FINITE ELEMENT QUASI-INTERPOLATION AND BEST APPROXIMATION

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ALEXANDRE ERN† AND JEAN-LUC GUERMOND‡

Abstract. This paper introduces a quasi-interpolation operator for scalar- and vector-valued finite element spaces constructed on affine, shape-regular meshes with some continuity across mesh interfaces. This operator is stable in L^1, is a projection, whether homogeneous boundary conditions are imposed or not, and, assuming regularity in the fractional Sobolev spaces $W^{s,p}$ where $p \in [1,\infty]$ and s can be arbitrarily close to zero, gives optimal local approximation estimates in any L^p-norm. The theory is illustrated on H^1-, $\boldsymbol{H}(\text{curl})$- and $\boldsymbol{H}(\text{div})$-conforming spaces.

Key words. Quasi-interpolation, Finite Elements, Best Approximation

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1. Introduction. Given a shape-regular sequence of affine meshes $(\mathcal{T}_h)_{h>0}$ approximating a bounded polyhedral domain D in $\mathbb{R}^d$, and given a sequence of finite element spaces $(P(\mathcal{T}_h))_{h>0}$ based on the mesh sequence $(\mathcal{T}_h)_{h>0}$, either scalar- or vector-valued, and conforming in some functional space W where some continuity across mesh interfaces is enforced, the question addressed in this paper is that of constructing a quasi-interpolation operator that is minimally stable, delivers the best approximation error up to a uniform constant, is a projection, and satisfies homogeneous boundary conditions that are legitimate in W.

The above question has been solved in two space dimensions on triangular meshes for H^1-conforming scalar-valued finite elements in Clément [7]. The main ingredient for the construction of the so-called Clément operator is a regularization based on macroelements consisting of patches of elements. One difficulty with the Clément interpolation operator is that it is not a projection and does not preserve homogeneous boundary conditions. These two issues have been solved in Scott and Zhang [15], where an alternative quasi-interpolation operator is proposed. The so-called Scott-Zhang operator uses averages on the interfaces and on the boundary faces, which makes it a projection independently of the boundary conditions. The drawback of this approach though is that it is stable only in functional spaces with integrable traces; the lower limit is $W^{1,1}(D)$ (or $W^{s,p}(D)$ with $sp > 1$ and $p > 1$). The Clément operator has been generalized to meshes composed of non-affine simplices in arbitrary space dimensions by Bernardi [3]. The Clément construction has also been revisited by Bernardi and Girault [4] to make it a projection in $\mathbb{R}^2$ on triangular meshes. The key difference between the construction in [4] and that in [7] is that the local L^2-projection on patches is piecewise polynomial in [4] instead of being globally polynomial in [7].

In the present paper we restrict ourselves to affine meshes, but we construct a quasi-interpolation operator for scalar- and vector-valued finite element spaces that is a projection, whether homogeneous boundary conditions are imposed or not, is stable in L^1, and gives optimal local approximation estimates, up to a uniform constant, in

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any L^p-norm assuming regularity in the fractional Sobolev spaces $W^{s,p}(D)$, where s can be arbitrarily close to zero.

The paper is organized as follows. In §2, we introduce the notation and construct a sequence of abstract finite element spaces conforming in some functional space W. Key assumptions are identified and listed. The theory is illustrated on usual H^1-, $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming finite element spaces. A local interpolation operator stable in L^1 is constructed in §3. The key difference with all the previous approaches [7, 3, 15, 4] is that no patch or interfacial patch is involved at this stage since the operator in question is constructed on each element. An averaging operator acting only on discrete functions is introduced and analyzed in §4. This operator, loosely inspired from Oswald [14, Eqs. (25)-(26)], consists of averaging the local degrees of freedoms that are shared by two or more elements, see also [1, 12, 5]. The final quasi-interpolation operator is constructed in §5 without enforcing any boundary condition. The main result of this section is Theorem 5.5. The question of the boundary conditions is addressed in §6. Homogeneous boundary conditions are simply enforced by removing the degrees of freedom located at the boundary from the quasi-interpolation operator constructed in §5. The resulting operator is still a projection and still has optimal local approximation properties as stated in Theorem 7.1.

The results of the present paper may be useful whenever finite element best approximation estimates are invoked. For instance, Theorem 5.5 and Theorem 7.1 complete the proof of Theorem 5.6 from Arnold et al. [2, p. 67] where a best approximation result is stated in the context of finite-element-based exterior calculus. The argument therein is very elegant but somewhat incomplete since it invokes the Clément interpolant, which is a two-dimensional construction for scalar-valued functions.

2. Finite elements. In this section we introduce some notation and construct a sequence of abstract finite element spaces, conforming in some functional space W. In the entire paper the space dimension is denoted d and the domain D is a bounded polyhedron in $\mathbb{R}^d$.

2.1. Meshes. Let $(\mathcal{T}_h)_{h>0}$ be a mesh sequence that we assume to be affine and shape-regular in the sense of Ciarlet. We also assume for the sake of simplicity that the meshes cover D exactly and that they are matching, i.e., for all cells $K, K' \in \mathcal{T}_h$ such that $K \neq K'$ and $K \cap K' \neq \emptyset$, the set $K \cap K'$ is a common vertex, edge, or face of both K and K' (with obvious extensions in higher space dimensions). By convention, given a mesh $\mathcal{T}_h$, the elements in $K \in \mathcal{T}_h$ are closed sets in $\mathbb{R}^d$.

We assume that there is a reference element $\widehat{K}$ such that for any mesh $\mathcal{T}_h$ and any cell $K \in \mathcal{T}_h$, there is an bijective affine mapping between $\widehat{K}$ and K, which we henceforth denote $\boldsymbol{T}_K : \widehat{K} \longrightarrow K$. Since $\boldsymbol{T}_K$ is affine and bijective, there is an invertible matrix $\mathbb{J}_K \in \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) - \boldsymbol{T}_K(\widehat{\boldsymbol{y}}) = \mathbb{J}_K(\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}), \qquad \forall \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}} \in \widehat{K}. \tag{2.1}$$

In what follows, we denote points in $\mathbb{R}^d$ and $\mathbb{R}^d$-valued functions and mappings using bold face, and we denote the Euclidean norm in $\mathbb{R}^d$ by $\|\cdot\|_{\ell^2(\mathbb{R}^d)}$, or $\|\cdot\|_{\ell^2}$ when the context is unambiguous. We abuse the notation by using the same symbol for the induced matrix norm. Owing to the shape-regularity assumption of the mesh sequence, there are uniform constants $c^\sharp$, $c^\flat$ such that

$$|\det(\mathbb{J}_K)| = |K||\widehat{K}|^{-1}, \qquad \|\mathbb{J}_K\|_{\ell^2} \leq c^\sharp h_K, \qquad \|\mathbb{J}_K^{-1}\|_{\ell^2} \leq c^\flat h_K^{-1}, \tag{2.2}$$

where h_K is the diameter of K. Recall that $c^\sharp = \frac{1}{\rho_{\widehat{K}}}$ and $c^\flat = \frac{h_K}{\rho_K} h_{\widehat{K}}$ for meshes composed of simplices, where ρ_K is the diameter of the largest ball that can be inscribed in K, $h_{\widehat{K}}$ is the diameter of $\widehat{K}$, and $\rho_{\widehat{K}}$ is the diameter of the largest ball that can be inscribed in $\widehat{K}$.

2.2. Finite element generation. We are going to consider various approximation spaces based on the mesh sequence $(\mathcal{T}_h)_{h>0}$. Again for the sake of simplicity, we assume that each approximation space is constructed from a fixed reference finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$. The reference degrees of freedom are denoted $\{\widehat{\sigma}_1, \dots, \widehat{\sigma}_{n_{\mathrm{sh}}}\}$ and the associated reference shape functions are denoted $\{\widehat{\theta}_1, \dots, \widehat{\theta}_{n_{\mathrm{sh}}}\}$; by definition $\widehat{\sigma}_i(\widehat{\theta}_j) = \delta_{ij}$, $\forall i, j \in \{1{:}n_{\mathrm{sh}}\}$. We denote $\mathcal{N} := \{1{:}n_{\mathrm{sh}}\}$ to alleviate the notation. The shape functions are $\mathbb{R}^q$-valued for some integer $q \geq 1$. We henceforth assume that $\widehat{P} \subset W^{1,\infty}(\widehat{K}; \mathbb{R}^q)$ (recall that $\widehat{P}$ is a space of polynomial functions in general).

We assume that there exists a Banach space $V(\widehat{K}) \subset L^1(\widehat{K}; \mathbb{R}^q)$ such that the linear forms $\{\widehat{\sigma}_1, \dots, \widehat{\sigma}_{n_{\mathrm{sh}}}\}$ can be extended to $\mathcal{L}(V(\widehat{K}); \mathbb{R})$, i.e., $V(\widehat{K})$ is the domain of the degrees of freedom; see [10, p. 39]. Then, we define $\mathcal{I}_{\widehat{K}} : V(\widehat{K}) \to \widehat{P}$, the interpolation operator associated with the reference finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$, by

$$\mathcal{I}_{\widehat{K}}(\widehat{v})(\widehat{\boldsymbol{x}}) = \sum_{i \in \mathcal{N}} \widehat{\sigma}_i(v) \widehat{\theta}_i(\widehat{\boldsymbol{x}}), \qquad \forall \widehat{\boldsymbol{x}} \in \widehat{K}, \quad \forall \widehat{v} \in V(\widehat{K}). \tag{2.3}$$

By construction, $\mathcal{I}_{\widehat{K}} \in \mathcal{L}(V(\widehat{K}); \widehat{P})$, and $\widehat{P}$ is point-wise invariant under $\mathcal{I}_{\widehat{K}}$.

We now address the question of constructing finite elements for any mesh cell $K \in \mathcal{T}_h$. We assume that there exists a Banach space $V(K) \subset L^1(K; \mathbb{R}^q)$ and a bounded, bijective, linear mapping between $V(K)$ and $V(\widehat{K})$:

$$\psi_K : V(K) \ni v \longmapsto \psi_K(v) \in V(\widehat{K}). \tag{2.4}$$

We then set

$$P_K := \{p = \psi_K^{-1}(\widehat{p}) \mid \widehat{p} \in \widehat{P}\}, \tag{2.5a}$$

$$\Sigma_K := \{\sigma_{K,i}\}_{i \in \mathcal{N}} \text{ s.t. } \sigma_{K,i} = \widehat{\sigma}_i \circ \psi_K. \tag{2.5b}$$

Proposition 2.1 (Finite element). *The triple (K, P_K, Σ_K) is a finite element.*

Proof. Note first that $\dim(P_K) = \dim(\widehat{P}) = n_{\mathrm{sh}}$ since ψ_K is bijective. Moreover, a function $p \in P_K$ such that $\sigma_{K,i}(p) = 0$ for all $i \in \mathcal{N}$ is such that $\psi_K(p) = 0$ by the unisolvence property of the reference finite element; hence, $p = 0$. Finally, the linear forms $\sigma_{K,i}$ are in $\mathcal{L}(V(K); \mathbb{R})$ since $|\sigma_{K,i}(v)| \leq \|\widehat{\sigma}_i\|_{\mathcal{L}(V(\widehat{K});\mathbb{R})} \|\psi_K\|_{\mathcal{L}(V(K);V(\widehat{K}))} \|v\|_{V(K)}$, for all $v \in V(K)$. ◻

The above definitions lead us to consider the canonical interpolation operator associated with the finite element (K, P_K, Σ_K):

$$\mathcal{I}_K(v)(\boldsymbol{x}) = \sum_{i \in \mathcal{N}} \sigma_{K,i}(v) \theta_{K,i}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in K, \quad \forall v \in V(K), \tag{2.6}$$

where we have set $\theta_{K,i} := \psi_K^{-1}(\widehat{\theta}_i)$. Note that $\mathcal{I}_K \in \mathcal{L}(V(K); P_K)$ and that P_K is point-wise invariant under $\mathcal{I}_K$.

Since the mesh is affine, we assume that ψ_K has a simple structure; more precisely, we assume that there is a $q \times q$ invertible matrix $\mathbb{A}_K$ such that

$$\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K), \tag{2.7}$$

so that we can extend ψ_K to $L^1(K;\mathbb{R}^q)$. The following classical result shows that ψ_K maps $W^{l,p}(K;\mathbb{R}^q)$ to $W^{l,p}(\widehat{K};\mathbb{R}^q)$ for all $l\in\mathbb{N}$ and all $p\in[1,\infty]$. In particular, this implies that $P_K\subset W^{1,\infty}(K;\mathbb{R}^q)$.

