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A MAX-CUT FORMULATION OF 0/1 PROGRAMS

JEAN B. LASSERRE

ABSTRACT. We consider the linear or quadratic 0/1 program

\[ P: \quad f^* = \min \{ c^T x + x^T F x : A x = b; \quad x \in \{0,1\}^n \}, \]

for some vectors \( c \in \mathbb{R}^n \), \( b \in \mathbb{Z}^m \), some matrix \( A \in \mathbb{Z}^{m \times n} \) and some real symmetric matrix \( F \in \mathbb{R}^{n \times n} \). We show that \( P \) can be formulated as a MAX-CUT problem whose quadratic form criterion is explicit from the data of \( P \). In particular, to \( P \) one may associate a graph \((G, V)\) whose connectivity is related to the connectivity of the matrix \( F \) and \( A^T A \), and \( P \) reduces to finding a maximum (weighted) cut in such a graph. Hence the whole arsenal of approximation techniques for MAX-CUT can be applied. On a sample of 0/1 knapsack problems, we compare the lower bound on \( f^* \) of the associated standard (Shor) SDP-relaxation with the standard linear relaxation where \( \{0,1\}^n \) is replaced with \([0,1]^n\) (resulting in an LP when \( F = 0 \) and a quadratic program when \( F \) is positive definite). We also compare our lower bound with that of the first SDP-relaxation associated with the copositive formulation of \( P \).

1. Introduction

Consider the linear or quadratic 0/1 program \( P \) defined by:

\[ \min_{x} \{ c^T x + x^T F x : A x = b; \quad x \in \{0,1\}^n \} \]

for some cost vector \( c \in \mathbb{R}^n \), some matrix \( A \in \mathbb{Z}^{m \times n} \), some vector \( b \in \mathbb{Z}^m \), and some real symmetric matrix \( F \in \mathbb{R}^{n \times n} \). If \( F = 0 \) then \( P \) is a 0/1 linear program and a quadratic 0/1 program otherwise. Obtaining good quality lower bounds on \( f^* \) is highly desirable since the efficiency of Branch & Bound algorithms to solve large scale problems \( P \) heavily depends on the quality of bounds of this form computed at nodes of the search tree.

To obtain lower bounds for 0/1 programs (1.1) one may solve a relaxation of \( P \) where the integrality constraints \( x \in \{0,1\}^n \) are replaced with the box constraints \( x \in [0,1]^n \). If \( F = 0 \) the resulting relaxation is linear whereas if \( F \) is positive definite it is a (convex) quadratic program. If \( F \) is not positive semidefinite then one may also solve a convex quadratic program but now with with an appropriate convex quadratic underestimator \( x^T \tilde{F} x \) of \( x^T F x \) on \([0,1]^n\). An alternative is to consider an equivalent formulation of \( P \) as a copositive conic program as advocated by Burer [3] and compute...
a sequence of lower bounds by solving an appropriate hierarchy of LP- or SDP-relaxations associated with the copositive cone (or its dual). For more details on the latter approach the interested reader is referred to e.g. De Klerk and Pasechnik [4], Dürr [5], Bonze [1], and Bonze and de Klerk [2].

**Contribution.** The purpose of this note is to show that solving \( P \) is equivalent to minimizing a quadratic form in \( n+1 \) variables on the hypercube \( \{-1, 1\}^{n+1} \) (and the quadratic form is explicit from the data of \( P \)). In other words \( P \) can be viewed as an explicit instance of the MAX-CUT problem. Hence the MAX-CUT problem which at first glance seems to be a very specific combinatorial optimization problem, in fact can be considered as a canonical model of linear and quadratic 0/1 programs. In particular, to each linear or quadratic 0/1 program (1.1) one may associate a graph \((G, V)\) with \( n+1 \) nodes and \((i,j) \in V\) whenever a product \(x_i x_j\) has a nonzero coefficient in some quadratic form built upon the data \( c, b, F \) and \( A \) of (1.1). (Among other things, the sparsity of \((G, V)\) is related to the sparsity of the matrices \( F \) and \( A^T A \).) Then solving (1.1) reduces to finding a maximum (weighted) cut of \( G \).

