < t_{i+1}$. We define $r_i' := \sqrt{t_i'}$.

6.1 Critical zeros

Clearly, the origin is a critical zero of the eigenfunction $F_a = A + a\,B$, while $G_{a,b} = A + a\,B + b\,C$ does not vanish at the origin.

Away from the origin, the critical zeros of F_a are given by the system

$$\begin{array}{ll} F_a(r,\varphi) & = 0\,, \\ \partial_r F_a(r,\varphi) & = 0\,, \\ \partial_\varphi F_a(r,\varphi) & = 0\,. \end{array} \tag{6.5}$$

The first and second conditions imply that a critical zero (r,φ) satisfies

$$\sin(2k\varphi)\,\sin(4k\varphi - \epsilon\pi)\,\Big(kL_k^{(2k)}(r^2) - r^2\dot{L}_k^{(2k)}(r^2)\Big) = 0\,, \tag{6.6}$$

where $\dot{L}$ is the derivative of the polynomial L.

The first and third conditions imply that a critical zero (r,φ) satisfies

$$2\sin(2k\varphi)\,\cos(4k\varphi - \epsilon\pi) - \cos(2k\varphi)\,\sin(4k\varphi - \epsilon\pi) = 0\,. \tag{6.7}$$

It is easy to deduce from (6.5) that when (r,φ) is a critical zero, $\sin(2k\varphi)\,\sin(4k\varphi - \epsilon\pi) \neq 0$. It follows that, away from the origin, a critical zero (r,φ) of F_a satisfies the system

$$\begin{array}{l} kL_k^{(2k)}(r^2) - r^2\dot{L}_k^{(2k)}(r^2) = 0\,, \\ 2\sin(2k\varphi)\,\cos(4k\varphi - \epsilon\pi) - \cos(2k\varphi)\,\sin(4k\varphi - \epsilon\pi) = 0\,. \end{array} \tag{6.8}$$

The first equation has precisely $(k-1)$ positive simple zeros $r_{c,i}$, one in each interval $]r_i, r'_i[$, for $1 \le i \le (k-1)$. An easy analysis of the second shows that it has $4k$ simple zeros $\varphi_{c,j}$, one in each interval $]\varphi_j, \psi_{2j+1}[$, for $0 \le j \le 4k-1$.

Property 6.2. *The only possible critical zeros of the function F_a, away from the origin, are the points $(r_{c,i}, \varphi_{c,j})$, for $1 \le i \le k-1$ and $0 \le j \le 4k-1$, with corresponding finitely many values of a given by (6.5). In particular, there exists some $a_0 > 0$ such that for $0 < a < a_0$, the eigenfunction F_a has no critical zero away from the origin.*

The function $G_{a,b}$ does not vanish at the origin (provided that $b \neq 0$). Its critical zeros are given by the system

$$\begin{array}{ll} G(r,\varphi) & = 0\,, \\ \partial_r G(r,\varphi) & = 0\,, \\ \partial_\varphi G(r,\varphi) & = 0\,. \end{array} \tag{6.9}$$

We look at the situation for r large. Write

$$
\begin{aligned}
L_k^{(2k)}(t) &= \tfrac{(-1)^k}{k!}\, t^k + P_k(t)\,, \\
L_{2k}(t) &= \tfrac{1}{(2k)!}\, t^{2k} + Q_k(t)\,,
\end{aligned}
\tag{6.10}
$$

where P_k and Q_k are polynomials with degree $(k-1)$ and $(2k-1)$ respectively.

The first and second equations in (6.9) are equivalent to the first and second equations of the system

$$
\begin{aligned}
0 &= \tfrac{(-1)^k}{k!}\, \sin(2k\varphi) + a\, \sin(4k\varphi - \epsilon\pi) + \tfrac{b}{(2k)!} + O(\tfrac{1}{r^2})\,, \\
0 &= \tfrac{(-1)^k}{k!}\, \cos(2k\varphi) + 2a \cos(4k\varphi - \epsilon\pi) + O(\tfrac{1}{r^2})\,,
\end{aligned}
\tag{6.11}
$$

where the $O(\frac{1}{r^2})$ are uniform in φ and a, b (provided they are initially bounded).

Property 6.3. *There exist positive numbers $a_1 \le a_0$, b_1, R_1, such that for $0 < a < a_1$, $0 < |b| < b_1$, and $r > R_1$, the function $G_{a,b}(r,\varphi)$ has no critical zero. It follows that for fixed $0 < a < a_1$, and b small enough (depending on a), the function $G_{a,b}$ has no critical zero in $\mathbb{R}^2$.*

Proof. Let $\alpha := \frac{(-1)^k}{k!}\, \sin(2k\varphi) + a\, \sin(4k\varphi - \epsilon\pi)$, $\beta := \frac{b}{(2k)!}$ and

$$
\gamma := \frac{(-1)^k}{k!}\, \cos(2k\varphi) + 2a \cos(4k\varphi - \epsilon\pi)\,.
$$

Compute $(\alpha+\beta)^2 + \gamma^2$. For $0 < a < \frac{1}{2k!}$, one has

$$
(\alpha+\beta)^2 + \gamma^2 \le \frac{1}{(2k!)^2} - \frac{4a}{k!} - \frac{4|b|}{k!(2k)!}\,.
$$

The first assertion follows. The second assertion follows from the first one and from Property 6.2. ☐

6.2 The checkerboard

Since a in positive, the nodal set of F_a satisfies

$$
\mathcal{L} \subset N(F_a) \subset \mathcal{L} \cup \{AB < 0\}\,, \tag{6.12}
$$

where $\mathcal{L}$ is the finite set $N(A) \cap N(B)$, more precisely,

$$
\mathcal{L} = \{(r_i, \psi_m) \mid 1 \le i \le k\,,\ 0 \le m \le 8k-1\}\,. \tag{6.13}
$$

Let $p_{i,m}$ denote the point with polar coordinates (r_i, ψ_m). It is easy to check that the points $p_{i,m}$ are regular points of the nodal set $N(F_a)$. More precisely the nodal set $N(F_a)$ at these points is a regular arc transversal to the lines $\{\varphi = \psi_m\}$ and $\{r = r_i\}$. Note also that the nodal set $N(F_a)$ can only cross the nodal sets $N(A)$ or $N(B)$ at the points in $\mathcal{L}$.