LEMMA 2.2 (Bound in Sobolev norms). *Let $l\in\mathbb{N}$. There is a uniform constant c depending on the shape-regularity of the mesh sequence $(\mathcal{T}_h)_{h>0}$ and on l such that the following holds:*

$$|\psi_K|_{\mathcal{L}(W^{l,p}(K;\mathbb{R}^q);W^{l,p}(\widehat{K};\mathbb{R}^q))}\le c\,\|\mathbb{A}_K\|_{\ell^2}\,\|\mathbb{J}_K\|_{\ell^2}^l\,|\det(\mathbb{J}_K)|^{-\frac{1}{p}}, \tag{2.8a}$$

$$|\psi_K^{-1}|_{\mathcal{L}(W^{l,p}(\widehat{K};\mathbb{R}^q);W^{l,p}(K;\mathbb{R}^q))}\le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\,\|\mathbb{J}_K^{-1}\|_{\ell^2}^l\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}, \tag{2.8b}$$

for all $K\in\mathcal{T}_h$ and all $p\in[1,\infty]$ (with $z^{\pm\frac{1}{p}}=1$, $\forall z>0$ if $p=\infty$).

Proof. For any multilinear map $A\in\mathcal{M}_l(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)$, let us set

$$\|A\|_{\mathcal{M}_l(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)}:=\sup_{(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_l)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d}\frac{\|A(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_l)\|_{\ell^2}}{\|\boldsymbol{y}_1\|_{\ell^2}\cdots\|\boldsymbol{y}_l\|_{\ell^2}}.$$

Then, denoting by $D^l\psi_K(v)$ the l-order Frechet derivative of ψ_K at v, the assumption (2.7) implies that $\|D^l\psi_K(v)\|_{\mathcal{M}_l(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)}\le\|\mathbb{A}_K\|_{\ell^2}\|D^l(v\circ\boldsymbol{T}_K)\|_{\mathcal{M}_l(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)}$ for all $l\in\mathbb{N}$. Then, standard results about the transformation of seminorms in the Sobolev space $W^{l,p}$ using the pullback by $\boldsymbol{T}_K$ lead to (2.8), see e.g., Ciarlet [6, Thm. 3.1.2] or Ern and Guermond [10, Lemma 1.101]. ☐

COROLLARY 2.3 (Bound on $\mathbb{A}_K$). *Assume that there is a uniform constant c so that*

$$\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2}\le c\,\|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}. \tag{2.9}$$

Then, for all $s,m\in\mathbb{N}$, the following holds:

$$|\psi_K^{-1}|_{\mathcal{L}(W^{m,p}(\widehat{K});W^{m,p}(K))}|\psi_K|_{\mathcal{L}(W^{s,p}(K);W^{s,p}(\widehat{K}))}\le c\,h_K^{s-m}, \tag{2.10}$$

for all $K\in\mathcal{T}_h$ and all $p\in[1,\infty]$.

Proof. Combine (2.8) with (2.2) and (2.9). ☐

2.3. Abstract finite element spaces. Let $\{(K,P_K,\Sigma_K)\}_{K\in\mathcal{T}_h}$ be a $\mathcal{T}_h$-based family of finite elements constructed as in Proposition 2.1. We introduce the broken finite element space

$$P^{\mathrm{b}}(\mathcal{T}_h)=\{v_h\in L^1(D;\mathbb{R}^q)\mid\psi_K(v_{h|K})\in\widehat{P},\,\forall K\in\mathcal{T}_h\}. \tag{2.11}$$

Note that the statement $\psi_K(v_{h|K})\in\widehat{P}$ in (2.11) is equivalent to $v_{h|K}\in P_K$. Recall that $P_K\subset L^1(K;\mathbb{R}^q)$ for all $K\in\mathcal{T}_h$, so that $P^{\mathrm{b}}(\mathcal{T}_h)$ is indeed a subspace of $L^1(D;\mathbb{R}^q)$. Actually, since $P_K\subset W^{1,\infty}(K;\mathbb{R}^q)$, we infer that $P^{\mathrm{b}}(\mathcal{T}_h)\subset W^{1,p}(\mathcal{T}_h;\mathbb{R}^q):=\{v\in L^p(D;\mathbb{R}^q)\mid v_{|K}\in W^{1,p}(K;\mathbb{R}^q),\ \forall K\in\mathcal{T}_h\}$ with $p=\infty$.

We further asume to have at hand a Banach space $W\hookrightarrow L^1(D;\mathbb{R}^q)$ (the symbol $\hookrightarrow$ means that the embedding is continuous) and we define

$$P(\mathcal{T}_h):=P^{\mathrm{b}}(\mathcal{T}_h)\cap W. \tag{2.12}$$

To better formalize this definition, we introduce the notion of interfaces and jump across interfaces. We say that a subset $F\subset\overline{D}$ is an interface if it has positive $(d{-}1)$-dimensional measure and if there are distinct mesh cells $K_l,K_r\in\mathcal{T}_h$ such that

$F = \partial K_l \cap \partial K_r$. The numbering of the two mesh cells is arbitrary, but kept fixed once and for all, and we let $\boldsymbol{n}_F$ be the unit normal vector to F pointing from K_l to K_r. We denote by $\boldsymbol{n}_{K_l}$ and $\boldsymbol{n}_{K_r}$ the outward unit normal of K_l and K_r. We say that a subset $F \subset \overline{D}$ is a boundary face if it has positive $(d-1)$-dimensional measure and if there is a mesh cell $K \in \mathcal{T}_h$ such that $F = \partial K \cap \partial D$, and we let $\boldsymbol{n}_F$ be the unit normal vector to F pointing outward D. Interfaces are collected in the set $\mathcal{F}_h^\circ$, boundary faces in the set $\mathcal{F}_h^\partial$, and we let $\mathcal{F}_h = \mathcal{F}_h^\circ \cup \mathcal{F}_h^\partial$. Let $F \in \mathcal{F}_h^\circ$ be a mesh interface, and let K_l, K_r be the two cells such that $F = \partial K_l \cap \partial K_r$; the jump of $v \in W^{1,1}(\mathcal{T}_h; \mathbb{R}^q)$ across F is defined to be

$$[\![v]\!]_F(\boldsymbol{x}) = v_{|K_l}(\boldsymbol{x}) - v_{|K_r}(\boldsymbol{x}) \qquad \text{a.e. } \boldsymbol{x} \in F, \tag{2.13}$$

and we recall that functions in $W^{1,1}(D; \mathbb{R}^q)$ are such that $[\![v]\!]_F = 0$ for all $F \in \mathcal{F}_h^\circ$, see, e.g., Di Pietro and Ern [9, Lemma 1.23]. We next define the notion of γ-jump across interfaces by means of a (bounded) linear operator $\gamma_K : W^{1,1}(K; \mathbb{R}^q) \longrightarrow L^1(\partial K; \mathbb{R}^t)$, for some $t \geq 1$, as follows:

$$[\![v]\!]_F^\gamma(\boldsymbol{x}) = \gamma_{K_l}(v_{|K_l})(\boldsymbol{x}) - \gamma_{K_r}(v_{|K_r})(\boldsymbol{x}) \qquad \text{a.e. } \boldsymbol{x} \in F. \tag{2.14}$$

We assume that $|[\![v]\!]_F^\gamma(\boldsymbol{x})| \leq |[\![v]\!]_F(\boldsymbol{x})|$, a.e. $\boldsymbol{x} \in F$, for all $v \in W^{1,1}(\mathcal{T}_h)$, so that

$$v \in W^{1,1}(D; \mathbb{R}^q) \implies [\![v]\!]_F^\gamma = 0, \ \forall F \in \mathcal{F}_h^\circ. \tag{2.15}$$

The notion of γ-jump is related to the space W by the assumption that

$$v \in W \cap W^{1,1}(\mathcal{T}_h; \mathbb{R}^q) \implies [\![v]\!]_F^\gamma = 0, \ \forall F \in \mathcal{F}_h^\circ, \tag{2.16}$$

and conversely that a function in $W^{1,\infty}(\mathcal{T}_h; \mathbb{R}^q)$ with zero γ-jumps across interfaces is in W. With this setting, definition (2.12) becomes

$$P(\mathcal{T}_h) = \{v_h \in P^{\mathrm{b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F^\gamma = 0, \ \forall F \in \mathcal{F}_h^\circ\}. \tag{2.17}$$

2.4. Finite element examples. The present theory is quite general and covers a large class of scalar- and vector-valued finite elements. For instance, it covers finite elements of Lagrange, Nédélec, and Raviart-Thomas type. To remain general, we denote the three reference elements corresponding to these three classes as follows: $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$, $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{c}}, \widehat{\Sigma}^{\mathrm{c}})$ and $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{d}}, \widehat{\Sigma}^{\mathrm{d}})$. We think of $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$ as a scalar-valued finite element ($q = 1$) that has some degrees of freedom which require point evaluation (i.e., evaluations over zero-dimensional manifolds), for instance $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$ could be a Lagrange element. We assume that the finite element $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{c}}, \widehat{\Sigma}^{\mathrm{c}})$ is vector-valued ($q = d$) and some of its degrees of freedom require to evaluate integrals over edges (i.e., evaluations over one-dimensional manifolds). Typically, $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{c}}, \widehat{\Sigma}^{\mathrm{c}})$ is a Nédélec-type or edge element. Likewise, the finite element $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{d}}, \widehat{\Sigma}^{\mathrm{d}})$ is assumed to be vector-valued ($q = d$) and some of its degrees of freedom are assumed to require evaluation of integrals over $(d-1)$-manifolds. Typically, $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{d}}, \widehat{\Sigma}^{\mathrm{d}})$ is a Raviart-Thomas-type element. It is not necessary to know the exact nature of the element that we are handling at the moment. We denote by $V^{\mathrm{g}}(\widehat{K})$, $\boldsymbol{V}^{\mathrm{c}}(\widehat{K})$, $\boldsymbol{V}^{\mathrm{d}}(\widehat{K})$ admissible domains of the degrees of freedom in the three cases. Let $p \in [1, \infty]$. The above assumptions imply that we can choose $V^{\mathrm{g}}(\widehat{K}) = W^{s,p}(\widehat{K})$ with $s > \frac{d}{p}$, $\boldsymbol{V}^{\mathrm{c}}(\widehat{K}) = \boldsymbol{W}^{s,p}(\widehat{K})$ with $s > \frac{d-1}{p}$, and $\boldsymbol{V}^{\mathrm{d}}(\widehat{K}) = \boldsymbol{W}^{s,p}(\widehat{K})$ with $s > \frac{1}{p}$. Actually, when $p = 1$ we can choose $V^{\mathrm{g}}(\widehat{K}) = W^{d,1}(\widehat{K})$ (since $W^{d,1}(\widehat{K}) \hookrightarrow \mathcal{C}^0(\widehat{K})$), $\boldsymbol{V}^{\mathrm{d}}(\widehat{K}) = \boldsymbol{W}^{1,1}(\widehat{K})$ (since functions in $W^{1,1}(\widehat{K})$ have a trace in $L^1(\partial\widehat{K})$), and $\boldsymbol{V}^{\mathrm{c}}(\widehat{K}) = \boldsymbol{W}^{d-1,1}(\widehat{K})$ (since functions in $W^{d-1,1}(\widehat{K})$ have traces in L^1 on the one-dimensional edges of $\widehat{K}$).

Let $\mathcal{T}_h$ be a mesh in the sequence $(\mathcal{T}_h)_{h>0}$ and let K be a cell in $\mathcal{T}_h$. We denote by ψ_K^{g}, $\boldsymbol{\psi}_K^{\mathrm{c}}$, $\boldsymbol{\psi}_K^{\mathrm{d}}$ the linear map introduced in (2.4) in each case. In practice ψ_K^{g} is the pullback by $\boldsymbol{T}_K$, and ψ_K^{c} and ψ_K^{d} are the contravariant and covariant Piola transformations, respectively, i.e.,

$$\mathbb{A}_K^{\mathrm{g}} = 1, \qquad \psi_K^{\mathrm{g}}(v) = v \circ \boldsymbol{T}_K, \tag{2.18a}$$

$$\mathbb{A}_K^{\mathrm{c}} = \mathbb{J}_K^{\mathsf{T}}, \qquad \boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) = \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v} \circ \boldsymbol{T}_K), \tag{2.18b}$$

$$\mathbb{A}_K^{\mathrm{d}} = \det(\mathbb{J}_K)\,\mathbb{J}_K^{-1}, \qquad \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) = \det(\mathbb{J}_K)\,\mathbb{J}_K^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_K). \tag{2.18c}$$

Note that $c = 1$ in (2.9) for the above examples.