Therefore the whole specialized arsenal of approximation techniques for MAX-CUT can be applied. In particular one may obtain a lower bound \( f_1^* \) on \( f^* \) by solving the standard (Shor) SDP-relaxation associated with the resulting MAX-CUT problem while solving higher levels of the associated Lasserre-SOS hierarchy [6, 7] would provide a monotone nondecreasing sequence of improved lower bounds \( f_1^* \leq f_d^* \leq f^* \), \( d = 2, \ldots \), but of course at a higher computational cost. Alternatively one may also apply the Handelman hierarchy of LP-relaxations as described and analyzed in Laurent and Sun [10]. For more details on recent developments on computational approaches to MAX-CUT the interested reader is referred to Wiegele and Rendl [13]. If \( F = 0 \) (i.e. when \( P \) is a linear 0/1 program) the lower bound \( f_1^* \) can be better than the standard LP-relaxation which consists in replacing the integrality constraints \( x \in \{0, 1\}^n \) with the box \([0, 1]^n\), as shown on a (limited) sample of 0/1-knapsack-type examples. On such examples \( f_1^* \) also dominates the one obtained from the first relaxation of the copositive formulation (where the dual cone \( C^* \) of completely positive matrices is replaced with \( S^+ \cap \mathcal{N} \supset C^* \)) in about 55% of cases and the maximum relative difference is bounded by 0.55% in all cases.

In addition one may also obtain performance guarantees à la Nesterov [12] in the form

\[
 f_1^* \leq f^* \leq \frac{2}{\pi} f_1^* + (1 - \frac{2}{\pi}) h_1^*,
\]

(where \( h_1^* \) is the optimal value of a similar SDP but with a max-criterion instead of a min-criterion) or their improvements by Marshall [11].

In fact, and still on the same sample of linear and quadratic 0/1 knapsack examples, one also observes that the resulting lower bound \( f_1^* \) is almost always better than the lower bound obtained by solving the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation
(1.1) of the problem (which is also an SDP of same size). This is good news since typically the SOS-hierarchy is known to produce good lower bounds for general polynomial optimization problems (discrete or not) even at the first level of the hierarchy. Even more, the first level SDP-relaxation has the celebrated Goemans & Williamson performance guarantee ($\approx 87\%$) when the matrix $Q$ (associated with the quadratic form) has nonnegative entries and a performance guarantee $\approx 64\%$ when $Q \succeq 0$. (However note that the matrix $Q$ associated with our MAX-CUT problem equivalent to the initial 0/1 program (1.1) does not have all its entries nonnegative.) This explains why in the linear 0/1 knapsack examples the lower bound $f_1^*$ is almost always better than the one obtained with the standard LP-relaxation and why for quadratic 0/1 knapsack problems (1.1), $f_1^*$ is also likely to be better than the lower bound obtained by relaxing $\{0,1\}$ to $[0,1]^n$, replacing $F$ with a convex quadratic underestimator of $F$ on $[0,1]^n$, and solving the resulting convex quadratic program.

Finally, the same methodology also works for general 0/1 optimization problems with feasible set as in (1.1) and polynomial criterion $f \in \mathbb{R}[x]$ of degree $d > 2$, except that now the problem reduces to minimizing a new criterion $\tilde{f}(x)$ on the hypercube $\{-1,1\}^n$.

### 2. Main result

Denote by $\mathbb{Z}$ the set of integer numbers and $\mathbb{N} \subset \mathbb{Z}$ the set of natural numbers. Let $P$ be the 0/1 program defined in (1.1) with $F^T = F \in \mathbb{R}^{n \times n}$, $A \in \mathbb{Z}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{Z}^m$. Let $|c| := (|c_i|) \in \mathbb{R}_+^n$. With $e \in \mathbb{Z}^n$ being the vector of all ones, notice first that $P$ has an equivalent formulation on the hypercube $\{-1,1\}^n$, by the change of variables $\tilde{x} := 2x - e$. Indeed, $A$, $b$, $c$ and $F$ now become $A/2$, $b - Ae/2$, $(c + e^T F)/2$ and $F/4$ respectively.

