EXPLICIT RIGIDITY OF ALMOST-UMBILICAL HYPERSURFACES

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Abstract. We give an explicit estimate of the distance of a closed, connected, orientable and immersed hypersurface of a space form to a geodesic sphere and show that the spherical closeness can be controlled by a power of an integral norm of the traceless second fundamental form, whenever the latter is sufficiently small. Furthermore we use the inverse mean curvature flow in the hyperbolic space to deduce the best possible order of decay in the class of C^∞-bounded hypersurfaces of the Euclidean space.

Contents

1. Introduction

In this paper we prove two stability theorems of almost-umbilicity type, which give an answer to a question raised in [?], generalise the result in [?] to spaceforms and thereby partially improve [?, Thm. 1.3, Thm. 1.4]. Furthermore we use a recent counterexample for the inverse mean curvature flow in the hyperbolic space, cf. [?], to provide a new counterexample for spherical closeness estimates.

Let us shortly introduce the relevant notation. For an oriented hypersurface of a Riemannian manifold, $M^n \hookrightarrow N^{n+1}$, $|M|$ denotes its surface area, g its induced metric, A its second fundamental form, $\mathring{A}$ the traceless part of A,

$$\mathring{A} = A - Hg, \tag{1.1}$$

x_M the center of mass of M and $d_{\mathcal{H}}$ the Hausdorff distance of sets.

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For a tensor $(T^{j_1\dots j_l}_{i_1\dots i_k})$ on M, we define its L^p-norm to be

$$\|T\|_p = \left(\int_M |T^{j_1\dots j_l}_{i_1\dots i_k} T^{i_1\dots i_k}_{j_1\dots j_l}|^{\frac{p}{2}}\right)^{\frac{1}{p}}, \tag{1.2}$$

where indices are raised or lowered with the help of g. Let us formulate our first main result.

1.1. **Theorem.** *Let $M \hookrightarrow \mathbb{R}^{n+1}$ be a closed, connected, oriented and isometrically immersed C^2-hypersurface with $|M| = 1$. Let $p > n \geq 2$. Then there exists a constant $\epsilon_0 > 0$ depending on n, p and $\|A\|_p$, as well as a constant $\alpha = \alpha(n,p) \leq 1$, such that whenever there holds*

$$\|\mathring{A}\|_p \leq \|H\|_p \epsilon_0, \tag{1.3}$$

then

$$d_{\mathcal{H}}(M, S_R(x_M)) \leq \frac{c^\alpha}{\|H\|_p^\alpha} \|\mathring{A}\|_p^\alpha \equiv \epsilon^\alpha \tag{1.4}$$

and M is ϵ^α-quasi-isometric to a sphere S_R with a certain radius R.

By ϵ^α-quasi-isometric, we mean that the diffeomorphism from M into $S_R(p)$ satisfies

$$\left||dF_x(u)|^2 - 1\right| \leqslant \epsilon^\alpha \tag{1.5}$$

for any $x \in M$, $u \in T_xM$ and $|u| = 1$.

The assumption $|M| = 1$ is only for simplification. By scaling it is easy to obtain a scale-invariant version of (1.4) for arbitrary M.

In section 3 we generalize this theorem to conformally flat ambient spaces.

The history of the problem to control the closeness to a sphere by curvature quantities is quite long, starting from the well known *Nabelpunktsatz*. We refer to the bibliography in [**?**] for a quite detailed overview. Let us only mention several results which have appeared recently. For surfaces, $n = 2$, a quite straightforward calculation due to Andrews yields an explicit C^0-estimate for convex hypersurfaces, cf. [**?**, Prop. 4, Lemma 5],

$$\left|\langle x - q, \nu\rangle - \frac{1}{8\pi}\int_M H\right| \leq C|M||\kappa_1 - \kappa_2|, \tag{1.6}$$

where x is the embedding vector and q is the Steiner point. In section 4 we use the inverse mean curvature flow (IMCF) in the hyperbolic space to prove that the power on the right-hand side of (1.6) can not be improved to $\alpha > 1$, which is in turn then not possible either in Theorem 1.1. The latter proof relies on a recent example due to Hung and Wang, [**?**, Thm. 1, Prop. 5], that the convergence after rescaling in the IMCF can not be too fast in the hyperbolic space.