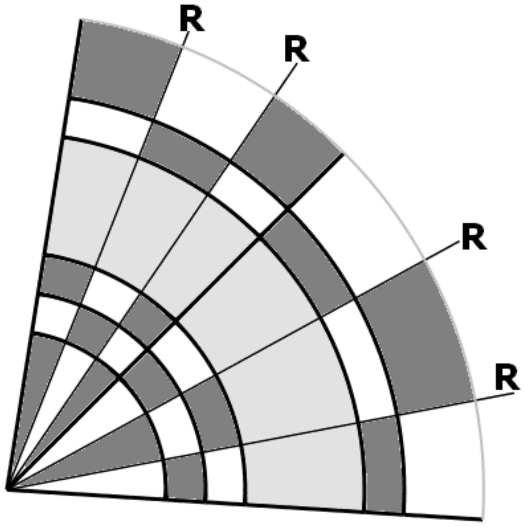

Figure 7: $\ell = 4k$, k even.

The connected components of the set $\{AB \neq 0\}$ form a "polar checkerboard" whose white boxes are the connected components in which $AB < 0$. The global aspect of the checkerboard depends on the parity of k. Recall that our assumption is that $\ell = 4k$, with k even. Figure 7 displays a partial view of the checkerboard, using the invariance under the rotation of angle $\frac{\pi}{k}$. The thin lines labelled "R" correspond to the angles ψ_m, with $m = 0, 1, 2, 3$. The thick lines to the angles φ_j, with $j = 0, 1, 2$. The thick arcs of circle correspond to the values r_i, with $i = 1, 2, 3$ and then $i = k-1, k$. The light grey part represents the zone r_i with $i = 4, \ldots, k-2$. The intersection points of the thin lines "R" with the thick arcs are the point in $\mathcal{L}$, in the sector $0 \leq \varphi \leq \varphi_2$. The outer arc of circle (in grey) represents the horizon.

6.3 Behavior at infinity

We now look at the behavior at infinity of the functions F_a and $G_{a,b}$. We restrict our attention to the sector $\{0 \leq \varphi \leq \varphi_2\}$.

Recall that k is even.

For $r > r_k$, the nodal set $N(F_a)$ can only visit the white sectors $\mathcal{S}_0 := \{\varphi_0 < \varphi < \psi_0\}$, $\mathcal{S}_1 := \{\psi_1 < \varphi < \varphi_1\}$, and $\mathcal{S}_2 := \{\psi_2 < \varphi < \psi_3\}$, issuing respectively from the points $p_{k,0}, p_{k,1}$ or $p_{k,2}, p_{k,3}$.

As above, we can write

$$F_a(r, \varphi) = r^{4k} g(\varphi) + \sin(2k\varphi) P_k(r^2)\,, \tag{6.14}$$

with

$$g(\varphi) = \frac{1}{k!} \sin(2k\varphi) + a \sin(4k\varphi - \epsilon\pi)\,,$$

where we have used the fact that k is even.

• **Analysis in $\mathcal{S}_0$**. We have $0 < \varphi < \frac{\epsilon\pi}{4k}$. Note that $g(0)\, g(\frac{\epsilon\pi}{4k}) < 0$. On the other-hand, $g'(\varphi)$ satisfies

$$g'(\varphi) \geq 2k \left\{ \frac{1}{k!} \cos(\frac{\epsilon\pi}{2}) - 2a \right\} .$$

It follows that provided that $0 < a < \frac{1}{2\,k!} \cos(\frac{\epsilon\pi}{2})$, the function g has exactly one zero θ_0 in the interval $]0, \frac{\epsilon\pi}{4k}[$.

It follows that for r big enough, the equation $F_a(r,\varphi) = 0$ has exactly one zero $\varphi(r)$ in the interval $]0, \frac{\epsilon\pi}{4k}[$, and this zero tends to θ_0 when r tends to infinity. Looking at (6.14) again, we see that $\varphi(r) = \theta_0 + O(\frac{1}{r^2})$. It follows that the nodal set in the sector $\mathcal{S}_0$ is a line issuing from $p_{k,0}$ and tending to infinity with the asymptote $\varphi = \theta_0$.

• **Analysis in $\mathcal{S}_1$**. The analysis is similar to the analysis in $\mathcal{S}_0$.

• **Analysis in $\mathcal{S}_2$**. In this case, we have that $\frac{(2+\epsilon)\pi}{4k} < \varphi < \frac{(3+\epsilon)\pi}{4k}$. It follows that $-\sin(2k\varphi) \geq \min\{\sin(\frac{\epsilon\pi}{2}), \cos(\frac{\epsilon\pi}{2})\} > 0$. If $0 < a < \frac{1}{k!} \min\{\sin(\frac{\epsilon\pi}{2}), \cos(\frac{\epsilon\pi}{2})\}$, then $F_a(r,\varphi)$ tends to negative infinity when r tends to infinity, uniformly in $\varphi \in]\psi_2, \psi_3[$. It follows that the nodal set of $N(F_a)$ is bounded in the sector $\mathcal{S}_2$.