Therefore from now on we consider the discrete program:

$$
\text{(2.1)} \quad P : \quad f^* = \min_{x \in \{-1,1\}^n} \left\{ c^T x + x^T F x : A x = b \right\},
$$
on the hypercube $\{-1,1\}^n$, with $A \in \mathbb{Z}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{Z}^m$, and $F^T = F \in \mathbb{R}^{n \times n}$. With $c$ and $F$, let us associate the scalars:

$$
\begin{align*}
 r^1_{c,F} &= \min \left\{ c^T x + \langle X, F \rangle : \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0; \ X_{ii} = 1, \ i = 1, \ldots, n \right\} \\
r^2_{c,F} &= \max \left\{ c^T x + \langle X, F \rangle : \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0; \ X_{ii} = 1, \ i = 1, \ldots, n \right\}
\end{align*}
$$

(with $X^T = X$) and let

$$
\rho(c, F) := \max_{i=1,2} |r^i_{c,F}|.
$$

It is straightforward to verify that

$$
\rho(c, F) \geq \max \left\{ |c^T x + x^T F x| : x \in \{-1,1\}^n \right\},
$$
and \( \rho(c, F) = |c| \) if \( F = 0 \). Moreover each scalar \( r^i_{c,F} \) can be computed by solving an SDP which is the Shor relaxation (or first level of the Lasserre-SOS hierarchy \([6, 7]\)) associated with the problems \( \min (\max) \{ c^T x + x^T F x : x \in \{-1,1\}^n \} \).

2.1. A MAX-CUT formulation of \( P \).

**Lemma 2.1.** Let \( P \) be as \((2.1)\) and let \( \rho(c, F) \) be as in \((2.2)\). Then \( f^* \) is the optimal value of the quadratic minimization problem:

\[
(2.3) \min_{x \in \{-1,1\}^n} c^T x + x^T F x + (2 \rho(c, F) + 1) \cdot \|Ax - b\|^2.
\]

**Proof.** Let \( \Delta := \{ x \in \{-1,1\}^n : Ax = b \} \) be the feasible set of \( P \) defined in \((1.1)\), and let \( f : \mathbb{R}^n \to \mathbb{R} \) be the function

\[
(2.4) \quad x \mapsto f(x) := c^T x + x^T F x + (2 \rho(c, F) + 1) \cdot \|Ax - b\|^2.
\]

On \( \{-1,1\}^n \) one has \( \max\{c^T x + x^T F x : x \in \{-1,1\}^n\} \leq \rho(c, F) \), and

\[
\|Ax - b\|^2 \geq 1, \quad \forall x \in \{-1,1\}^n \setminus \Delta,
\]

because \( A \in \mathbb{Z}^{m \times n} \) and \( b \in \mathbb{Z}^m \). Therefore,

\[
f(x) \left\{ \begin{array}{l}
c^T x + x^T F x \text{ on } \Delta \\
\geq c^T x + x^T F x + 2 \rho(c, F) + 1 > \rho(c, F) \text{ on } \{-1,1\}^n \setminus \Delta.
\end{array} \right.
\]

From this and \( c^T x + x^T F x < \rho(c, F) \) on \( \Delta \), the result follows. \qed

Next, let \( Q : \mathbb{R}^{n+1} \to \mathbb{R} \) be the homogenization of the quadratic polynomial \( f \), i.e., the quadratic form \( Q(x, x_0) := x_0^2 f(\frac{x}{x_0}) \), or in explicitly form:

\[
(2.5) \quad Q(x, x_0) = x_0 c^T x + x^T F x + (2 \rho(c, F) + 1) \cdot \|Ax - x_0 b\|^2.
\]

Observe that \( Q(x, 1) = f(x) \).

**Theorem 2.2.** Let \( f^* = \min\{c^T x + x^T F x : Ax = b; x \in \{-1,1\}^n \} \) and let \( Q \) be the quadratic form in \((2.5)\). Then

\[
(2.6) \quad f^* = \min_{(x,x_0) \in \{-1,1\}^{n+1}} Q(x, x_0),
\]

that is, \( f^* \) is the optimal value of the MAX-CUT problem associated with the quadratic form \( Q \).