For strictly convex hypersurfaces of $\mathbb{R}^{n+1}$ there is the following estimate of circumradius R minus inradius r due to Leichtweiß, cf. [**?**, Thm. 1.4, eq. (38)]:

$$R - r \leq c_n \max_{x\in M}(R_n(x) - R_1(x)), \tag{1.7}$$

where $R_1 \leq \dots \leq R_n$ are the ordered radii of curvature. However, Theorem 1.1 deals with estimates in dependence of integral pinching. For the case $n = 2$, an estimate similar to (1.1) with a better constant was obtained by De Lellis and Müller, cf. [?]

In [?, Cor. 1.2] Perez derived a *qualitative* solution and obtained under certain assumptions, for given $\epsilon > 0$, a $\delta > 0$, such that

$$\|\mathring{A}\|_p < \delta \tag{1.8}$$

implies

$$d_{\mathcal{H}}(M, S_{r_0}(x)) < \epsilon. \tag{1.9}$$

In [?, p. xvi] the author posed the derivation of an explicit δ as a question of interest.

The second author of the present paper derived a *quantitative* δ within the class of strictly convex hypersurfaces in [?, Thm. 1.1]. There it is shown that a δ which has the order $\epsilon^{2+\alpha}$, $\alpha > 0$ provides (1.9). However, the proof in [?] relies on the convexity of the hypersurface. Unfortunately, also in the paper at hand we did not achieve a constant independent on the size of the curvature itself. The constant is only uniform in the class of hypersurfaces with a fixed bound on the curvature of the scaled hypersurface.

The following theorem, due to Grosjean and the first author, [?, Thm. 1.4], already provides this conclusion, however only with the additional assumption of smallness of the oscillation of the mean curvature itself:

1.2. **Theorem.** [?, Thm. 1.4]
Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Let p be the center of mass of M. Let $\epsilon < 1$, $r, q > n$, $s \geq r$ and $c > 0$. Let us assume that $|M|^{\frac{1}{n}}\|H\|_q \leq c$. Then there exist positive constants $C = C(n, q, c)$, $\alpha = \alpha(q, n)$, such that if $\epsilon^\alpha \leq \frac{1}{C}$,

$$\|\mathring{A}\|_r \leq \|H\|_r \epsilon \tag{1.10}$$

and

$$\|H^2 - \|H\|_s^2\|_{\frac{r}{2}} \leq \|H\|_r^2 \epsilon, \tag{1.11}$$

then M is ϵ^α-Hausdorff close to $S_{\frac{1}{\|H\|_2}}(p)$. Moreover if $|M|^{\frac{1}{n}}\|A\|_q \leq c$, then M is diffeomorphic and ϵ^α-quasi-isometric to $S_{\frac{1}{\|H\|_2}}(p)$.

Note that in this theorem, L^p-norms are defined slightly different, namely such that the L^p-norms of scale-invariant functions are scale-invariant. Our notation corresponds to the one in [?]. This ambiguity does not cause any problems, since we prove Theorem 1.1 for $|M| = 1$. Also note the typo in [?, Thm. 1.4], where the α is missing in the conclusion.

In [?, Thm. 3.1], which also covers other ambient spaces, (1.11) was replaced by an assumption on the gradient of H. However, with the help of the following theorem due to Perez it is possible to get rid of (1.11) completely.

1.3. **Theorem.** [?, Thm. 1.1]
Let $p > n \geq 2$ and $c_0 > 0$ be given. Then there is a constant $C > 0$, depending only

on n, p and c_0, such that:
If $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed and connected n-dimensional hypersurface with

$$|\Sigma| = 1 \tag{1.12}$$

and

$$\|A\|_p \leq c_0, \tag{1.13}$$

then

$$\min_{\lambda\in\mathbb{R}} \|A - \lambda g\|_p \leq C\|\mathring{A}\|_p. \tag{1.14}$$

The proof of Theorem 1.1 is a combination of Theorem 1.2 and Theorem 1.3.

2. Proof of Theorem 1.1

Without loss of generality we may suppose that M is of class C^∞, since both sides of the inequality are continuous with respect to the C^2-norm and hence the general result can then be achieved by approximation.