6.4 The nodal set $N(F_a)$ and $N(G_{a,b})$

Proposition 6.4. *For $\ell = 4k$, k even, and a positive small enough, the nodal set of F_a consists of three sets of "ovals"*

1. *a cluster of $2k$ closed (singular) curves, with a common singular point at the origin,*
2. *$2k$ curves going to infinity, tangentially to lines $\varphi = \vartheta_{a,j}$ (in the case of the sphere, they would correspond to a cluster of closed curves at the south pole),*
3. *$2k(k-1)$ disjoint simple closed curves (which correspond to the white cases at finite distance of Stern's checkerboard for A and B).*

Proof.

Since B vanishes at higher order than A at the origin, the behavior of the nodal set of F_a is well determined at the origin. More precisely, the nodal set of F_a at the origin consists of $4k$ semi-arcs, issuing from the origin tangentially to the lines $\varphi = j\pi/2k$, for $0 \leq j \leq 4k-1$. At infinity, the behavior of the nodal set of F_a is determined for a small enough in Subsection 6.3. An analysis à la Stern, then shows that for a small enough there is a cluster of ovals in the intermediate region $\{r_2 < r < r_{k-1}\}$, when $k \geq 4$. □

Fixing so a small enough so that the preceding proposition holds, in order to obtain a regular nodal set, it suffices to perturb F_a into $G_{a,b}$, with b small enough, choosing its sign so that the nodal set F_a is desingularized at the origin, creating $2k$ ovals.

Figure 8 displays the cases $\ell = 8$ (i.e. $k = 2$).

Finally, we have constructed an eigenfunction $G_{a,b}$ with $2k(k+1)$ nodal component so that $\mu(G_{a,b}) \sim 2k^2 = \frac{\ell^2}{8}$.

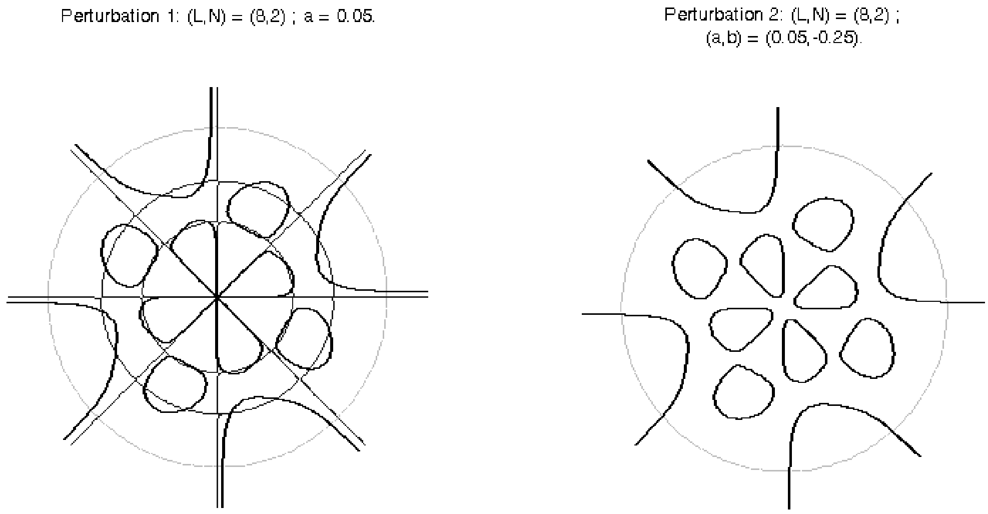

Figure 8: Ovals for $\ell = 8$.

7 On bounds for the length of the nodal set

In Subsection 7.1, we obtain Theorem 1.7 as a corollary of a more general result, Proposition 7.2, which sheds some light on the exponent $\frac{3}{2}$. The proof is typically 2-dimensional, à la Brüning-Gromes [6, 7]. We consider more general potentials in Subsection 7.3. In Subsection 7.3, we extend the methods of Long Jin [29] to some Schrödinger operators. We obtain both lower and upper bounds on the length of the nodal sets in the classically permitted region, Proposition 7.10.

7.1 Lower bounds, proof à la Brüning-Gromes

Consider the eigenvalue problem on $L^2(\mathbb{R}^2)$

$$H_V := -\Delta + V(x)\,, \quad H_V\, u = \lambda\, u\,, \tag{7.1}$$

for some suitable non-negative potential V such that the operator has discrete spectrum (see [24, Chapter 8]). More precisely, we assume:

Assumption 7.1. *The potential V is positive, continuous and tends to infinity at infinity.*

Introduce the sets

$$B_V(\lambda) := \left\{x \in \mathbb{R}^2 \mid V(x) < \lambda\right\}\,, \tag{7.2}$$

and, for $r > 0$,

$$B_V^{(-r)}(\lambda) := \left\{x \in \mathbb{R}^2 \mid B(x,r) \subset B_V(\lambda)\right\}, \tag{7.3}$$

where $B(x,r)$ is the open ball with center x and radius r.

Proposition 7.2. *Fix $\delta \in]0,1[$ and $\rho \in]0,1]$. Under Assumption 7.1, for λ large enough, and for any nonzero eigenfunction u of H_V, $H_V u = \lambda u$, the length of $N(u) \cap B_V(\delta\lambda)$ is not less than*

$$\frac{2(1-\delta)}{9\pi^2 j_{0,1}} \sqrt{\lambda}\, A\left(B_V^{(-2\rho)}(\delta\lambda)\right). \tag{7.4}$$

Proof of Proposition 7.2.

Lemma 7.3. *Choose some radius $0 < \rho \le 1$, and let*

$$\rho_\delta := \frac{j_{0,1}}{\sqrt{1-\delta}}\,. \tag{7.5}$$

Then, for $\lambda > \left(\frac{\rho_\delta}{\rho}\right)^2$, and for any $x \in B_V^{(-\rho)}(\delta\lambda)$, the ball $B(x, \frac{\rho_\delta}{\sqrt{\lambda}})$ intersects the nodal set $N(u)$ of the function u.