The corresponding broken finite element spaces are:

$$P^{\mathrm{g,b}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in L^1(D) \mid \psi_K^{\mathrm{g}}(\boldsymbol{v}_{h|K}) \in \widehat{P}^{\mathrm{g}},\ \forall K \in \mathcal{T}_h\}, \tag{2.19a}$$

$$\boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in \boldsymbol{L}^1(D) \mid \boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{c}},\ \forall K \in \mathcal{T}_h\}, \tag{2.19b}$$

$$\boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in \boldsymbol{L}^1(D) \mid \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{d}},\ \forall K \in \mathcal{T}_h\}. \tag{2.19c}$$

We introduce $V^{\mathrm{g}} := \{v \in L^1(D) \mid \nabla v \in \boldsymbol{L}^1(D)\}$, $\boldsymbol{V}^{\mathrm{c}} := \{\boldsymbol{v} \in \boldsymbol{L}^1(D) \mid \nabla{\times}\boldsymbol{v} \in \boldsymbol{L}^1(D)\}$, $\boldsymbol{V}^{\mathrm{d}} := \{\boldsymbol{v} \in \boldsymbol{L}^1(D) \mid \nabla{\cdot}v \in L^1(D)\}$. This leads us to consider the following γ-traces:

$$\gamma_K^{\mathrm{g}}(v_{|K})(\boldsymbol{x}) = v_{|K}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in F, \tag{2.20a}$$

$$\gamma_K^{\mathrm{c}}(\boldsymbol{v}_{|K})(\boldsymbol{x}) = \boldsymbol{v}_{|K}(\boldsymbol{x}){\times}\boldsymbol{n}_K, \quad \forall \boldsymbol{x} \in F, \tag{2.20b}$$

$$\gamma_K^{\mathrm{d}}(\boldsymbol{v}_{|K})(\boldsymbol{x}) = \boldsymbol{v}_{|K}(\boldsymbol{x}){\cdot}\boldsymbol{n}_K, \qquad \forall \boldsymbol{x} \in F, \tag{2.20c}$$

and the following conforming finite element spaces:

$$P^{\mathrm{g}}(\mathcal{T}_h) := P^{\mathrm{g,b}}(\mathcal{T}_h) \cap V^{\mathrm{g}} = \{v_h \in P^{\mathrm{g,b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F^{\mathrm{g}} = 0,\ \forall F \in \mathcal{F}_h^{\circ}\}, \tag{2.21a}$$

$$\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) := \boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) \cap \boldsymbol{V}^{\mathrm{c}} = \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} = 0,\ \forall F \in \mathcal{F}_h^{\circ}\}, \tag{2.21b}$$

$$\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) := \boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) \cap \boldsymbol{V}^{\mathrm{d}} = \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}} = 0,\ \forall F \in \mathcal{F}_h^{\circ}\}, \tag{2.21c}$$

where we slightly simplified the notation by using $[\![v_h]\!]_F^{\mathrm{g}}$ instead of $[\![v_h]\!]_F^{\gamma^{\mathrm{g}}}$, etc.

Let us introduce the canonical interpolation operators $\mathcal{I}_h^{\mathrm{g}}$, $\mathcal{I}_h^{\mathrm{d}}$, $\mathcal{I}_h^{\mathrm{c}}$ such that $\mathcal{I}_h^{\mathrm{g}}(v)_{|K} = \mathcal{I}_K^{\mathrm{g}}(v_{|K})$, $\mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v})_{|K} = \mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}_{|K})$, $\mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v})_{|K} = \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v}_{|K})$. The considerations in §2.4 show that it is legitimate to take $W^{s,p}(D)$, $s > \frac{d}{p}$, for the domain of $\mathcal{I}_h^{\mathrm{g}}$, $\boldsymbol{W}^{s,p}(D)$, $s > \frac{d-1}{p}$, for the domain of $\mathcal{I}_h^{\mathrm{c}}$, and $\boldsymbol{W}^{s,p}(D)$, $s > \frac{1}{p}$, for the domain of $\mathcal{I}_h^{\mathrm{d}}$, i.e., the canonical interpolation operators $\mathcal{I}_h^{\mathrm{g}}$, $\mathcal{I}_h^{\mathrm{c}}$ and $\mathcal{I}_h^{\mathrm{d}}$ are not stable in any $L^p(D)$ (or $\boldsymbol{L}^p(D)$). The objective of this paper is to construct quasi-interpolation operators mapping onto the spaces $P^{\mathrm{g}}(\mathcal{T}_h)$, $\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h)$ and $\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h)$ that are stable in $L^1(D)$ (or $\boldsymbol{L}^1(D)$) and have optimal approximation properties with and without boundary conditions.

2.5. Summary of the assumptions. Let us now summarize the assumptions that will be used in the rest of the paper. Henceforth $(\mathcal{T}_h)_{h>0}$ is a shape-regular sequence of affine, matching, simplicial meshes so that (2.1) and (2.2) hold. We also assume that the map ψ_K satisfies (2.7) and (2.9). $\{(K, P_K, \Sigma_K)\}_{K\in\mathcal{T}_h}$ is a $\mathcal{T}_h$-based sequence of finite elements constructed as in Proposition 2.1. In view of approximation, we let k be the largest natural number such that $[\mathbb{P}_{k,d}]^q \subset \widehat{P}$, where $\mathbb{P}_{k,d}$ is the real vector space of d-variate polynomials functions of degree at most k, and we assume that $\widehat{P} \subset W^{k+1,\infty}(\widehat{K};\mathbb{R}^q)$.

We assume that we have at hand a Banach space W and a notion of γ-jump across mesh interfaces as described in §2.3. The W-conforming finite element space $P(\mathcal{T}_h)$ is the subspace of the broken finite element space $P^{\mathrm{b}}(\mathcal{T}_h)$ characterized by zero γ-jumps across interfaces, see (2.17). Finally, two important assumptions relating the degrees of freedom to the γ-jump and γ-trace are the estimates (4.2) and (6.5) below.

In what follows, c denotes a generic positive constant whose value may depend on the shape-regularity of the mesh sequence $(\mathcal{T}_h)_{h>0}$ and on the reference finite element $(\widehat{K},\widehat{P},\widehat{\Sigma})$. The value of this constant may vary from one occurrence to the other.

3. L^1-stable local interpolation. In this section we extend the degrees of freedom in order to be able to approximate functions that are only integrable.

3.1. Extension of degrees of freedom. Let us consider $\widehat{\rho}_i \in \widehat{P}$, $i \in \mathcal{N}$, be such that

$$\frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{\rho}_i{\cdot}\widehat{p}\,\mathrm{d}\widehat{x} = \widehat{\sigma}_i(\widehat{p}), \quad \forall \widehat{p} \in \widehat{P}. \tag{3.1}$$

Note that $\widehat{\rho}_i$ is well defined since it is the Riesz representative of $\widehat{\sigma}_i$ in $\widehat{P}$ when $\widehat{P}$ is equipped with the L^2-scalar product weighted by $1/|\widehat{K}|$. This leads us to define

$$\widehat{\sigma}_i^\sharp(\widehat{v}) := \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{\rho}_i{\cdot}\widehat{v}\,\mathrm{d}\widehat{x}, \quad \forall \widehat{v} \in L^1(\widehat{K};\mathbb{R}^q). \tag{3.2}$$

Note that the assumption $\widehat{P} \subset L^\infty(\widehat{K};\mathbb{R}^q)$ implies that $\widehat{\rho}_i \in L^\infty(\widehat{K};\mathbb{R}^q)$, which in turn implies that all the extended degrees of freedom $\{\widehat{\sigma}_i^\sharp\}_{i\in\mathcal{N}}$ are indeed bounded over $L^1(\widehat{K};\mathbb{R}^q)$ since $\|\widehat{\sigma}_i^\sharp\|_{\mathcal{L}(L^1(\widehat{K};\mathbb{R}^q);\mathbb{R})} \le |\widehat{K}|^{-1}\|\widehat{\rho}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}$. In passing we have also proved that $\|\widehat{\sigma}_i^\sharp\|_{\mathcal{L}(L^p(\widehat{K};\mathbb{R}^q);\mathbb{R})} \le |\widehat{K}|^{-1}\|\widehat{\rho}_i\|_{L^{p'}(\widehat{K};\mathbb{R}^q)}$ for all $p \in [1,\infty]$, where $\frac{1}{p}+\frac{1}{p'}=1$. We then define

$$\mathcal{I}_{\widehat{K}}^\sharp(\widehat{v}) := \sum_{i\in\mathcal{N}} \widehat{\sigma}_i^\sharp(\widehat{v})\widehat{\theta}_i, \qquad \forall \widehat{v} \in L^1(\widehat{K};\mathbb{R}^q). \tag{3.3}$$

$\widehat{P}$ is point-wise invariant under $\mathcal{I}_{\widehat{K}}^\sharp$ since $\widehat{\sigma}_i^\sharp(\widehat{p}) = \widehat{\sigma}_i(\widehat{p})$ for all $\widehat{p} \in \widehat{P}$ and all $i \in \mathcal{N}$.

Let $K \in \mathcal{T}_h$ and let (K, P_K, Σ_K) be a finite element constructed as in (2.5). Note that the assumption (2.7) implies that $\psi_K(L^1(K;\mathbb{R}^q)) = L^1(\widehat{K};\mathbb{R}^q)$. We then extend the degrees of freedom in Σ_K over $L^1(K;\mathbb{R}^q)$ by setting

$$\sigma_{K,i}^\sharp(v) := \widehat{\sigma}_i^\sharp(\psi_K(v)). \tag{3.4}$$

The Riesz representative of $\sigma_{K,i}^\sharp$ in P_K equipped with the L^2-scalar product weighted by $1/|K|$ is $\mathbb{A}_K^{\mathsf{T}}(\widehat{\rho}_i \circ \boldsymbol{T}_K^{-1})$. The above definition leads us to define

$$\mathcal{I}_K^\sharp(v) := \sum_{i\in\mathcal{N}} \sigma_{K,i}^\sharp(v)\theta_{K,i}, \qquad \forall v \in L^1(K;\mathbb{R}^q). \tag{3.5}$$

Proposition 3.1 (Stability, commutation, invariance). *(i) There exists a uniform constant c such that $\|\mathcal{I}_K^\sharp\|_{\mathcal{L}(L^p(K;\mathbb{R}^q);L^p(K;\mathbb{R}^q))} \le c$, for all $p \in [1,\infty]$ and all $K \in \mathcal{T}_h$; (ii) $\mathcal{I}_K^\sharp$ commutes with ψ_K; (iii) P_K is point-wise invariant under $\mathcal{I}_K^\sharp$.*

Proof. Using (2.8) with $l=0$ and recalling that $\theta_{K,i} := \psi_K^{-1}(\widehat{\theta}_i)$ by definition and using the assumption (2.9), we infer that

$$\begin{aligned}\|\mathcal{I}_K^\sharp\|_{\mathcal{L}(L^p(K;\mathbb{R}^q);L^p(K;\mathbb{R}^q))} &\le \|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2}\sum_{i\in\mathcal{N}}\|\widehat{\sigma}_i^\sharp\|_{\mathcal{L}(L^p(\widehat{K};\mathbb{R}^q);\mathbb{R})}\|\widehat{\theta}_i\|_{L^p(\widehat{K};\mathbb{R}^q)}\\ &\le c\,\|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}|\widehat{K}|^{-1}\sum_{i\in\mathcal{N}}\|\widehat{\rho}_i\|_{L^{p'}(\widehat{K};\mathbb{R}^q)}.\end{aligned}$$

The conclusion readily follows from the shape-regularity assumptions. To prove the second statement, we use again that $\theta_{K,i} = \psi_K^{-1}(\widehat{\theta}_i)$ to infer that

$$\psi_K\left(\mathcal{I}_K^\sharp(v)\right) := \psi_K\left(\sum_{i\in\mathcal{N}}\sigma_{K,i}^\sharp(v)\psi_K^{-1}(\widehat{\theta}_i)\right) = \sum_{i\in\mathcal{N}}\widehat{\sigma}_i^\sharp(\psi_K(v))\widehat{\theta}_i = \mathcal{I}_{\widehat{K}}^\sharp(\psi_K(v)),$$

for all $v\in L^1(K;\mathbb{R}^q)$. To prove the third statement, let us consider any $p\in P_K$; then using the above definitions we have

$$\sigma_{K,i}^\sharp(p) = \widehat{\sigma}_i^\sharp(\psi_K(p)) = \frac{1}{|\widehat{K}|}\int_{\widehat{K}}\widehat{\rho}_i\cdot\psi_K(p)\,\mathrm{d}\widehat{x} = \widehat{\sigma}_i(\psi_K(p)) = \sigma_{K,i}(p).$$

This proves that $\mathcal{I}_K^\sharp(p) = \mathcal{I}_K(p)$, so that $\mathcal{I}_K^\sharp(p) = p$. ∎

3.2. Error estimates for $\mathcal{I}_K^\sharp$. We establish in this section error estimates for the operator $\mathcal{I}_K^\sharp$.