**Proof.** Let \( f \) be as in \((2.4)\). By definition of \( Q \),

\[
(2.7) \quad \min_{x \in \{-1,1\}^n} f(x) = \min_{(x,x_0) \in \{-1,1\}^{n+1}} \{ Q(x, x_0) : x_0 = 1 \}.
\]

On the other hand, let \( (x^*, x^*_0) \in \{-1,1\}^{n+1} \) be a global minimizer of \( \min\{Q(x, x_0) : (x, x_0) \in \{-1,1\}^{n+1} \} \). Then by homogeneity of \( Q \), \( (-x^*, -x_0) \) is also a global minimizer and so one may decide arbitrarily to fix \( x_0 = 1 \). That is,

\[
\min_{(x,x_0) \in \{-1,1\}^{n+1}} Q(x, x_0) = \min_{(x,x_0) \in \{-1,1\}^{n+1}} \{ Q(x, x_0) : x_0 = 1 \}.
\]

which combined with \((2.7)\) yields the desired result. \qed
Next, write $Q(x, x_0) = (x, x_0)Q(x, x_0)^T$ for an appropriate real symmetric matrix $Q \in \mathbb{R}^{(n+1) \times (n+1)}$, and introduce the semidefinite programs

(2.8) \[ \min_X \{ \langle Q, X \rangle : X \succeq 0; X_{ii} = 1, \ i = 1, \ldots, n+1 \} \]

with optimal value denoted by $\min Q_+$, and

(2.9) \[ \max_X \{ \langle Q, X \rangle : X \succeq 0; X_{ii} = 1, \ i = 1, \ldots, n+1 \} \]

with optimal value denoted by $\max Q^+$. 

Proposition 2.3. Let $P$ be the problem defined in (2.1) with optimal value $f^*$. Then

(2.10) \[ \min Q_+ \leq f^* \leq \frac{2}{\pi} \min Q_+ + (1 - \frac{2}{\pi}) \max Q^+ \]

where $Q$ is the real symmetric matrix associated with the quadratic form (2.5) and $\min Q_+$ (resp. $\max Q^+$) is the optimal value of the semidefinite program (2.8) (resp. (2.9)).

Proof. The bounds in (2.10) are from Nesterov [12]. In addition, one may also use the bounds provided in Marshall [11] which sometimes improve those in (2.10). \qed

The quality of the upper bound in (2.10) depends strongly on the magnitude of the “penalty coefficient" $2\rho(c, F) + 1$ in the definition of the function $f$ in (2.4). However for a practical use of relaxations what matters most is the quality of the lower bound $\min Q_+$ which in principle is very good for MAX-CUT problems (even if $Q \not\succeq 0$ or $Q \not\succeq 0$). For instance in a Branch & Bound algorithm the lower bound $\min Q_+$ has an important impact in the pruning of nodes in the search tree.

Sparsity. Hence to each 0/1 program (1.1) one may associate a graph $(G, V)$ with $n + 1$ nodes and an arc $(i, j) \in V$ connects the nodes $i, j \in G$ if and only if the coefficient $Q_{ij}$ of the quadratic form $Q(x, x_0)$ does not vanish. Sparsity properties of $(G, V)$ are of primary interest, e.g. for computational reasons. From the definition of the matrix $Q$, this sparsity is in turn related to sparsity of the matrix $F + (2\rho(c, F) + 1) A^T A$, hence of sparsity of $F$ and $A^T A$. In particular two nodes $i, j$ are not connected if $F_{ij} = 0$ and $A_{ki} A_{kj} = 0$ for all $k = 1, \ldots, m$.