Proof of Lemma 7.3. Let $r := \frac{\rho_\delta}{\sqrt{\lambda}}$. If the ball $B(x,r)$ did not intersect $N(u)$, then it would be contained in a nodal domain D of the eigenfunction u. Denoting by $\sigma_1(\Omega)$ the least Dirichlet eigenvalue of the operator H_V in the domain Ω, by monotonicity, we could write

$$\lambda = \sigma_1(D) \le \sigma_1\left(B(x,r)\right).$$

Since $x \in B_V^{(-\rho)}(\delta\lambda)$ and $\lambda > \left(\frac{\rho_\delta}{\rho}\right)^2$, the ball $B(x,r)$ is contained in $B_V(\delta\lambda)$, and we can bound V from above by $\delta\lambda$ in this ball. It follows that $\sigma_1\left(B(x,r)\right) < \frac{j_{0,1}^2}{r^2} + \delta\lambda$. This leads to a contradiction with the definition of ρ_δ. $\square$

Consider the set $\mathcal{F}$ of finite subsets $\{x_1, \dots, x_n\}$ of $\mathbb{R}^2$ with the following properties,

$$\begin{cases} x_i \in N(u) \cap B_V^{(-\rho)}(\delta\lambda),\ 1 \le i \le n\,, \\ B(x_i, \frac{\rho_\delta}{\sqrt{\lambda}})\,,\ 1 \le i \le n\,,\ \text{pairwise disjoint.} \end{cases} \tag{7.6}$$

For λ large enough, the set $\mathcal{F}$ is not empty, and can be ordered by inclusion. It admits a maximal element $\{x_1, \dots, x_N\}$, where N depends on δ, ρ, λ and u.

Lemma 7.4. *The balls $B(x_i, \frac{3\rho_\delta}{\sqrt{\lambda}})\,, 1 \le i \le N$, cover the set $B_V^{(-2\rho)}(\delta\lambda)$.*

Proof of Lemma 7.4. Assume the claim in not true, i.e. that there exists some $y \in B_V^{(-2\rho)}(\delta\lambda)$ such that $|y - x_i| > \frac{3\rho_\delta}{\sqrt{\lambda}}$ for all $i \in \{1, \dots, N\}$. Since $y \in B_V^{(-\rho)}(\delta\lambda)$, by Lemma 7.3, there exists some $x \in N(u) \cap B(y, \frac{\rho_\delta}{\sqrt{\lambda}})$, and we have $x \in B_V^{(-\rho)}(\delta\lambda)$. Furthermore, for all $i \in \{1, \dots, N\}$, we have $|x - x_i| \ge \frac{2\rho_\delta}{\sqrt{\lambda}}$. The set $\{x, x_1, x_2, \dots, x_N\}$ would belong to $\mathcal{F}$, contradicting the maximality of $\{x_1, x_2, \dots, x_N\}$. $\square$

Lemma 7.4 gives a lower bound on the number N,

$$N \ge \frac{\lambda}{9\pi^2 \rho_\delta^2} A\left(B_V^{(-2\rho)}(\delta\lambda)\right), \tag{7.7}$$

where $A(\Omega)$ denotes the area of the set Ω.

Lemma 7.5. *For any $\alpha < j_{0,1}$, the ball $B(x, \frac{\alpha}{\sqrt{\lambda}})$ does not contain any closed connected component of the nodal set $N(u)$.*

Proof of Lemma 7.5. Indeed, any closed connected component of $N(u)$ contained in $B(x, \frac{\alpha}{\sqrt{\lambda}})$ would bound some nodal domain D of u, contained in $B(x, \frac{\alpha}{\sqrt{\lambda}})$, and we would have

$$\lambda = \sigma_1(D) \geq \sigma_1 \left(B(x, \frac{\alpha}{\sqrt{\lambda}} \right) \geq \frac{j_{0,1}^2}{\alpha^2} \lambda \,,$$

contradicting the assumption on α. □

Take the maximal set $\{x_1, \ldots, x_N\} \subset N(u) \cap B_V^{(-\rho)}(\delta\lambda)$ constructed above. The balls $B(x_i, \frac{\rho_\delta}{\sqrt{\lambda}})$ are pairwise disjoint, and so are the balls $B(x_i, \frac{\alpha}{\sqrt{\lambda}})$ for any $0 < \alpha < j_{0,1}$. There are at least two nodal arcs issuing from a point x_i, and they must exit $B(x_i, \frac{\alpha}{\sqrt{\lambda}})$, otherwise we could find a closed connected component of $N(u)$ inside this ball, contradicting Lemma 7.5. The length of $N(u) \cap B(x_i, \frac{\alpha}{\sqrt{\lambda}})$ is at least $\frac{2\alpha}{\sqrt{\lambda}}$. Finally, the length of $N(u) \cap B_V(\delta\lambda)$ is at least $N \frac{2\alpha}{\sqrt{\lambda}}$ which is bigger than

$$\frac{2\alpha}{9\pi^2 \rho_\delta^2} \sqrt{\lambda} A \left(B_V^{(-2\rho)}(\delta\lambda) \right) \,.$$

Since this is true for any $\alpha < j_{0,1}$. □

Proof of Theorem 1.7.

We apply the preceding proposition with $V(x) = |x|^{2k}$ and $\rho = 1$. Then, $B_V(\lambda) = B(\lambda^{\frac{1}{2k}})$ and $B_V^{(-r)}(\lambda) = B(\lambda^{\frac{1}{2k}} - r)$. In this case, the length of the nodal set is bounded by some constant times $\lambda^{\frac{1}{2}+\frac{1}{k}} \approx \lambda^{\frac{1}{2}} \, A\left(B_V(\delta\lambda)\right)$. When $k = 1$, we obtain Proposition 1.7. □

Remark. The above proof sheds some light on the exponent $\frac{3}{2}$ in Proposition 1.7.