THEOREM 3.2 (Local interpolation). *There exists a uniform constant c such that the following local error estimate holds:*

$$|v-\mathcal{I}_K^\sharp v|_{W^{m,p}(K;\mathbb{R}^q)} \le c\,h_K^{l-m}|v|_{W^{l,p}(K;\mathbb{R}^q)}, \tag{3.6}$$

for all $m\in\{0{:}k+1\}$, all $l\in\{m{:}k+1\}$, all $p\in[1,\infty]$, all $v\in W^{l,p}(K;\mathbb{R}^q)$, and all $K\in\mathcal{T}_h$.

Proof. Let $m\in\{0{:}k+1\}$ and $l\in\{m{:}k+1\}$.
(1) Let us set $\mathcal{G}(\widehat{w}) := \widehat{w}-\mathcal{I}_{\widehat{K}}^\sharp(\widehat{w})$ for all $\widehat{w}\in W^{l,p}(\widehat{K};\mathbb{R}^q)$. The operator $\mathcal{G}$ is well-defined since $W^{l,p}(\widehat{K};\mathbb{R}^q)\hookrightarrow L^1(\widehat{K};\mathbb{R}^q)$. Since all the norms are equivalent in $\widehat{P}$ and $\widehat{P}\subset W^{k+1,p}(\widehat{K};\mathbb{R}^q)\hookrightarrow W^{m,p}(\widehat{K};\mathbb{R}^q)$, there exists c depending only on n_{sh} and $\widehat{K}$ such that $\|\mathcal{I}_{\widehat{K}}^\sharp(\widehat{w})\|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \le c\|\mathcal{I}_{\widehat{K}}^\sharp(\widehat{w})\|_{\boldsymbol{L}^1(\widehat{K};\mathbb{R}^q)}$, which in turns implies that $\|\mathcal{I}_{\widehat{K}}^\sharp(\widehat{w})\|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \le c\|\widehat{w}\|_{L^1(\widehat{K};\mathbb{R}^q)}$, since we have already established that $\mathcal{I}_{\widehat{K}}^\sharp$ is uniformly bounded over $L^1(\widehat{K};\mathbb{R}^q)$; hence, $\mathcal{G}\in\mathcal{L}(W^{l,p}(\widehat{K};\mathbb{R}^q);W^{m,p}(\widehat{K};\mathbb{R}^q))$. Assume first that $l\ge 1$, then $[\mathbb{P}_{l-1}]^q$ is point-wise invariant under $\mathcal{I}_{\widehat{K}}^\sharp$ since $l-1\le k$ and $[\mathbb{P}_{l-1}]^q\subset[\mathbb{P}_k]^q\subset\widehat{P}$; this in turn implies that the operator $\mathcal{G}$ vanishes on $[\mathbb{P}_{l-1}]^q$. As a consequence, we infer that

$$\begin{aligned}|\widehat{w}-\mathcal{I}_{\widehat{K}}^\sharp\widehat{w}|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} = |\mathcal{G}(\widehat{w})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} &= \inf_{\widehat{p}\in[\mathbb{P}_{l-1}]^q}|\mathcal{G}(\widehat{w}+\widehat{p})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)}\\ &\le \|\mathcal{G}\|_{\mathcal{L}(W^{l,p}(\widehat{K};\mathbb{R}^q);W^{m,p}(\widehat{K};\mathbb{R}^q))}\inf_{\widehat{p}\in[\mathbb{P}_{l-1}]^q}\|\widehat{w}+\widehat{p}\|_{W^{l,p}(\widehat{K};\mathbb{R}^q)}\\ &\le c\inf_{\widehat{p}\in[\mathbb{P}_{l-1}]^q}\|\widehat{w}+\widehat{p}\|_{W^{l,p}(\widehat{K};\mathbb{R}^q)} \le c\,|\widehat{w}|_{W^{l,p}(\widehat{K};\mathbb{R}^q)},\end{aligned}$$

for all $\widehat{w}\in W^{l,p}(\widehat{K};\mathbb{R}^q)$, where the last estimate is a consequence of the Bramble–Hilbert/Deny–Lions Lemma. Finally, the above inequality is trivial if $l=m=0$.

(2) Now let $v \in W^{l,p}(K;\mathbb{R}^q)$. Using the above argument together with the fact that $\mathcal{I}_K^\sharp$ commutes with ψ_K (see Proposition 3.1), we have

$$\begin{aligned}
|v - \mathcal{I}_K^\sharp v|_{W^{m,p}(K;\mathbb{R}^q)} &\leq |\psi_K^{-1}|_{\mathcal{L}(W^{m,p}(\widehat{K};\mathbb{R}^q);W^{m,p}(K;\mathbb{R}^q))}\, |\psi_K(v) - \psi_K(\mathcal{I}_K^\sharp v)|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq |\psi_K^{-1}|_{\mathcal{L}(W^{m,p}(\widehat{K};\mathbb{R}^q);W^{m,p}(K;\mathbb{R}^q))}\, |\psi_K(v) - \mathcal{I}_{\widehat{K}}^\sharp(\psi_K(v))|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c\, |\psi_K^{-1}|_{\mathcal{L}(W^{m,p}(\widehat{K};\mathbb{R}^q);W^{m,p}(K;\mathbb{R}^q))}\, |\psi_K(v)|_{W^{l,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c\, |\psi_K^{-1}|_{\mathcal{L}(W^{m,p}(\widehat{K};\mathbb{R}^q);W^{m,p}(K;\mathbb{R}^q))} |\psi_K|_{\mathcal{L}(W^{l,p}(K;\mathbb{R}^q);W^{l,p}(\widehat{K};\mathbb{R}^q))} |v|_{W^{l,p}(K;\mathbb{R}^q)}.
\end{aligned}$$

The estimate (3.6) follows by using (2.10). ☐

4. Averaging operator. In this section, we introduce a bounded linear operator $\mathcal{J}_h^{\mathrm{av}} : P^{\mathrm{b}}(\mathcal{T}_h) \longrightarrow P(\mathcal{T}_h)$ based on averaging.

4.1. Connectivity array. Let $\{\varphi_a\}_{a\in\mathcal{A}_h}$ be a basis of $P(\mathcal{T}_h)$; the functions φ_a are called global shape functions. We assume that this basis is constructed so that for any $K \in \mathcal{T}_h$, there a unique $i \in \mathcal{N}$ such that $\varphi_{a|K} = \theta_{K,i}$. (Recall that this is the usual way of constructing finite element bases.) We denote by $\mathsf{a} : \mathcal{T}_h \times \mathcal{N} \longrightarrow \mathcal{A}_h$ the mapping such that $\varphi_{\mathsf{a}(K,i)|K} = \theta_{K,i}$; this mapping is henceforth called connectivity array. This leads us to introduce the connectivity set $\mathcal{C}_a \subset \mathcal{T}_h \times \mathcal{N}$ for any $a \in \mathcal{A}_h$ such that

$$\mathcal{C}_a := \mathsf{a}^{-1}(a) = \{(K,i) \in \mathcal{T}_h \times \mathcal{N} \mid a = \mathsf{a}(K,i)\}. \tag{4.1}$$

We denote $\{\varsigma_a\}_{a\in\mathcal{A}_h}$ the global degrees of freedom associated with the basis $\{\varphi_a\}_{a\in\mathcal{A}_h}$, i.e., $\varsigma_{\mathsf{a}(K,i)}(\boldsymbol{v}) := \sigma_{K,i}(\boldsymbol{v}_{|K})$.

Remark 4.1. *(Particular case* $\mathrm{card}(\mathcal{C}_a) = 1$*)* Assume that $\mathrm{card}(\mathcal{C}_a) = 1$, i.e., $\mathcal{C}_a = \{(K_0, i_0)\}$, then $\varphi_{a|K} = 0$ for any $K \neq K_0$, since it is not possible to find an index $i \in \mathcal{N}$ such that $a = \mathsf{a}(K,i)$. This means that φ_a is supported on one element only, i.e., φ_a is the zero extension of θ_{K_0,i_0}. Given the characterization of $P(\mathcal{T}_h)$ assumed in (2.17), this means that the γ-trace φ_a on the interior faces of K is zero.

For any $a \in \mathcal{A}_h$, we set $\mathcal{F}_a^\circ = \emptyset$ if $\mathrm{card}(\mathcal{C}_a) = 1$. If $\mathrm{card}(\mathcal{C}_a) \geq 2$ we define $\mathcal{F}_a^\circ \subset \mathcal{F}_h^\circ$ to be the set of the interfaces F such that there are $(K,i),(K',i') \in \mathcal{C}_a$ so that $F = K \cap K'$. We henceforth denote $\mathcal{F}_K^\circ := \cup_{i\in\mathcal{N}} \mathcal{F}_{\mathsf{a}(K,i)}^\circ$. We now relate the γ-traces to the degrees of freedom by making the following assumption: there exists a uniform constant c such that the following holds for all v in $P^{\mathrm{b}}(\mathcal{T}_h)$ and all $a \in \mathcal{A}_h$ such that $\mathrm{card}(\mathcal{C}_a) \geq 2$:

$$|\sigma_{K,i}(v) - \sigma_{K',i'}(v)| \leq c \max(\|A_K\|_{\ell^2}, \|A_{K'}\|_{\ell^2}) \|[\![v]\!]_F^\gamma\|_{L^\infty(F;\mathbb{R}^t)}, \tag{4.2}$$

for all $F \in \mathcal{F}_a^\circ$ and all pairs $(K,i),(K',i') \in \mathcal{C}_a$ such that $F = K \cap K'$. Owing to (2.17), this assumption immediately implies that

$$|\sigma_{K,i}(v) - \sigma_{K',i'}(v)| = 0, \qquad \forall v \in P(\mathcal{T}_h), \tag{4.3}$$

Note that estimate (4.2) holds true for all the finite elements considered in §2.4.

4.2. Averaging operator. We now define the operator $\mathcal{J}_h^{\mathrm{av}} : P^{\mathrm{b}}(\mathcal{T}_h) \longrightarrow P(\mathcal{T}_h)$ such that the following holds for all $v \in P^{\mathrm{b}}(\mathcal{T}_h)$ and all $a \in \mathcal{A}_h$:

$$\varsigma_a(\mathcal{J}_h^{\mathrm{av}}(v)) = \frac{1}{\mathrm{card}(\mathcal{C}_a)} \sum_{(K,i)\in\mathcal{C}_a} \sigma_{K,i}(v_{|K}). \tag{4.4}$$

Note that this definition is sufficient to define $\mathcal{J}_h^{\mathrm{av}}(v) \in P^{\mathrm{b}}(\mathcal{T}_h)$ since the property $\varsigma_a(\varphi_{a'}) = \delta_{aa'}$ implies that $\mathcal{J}_h^{\mathrm{av}}(v) = \sum_{a\in\mathcal{A}_h} \varsigma_a(\mathcal{J}_h^{\mathrm{av}}(v))\varphi_a$.

LEMMA 4.1 (Bound on L^∞-norm). *There exists a uniform constant c such that*

$$\|w\|_{L^\infty(K;\mathbb{R}^q)} \le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2} \sum_{i\in\mathcal{N}} |\sigma_{K,i}(w)|, \qquad \forall w \in P_K,\ \forall K \in \mathcal{T}_h. \tag{4.5}$$

Proof. Recall first that the degrees of freedom of the finite element (K, P_K, Σ_K) are defined by $\sigma_{K,i}(w) := \widehat{\sigma}_i(\psi_K(w))$ for all $w \in P_K$, and the local shape functions are $\theta_{K,i} = \psi_K^{-1}(\widehat{\theta}_i)$. This implies that $w_{|K} = \sum_{i\in\mathcal{N}} \sigma_{K,i}(w)\theta_{K,i} = \sum_{i\in\mathcal{N}} \sigma_{K,i}(w)\psi_K^{-1}(\widehat{\theta}_i)$, for all $w \in P_K$. Then,

$$\|w\|_{L^\infty(K;\mathbb{R}^q)} \le \sum_{i\in\mathcal{N}} |\sigma_{K,i}(w)|\|\psi_K^{-1}\|_{\mathcal{L}(L^\infty(\widehat{K};\mathbb{R}^q);L^\infty(K;\mathbb{R}^q))}\|\widehat{\theta}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}.$$

The quantity $c_0 := \max_{i\in\mathcal{N}} \|\widehat{\theta}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}$ is a uniform constant that depends only on the reference element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$. Using (2.8b) with $l = 0$ and $p = \infty$, we infer that $\|\psi_K^{-1}\|_{\mathcal{L}(L^\infty(\widehat{K};\mathbb{R}^q);L^\infty(K;\mathbb{R}^q))} \le \|\mathbb{A}_K^{-1}\|_{\ell^2}$. The conclusion follows readily. ∎

LEMMA 4.2 (Approximation by averaging). *There exists a uniform constant c such that the following holds:*

$$|w - \mathcal{J}_h^{\mathrm{av}}(w)|_{W^{m,p}(K;\mathbb{R}^q)} \le c h_K^{d\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{r}-m} \sum_{F\in\mathcal{F}_K^\circ} \|[\![w]\!]_F^\gamma\|_{L^r(F;\mathbb{R}^t)} \tag{4.6}$$

for all $m \in \{0{:}k+1\}$, all $p, r \in [1,\infty]$, all $w \in P^{\mathrm{b}}(\mathcal{T}_h)$, and all $K \in \mathcal{T}_h$.