Example 2.4. To evaluate the quality of the lower bound obtained with the MAX-CUT formulation consider the following simple linear knapsack-type examples:

(2.11) \[ \min \{ c^T x : a^T x = b; x \in \{-1, 1\}^n \}, \]

on $\{-1, 1\}^n$, with 4 and 10 variables. For $n = 4$, $c = (13, 11, 7, 3)$ and $a = (3, 7, 11, 13)$, while for $n = 10$, $c = (37, 31, 29, 23, 19, 17, 13, 11, 7, 3)$, and $a = (3, 7, 11, 13, 17, 19, 23, 29, 31, 37)$.
The right-hand-side $b$ is taken into $[-|a|, |a|] \cap \mathbb{Z}$. Figure 1 displays the difference $\min Q_+ - \min \text{LP}$ where the lower bound $\min \text{LP}$ is obtained by relaxing the integrality constraints $x \in \{-1,1\}^n$ to the box constraint $x \in [-1,1]^n$ and solving the resulting LP. As expected the lower bound $\min Q_+$ is much better than $\min \text{LP}$. In fact the cases where the LP-bound is slightly better is for right-hand-side $b$ such that the relaxation provides the optimal value $f^*$. 

![Figure 1](image1.png)

**Figure 1.** Difference $\min Q_+ - \min \text{LP}$ with $n=(4,10,15)$

Moreover Figure 2 displays the difference $\min Q_+ - \min \hat{Q}_+$ where $\min \hat{Q}_+$ is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (2.11) of the knapsack problem where one has even included the redundant constraints $x_i (a^T x - b) = 0$, $i = 1, \ldots, n$. One observes that in most cases the lower bound $\min Q_+$ is slightly better than $\min \hat{Q}_+$.

This is encouraging since the Lasserre-SOS hierarchy [6, 7] is known to produce good lower bounds in general, and especially at the first level of the hierarchy for MAX-CUT problems whose matrix $Q$ of the associated quadratic form has certain properties, e.g., $Q_{ij} \geq 0$ for all $i, j$ or $Q \succeq 0$ (in the maximizing case); see e.g. Marshall [11].

![Figure 2](image2.png)

**Figure 2.** Difference $\min Q_+ - \min \hat{Q}_+$ (n=10) (left) and relative difference $100 \times (\min Q_+ - \min \hat{Q}_+)/\min \hat{Q}_+$

**Example 2.5.** In a second sample of linear knapsack problems (2.11) with $n = 10, 15$, we have chosen the same vector $a$ as in Example 2.4 but now with
a cost criterion of the form  

\[ c(i) = a(i) + s \eta, \quad i = 1, \ldots, n \]

where  \( \eta \) is a random variable uniformly distributed on \([0, 1]\), and  \( s = 20, 10, 1 \) is a weighting factor. The reason is that knapsack problems with ratios  \( c(i)/a(i) \approx 1 \) for all  \( i \), can be difficult to solve. As before the right-hand-side  \( b \) is taken into \([-|a|, |a|] \cap \mathbb{Z}\). Figure 3 displays the results obtained for  \( s = 20 \), and  \( n = 10, 15 \). Figure 4 displays the same example for another sample with cost  \( c \) and  \( s = 10 \).

![Figure 3](image1.png)  
Figure 3. Difference  \( \min Q_+ - \min LP, \quad c = a + 20 \star \eta \) (n=10,15)  

![Figure 4](image2.png)  
Figure 4. Difference  \( \min Q_+ - \min LP, \quad c = a + 10 \star \eta \) (n=10,15)  

Finally, as for Example 2.4, Figure 5 displays the difference  \( \min Q_+ - \min \tilde{Q}_+ \) (where  \( \min Q_+ \) is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (2.11) of the knapsack problem where one has even included the redundant constraints  \( x_i (a_i^T x - b) = 0, \quad i = 1, \ldots, n \). Again one observes that in most cases the lower bound  \( \min Q_+ \) is slightly better than  \( \min \tilde{Q}_+ \).

**Example 2.6.** We next consider the same knapsack problems (2.11) as in Example 2.5 but now with quadratic criterion  \( c^T x + x^T Fx \), again with a cost criterion of the form  \( c(i) = a(i) + s \eta, \quad i = 1, \ldots, n \), where  \( \eta \) is a random variable uniformly distributed in \([0, 1]\) and  \( s \) is some weighting factor. The real symmetric matrix  \( F \) is also randomly generated and is not positive definite in general. Again in Figures 6 and 7 one observes that the lower
bound $\min Q_+$ is almost always better than the optimal value $\min \hat{Q}_+$ of the first level of the Lasserre-SOS hierarchy applied to the original formulation (2.11) of the problem.