Using [**?**, Thm. 1.1], we obtain a $\lambda_0 \in \mathbb{R}$, such that

$$\|A - \lambda_0 g\|_p \leq C'\|\mathring{A}\|_p, \tag{2.1}$$

where $C' = C'(n, p, \|A\|_p)$. Let us calculate

$$\begin{aligned} \|H^2 - \|H\|_p^2\|_{\frac{p}{2}} &\leq \|H^2 - \lambda_0^2\|_{\frac{p}{2}} + \|\lambda_0^2 - \|H\|_p^2\|_{\frac{p}{2}} \\ &= \left(\int_M |H - \lambda_0|^{\frac{p}{2}}|H + \lambda_0|^{\frac{p}{2}}\right)^{\frac{2}{p}} + |\lambda_0^2 - \|H\|_p^2| \\ &\leq 2(\|H\|_p + |\lambda_0|)\|H - \lambda_0\|_p \\ &\leq \frac{2}{\sqrt{n}}(\|H\|_p + |\lambda_0|)\|A - \lambda_0 g\|_p \\ &\leq c'\|H\|_p\|\mathring{A}\|_p, \end{aligned} \tag{2.2}$$

where $c' = c'(n, p, \|A\|_p)$. The last inequality is due to the fact that

$$|\lambda_0 - \|H\|_p| \leq c''\|\mathring{A}\|_p. \tag{2.3}$$

Defining

$$c = \max(1, c'), \tag{2.4}$$

$$\epsilon = \frac{c\|\mathring{A}\|_p}{\|H\|_p}, \tag{2.5}$$

and

$$\epsilon_0 := \frac{\min\left(1, C^{-\frac{1}{\alpha}}\right)}{2c} \tag{2.6}$$

then by (1.3),

$$\epsilon \leq c\epsilon_0 = \frac{1}{2}\min\left(1, C^{-\frac{1}{\alpha}}\right), \tag{2.7}$$

where α and C are the constants from [**?**, Thm. 1.4]. Furthermore we have

$$\|\mathring{A}\|_p \leq \|H\|_p \epsilon \tag{2.8}$$

and

$$\|H^2 - \|H\|_p^2\|_{\frac{p}{2}} \leq \|H\|_p^2 \epsilon. \tag{2.9}$$

Thus we may apply [**?**, Thm. 1.4] to conclude that M is ϵ^α-close to a sphere.

3. Generalization to conformally flat manifolds

Using that the property of a submanifold to be totally umbilic is conformally invariant, we easily obtain the following generalization to conformally flat manifolds, which in particular include the space forms and improves the ϵ^α-proximity statement in [**?**, Thm. 1.3] in the sense that it removes assumption (2).

3.1. **Theorem.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $N^{n+1} = (\Omega, \bar{g})$ be a conformally flat Riemannian manifold, i.e.*

$$\bar{g} = e^{2\psi}\tilde{g}, \tag{3.1}$$

where $\tilde{g}$ is the Euclidean metric and $\psi \in C^\infty(\Omega)$. Let $M^n \hookrightarrow N^{n+1}$ be a closed, connected, oriented and isometrically immersed C^2-hypersurface. Let $p > n \geq 2$. Then there exist constants c and ϵ_0, depending on n, p, $|M|$, $\|A\|_p$ and $\|\psi\|_{\infty,M}$, as well as a constant $\alpha = \alpha(n,p)$, such that whenever there holds

$$\|\mathring{A}\|_p \leq \epsilon_0, \tag{3.2}$$

there also holds

$$d_{\mathcal{H}}(M, \tilde{S}_R) \leq c\|\mathring{A}\|_p^\alpha, \tag{3.3}$$

where $\tilde{S}_R$ is the image of a Euclidean sphere considered as a hypersurface in N^{n+1} and the Hausdorff distance is also measured with respect to the metric $\bar{g}$.

3.2. *Remark.* Since in conformally flat spaces the scaling behaviour of the second fundamental form with respect to homotheties heavily depends on the nature of the ambient space, in this case there seems to be no way to give a general scale invariant estimate. This is the reason why this closeness estimate is only uniformly valid in the class of C^2-bounded hypersurfaces.

Furthermore note that for example in all space forms the hypersurface $\tilde{S}_R$ is actually a geodesic sphere. This follows from the fact that in those spaces totally umbilical hypersurfaces are spheres and total umbilicity is conformally invariant, as will be appearant from the following proof of Theorem 3.1.