Proof. We only prove the bound for $m = 0$ and $p = r = \infty$, the other cases follow by invoking standard inverse inequalities. Let $w \in P^{\mathrm{b}}(\mathcal{T}_h)$, set $e = w - \mathcal{J}_h^{\mathrm{av}}(w)$ and observe that $e \in P^{\mathrm{b}}(\mathcal{T}_h)$. Let $K \in \mathcal{T}_h$. Then, using the result of Lemma 4.1, we infer that

$$\|e\|_{L^\infty(K;\mathbb{R}^q)} \le c\|\mathbb{A}_K^{-1}\|_{\ell^2} \sum_{i\in\mathcal{N}} |\sigma_{K,i}(e_{|K})|.$$

Owing to definition (4.4), we first observe that

$$\sigma_{K,i}(e_{|K}) = \frac{1}{\operatorname{card}(\mathcal{C}_{\mathsf{a}(K,i)})} \sum_{(K',i')\in\mathcal{C}_{\mathsf{a}(K,i)}} \left(\sigma_{K,i}(w_{|K}) - \sigma_{K',i'}(w_{|K'})\right).$$

Note that $\sigma_{K,i}(e_{|K}) = 0$ if $\operatorname{card}(\mathcal{C}_{\mathsf{a}(K,i)}) = 1$ (see Remark 4.1). Let us now consider the degrees of freedom such that $\operatorname{card}(\mathcal{C}_{\mathsf{a}(K,i)}) \ge 2$. Let $\mathcal{F}_{\mathsf{a}(K,i)}^\circ$ be the set of internal faces in $\mathcal{C}_{\mathsf{a}(K,i)}$. For all $K' \in \mathcal{C}_{\mathsf{a}(K,i)}$, there is a path of mesh cells in $\mathcal{C}_{\mathsf{a}(K,i)}$ linking K to K' so that any two consecutive mesh cells in the path share a common face $F \in \mathcal{F}_{\mathsf{a}(K,i)}^\circ$, and each face is crossed only once. Furthermore, if $(K_l, i_l), (K_r, i_r) \in \mathcal{C}_{\mathsf{a}(K,i)}$ are such that $K_l \cap K_r = F \in \mathcal{F}_{\mathsf{a}(K,i)}^\circ$, then (4.2) implies that there exists a uniform constant c such that

$$|\sigma_{K_l,i_l}(w_{|K_l}) - \sigma_{K_r,i_r}(w_{|K_r})| \le c\max(\|\mathbb{A}_{K_l}\|_{\ell^2}, \|\mathbb{A}_{K_r}\|_{\ell^2})\|[\![w]\!]_F^\gamma\|_{L^\infty(F;\mathbb{R}^t)}.$$

As a result,

$$\|e\|_{L^\infty(K;\mathbb{R}^q)} \le c \max_{i\in\mathcal{N}} \max_{(K',i')\in\mathcal{C}_{\mathsf{a}(K,i)}} \left(\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{A}_{K'}\|_{\ell^2}\right) \sum_{F\in\mathcal{F}_{\mathsf{a}(K,i)}^\circ} \|[\![w]\!]_F^\gamma\|_{L^\infty(F;\mathbb{R}^t)}.$$

whence the estimate (4.6) readily follows since $\mathcal{F}_K^\circ := \cup_{i\in\mathcal{N}}\mathcal{F}_{\mathsf{a}(K,i)}$, $\operatorname{card}(\mathcal{N})$ is uniformly bounded, and the mesh sequence $(\mathcal{T}_h)_{h>0}$ is shape-regular. ∎

5. Quasi-interpolation operator. Let $\mathcal{I}_h^\sharp : L^1(D;\mathbb{R}^q) \longrightarrow P^{\mathrm{b}}(\mathcal{T}_h)$ be such that $\mathcal{I}_h^\sharp(v)_{|K} = \mathcal{I}_K^\sharp(v_{|K})$ for all $K \in \mathcal{T}_h$. We now construct the global quasi-interpolation operator $\mathcal{I}_h^{\mathrm{av}} : L^1(D;\mathbb{R}^q) \longrightarrow P(\mathcal{T}_h)$ by setting

$$\mathcal{I}_h^{\mathrm{av}} := \mathcal{J}_h^{\mathrm{av}} \circ \mathcal{I}_h^\sharp. \tag{5.1}$$

For any $K \in \mathcal{T}_h$, we introduce the notation

$$\mathcal{T}_K := \cup_{i\in\mathcal{N}}\{K' \in \mathcal{T}_h \mid \exists i' \in \mathcal{N},\ (K',i') \in \mathcal{C}_{\mathsf{a}(K,i)}\}, \tag{5.2}$$

$$D_K := \mathrm{int}\{\boldsymbol{x} \in \overline{D} \mid \exists K' \in \mathcal{T}_K,\, \boldsymbol{x} \in K'\}. \tag{5.3}$$

The set $\mathcal{T}_K$ is the union of all the cells that share global shape functions with K and D_K is the interior of the collection of the points composing the cells in $\mathcal{T}_K$.

Lemma 5.1 (L^p-stability). *There exists a uniform constant c such that*

$$\|\mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c\,\|v\|_{L^p(D_K;\mathbb{R}^q)}, \tag{5.4}$$

for all $p \in [1,\infty]$, all $v \in L^p(D;\mathbb{R}^q)$, and all $K \in \mathcal{T}_h$.

Proof. Using the triangle inequality and the shape-regularity of the mesh sequence $(\mathcal{T}_h)_{h>0}$, we infer that (recall that the value of c can change at each occurrence)

$$\begin{aligned}
\|\mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} &\le \sum_{i\in\mathcal{N}} \frac{\|\theta_{K,i}\|_{L^p(K;\mathbb{R}^q)}}{\mathrm{card}(\mathcal{C}_{\mathsf{a}(K,i)})} \sum_{(K',i')\in\mathcal{C}_{\mathsf{a}(K,i)}} \left|\sigma_{K',i'}(\mathcal{I}_{K'}^\sharp(v))\right| \\
&\le c\sum_{i\in\mathcal{N}} \frac{|K|^{\frac1p}\|\mathbb{A}_K^{-1}\|_{\ell^2}}{\mathrm{card}(\mathcal{C}_{\mathsf{a}(K,i)})} \sum_{(K',i')\in\mathcal{C}_{\mathsf{a}(K,i)}} \left|\sigma_{K',i}(\mathcal{I}_{K'}^\sharp(v))\right| \\
&\le c\sum_{i\in\mathcal{N}} \frac{1}{\mathrm{card}(\mathcal{C}_{\mathsf{a}(K,i)})} \sum_{(K',i')\in\mathcal{C}_{\mathsf{a}(K,i)}} \left|\sigma_{K',i'}(\mathcal{I}_{K'}^\sharp(v))\right| |K'|^{\frac1p}\|\mathbb{A}_{K'}^{-1}\|_{\ell^2}. \\
&\le c\sum_{K'\in\mathcal{T}_K}\sum_{i'\in\mathcal{N}} \left|\sigma_{K',i'}(\mathcal{I}_{K'}^\sharp(v))\right| |K'|^{\frac1p}\|\mathbb{A}_{K'}^{-1}\|_{\ell^2}.
\end{aligned}$$

The conclusion follows by invoking the L^p-stability of $\mathcal{I}_{K'}^\sharp$ (see Proposition 3.1) and by observing that the equivalence of norms in $\widehat{P}$ implies that

$$\sum_{i'\in\mathcal{N}} \left|\sigma_{K',i}(\mathcal{I}_{K'}^\sharp(v))\right| |K'|^{\frac1p}\|\mathbb{A}_{K'}^{-1}\|_{\ell^2} \le c\,\|\mathcal{I}_{K'}^\sharp(v)\|_{L^p(K';\mathbb{R}^q)} \le c\|v\|_{L^p(K';\mathbb{R}^q)}.$$

This completes the proof. ∎

Theorem 5.2 (Approximation). *There exists a uniform constant c such that*

$$|v - \mathcal{I}_h^{\mathrm{av}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c h_K^{l-m}|v|_{W^{l,p}(D_K;\mathbb{R}^q)}, \tag{5.5}$$

for all $m \in \{0{:}k+1\}$, all $l \in \{m{:}k+1\}$, all $p \in [1,+\infty]$, all $v \in W^{l,p}(D_K;\mathbb{R}^q)$, and all $K \in \mathcal{T}_h$. Consequently, denoting $h = \max_{K\in\mathcal{T}_h} h_K$, the following holds for all $v \in W^{l,p}(D;\mathbb{R}^q)$:

$$|v - \mathcal{I}_h^{\mathrm{av}}(v)|_{W^{m,p}(\mathcal{T}_h;\mathbb{R}^q)} \le c h^{l-m}|v|_{W^{l,p}(D;\mathbb{R}^q)}. \tag{5.6}$$

Proof. Let $K \in \mathcal{T}_h$ and $v \in W^{l,p}(D_K;\mathbb{R}^q)$. The triangle inequality implies that

$$|v - \mathcal{I}_h^{\mathrm{av}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le |v - \mathcal{I}_K^\sharp(v)|_{W^{m,p}(K;\mathbb{R}^q)} + |\mathcal{I}_K^\sharp(v) - \mathcal{J}_h^{\mathrm{av}}(\mathcal{I}_K^\sharp(v))|_{W^{m,p}(K;\mathbb{R}^q)}.$$

Let $\mathfrak{T}_1$ and $\mathfrak{T}_1$ be the two terms on the right-hand side of the above inequality. $\mathfrak{T}_1$ is estimated using Theorem 3.2, $|v - \mathcal{I}_K^\sharp v|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{s-m} |v|_{W^{s,p}(K;\mathbb{R}^q)}$. $\mathfrak{T}_2$ is estimated using Lemma 4.2 and the fact that $v \in W^{l,p}(D_K;\mathbb{R}^q) \subset W^{1,1}(D_K;\mathbb{R}^q)$ has zero γ-jumps across interfaces (see (2.15)):

$$\begin{aligned} h_K^m |\mathfrak{T}_2| &\le c h_K^{\frac{1}{p}} \sum_{F \in \mathcal{F}_K^\circ} \|[\![\mathcal{I}_K^\sharp(v)]\!]_F^\gamma\|_{L^r(F;\mathbb{R}^t)} = c h_K^{\frac{1}{p}} \sum_{F \in \mathcal{F}_K^\circ} \|[\![v - \mathcal{I}_K^\sharp(v)]\!]_F^\gamma\|_{L^r(F;\mathbb{R}^t)} \\ &\le c h_K^{\frac{1}{p}} \sum_{K' \in \mathcal{T}_K} \sum_{F \subset \partial K' \cap \mathcal{F}_K^\circ} \|(v - \mathcal{I}_K^\sharp(v))_{|K'}\|_{L^p(F;\mathbb{R}^q)} \le c h_K^l \sum_{K' \in \mathcal{T}_K} |v|_{W^{l,p}(K';\mathbb{R}^q)}, \end{aligned}$$

where we have used the triangle inequality to bound the jump by the values over the two adjacent mesh cells, the multiplicative trace inequality from Lemma 5.3 below, the approximation result of Theorem 3.2, and the mesh regularity. Combining the bounds on $\mathfrak{T}_1$ and $\mathfrak{T}_2$ gives (5.5). Finally, (5.6) results from (5.5) and mesh regularity since $\mathrm{card}(\mathcal{T}_K)$ is uniformly bounded with respect to h. ∎

Lemma 5.3 (Multiplicative trace inequality). *Let $(\mathcal{T}_h)_{h>0}$ be a shape-regular sequence of affine meshes in $\mathbb{R}^d$. Consider a cell $K \in \mathcal{T}_h$ and let F be a face of K. Then, there is c, uniform with respect to K, F, and h, such that the following holds:*

$$\|v\|_{L^p(F)} \le c\, \|v\|_{L^p(K)}^{\frac{p-1}{p}} \left(h_K^{-\frac{1}{p}} \|v\|_{L^p(K)}^{\frac{1}{p}} + \|\nabla v\|_{\boldsymbol{L}^p(K)}^{\frac{1}{p}} \right). \tag{5.7}$$

for all $p \in [1,\infty]$ and all $v \in W^{1,p}(K)$.