2.2. Extension to inequalities. Let $f(x) := c^T x + x^T F x$ for some $c \in \mathbb{R}^n$ and some $F^T = F \in \mathbb{R}^{n \times n}$, and consider the problem:

$$(2.12) \quad P: \quad f^* = \min_x \{ f(x) : A x \leq b; \ x \in \{0,1\}^n \},$$

for some cost vector $c \in \mathbb{Z}^n$, some matrix $A \in \mathbb{Z}^{m \times n}$, and some vector $b \in \mathbb{Z}^m$. We may and will replace (2.12) with the equivalent pure integer program:

$$P': \quad f^* = \min_{x,y} \{ f(x) : A x + y = b; \ x \in \{0,1\}^n; y \in \mathbb{N}^m \}.$$ 

Next, as $x \in \{0,1\}^n$ we can bound each integer variable $y_j$ by $M_j := b_j - \min \{ A_j x : x \in \{0,1\}^n \}$, $j = 1, \ldots, m$, where $A_j$ denotes the $j$-th row vector of the matrix $A$; and in fact $M_j = b_j - \sum_{i \in [n]} \min\{0, A_{ji}\}, j = 1, \ldots, m$. Then we may use the standard decomposition of $y_j$ into a weighted sum of boolean
variables:

\[ y_j = \sum_{k=0}^{s_j} 2^k z_{jk}, \quad z_{jk} \in \{0,1\}^n, \quad j = 1, \ldots, m, \]

(where \( s_j := \lceil \log(M_j) \rceil \)) and replace (2.12) with the equivalent 0/1 program:

\[ f^* = \min_{x,z} \{ f(x) : A^T x + \sum_{k=0}^{s_j} 2^k z_{jk} = b_j, j \leq m; (x,z) \in \{0,1\}^{n+s} \}, \]

(where \( s := \sum_j (1 + s_j) \)) which is of the form (1.1).

2.3. **Extension to polynomial programs.** Let \( f \in \mathbb{R}[x] \) be a polynomial of even degree \( d > 2 \) and consider the polynomial program:

\[ (2.13) \quad f^* = \min_{x} \{ f(x) : A^T x + \sum_{k=0}^{s_j} 2^k z_{jk} = b_j, j \leq m; (x) \in \{-1,1\}^n \}, \]

on the hyper cube \( \{-1,1\}^n \). Let \( d' := \lceil d/2 \rceil, \quad \mathbf{x} \mapsto g_j(\mathbf{x}) := 1 - x_j^2, \quad j = 1, \ldots, n, \) and with \( f \) let us associate the scalars:

\[ r_j^1 = \min \{ L_y(f) : M_{d'}(y) \succeq 0; M_{d'-1}(g_j(y)) = 0, j = 1, \ldots, n \} \]

\[ r_j^2 = \max \{ L_y(f) : M_{d'}(y) \succeq 0; M_{d'-1}(g_j(y)) = 0, j = 1, \ldots, n \} \]

where \( M_{d'}(y) \) (resp. \( M_{d'-1}(g_j(y)) \)) is the moment matrix (resp. localizing matrix) of order \( d' \) associated with the real sequence \( y = (y_\alpha), \alpha \in \mathbb{N}^n \), (resp. with the sequence \( y \) and the polynomial \( g_j \)). It turns out that \( r_j^1 \) (resp. \( r_j^2 \)) is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy associated with the optimization problem \( \min \{ f(x) : x \in \mathbb{R}^n \} \).
and so \( r_f^1 \leq \min \{ f(x) : x \in \{-1,1\}^n \} \) whereas \( r_f^2 \geq \max \{ f(x) : x \in \{-1,1\}^n \} \). For more details, see e.g. [8, 9].

Next, if we define

\[
\rho_f := \max_{i=1,2} |r_f^i|,
\]

then it is straightforward to verify that \( \rho_f \geq \max \{ |f(x)| : x \in \{-1,1\}^n \} \).