7.2 More general potentials

We reinterpret Proposition 7.2 for more general potentials $V(x)$, under natural assumptions which appear in the determination of the Weyl's asymptotics of H_V (see [32], [23]). After renormalization, we assume:

Assumption 7.6. *V is of class C^1, $V \geq 1$, and there exist some positive constants ρ_0 and C_1 such that for all $x \in \mathbb{R}^2$,*

$$|\nabla V(x)| \leq C_1 V(x)^{1-\rho_0} \,. \tag{7.8}$$

Note that under this assumption there exist positive constants r_0 and C_0 such that

$$x, y \text{ satisfy } |x - y| \leq r_0 \Rightarrow V(x) \leq C_0 \, V(y) \,. \tag{7.9}$$

The proof is easy. We first write

$$V(x) \leq V(y) + |x - y| \sup_{z \in [x,y]} |\nabla V(z)| \,.$$

Applying (7.8) (here we only use $\rho_0 \geq 0$), we get

$$V(x) \leq V(y) + C_1|x-y| \sup_{z\in[x,y]} V(z)\,.$$

We now take $x \in B(y,r)$ for some $r > 0$ and get

$$\sup_{x\in B(y,r)} V(x) \leq V(y) + C_1\, r \sup_{x\in B(y,r)} V(x)\,,$$

which we can rewrite, if $C_1 r < 1$, in the form

$$V(y) \leq \sup_{x\in B(y,r)} V(x) \leq V(y)(1-C_1 r)^{-1}\,.$$

This is more precise than (7.9) because we get $C_0(r_0) = (1-C_1r_0)^{-1}$, which tends to 1 as $r_0 \to 0$.

We assume

Assumption 7.7. *For any $\delta \in]0,1[$, there exists some positive constants A_δ and λ_δ such that*

$$1 < A(B_V(\lambda))/A(B_V(\delta\lambda)) \leq A_\delta\,,\ \forall \lambda \geq \lambda_\delta\,. \tag{7.10}$$

Proposition 7.8. *Fix $\delta \in (0,1)$, and assume that V satisfies the previous assumptions. Then, there exists a positive constant C_δ (depending only on the constants appearing in the assumptions on V) and λ_δ such that for any eigenpair (u,λ) of H_V with $\lambda \geq \lambda_\delta$, the length of $N(u) \cap B_V(\delta\lambda)$ is larger than $C_\delta \lambda^{\frac{1}{2}} A(B_V(\lambda))$.*

Proof.

Using (7.10), it is enough to prove the existence of r_1 such that, for $0 < r < r_1$, there exists $C_2(r)$ and $M(r)$ s.t.

$$B_V(\mu - C_2\mu^{1-\rho_0}) \subset B_V^{-r}(\mu)\,,\ \forall \mu > M(r)\,.$$

But, if $x \in B_V(\mu - C_2\mu^{1-\rho_0})$, and $y \in B(x,r)$, we have

$$V(y) \leq V(x) + C_1C_0(r)^{1-\rho_0} r V(x)^{1-\rho_0} \leq \mu - C_2\mu^{1-\rho_0} + C_1C_0(r)^{1-\rho_0} r \mu^{1-\rho_0}\,.$$

Taking $C_2(r) = C_1C_0(r)^{1-\rho_0} r$ and $M(r) \geq (C_2(r)+1)^{\frac{1}{\rho_0}}$ gives the result. □

Remarks.

1. The method of proof of Proposition 7.2, which is reminiscent of the proof by Brüning [7] (see also [6]) is typically 2-dimensional.
2. The same method could be applied to a Schrödinger operator on a complete non-compact Riemannian surface, provided one has some control on the geometry, the first eigenvalue of small balls, etc..
3. If we assume that there exist positive constants $m_0 \leq m_1$ and C_3 such that for any $x \in \mathbb{R}^2$,
$$\frac{1}{C_3} < x >^{m_0} \leq V(x) \leq C_3 < x >^{m_1}\,, \tag{7.11}$$
where $< x > := \sqrt{1+|x|^2}$, then $A(B_V(\lambda))$ has a controlled growth at ∞.
4. If $m_0 = m_1$ in (7.11), then (7.10) is satisfied. The control of A_δ as $\delta \to +1$ can be obtained under additional assumptions.

7.3 Upper and lower bounds on the length of the nodal set: the semi-classical approach of Long Jin

In [29], Long Jin analyzes the same question in the semi-classical context for a Schrödinger operator

$$H_{W,h} := -h^2\Delta_g + W(x)\,,$$

where Δ_g is the Laplace-Beltrami operator on the compact connected analytic Riemannian surface (M,g), with W analytic. In this context, he shows that if (u_h,λ_h) is an h-family of eigenpairs of $H_{W,h}$ such that $\lambda_h \to E$, then the length of the zero set of u_h inside the classical region $W^{-1}(]-\infty,E])$ is of order h^{-1}.

Although not explicitly done in [29], the same result is also true in the case of $M=\mathbb{R}^2$ under the condition that $\liminf W(x) > E \geq \inf W$, keeping the assumption that W is analytic. Let us show how we can reduce the case $M=\mathbb{R}^2$ to the compact situation.

Proposition 7.9. *Let us assume that W is continuous and that there exists E_1 such that $W^{-1}(]-\infty,E_1])$ is compact. Then the bottom of the essential spectrum of $H_{W,h}$ is bigger than E_1. Furthermore, if (λ_h,u_h) is a family ($h\in]0,h_0]$) of eigenpairs of $H_{W,h}$ such that $\lim_{h\to 0}\lambda_h = E_0$ with $E_0<E_1$ and $||u_h||=1$, then given K a compact neighborhood of $W^{-1}(]-\infty,E_0])$, there exists $\epsilon_K>0$ such that*

$$||u_h||_{L^2(K)} = 1+\mathcal{O}\left(\exp(-\epsilon_K/h)\right)\,,$$

as $h\to 0$.