Proof. See [9, Lemma 1.49]. ∎

We are now in a position to establish error estimates in fractional Sobolev spaces; recall that assuming that $r = m + s$, $m \in \mathbb{N}$, $s \in (0,1)$, we have

$$\|v\|_{W^{r,p}(D;\mathbb{R}^q)} = \left(\|v\|_{W^{m,p}(D;\mathbb{R}^q)}^p + \sum_{|\alpha|=m} \int_D \int_D \frac{\|\partial_\alpha v(\boldsymbol{x}) - \partial_\alpha v(\boldsymbol{y})\|_{\ell^2(\mathbb{R}^q)}^p}{\|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2(\mathbb{R}^d)}^{sp+d}} \,\mathrm{d}x \,\mathrm{d}y \right)^{\frac{1}{p}}. \tag{5.8}$$

Corollary 5.4 (Global approximation in fractional Sobolev spaces). *There exists a uniform constant c such that*

$$\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(D;\mathbb{R}^q)} \le c\, h^r |v|_{W^{r,p}(D;\mathbb{R}^q)}, \tag{5.9}$$

for any real number $r \in [0, k+1]$, all $p \in [1,\infty]$, and all $v \in W^{r,p}(D;\mathbb{R}^q)$.

Proof. Apply the generalized Riesz–Thorin Theorem between $l = [r]$ and $l = [r]+1$ to (5.6) with $m = 0$. ∎

The above result can be localized by using the definition of the $W^{s,p}$-norm.

Theorem 5.5 (Local approximation in fractional Sobolev spaces). *There exists a uniform constant c such that*

$$\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c h_K^r |v|_{W^{r,p}(D_K;\mathbb{R}^q)}. \tag{5.10}$$

for all $r \in [0, k+1]$, all $p \in [1,\infty]$, all $v \in W^{r,p}(D_K;\mathbb{R}^q)$, and all $K \in \mathcal{T}_h$.

Proof. The result has already been proved when r is an integer. Let us assume for the time being that $r = s \in (0,1)$. Let $\overline{v}_{D_K}$ be the average of v over D_K, i.e., $\overline{v}_{D_K} = \frac{1}{|D_K|}\int_{D_K} v(\boldsymbol{x})\,\mathrm{d}x$, then using that $\mathcal{I}_h^{\mathrm{av}}(\overline{v}_{D_K}) = \overline{v}_{D_K}$ together with the L^p-stability of $\mathcal{I}_h^{\mathrm{av}}$ (see Lemma 5.1), we have

$$\begin{aligned}
\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} &= \|v - \overline{v}_{D_K} - \mathcal{I}_h^{\mathrm{av}}(v - \overline{v}_{D_K})\|_{L^p(K;\mathbb{R}^q)} \\
&\le \|v - \overline{v}_{D_K}\|_{L^p(K;\mathbb{R}^q)} + \|\mathcal{I}_h^{\mathrm{av}}(v - \overline{v}_{D_K})\|_{L^p(K;\mathbb{R}^q)} \\
&\le c\,\|v - \overline{v}_{D_K}\|_{L^p(D_K;\mathbb{R}^q)}.
\end{aligned}$$

Using Lemma 5.6 below component-wise, we infer that

$$\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c h_{D_K}^s \left(\frac{h_{D_K}^d}{|D_K|}\right)^{\frac{1}{p}} |v|_{W^{s,p}(D_K;\mathbb{R}^q)}.$$

We conclude by using the shape-regularity of the mesh sequence.

We bootstrap the above argument when $r > 1$. For instance assume that $r \in (1,2)$ and $k \ge 1$. Let $\boldsymbol{x}_G$ be the barycenter of D_K, and let $\overline{v}_{D_K}$ and $\overline{Dv}_{D_K}$ be the averages of v and Dv over D_K, respectively. By proceeding as above, we have

$$\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c\|v - \overline{v}_{D_K} - \overline{Dv}_{D_K}(\boldsymbol{x} - \boldsymbol{x}_G)\|_{L^p(D_K;\mathbb{R}^q)}.$$

Upon observing that $\overline{v - \overline{Dv}_{D_K}(\boldsymbol{x} - \boldsymbol{x}_G)} = \overline{v}_{D_K}$, we infer that

$$\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c h_K \|Dv - \overline{Dv}_{D_K}\|_{L^p(D_K;\mathbb{R}^{d\times q})}.$$

Using again Lemma 5.6 component-wise in $\mathbb{R}^{d\times q}$ with $r - 1 = s \in (0,1)$, we deduce that

$$\|v - \mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c h_K^{1+r-1} |v|_{W^{s,p}(D_K;\mathbb{R}^q)}.$$

The argument is now clear for any non-integer value of r. ◻

Lemma 5.6 (Poincaré inequality in fractional Sobolev spaces). *Let O be an open set in $\mathbb{R}^d$ and let $\overline{v}_O$ be the average of v over O, for any $v \in L^1(O)$. Let $h_O := \mathrm{diam}(O)$. Then, for all $v \in W^{s,p}(O)$ with $s \in (0,1)$ and $p \in [1,\infty]$, the following holds:*

$$\|v - \overline{v}_O\|_{L^p(O)} \le h_O^s \left(\frac{h_O^d}{|O|}\right)^{\frac{1}{p}} |v|_{W^{s,p}(O)}. \tag{5.11}$$

Proof. Using the definitions, we have

$$\begin{aligned}
\int_O |v(\boldsymbol{x}) - \overline{v}_O|^p \mathrm{d}x &= \int_O |O|^{-p} \left|\int_O (v(\boldsymbol{x}) - v(\boldsymbol{y}))\,\mathrm{d}y\right|^p \mathrm{d}x \\
&\le \int_O |O|^{-p} \left(\int_O \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2}^{s+\frac{d}{p}}} \|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2}^{s+\frac{d}{p}}\,\mathrm{d}y\right)^p \mathrm{d}x \\
&\le \int_O |O|^{-p} \int_O \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^p}{\|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2}^{sp+d}}\,\mathrm{d}y \left(\int_O \|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2}^{(s+\frac{d}{p})p'}\,\mathrm{d}y\right)^{\frac{p}{p'}} \mathrm{d}x,
\end{aligned}$$

where $p' := \frac{p}{p-1}$. Then using that $\|\boldsymbol{x}-\boldsymbol{y}\|_{\ell^2} \le h_O$ for all $\boldsymbol{x},\boldsymbol{y}\in O$, we infer that

$$\begin{aligned}
\|v-\overline{v}_O\|^p_{L^p(O)} &\le \int_O |O|^{-p}\int_O \frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|^p}{\|\boldsymbol{x}-\boldsymbol{y}\|^{sp+d}_{\ell^2}}\,\mathrm{d}y\,\mathrm{d}x \left(\max_{\boldsymbol{x}\in O}\int_O \|\boldsymbol{x}-\boldsymbol{y}\|_{\ell^2}^{(s+\frac{d}{p})p'}\,\mathrm{d}y\right)^{\frac{p}{p'}} \\
&\le |v|^p_{W^{s,p}(O)}|O|^{-p}\left(\int_O h_O^{(s+\frac{d}{p})p'}\,\mathrm{d}y\right)^{\frac{p}{p'}} \\
&\le |v|^p_{W^{s,p}(O)}|O|^{-p}|O|^{\frac{p}{p'}}h_O^{sp+d} \le |v|^p_{W^{s,p}(O)}h_O^{sp+d}|O|^{-1}.
\end{aligned}$$

Hence $\|v-\overline{v}_O\|_{L^p(O)} \le h_O^s\left(\frac{h_O^d}{|O|}\right)^{\frac{1}{p}}|v|_{W^{s,p}(O)}$. $\square$

Remark 5.1. *(Best approximation)* Theorem 5.5 immediately implies the following bound for the best approximation in $P(\mathcal{T}_h)$:

$$\inf_{w_h\in P(\mathcal{T}_h)}\|v-w_h\|_{L^p(K;\mathbb{R}^q)} \le c\,h^r|v|_{W^{r,p}(D_K;\mathbb{R}^q)}, \tag{5.12}$$

for all $r\in[0,k+1]$, all $p\in[1,\infty]$, all $v\in W^{r,p}(D_K;\mathbb{R}^q)$, and all $K\in\mathcal{T}_h$.

Remark 5.2. *(Approximation for $\mathcal{I}_K^\sharp$)* Note in passing that Theorem 5.5 can be re-written with the operator $\mathcal{I}_K^\sharp$, i.e., the following also holds:

$$\|v-\mathcal{I}_K^\sharp(v)\|_{L^p(K;\mathbb{R}^q)} \le ch_K^r|v|_{W^{r,p}(K;\mathbb{R}^q)}, \tag{5.13}$$

for all $r\in[0,k+1]$, all $p\in[1,\infty]$, all $v\in W^{r,p}(K;\mathbb{R}^q)$, and all $K\in\mathcal{T}_h$.

6. Quasi-interpolation with boundary prescription. Our goal in this section is to construct a variant of the quasi-interpolation operator $\mathcal{I}_h^{\mathrm{av}}$ that prescribes homogeneous boundary values.

6.1. Trace operator. Let $F\in\mathcal{F}_h^\partial$ be a boundary face. We denote by K_F the unique cell such that $F\in\partial K_F$. We consider the global trace operator $\gamma: W^{1,1}(D;\mathbb{R}^q)\longrightarrow L^1(\partial D;\mathbb{R}^t)$ such that

$$\gamma(v)_{|F} = \gamma_{K_F}(v_{|K_F}), \qquad \forall F\in\mathcal{F}_h^\partial. \tag{6.1}$$

We assume that γ can be extended to W into a bounded linear operator $\gamma: W\longrightarrow W^\partial$ where W^∂ is an appropriate Banach space. (The knowledge of the exact structure of W^∂ is not important for our purpose.) We define $W_0=\ker(\gamma)$, i.e., $W_0=\{v\in W\mid \gamma(v)=0\}$. Let us introduce $P_0(\mathcal{T}_h)=P(\mathcal{T}_h)\cap W_0$:

$$P_0(\mathcal{T}_h) := \{v_h\in P(\mathcal{T}_h)\mid \gamma(v_h)=0\}. \tag{6.2}$$

The typical examples we have in mind are

$$P_0^{\mathrm{g}}(\mathcal{T}_h) := P^{\mathrm{g}}(\mathcal{T}_h)\cap W_0 = \{v_h\in P^{\mathrm{g}}(\mathcal{T}_h)\mid v_{h|\partial D}=0\}, \tag{6.3a}$$
$$P_0^{\mathrm{c}}(\mathcal{T}_h) := P^{\mathrm{c}}(\mathcal{T}_h)\cap \boldsymbol{V}_0^{\mathrm{c}} = \{\boldsymbol{v}_h\in P^{\mathrm{c}}(\mathcal{T}_h)\mid \boldsymbol{v}_h\times\boldsymbol{n}_{|\partial D}=\boldsymbol{0}\}, \tag{6.3b}$$
$$P_0^{\mathrm{d}}(\mathcal{T}_h) := P^{\mathrm{d}}(\mathcal{T}_h)\cap \boldsymbol{V}_0^{\mathrm{d}} = \{\boldsymbol{v}_h\in P^{\mathrm{d}}(\mathcal{T}_h)\mid \boldsymbol{v}_h\cdot\boldsymbol{n}_{|\partial D}=0\}, \tag{6.3c}$$

with $V_0^{\mathrm{g}}=\{v\in V^{\mathrm{g}}\mid v_{|\partial D}=0\}$, $\boldsymbol{V}_0^{\mathrm{c}}=\{\boldsymbol{v}\in\boldsymbol{V}^{\mathrm{c}}\mid \boldsymbol{v}\times\boldsymbol{n}_{|\partial D}=\boldsymbol{0}\}$, $\boldsymbol{V}_0^{\mathrm{d}}=\{\boldsymbol{v}\in\boldsymbol{V}^{\mathrm{d}}\mid \boldsymbol{v}\cdot\boldsymbol{n}_{|\partial D}=0\}$.