Thus Theorem 3.1 gives an explicit spherical closeness estimate of almost-umbilical hypersurfaces in the hyperbolic space as well as in the half-sphere of constant positive sectional curvature.

Proof. Under a conformal relation of the metrics as in (3.1) the corresponding induced geometric quantities of the the embedded hypersurface M are related as follows.

$$g_{ij} = e^{2\psi}\tilde{g}_{ij} \tag{3.4}$$

and

$$h_{ij}e^{-\psi} = \tilde{h}_{ij} + \psi_\beta \tilde{\nu}^\beta \tilde{g}_{ij}. \tag{3.5}$$

Those formulae can be found in [?, Prop. 1.1.11]. Hence

$$h_{ij} - Hg_{ij} = e^{\psi}(\tilde{h}_{ij} - \tilde{H}\tilde{g}_{ij}) \tag{3.6}$$

and hence

$$c\|\mathring{\tilde{A}}\|_p \le \|\mathring{A}\|_p \le C\|\mathring{\tilde{A}}\|_p, \tag{3.7}$$

where the constants depend on $\|\psi\|_{\infty,M}$. Since the Euclidean and the conformal Hausdorff distances are equivalent whenever $|\psi|$ is bounded, we obtain the result after applying Theorem 1.1. ☐

Due to a well known interpolation theorem for convex hypersurfaces of Riemannian manifolds we obtain the following gradient stability estimate in space forms.

3.3. **Corollary.** *Let N^{n+1} be a space form and M as in Theorem 3.1 be additionally strictly convex, where we also assume that $\bar{g}$ is given in geodesic polar coordinates*

$$\bar{g} = dr^2 + \vartheta^2(r)\sigma_{ij}dx^i dx^j \equiv dr^2 + \bar{g}_{ij}dx^i dx^j \tag{3.8}$$

with suitable ϑ depending on the space form. Let $p > n$. Then there exist constants c and ϵ_0 depending on n, p, $|M|$, $\|A\|_p$ and $\|\psi\|_\infty$, as well as a constant $\alpha = \alpha(p,n)$, such that

$$\|\mathring{A}\|_p \le \epsilon_0 \tag{3.9}$$

implies

$$v = \sqrt{1 + \bar{g}^{ij}u_i u_j} \le e^{c\|\mathring{A}\|_p^\alpha}, \tag{3.10}$$

where

$$M = \{(x^0, x^i) \colon x^0 = u(x^i), (x^i) \in \mathcal{S}_0\} \tag{3.11}$$

is a suitable graph representation over a geodesic sphere $\mathcal{S}_0 \hookrightarrow N^{n+1}$ and $(\bar{g}^{ij})$ is the inverse of $(\bar{g}_{ij})$.

Proof. It is well known that a strictly convex hypersurface of $\mathbb{S}^{n+1}$ is contained in an open hemisphere, cf. [?] for the smooth case and also [?, Cor. 1.2] for the C^2-case. Thus in any space form a strictly convex M is covered by a conformally flat coordinate system as in Theorem 3.1, which is thus applicable. Let $\mathcal{S}_0$ be the corresponding sphere with center p, then we can write M as a graph over $\mathcal{S}_0$ due to the strict convexity. Thus we may apply the well-known interpolation estimate

$$v \le e^{\bar{\kappa} \operatorname{osc} u}, \tag{3.12}$$

cf. [?, Thm. 2.7.10], where $\bar{\kappa}$ is a lower bound for the principal curvatures of the coordinate slices $\{r = \text{const}\}$. The latter, however, only depends on $\|\psi\|_\infty$ as well. ☐

3.4. *Remark.* Note that in the Euclidean case the assumption of the strict convexity is redundant, if (3.9) is satisfied with $p = \infty$, since then it is possible to deduce strict convexity from (3.9), cf. [?, Lemma 2.2].

4. An optimality result

We prove the optimality of the the statement in Theorem 1.1 in the sense that there is no hope to derive a uniform estimate of the form

$$d_{\mathcal{H}}(M, S_R(x_0)) \leq c\|\mathring{A}\|_\infty^\alpha, \quad \alpha > 1, \tag{4.1}$$

in the class of uniformly C^∞-bounded hypersurfaces M. To be precise, for $\alpha > 1$ we get the following negation of (1.6) in the class of uniformly convex hypersurfaces and for all $n \geq 2$.