**Lemma 2.7.** Let \( f^* \) be as (2.13) and let \( \rho_f \) be as in (2.14). Then \( f^* \) is the optimal value of the polynomial minimization problem:

\[
(2.15) \min_{x \in \{-1,1\}^n} f(x) + (2 \rho_f + 1) \cdot \|Ax - b\|^2.
\]

The proof being almost a verbatim copy of that of Lemma 2.1, is omitted.

As for the quadratic case and with same arguments, one may also show that if \( d \) is even, the polynomial optimization problem (2.15) is equivalent to minimizing the homogeneous polynomial \( \tilde{f} \) of degree \( d \) on the hypercube \( \{-1,1\}^{n+1} \), where

\[
(x, x_0) \mapsto \tilde{f}(x) := x_0^d f(x/x_0) + (2r_f + 1) x_0^{d-2} \cdot \|Ax - x_0 b\|^2.
\]

But since it is not a MAX-CUT problem, to obtain a lower bound on \( f^* \) one may just as well consider solving the first level of the Lasserre-SOS hierarchy associated with (2.15) or even directly with (2.13). The advantage of using the formulation (2.15) is that one always minimizes on the hypercube \( \{-1,1\}^n \) instead of minimizing on the subset \( \{-1,1\}^n \cap \{x : Ax = b\} \) of the hypercube which is problem dependent.

### 2.4. Comparing with the copositive formulation.

As already mentioned in the introduction, the 0/1 program (1.1) also has a copositive formulation. Namely, let \( e_i = (\delta_{i=j}) \in \mathbb{R}^n \), \( i = 1, \ldots, n \), and \( e = (1, \ldots, 1) \in \mathbb{R}^n \). Following Burer [3, p. 481–482], introduce \( n \) additional variables \( z = (z_1, \ldots, z_n) \) and the \( n \) additional equality constraints \( x_i + z_i = 1 \), \( i = 1, \ldots, n \), with \( z \geq 0 \) (which are necessary to obtain an equivalent formulation). So let \( \tilde{x}^T = (x^T, z^T) \in \mathbb{R}^{2n} \) and with \( I \in \mathbb{R}^{n \times n} \) being the identity matrix, introduce the real matrices

\[
\tilde{F} := \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}, \quad S := \begin{bmatrix} A & 0 \\ I & I \end{bmatrix}
\]

and the real vectors \( \tilde{c}^T := (c^T, 0) \in \mathbb{R}^{2n} \) and \( \tilde{b}^T = (b^T, e^T) \in \mathbb{R}^{m+n} \). Let \( S_i \) denote the \( i \)-th row vector of \( S \), \( i = 1, \ldots, 2n \). Then the copositive
formulation of (1.1) reads:

\[
\begin{align*}
\min \quad & c^T \hat{x} + \langle \hat{F}, X \rangle \\
\text{s.t.} \quad & S_i \hat{x} = \hat{b}_i, \quad i = 1, \ldots, m + n \\
& S_i X S_i^T = \hat{b}_i^2, \quad i = 1, \ldots, m + n \\
& X_{ii} = x_i, \quad i = 1, \ldots, n \\
& \begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & X \end{bmatrix} \in C_{2n+1}^*, 
\end{align*}
\]  

(2.16)

where \( C_{2n+1} \) is the convex cone of \((2n+1) \times (2n+1)\) copositive matrices and \( C_{2n+1}^* \) is its dual, i.e., the convex cone of \( \text{completely positive matrices} \).