We say that a global degree of freedom ς_a, $a\in\mathcal{A}_h$, is a boundary degree of freedom if $\varsigma_a(v)=0$ for all $v\in W_0$. The collection of all the boundary degrees of freedom is denoted $\mathcal{A}_h^\partial$; the degrees of freedom in $\mathcal{A}_h^\circ = \mathcal{A}_h \backslash \mathcal{A}_h^\partial$ are called interior degrees of freedom. For all $a \in \mathcal{A}_h^\partial$, we define $\mathcal{F}_a^\partial$ to be the collection of all the boundary faces F such that there is $(K,i) \in \mathcal{C}_a$ and $F \subset \partial K \cap \mathcal{F}_h^\partial$; we set $\mathcal{F}_a^\partial = \emptyset$ if $a \in \mathcal{A}_h^\circ$. We henceforth denote $\mathcal{F}_K^\partial = \cup_{i\in\mathcal{N}} \mathcal{F}_{\mathsf{a}(K,i)}^\partial$. We abuse the notation by setting

$$[\![v]\!]_F^\gamma(\boldsymbol{x}) = \gamma_{K_F}(v_{|K_F})(\boldsymbol{x}), \quad \text{and} \quad [\![v]\!]_F(\boldsymbol{x}) = v_{|K}(\boldsymbol{x}), \quad \text{a.e. } \boldsymbol{x} \in F,\ \forall F \in \mathcal{F}_h^\partial, \tag{6.4}$$

and assume that $|[\![v]\!]_F^\gamma(\boldsymbol{x})| \le |[\![v]\!]_F(\boldsymbol{x})|$, a.e. $\boldsymbol{x} \in F$, for all $F \in \mathcal{F}_h^\partial$. In coherence with assumption (4.2), we assume that there is a uniform constant c such that the following holds for all the boundary degrees of freedom $a \in \mathcal{A}_h^\partial$:

$$|\sigma_{K,i}(v)| \le c\,\|\mathbb{A}_K\|_{\ell^2} \|\gamma_K(v_{|K})\|_{L^\infty(F;\mathbb{R}^t)}, \tag{6.5}$$

for all $F \in \mathcal{F}_a^\partial$, all $(K,i) \in \mathcal{C}_a$ such that $F \subset \partial K$, and all $v \in P^{\mathrm{b}}(\mathcal{T}_h)$. Note that this assumption is satisfied by all the finite elements considered in §2.4.

6.2. Averaging operator revisited. We are going to modify the averaging operator $\mathcal{J}_h^{\mathrm{av}}$ to prescribe homogeneous boundary conditions. We define $\mathcal{J}_{h0}^{\mathrm{av}} : P^{\mathrm{b}}(\mathcal{T}_h) \to P_0(\mathcal{T}_h)$ by setting

$$\varsigma_a(\mathcal{J}_{h0}^{\mathrm{av}}(v)) = \begin{cases} \frac{1}{\mathrm{card}(\mathcal{C}_a)} \sum_{(K,i)\in\mathcal{C}_a} \sigma_{K,i}(v_{|K}) & \text{if } a \in \mathcal{A}_h^\circ, \\ 0 & \text{if } a \in \mathcal{A}_h^\partial, \end{cases} \tag{6.6}$$

i.e., $\mathcal{J}_{h0}^{\mathrm{av}}(v) = \sum_{a\in\mathcal{A}_h^\circ} \varsigma_a(\mathcal{J}_{h0}^{\mathrm{av}}(v))\varphi_a$ for all $v \in P^{\mathrm{b}}(\mathcal{T}_h)$.

Lemma 6.1 (Approximation by averaging). *There exists a uniform constant c such that the following holds:*

$$|w - \mathcal{J}_{h0}^{\mathrm{av}}(w)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{d\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{q}-m} \sum_{F\in\mathcal{F}_K^\circ\cup\mathcal{F}_K^\partial} \|[\![w]\!]_F^\gamma\|_{L^q(F;\mathbb{R}^t)}, \tag{6.7}$$

for all $m \in \{0{:}k+1\}$, all $p,q \in [1,\infty]$, all $w \in P^{\mathrm{b}}(\mathcal{T}_h)$, and all $K \in \mathcal{T}_h$.

Proof. This is a straightforward adaptation of the proof of Lemma 4.2 based on the assumption (6.5) and the observation that $\sigma_{K,i}(w - \mathcal{J}_{h0}^{\mathrm{av}}(w)) = \sigma_{K,i}(w)$ if $\mathsf{a}(K,i)$ is a boundary degree of freedom. ∎

6.3. Quasi-interpolation operator revisited. A global W_0-conforming quasi-interpolation operator $\mathcal{I}_{h0}^{\mathrm{av}} : L^1(D;\mathbb{R}^q) \to P_0(\mathcal{T}_h)$ is defined by setting

$$\mathcal{I}_{h0}^{\mathrm{av}} = \mathcal{J}_{h0}^{\mathrm{av}} \circ \mathcal{I}_h^\sharp. \tag{6.8}$$

Note that $P_0(\mathcal{T}_h)$ is point-wise invariant under $\mathcal{I}_{h0}^{\mathrm{av}}$ since (6.7) implies that $P_0(\mathcal{T}_h)$ is point-wise invariant under $\mathcal{J}_{h0}^{\mathrm{av}}$. Hence, $\mathcal{I}_{h0}^{\mathrm{av}}$ is a projection, i.e., $(\mathcal{I}_{h0}^{\mathrm{av}})^2 = \mathcal{I}_{h0}^{\mathrm{av}}$.

Lemma 6.2 (L^p-stability of $\mathcal{I}_{h0}^{\mathrm{av}}$). *There is a uniform constant c such that*

$$\|\mathcal{I}_{h0}^{\mathrm{av}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c\,\|v\|_{L^p(D_K;\mathbb{R}^q)}, \tag{6.9}$$

for all $p \in [1,\infty]$, all $v \in L^p(D;\mathbb{R}^q)$, and all $K \in \mathcal{T}_h$.

Proof. Proceed as for the proof of Lemma 5.1. ∎

Theorem 6.3 (Approximation). *There exists a uniform constant c such that*

$$|v - \mathcal{I}_{h0}^{\mathrm{av}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{l-m} |v|_{W^{l,p}(D_K;\mathbb{R}^q)}, \tag{6.10}$$

for all $m \in \{0{:}k+1\}$, *all* $l \in \{m{:}k+1\}$, *all* $p \in [1,\infty]$, *and all* $v \in\in W^{l,p}(D_K;\mathbb{R}^q)$ *with* $\gamma(v)_{|\partial D_K \cap \partial D} = 0$. *Consequently the following holds for all* $v \in W^{l,p}(D;\mathbb{R}^q) \cap W_0$:

$$|v - \mathcal{I}_{h0}^{\mathrm{av}}(v)|_{W^{m,p}(\mathcal{T}_h;\mathbb{R}^q)} \le c\, h^{l-m} |v|_{W^{l,p}(D;\mathbb{R}^q)}. \tag{6.11}$$

Proof. This is an adaptation of the proof of Theorem 5.2. The only difference is in the handling of the term $\mathfrak{T}_2$; in the present case, (6.7) implies that

$$h_K^m |\mathfrak{T}_2| \le c h_K^{\frac{1}{p}} \sum_{F\in\mathcal{F}_K^\circ} \|[\![\mathcal{I}_K^\sharp(v)]\!]_F^\gamma\|_{L^r(F;\mathbb{R}^t)} + c' h_K^{\frac{1}{p}} \sum_{F\in\mathcal{F}_K^\partial} \|\gamma(\mathcal{I}_K^\sharp(v))\|_{L^r(F;\mathbb{R}^t)}.$$

The sum over $\mathcal{F}_K^\circ$ is handled as in the proof of Theorem 5.2. For the sum over $\mathcal{F}_K^\partial$, we use that $\gamma(\mathcal{I}_h^\sharp(v)) = \gamma(\mathcal{I}_h^\sharp(v) - v)$ and conclude as in the proof of Theorem 5.2. ◻

7. Boundary conditions in fractional Sobolev spaces. The purpose of this section is to establish error estimates for the quasi-interpolation operator $\mathcal{I}_{h0}^{\mathrm{av}}$ in the fractional $W^{r,p}$-norm for any real number $r \in [0,k+1]$.

7.1. Overview of the difficulty. Given F_0, F_1, two normed spaces continuously embedded into a topological vector space $\mathcal{E}$, and given $t \in [1,\infty]$, $s \in (0,1)$, we denote by $[F_0,F_1]_{s,t}$ the interpolation space obtained by the real interpolation method (i.e., the K-method), see e.g., Tartar [16, Chap. 22]. Recall that $W^{s,p}(D) = [L^p(D), W^{1,p}(D)]_{s,p}$ since D is Lipschitz, see Tartar [16, Lem. 36.1]. Let us define

$$W^{s,p}_{00,\gamma}(D;\mathbb{R}^q) := [L^p(D;\mathbb{R}^q), W^{1,p}_{0,\gamma}(D;\mathbb{R}^q)]_{s,p}, \tag{7.1}$$

with $W^{1,p}_{0,\gamma}(D;\mathbb{R}^q) := W^{1,p}(D;\mathbb{R}^q) \cap W_0$. Then, using Theorem 6.3 with $l \in \{0,1\}$ and $m = 0$, the interpolation theorem implies that

$$\|v - \mathcal{I}_{h0}^{\mathrm{av}}(v)\|_{L^p(D;\mathbb{R}^q)} \le c\, h^s \|v\|_{W^{s,p}_{00,\gamma}(D;\mathbb{R}^q)}, \tag{7.2}$$

for all $p \in [1,\infty]$ and all $v \in W^{s,p}_{00,\gamma}(D;\mathbb{R}^q)$. This estimate is not fully satisfactory for two reasons. First it is not local. Second it is not really clear what $W^{\frac{1}{p},p}_{00,\gamma}(D;\mathbb{R}^q)$ is. For instance, let us define

$$W^{1,p}_0(D) := \{v \in W^{1,p}(D) \mid v_{|\partial D} = 0\}, \tag{7.3a}$$
$$\boldsymbol{W}^{1,p}_T(D) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(D) \mid \boldsymbol{v}{\times}\boldsymbol{n}_{|\partial D} = \boldsymbol{0}\}, \tag{7.3b}$$
$$\boldsymbol{W}^{1,p}_N(D) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(D) \mid \boldsymbol{v}{\cdot}\boldsymbol{n}_{|\partial D} = 0\}. \tag{7.3c}$$

One then realizes that characterizing $[\boldsymbol{L}^p(D), \boldsymbol{W}^{1,p}_T(D)]_{s,p}$ and $[\boldsymbol{L}^p(D), \boldsymbol{W}^{1,p}_N(D)]_{s,p}$ in terms of Sobolev regularity is (possible but) not straightforward, and to the best of our knowledge, a full characterization of these spaces is not yet available. To better appreciate the difficulty, let us consider the more traditional spaces $W^{s,p}_{00}(D) := [L^p(D), W^{1,p}_0(D)]_{s,p}$, $s \in (0,1)$, and $W^{s,p}_0(D) := \overline{\mathcal{C}^\infty_0(D)}^{W^{s,p}}$. Let $E_0 : W^{s,p}(D) \to L^1(\mathbb{R}^d)$ be the zero-extension operator outside D, i.e., $E_0 u|_D = u$ and $(E_0 u)|_{\mathbb{R}^d\setminus D} = 0$, and let us define the space $\tilde{W}^{s,p}(D) = \{u \in W^{s,p}(D) \mid E_0 u \in W^{s,p}(\mathbb{R}^d)\}$. It is known that $\tilde{W}^{s,p}(D) = W^{s,p}_0(D)$ for $sp \ne 1$, and $W^{s,p}(D) = \tilde{W}^{s,p}(D) = W^{s,p}_0(D)$ for $sp < 1$, see Grisvard [11, Cor. 1.4.4.5] (see also Lions and Magenes [13, Thm 11.1] and Tartar

[16, Chap. 33]). It is also known for $p = 2$ (and probably true in general) that $\tilde{W}^{s,p}(D) = W^{s,p}_{00}(D)$ but $W^{\frac{1}{p},p}_0(D) \neq W^{\frac{1}{p},p}_{00}(D)$, see Tartar [16, p. 160].

Hence, the interpolation theory shows that, in addition to the problem of the characterization of the interpolation spaces, there is a key difficulty when $sp = 1$. The purpose of this section is to go around the first problem and to establish a counterpart of Theorem 6.3 for $rp \neq 1$ when $r \in [0, k+1]$.

7.2. Error estimates. Let $r \in [0, k+1]$ and $p \in [1, \infty]$. If $r > \frac{1}{p}$, then functions in $W^{r,p}(D)$ have traces on ∂D, and therefore it makes sense to define

$$W^{r,p}_{0,\gamma}(D;\mathbb{R}^q) := \{v \in W^{r,p}(D;\mathbb{R}^q) \mid \gamma(v) = 0\}. \tag{7.4}$$

Let us denote by L_h the layer of cells that touch the boundary of D, i.e., $K \in L_h$ if $K \cap \partial D \neq \emptyset$. We also introduce $L^\flat_h$ the layer of cells with at least one face on ∂D, i.e., $K \in L^\flat_h$ if there if $F \in \mathcal{F}_h$ such that $K \cap F \subset \partial D$. Finally, we define $L^\sharp_h = \cup_{K \in L^\flat_h} \mathcal{T}_K$.