This proposition is a consequence of Agmon estimates (see Helffer-Sjöstrand [21] or Helffer-Robert [20] for a weaker result with a remainder in $\mathcal{O}_K(h^\infty)$) measuring the decay of the eigenfunctions in the classically forbidden region. This can also be found in a weaker form in the recent book of M. Zworski [38] (Chapter 7), which also contains a presentation of semi-classical Carleman estimates.

Observing that in the proof of Long Jin the compact manifold M can be replaced by any compact neighborhood of $W^{-1}(]-\infty,E_0])$, we obtain:

Proposition 7.10. *Let us assume in addition that W is analytic in some compact neighborhood of $W^{-1}(]-\infty,E_0])$, then the length of the zero set of u_h inside the classical region $W^{-1}(]-\infty,E_0])$ is of order h^{-1}. More precisely, there exist $C>0$ and $h_0>0$ such that for all $h\in]0,h_0]$ we have*

$$\frac{1}{C}h^{-1} \leq \text{length}\left(N(u_h)\cap W^{-1}(]-\infty,E_0])\right) \leq C\,h^{-1}\,. \tag{7.12}$$

Remark 7.11. *As observed in [29] (Remark 1.3), the results of [18] suggest that the behavior of the nodal sets in the classically forbidden region could be very different from the one in the classically allowed region.*

We can by scaling recover Proposition 1.7, and more generally treat the eigenpairs of $-\Delta_x+|x|^{2k}$. Indeed, assume that $(-\Delta_x+|x|^{2k})u(x)=\lambda u(x)$. Write $x=\rho\,y$. Then,

$(-\rho^{-2}\Delta_y + \rho^{2k}|y|^{2k} - \lambda)u(\rho y) = 0$. If we choose $\rho^{2k} = \lambda$, $h = \rho^{-k-1} = \lambda^{-\frac{k+1}{2k}}$ and let $v_h(y) = h^{\frac{1}{2(k+1)}} y)\, u(h^{\frac{1}{k+1}} y)$, then, $(-h^2\Delta_y + |y|^{2k} - 1)v_h(y) = 0$. Applying (7.12) to the family v_h and rescaling back to the variable x, we find that

$$\frac{1}{C}\lambda^{\frac{k+2}{2k}} \leq \text{length}\left(N(u) \cap \{x \in \mathbb{R}^2 \mid |x|^{2k} < \lambda\}\right) \leq C\,\lambda^{\frac{k+2}{2k}}\,. \tag{7.13}$$

With this extension of Long Jin's statement, when $V = |x|^{2k}$, we also obtain an upper bound of the length of $N(u)$ in $B_V(\lambda)$. Note that when $k \to +\infty$, the problem tends to the Dirichlet problem in a ball of size 1. We then recover that the length of $N(u)$ is of order $\sqrt{\lambda}$.

The above method can also give results in the non-homogeneous case, at least when (7.11) is satisfied with $m_0 = m_1$. We can indeed prove the following generalization.

Proposition 7.12.
Let us assume that there exist $m \geq 1$, $\epsilon_0 > 0$ and $C > 0$ such that V is holomorphic in

$$\mathcal{D} := \{z = (z_1, z_2) \in \mathbb{C}^2\,,\ |\Im z| \leq \epsilon_0 < \Re z >\}$$

and satisfies

$$|V(z)| \leq C < \Re z >^m\,,\ \forall z \in \mathcal{D}\,. \tag{7.14}$$

Suppose in addition that we have the ellipticity condition

$$\frac{1}{C'} < x >^m \leq V(x)\,,\ \forall x \in \mathbb{R}^2\,. \tag{7.15}$$

Then, for any $\epsilon > 0$, the length $N(u) \cap (B_V(\lambda) \setminus B_V(\epsilon\lambda))$ for an eigenpair (u, λ) of H_V, is of the order of $\lambda^{\frac{1}{2}+\frac{2}{m}}$ as $\lambda \to +\infty$. Moreover, one can take $\epsilon = 0$ when V is a polynomial.

Proof

The lower bound was already obtained by a more general direct approach in Proposition 7.8. One can indeed verify using Cauchy estimates that (7.14) and (7.15) imply (7.8) and (7.11), with $\rho_0 = 1/2m$. Under the previous assumptions, we consider

$$W_\lambda(y) = \lambda^{-1} V(\lambda^{\frac{1}{m}} y)\,,\ v_\lambda(y) = \lambda^{\frac{1}{4m}}\, u(\lambda^{\frac{1}{m}} y)\,.$$

We observe that with

$$h = \lambda^{-\frac{1}{2}-\frac{1}{m}}\,, \tag{7.16}$$

the pair $(v_\lambda, 1)$ is an eigenpair for the semi-classical Schrödinger operator $-h^2\Delta_y + W_\lambda(y)$:

$$(-h^2\Delta + W_\lambda)v_\lambda = v_\lambda\,.$$

It remains to see if we can extend the result of Long Jin to this situation. We essentially follow his proof, whose basic idea goes back to Donnelly-Feffermann [15]. The difference being that W_λ depends on h through (7.16).

The inspection of the proof[2] shows that there are three points to control.