The hard constraint being membership in \( C_{2n+1}^* \), a strategy is to use hierarchies of tractable approximations (of increasing size) of \( C_{2n+1}^* \), as described in e.g. Dürr [5]. In particular a possible choice for the first relaxation in such hierarchies is to replace (2.16) with the semidefinite program:

\[
\begin{align*}
\min \quad & c^T \hat{x} + \langle \hat{F}, X \rangle \\
\text{s.t.} \quad & S_i \hat{x} = \hat{b}_i, \quad i = 1, \ldots, m + n \\
& S_i X S_i^T = \hat{b}_i^2, \quad i = 1, \ldots, m + n \\
& X_{ii} = x_i, \quad i = 1, \ldots, n \\
& \begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & X \end{bmatrix} \in S_{2n+1}^+ \cap \mathcal{N}_{2n+1},
\end{align*}
\]  

(2.17)

where \( S_{2n+1}^+ \) (resp. \( \mathcal{N}_{2n+1} \)) is the convex cone of real symmetric positive semidefinite (resp. entrywise nonnegative) matrices. Then (2.17) is a semidefinite relaxation of (2.16) because \( C_{2n+1}^* \subset S_{2n+1}^+ \cap \mathcal{N}_{2n+1} \), and so \( f_{\text{copo}}^* \leq f^* \). In fact if one considers the problem

\[
\min_{x \in \{0,1\}^n} \left\{ c^T x + x^T F x : A x = b; \ (A_i^T x)^2 = b_i^2, \ i = 1, \ldots, m \right\},
\]  

(2.18)

which is clearly equivalent to (1.1), then the first SDP-relaxation of the Lasserre-SOS hierarchy associated with (2.18) reads:

\[
\begin{align*}
\min \quad & c^T x + \langle F, X \rangle \\
\text{s.t.} \quad & A x = b \\
& A_i X A_i^T = b_i^2, \quad i = 1, \ldots, n \\
& X_{ii} = x_i, \quad i = 1, \ldots, n \\
& \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0,
\end{align*}
\]  

(2.19)

which is of the same flavor as the semidefinite program (2.17) but with dimension twice as less than (2.17).

**Example 2.8.** We have compared the first SDP-relaxation of the MAX-CUT formulation with the first SDP-relaxation (2.17) of the copositive formulation for the linear 0/1 knapsack problems (1.1) with \( F = 0, \ n = 10,15 \) and vector \( a \) as in Example 2.5.

We have kept the formulation on the hypercube \( \{0,1\}^n \) rather than on the hypercube \( \{-1,1\}^n \) and so in fact the first SDP relaxation is for problem
(2.3) with a quadratic cost function (and not a quadratic form as in the MAX-CUT formulation on \{-1,1\}^n).

In each case \( n = 10 \) (resp. \( n = 15 \)), we have chosen 19 (resp. 18) values of the right-hand-side \( b = 10s, s = 1, \ldots, 19 \) (resp. \( b = 20s, s = 1, \ldots, 18 \)), and for each problem we have run a sample of 10 problems with cost vector \( c = a + 10 \eta \) where \( \eta \) is a random variable uniformly distributed in [0, 1].

For \( n = 10 \), the lower bound \( f_{\text{maxcut}}^* \) from the MAX-CUT formulation dominates the lower bound \( f_{\text{copo}}^* \) in (2.17), in 111 out of 190 problems (\( \approx 58\%) \) and the relative difference \( 100 \cdot |f_{\text{maxcut}}^* - f_{\text{copo}}^*|/\max[f_{\text{maxcut}}^*, f_{\text{copo}}^*] \) never exceeds 0.05\% over all 190 problems!

For \( n = 15 \), \( f_{\text{maxcut}}^* > f_{\text{copo}}^* \) in 94 out of 180 problems (\( \approx 52\%) \) and \( 100 \cdot |f_{\text{maxcut}}^* - f_{\text{copo}}^*|/\max[f_{\text{maxcut}}^*, f_{\text{copo}}^*] \) never exceeds 0.55\% in all 180 problems!

3. Conclusion

In this paper we have shown that a linear or quadratic 0/1 program has an equivalent MAX-CUT formulation and so the whole arsenal of approximation techniques for the latter can be applied. In particular, and as suggested by some preliminary tests on a (limited sample) of 0/1 knapsack examples, it is expected that the lower bound obtained from the Shor relaxation of MAX-CUT will be in general better than the one obtained from the standard LP-relaxation (for linear 0/1 programs) of the original problem. The situation might be even better for quadratic 0/1 programs since to obtain a lower bound there is no need to first compute a convex quadratic underestimator of the criterion before applying a convex quadratic relaxation.

References


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