Theorem 7.1 (Approximation). *Let $r \in [0, k+1]$ and $p \in [1, \infty]$. There exists a uniform constant c such that*

$$\|v - \mathcal{I}^{\rm av}_{h0}(v)\|_{L^p(K;\mathbb{R}^q)} \le ch^r_K |v|_{W^{r,p}(D_K;\mathbb{R}^q)}, \quad \forall v \in W^{r,p}(D;\mathbb{R}^q), \forall K \in \mathcal{T}_h \backslash L_h \tag{7.5a}$$

$$\|v - \mathcal{I}^{\rm av}_{h0}(v)\|_{L^p(K;\mathbb{R}^q)} \le ch^r_K |v|_{W^{r,p}(D_K;\mathbb{R}^q)}, \quad \begin{cases} \forall v \in W^{r,p}_{0,\gamma}(D;\mathbb{R}^q), \forall K \in L_h, \\ \textit{if } rp > 1, \end{cases} \tag{7.5b}$$

$$\|v - \mathcal{I}^{\rm av}_{h0}(v)\|_{L^p(L_h;\mathbb{R}^q)} \le ch^r \|v\|_{W^{r,p}(D;\mathbb{R}^q)}, \quad \forall v \in W^{r,p}(D;\mathbb{R}^q),\ \textit{if } rp < 1. \tag{7.5c}$$

Proof. Let K be a cell in $\mathcal{T}_h$ and let $v \in W^{r,p}(D;\mathbb{R}^q)$. Then the triangle inequality implies that

$$\|\mathcal{I}^{\rm av}_{h0}(v) - v\|_{L^p(K;\mathbb{R}^q)} \le \|\mathcal{I}^{\rm av}_{h}(v) - v\|_{L^p(K;\mathbb{R}^q)} + \|\mathcal{I}^{\rm av}_{h0}(v) - \mathcal{I}^{\rm av}_{h}(v)\|_{L^p(K;\mathbb{R}^q)}.$$

Since we have already established that $\|\mathcal{I}^{\rm av}_{h}(v) - v\|_{L^p(K;\mathbb{R}^q)} \le ch^r_K \|v\|_{W^{r,p}(D_K;\mathbb{R}^q)}$ in Theorem 5.5, we just need to estimate $\|\mathcal{I}^{\rm av}_{h0}(v) - \mathcal{I}^{\rm av}_{h}(v)\|_{L^p(K;\mathbb{R}^q)}$. By proceeding as in the proof of Theorem 5.2, we have

$$\|\mathcal{I}^{\rm av}_{h0}(v) - \mathcal{I}^{\rm av}_{h}(v)\|_{L^p(K;\mathbb{R}^q)} \le c\, h^{\frac{1}{p}}_K \sum_{F \in \mathcal{F}^\partial_K} \|\gamma(\mathcal{I}^\sharp_K(v))\|_{L^p(F;\mathbb{R}^t)}.$$

Note that the right-hand side is zero when K does not touch the boundary ($K \cap \partial D = \emptyset$). This proves (7.5a). The rest of the proof consists of evaluating $\|\gamma(\mathcal{I}^\sharp_K(v))\|_{L^p(F;\mathbb{R}^t)}$ when K touches the boundary ($K \cap \partial D \neq \emptyset$), i.e., when $K \in L_h$.

Case 1, $rp > 1$: Let us assume that $rp > 1$ and assume in addition that $v \in W^{r,p}_{0,\gamma}(D;\mathbb{R}^q)$. The boundary condition $\gamma(v) = 0$ implies that

$$\begin{aligned}\|\mathcal{I}^{\rm av}_{h0}(v) - \mathcal{I}^{\rm av}_{h}(v)\|_{L^p(K;\mathbb{R}^q)} &\le c\, h^{\frac{1}{p}}_K \sum_{F \in \mathcal{F}^\partial_K} \|\gamma(\mathcal{I}^\sharp_K(v) - v)\|_{L^p(F;\mathbb{R}^t)} \\ &\le c'\, h^{\frac{1}{p}}_K \sum_{F \in \mathcal{F}^\partial_K} \|\mathcal{I}^\sharp_K(v) - v\|_{L^p(F;\mathbb{R}^t)}\end{aligned}$$

The conclusion follows readily by invoking Lemma 7.3 below with $s = r - [r]$ and the approximation properties of $\mathcal{I}^\sharp_K$ stated in Remark 5.2.

Case 2, $rp<1$: Assume now that $rp<1$, i.e., $r\in[0,1)$. An inverse inequality implies that

$$\|\mathcal{I}_{h0}^{\mathrm{av}}(v)-\mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(L_h;\mathbb{R}^q)}^p \le c\sum_{K\in L_h^\flat}\|\mathcal{I}_h^\sharp(v))\|_{L^p(K;\mathbb{R}^q)}^p = c\,\|\mathcal{I}_h^\sharp(v))\|_{L^p(L_h^\flat;\mathbb{R}^q)}^p.$$

Let ρ be the distance to ∂D; then there is c uniform with respect to the mesh sequence such that $\|\rho\|_{L^\infty(L_h^\flat)}\le ch$ and

$$\begin{aligned}\|\mathcal{I}_{h0}^{\mathrm{av}}(v)-\mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(L_h;\mathbb{R}^q)} &\le c\|\mathcal{I}_h^\sharp(v)-v\|_{L^p(L_h^\flat;\mathbb{R}^q)}+\|v\|_{L^p(L_h^\flat;\mathbb{R}^q)}\\ &\le c\left(h^r\|v\|_{W^{r,p}(L_h^\sharp;\mathbb{R}^q)}+\|\rho^r\rho^{-r}v\|_{L^p(L_h^\flat;\mathbb{R}^q)}\right)\\ &\le c\left(h^r\|v\|_{W^{r,p}(L_h^\sharp;\mathbb{R}^q)}+\|\rho\|_{L^\infty(L_h^\flat)}^r\|\rho^{-r}v\|_{L^p(L_h^\flat;\mathbb{R}^q)}\right).\end{aligned}$$

Observing that $\tilde{W}^{r,p}(D)=W^{r,p}(D)$ since $rp<1$ (see Grisvard [11, Cor. 1.4.4.5]), we infer that

$$\|\rho^{-r}v\|_{L^p(L_h^\flat;\mathbb{R}^q)}\le\|\rho^{-r}v\|_{L^p(D;\mathbb{R}^q)}\le c\|v\|_{W^{r,p}(D;\mathbb{R}^q)}.$$

It seems that it is the best that can be done, i.e., $\|\rho^{-r}v\|_{L^p(L_h^\flat;\mathbb{R}^q)}$ cannot be controlled locally. In conclusion, $\|\mathcal{I}_{h0}^{\mathrm{av}}(v)-\mathcal{I}_h^{\mathrm{av}}(v)\|_{L^p(L_h;\mathbb{R}^q)}\le ch^r\|v\|_{W^{r,p}(D;\mathbb{R}^q)}$. □

Note that the estimate $\|v-\mathcal{I}_{h0}^{\mathrm{av}}(v)\|_{L^p(L_h;\mathbb{R}^q)}\le ch^r\|v\|_{W^{r,p}(D;\mathbb{R}^q)}$ for $rp<1$ in Theorem 7.1 just says that the difference $v-\mathcal{I}_{h0}^{\mathrm{av}}(v)$ does not blow up too fast close to the boundary. A better result is not expected since $\mathcal{I}_{h0}^{\mathrm{av}}(v)$ is forced to be zero at ∂D whereas v can blow up like $\rho^{-s}w$ where $w\in L^p(D;\mathbb{R}^q)$. In conclusion, Theorem 7.1 implies the following best approximation result.

Corollary 7.2 (Global best approximation). *There exists a uniform constant c, additionally depending on $|rp-1|$, such that*

$$\inf_{w_h\in P_0(\mathcal{T}_h)}\|v-w_h\|_{L^p(D;\mathbb{R}^q)}\le\begin{cases}ch^r|v|_{W^{r,p}(D;\mathbb{R}^q)}, & \forall v\in W_{0,\gamma}^{r,p}(D;\mathbb{R}^q)\ \textit{if } rp>1\\ ch^r\|v\|_{W^{r,p}(D;\mathbb{R}^q)}, & \forall v\in W^{r,p}(D;\mathbb{R}^q)\ \textit{if } rp<1.\end{cases}\tag{7.6}$$

Lemma 7.3 (Multiplicative trace inequality in fractional Sobolev spaces). *Assume $s\in(0,1]$ and $sp>1$. Then there exists a uniform constant c, additionally depending on $|sp-1|$, such that the following holds for all $v\in W^{t,p}(K)$ and all $K\in\mathcal{T}_h$:*

$$\|v\|_{L^p(F)}\le c(h_K^{-\frac{1}{p}}\|v\|_{L^p(F)}+h_K^{s-\frac{1}{p}}|v|_{W^{s,p}(K)}).\tag{7.7}$$

Proof. We prove the statement in $F\times(0,a)\subset\mathbb{R}^d$ where $a>0$ and F is considered as a subset of $\mathbb{R}^{d-1}$; the extension to K is done by invoking appropriate mappings. The details are omitted for brevity. We also restrict ourselves to $s\in(0,1)$ since the statment is a consequence of Lemma 5.3 when $s=1$.

Let first $v\in W^{s,p}(0,a)$ and following Grisvard [11, pp. 29-30], consider

$$w(x):=\frac{1}{x}\int_0^x(v(t)-v(x))\,\mathrm{d}t.$$

Then the following "strange" identity holds:

$$\begin{aligned} v(0) &= v(x) + w(x) + \int_0^x \frac{w(y)}{y}\,\mathrm{d}y \\ &= v(x) + \frac{1}{x}\int_0^x (v(t)-v(x))\,\mathrm{d}t + \int_0^x \frac{1}{y^2}\int_0^y (v(t)-v(y))\,\mathrm{d}t\,\mathrm{d}y. \end{aligned}$$

Using Hölder's inequality repeatedly, we infer that

$$\begin{aligned} \frac{1}{a}\int_0^a v(x)\,\mathrm{d}x &\le a^{-\frac{1}{p}}\|v\|_{L^p(0,a)}, \\ \frac{1}{a}\int_0^a \frac{1}{x}\int_0^x (v(t)-v(x))\,\mathrm{d}t\,\mathrm{d}x &\le c_1(s,p)\,a^{s-\frac{1}{p}}|v|_{W^{s,p}(0,a)}, \\ \frac{1}{a}\int_0^a \int_0^x \frac{1}{y^2}\int_0^y (v(t)-v(y))\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x &\le c_2(s,p)\,a^{s-\frac{1}{p}}|v|_{W^{s,p}(0,a)}, \end{aligned}$$

where $c_1(s,p) = \left(\frac{p-1}{p(s+1)}\frac{p-1}{p(s+1)-1}\right)^{\frac{p-1}{p}}$, $c_2(s,p) = \left(\frac{p-1}{p(s+1)}\frac{p-1}{sp-1}\right)^{\frac{p-1}{p}}\frac{p}{p(s+1)-1}$. Hence, using that $v(0) = \frac{1}{a}\int_0^a v(0)\,\mathrm{d}x$, we infer that

$$|v(0)| \le a^{-\frac{1}{p}}\|v\|_{L^p(0,a)} + (c_1(s,p)+c_2(s,p))a^{s-\frac{1}{p}}|v|_{W^{s,p}(0,a)}.$$

Let now $v \in W^{s,p}(F\times(0,a))$. Applying the above inequality to $v(\mathbf{0}_{\mathbb{R}^{d-1}},\cdot)$ and using the inequality $(\alpha+\beta)^p \le 2^{\frac{p-1}{p}}(|\alpha|^p + |\beta|^p)$, we infer that

$$\|v\|_{L^p(F)} \le c\,(a^{-\frac{1}{p}}\|v\|_{L^p(0,a)} + a^{s-\frac{1}{p}}I(v))$$

where

$$I(v)^p = \int_F \int_0^a \int_0^a \frac{|v(\boldsymbol{x}_{d-1},x_d) - v(\boldsymbol{x}_{d-1},y_d)|^p}{|x_d-y_d|^{sp+1}}\,\mathrm{d}x_1 \ldots \mathrm{d}x_{d-1}\,\mathrm{d}x_d\,\mathrm{d}y_d.$$

The rest of the proof consists of proving that there is a constant c such that $I(v) \le c|v|_{W^{s,p}(F\times(0,a))}$. This is actually (a slightly modified version of) Lemma 4.33 in Demengel and Demengel [8, p. 200]. ☐

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