Analyticity
What we need is to have for any y_0 in $\mathbb{R}^2 \setminus \{0\}$ a complex neighborhood $\mathcal{V}$ of y_0, $h_0 > 0$ and C such that, for any $h \in]0, h_0]$, v_λ admits an holomorphic extension in $\mathcal{V}$ with

$$\sup_{\mathcal{V}} |v_\lambda| \leq C \exp\left(\frac{C}{h}\right) ||v_\lambda||_{L^\infty(\mathbb{R}^2)} \,. \tag{7.17}$$

This can be done by using the FBI transform, controlling the uniformity when W is replaced by W_λ. But this is exactly what is given by Assumption (7.14). Notice that this is not true in general for $y_0 = 0$. We cannot in general find a λ-independent neighborhhod of 0 in $\mathbb{C}^2$ where W_λ is defined and bounded.
Note here that $||v_\lambda||_{L^\infty(\mathbb{R}^2)}$ is by standard semiclassical analysis $\mathcal{O}(h^{-N})$ for some N.
When V is in addition a polynomial:

$$V(x) = \sum_{j=0}^{m} P_j(x)$$

where P_j is an homogeneous polynomial of degree j, we get

$$W_\lambda(y) = P_m(y) + \sum_{\ell=1}^{m} \lambda^{-\frac{\ell}{m}} P_{m-\ell}(y) \,,$$

and we can verify the uniform analyticity property for any y_0.

Uniform confining
As we have mentioned before, Long Jin's paper was established in the case of a compact manifold (in this case and for Laplacians, it is worth to mention the papers of Sogge-Zelditch [34, 35]) but it can be extended to the case of $\mathbb{R}^2$ under the condition that the potential is confining, the length being computed in a compact containing the classically permitted region. This is the case with W_λ. Note that if $W_\lambda(y) \leq C_1$, then we get

$$\lambda^{-1} V(\lambda^{\frac{1}{m}} y) \leq C_1 \,,$$

which implies by the ellipticity condition $\frac{1}{C'} \lambda^{-1} |\lambda^{\frac{1}{m}}|^m |y|^m \leq C_1$, that is

$$|y| \leq (C' C_1)^{\frac{1}{m}} \,.$$

Uniform doubling property
Here instead of following Long Jin's proof, it is easier to refer to the results of Bakri-Casteras [1], which give an explicit control in term of the C^1 norm of W_λ. As before, we have to use our confining assumption in order to establish our result in any bounded domain Ω in $\mathbb{R}^2$ containing uniformly the classically permitted area $W_\lambda^{-1}(]-\infty, +1])$. This last assumption permits indeed to control the L^2-norm of v_λ from below in Ω. We actually need the two following estimates (we assume (7.16)):

[2] We refer here to the proof of (2.20) in [29].

Given Ω like above, for any $R > 0$, there exists C_R such that, for any (x, R) such that $B(x, R) \subset \Omega$,

$$||v_\lambda||_{L^2(B(x,R))} \geq \exp\left(-\frac{C_R}{h}\right). \tag{7.18}$$

Given Ω like above, there exists C such that, for any (x, r) such that $B(x, 2r) \subset \Omega$,

$$||v_\lambda||_{L^2(B(x,2r))} \leq \exp\left(\frac{C}{h}\right) ||v_\lambda||_{L^2(B(x,r))}. \tag{7.19}$$

Here we have applied Theorem 3.2 and Proposition 3.3 in [1] with electric potential $h^{-2}(W_\lambda - 1)$. These two statements involve the square root of the C^1 norm of the electric potential in $\overline{\Omega}$, which is $\mathcal{O}(h^{-1})$ in our case.

End of the proof
Hence, considering an annulus $A(\epsilon_0, R_0)$ we get following Long Jin that the length of the nodal set of v_λ in this annulus is indeed of order $\mathcal{O}(h^{-1})$ and after rescaling we get the proposition for the eigenpair (u, λ). In the polynomial case, we get the same result but in the ball $B(0, R_0)$. □

Remarks.

1. Long Jin's results hold in dimension n, not only in dimension 2. The above extensions work in any dimension as well, replacing the length by the $(n-1)$-Hausdorff measure.

2. As observed in [29], the results in [18] suggest that the behavior of nodal sets in the classically forbidden region could be very different from the one in the classically allowed region.

3. Under the assumptions of Proposition 7.12, one gets from Theorem 1.1 in [1] that the order of a critical point of the zero set of an eigenfunction of H_V associated with λ in the classically permitted region is at most of order $\lambda^{\frac{1}{2}+\frac{1}{m}}$. Let us emphasize that here no assumption of analyticity for V is used. On the other hand, note that using Courant's theorem and Euler's and Weyl's formulas, one can prove that the number of critical points in the classically allowed region is at most of order $\lambda^{1+\frac{2}{m}}$. When $m = 2$, we can verify from the results in Section 6 that this upper bound cannot be improved in general.

4. For nodal sets in forbidden regions, see [8].

References

[1] L. Bakri and J.-B. Casteras. Quantitative uniqueness for Schrödinger operator with regular potential. Math. Meth. Appl. Sci 37 (2014), 1992-2008. 30, 31

[2] P. Bérard and B. Helffer. Dirichlet eigenfunctions of the square membrane: Courant's property, and A. Stern's and Å. Pleijel's analyses. In Baklouti, Ali (ed.) et al., Analysis and geometry. MIMS-GGTM, Tunis, Tunisia, March 2427, 2014. Proceedings of the international conference. In honour of Mohammed Salah Baouendi. Springer Proceedings in Mathematics & Statistics 127 (2015), 69–114. 3, 4, 8, 10, 13, 15, 17

[3] P. Bérard and B. Helffer. On the number of nodal domains for the 2D isotropic quantum harmonic oscillator, an extension of results of A. Stern. arXiv:1409.2333v1 (September 8, 2014). 3

[4] P. Bérard and B. Helffer. A. Stern's analysis of the nodal sets of some families of spherical harmonics revisited. Monatshefte für Mathematik 180 (2016), 435–468. 3, 15

[5] P. Bérard and B. Helffer. Edited extracts from Antonie Stern's thesis. Séminaire de Théorie spectrale et géométrie, Institut Fourier, Grenoble, 32 (2014-2015), 39–72. 34

[6] J. Brüning and D. Gromes. Über die Länge der Knotenlinien schwingender Membranen, Math. Z. 124 (1972), 79–82. 6, 24, 27

[7] J. Brüning. Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators. Math. Z. 158 (1) (1978), 15–21. 6, 24, 27

[8] Y. Canzani and J. Toth. Nodal sets of Schrödinger eigenfunctions in forbidden regions. Ann. Henri Poincaré 17:11 (2016), 3063–3087. 31

[9] P. Charron. On Pleijel's theorem for the isotropic harmonic oscillator. Mémoire de Maîtrise, Université de Montréal (June 2015). 5, 6

[10] P. Charron. A Pleijel type theorem for the quantum harmonic oscillator. To appear in Journal of Spectral Theory. arXiv:1512.07880. 5

[11] P. Charron, B. Helffer and T. Hoffmann-Ostenhof. Pleijel's theorem for Schrödinger operators with radial potentials. To appear in Annales mathématiques du Québec. arXiv:1604.08372. 5

[12] R. Courant. Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke. Nachr. Ges. Göttingen (1923), 81-84. 2

[13] R. Courant and D. Hilbert. Methods of Mathematical Physics. Volume 1. John Wiley & Sons, 1989.

[14] H. Donnelly. Counting nodal domains in Riemannian manifolds. Ann. Glob. Anal. Geom. 46 (2014), 57–61. 4

[15] H. Donnelly and C. Fefferman. Nodal sets of eigenfunctions on Riemannian manifolds Inventiones Mathematicae 93(1) (1988), 161–183. 6, 29

[16] A. Eremenko, D. Jakobson, and N. Nadirashvili. On nodal sets and nodal domains on $\mathbb{S}^2$ and $\mathbb{R}^2$. Ann. Inst. Fourier 57 (2007), 2345–2360. 5

[17] G. Gauthier-Shalom and K. Przybytkowski. Description of a nodal set on $\mathbb{T}^2$. McGill University Research Report 2006 (unpublished). 3

[18] B. Hanin, S. Zelditch and Peng Zhou. Nodal sets of random eigenfunctions for the isotropic harmonic oscillator. Int. Math. Res. Not. 13 (2015), 4813–4839. 5, 28, 31

[19] B. Helffer, T. Hoffmann-Ostenhof and S. Terracini. Nodal domains and spectral minimal partitions. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 101-138. 5

[20] B. Helffer and D. Robert. Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles. J. Funct. Anal. 53 (3)(1983), 246–268. 28

[21] B. Helffer and J. Sjöstrand. Multiple wells in the semiclassical limit. I. Comm. Partial Differential Equations 9 (4) (1984), 337–408. 28

[22] D. Hinton. Sturm's 1836 oscillation results. Evolution of the theory. In *Sturm-Liouville theory: past and present.* W.O. Amrein, A.M. Hinz and D.B. Pearson, eds. Birkhäuser 2005, pp. 1–27. 2

[23] L. Hörmander. On the asymptotic distribution of the eigenvalues of pseudo-differential operators in $\mathbb{R}^n$. Ark. Mat. 17 (2) (1979), 297–313. 26

[24] R.S. Laugesen. Spectral Theory of Partial Differential Equations. University of Illinois, Urbana-Champain 2014. `https://wiki.cites.illinois.edu/wiki/display/MATH595STP/Math+595+STP` 24

[25] H. Lewy. On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere. Comm. Partial Differential Equations 2 (12) (1977), 1233–1244. 3

[26] J. Leydold. Knotenlinien und Knotengebiete von Eigenfunktionen. Diplomarbeit 1989, University of Vienna, Austria. Unpublished, available at `http://othes.univie.ac.at/34443/` 3, 4, 6, 10, 18

[27] J. Leydold. On the number of nodal domains of spherical harmonics. Topology 35 (1996), 301-321. 4

[28] A. Logunov and E. Malinnikova. Nodal sets of Laplace eigenfunctions: estimates of the Hausforff measure in dimension two and three. arXiv:1605.02595. 6

[29] Long Jin. Semiclassical Cauchy estimates and applications. Trans. Amer. Math. Soc. March 18, 2016. `http://dx.doi.org/10.1090/tran/6715` 6, 24, 28, 30, 31

[30] Å. Pleijel. Remarks on Courant's nodal theorem. Comm. Pure. Appl. Math. 9 (1956), 543-550. 4

[31] I. Polterovich. Pleijel's nodal domain theorem. Proc. Amer. Math. Soc. 137 (2009), 1021-1024.

[32] D. Robert. Propriétés spectrales d'opérateurs pseudo-différentiels. (French) Comm. Partial Differential Equations 3 (9) (1978), 755–826. 26

[33] J. Royo-Letelier. Segregation and symmetry breaking of strongly coupled two-component Bose-Einstein condensates in a harmonic trap. Calc. Var. Partial Differ. Equ. 49 (2014), 103–124. 5

[34] C.D. Sogge and S. Zelditch. Lower bounds on the Hausdorff measure of nodal sets. Math. Res. Lett. 18 (2011), 25–37. 6, 30

[35] C.D. Sogge and S. Zelditch. Lower bounds on the Hausdorff measure of nodal sets II. Math. Res. Lett. 19 (2012), 1361–1364. 6, 30

[36] A. Stern. Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunktionen. Inaugural-Dissertation zur Erlangung der Doktorwürde der Hohen Mathematisch-Naturwissenschaftlichen Fakultät der Georg August-Universität zu Göttingen (30 Juli 1924). Druck der Dieterichschen Universitäts-Buchdruckerei (W. Fr. Kaestner). Göttingen, 1925. Partial extracts in the above reference [5]. 2, 3, 8, 10

[37] G. Szegö. Orthogonal Polynomials. Fourth edition. AMS colloquium publications, Vol. XXIII, Amer. Math. Soc. Providence, R.I. (1975). 3, 6, 7, 18

[38] M. Zworski. Semi-classical analysis. Graduate Studies in Mathematics 138, AMS (2012). 28