4.1. **Theorem.** *Let $n \geq 2$ and $C = 2\max(|S_2(0)|, \|\bar{A}_{S_2}\|_\infty)$. For all $\alpha > 1$ and for all $k \in \mathbb{N}$ there exists a uniformly convex smooth hypersurface $M_k \hookrightarrow \mathbb{R}^{n+1}$ with*

$$\max(\|A_k\|_\infty, |M_k|) \leq C, \tag{4.2}$$

such that

$$\|\mathring{A}_k\|_\infty < \frac{1}{k} \tag{4.3}$$

and for all spheres $S \subset \mathbb{R}^{n+1}$ there holds

$$d_{\mathcal{H}}(M_k, S) > k\|\mathring{A}_k\|_\infty^\alpha. \tag{4.4}$$

Here $\bar{A}_{S_2}$ denotes the second fundamental form of the sphere with radius 2.

In a recent paper, Drach gave a counterexample to an improved spherical closeness estimate in the class of $C^{1,1}$ hypersurfaces, namely a special spindle shaped hypersurface, cf. [**?**, Thm. 1]. However since, we consider (1.4) in the space of at least C^2-hypersurfaces, we need to find a different contradiction to (4.1). This contradiction is deduced along the inverse mean curvature flow in the hyperbolic space.

Before we prove Theorem 4.1, let us for convenience recall the relevant facts about the inverse mean curvature flow in the hyperbolic space $\mathbb{H}^{n+1}$. There one considers a time parameter family of embeddings of closed, starshaped and mean-convex hypersurfaces

$$x\colon [0, T^*) \times M \hookrightarrow \mathbb{H}^{n+1}, \tag{4.5}$$

which solves

$$\dot{x} = \frac{1}{H}\nu, \tag{4.6}$$

where $H = g^{ij}h_{ij}$ and ν is the outward unit normal to $M_t = x(t, M)$. Note that we have switched the notation of H in this context due to a better comparability with the literature. It is known, cf. [**?**, Lemma 3.2], that for an initial starshaped and mean-convex hypersurface M_0 the flow exists for all times and all the flow hypersurfaces can be written as a graph over a fixed geodesic spere $\mathcal{S}_0$,

$$M_t = \{(x^0, x^i)\colon x^0(t,\xi) = u(t, x^i(t,\xi))\}, \tag{4.7}$$

where u describes the radial distance to the center of $\mathcal{S}_0$. In [**?**, Thm. 1.2] Gerhardt claimed to have shown convergence of the rescaled hypersurfaces

$$\hat{M}_t = \text{graph } \hat{u} \equiv \text{graph}\left(u - \frac{t}{n}\right) \tag{4.8}$$

to a geodesic sphere. However, as was pointed out in [?, Thm. 1] with the help of a concrete counterexample, the limit function of $\hat{u}$ is not constant in general. In particular the authors proved that there is a starshaped and mean-convex initial hypersurface M_0, such that the limit hypersurface is not of constant curvature, in particular not a geodesic sphere. However, there is a smooth limit function to which the $\hat{M}_t$ converge smoothly, compare the proof of [?, Thm. 6.11] and also compare [?, Thm. 1.2].

In order to relate the convergence results of the IMCF in the hyperbolic space with the rigidity estimate (1.4) in the Euclidean space, we have to look at the hyperbolic flow in the conformally flat model. In [?] the Poincaré ball model in the ball of radius two was considered. Let r denote the geodesic distance to the center of $\mathcal{S}_0$ in $\mathbb{H}^{n+1}$, then the by the coordinate change

$$\rho = 2 - \frac{4}{e^r + 1} \tag{4.9}$$

the representation of the hyperbolic metric transforms like

$$\bar{g} = dr^2 + \sinh^2(r)\sigma_{ij}dx^i dx^j = \frac{1}{\left(1 - \frac{1}{4}\rho^2\right)^2}(d\rho^2 + \rho^2\sigma_{ij}dx^i dx^j) \equiv e^{2\psi}\tilde{g}, \tag{4.10}$$

where σ_{ij} is the standard round metric of the sphere $\mathcal{S}_0$. Then the convergence

$$u - \frac{t}{n} \to \hat{u}_\infty \tag{4.11}$$

in the original coordinates is equivalent to the convergence of

$$(2 - w)e^{\frac{t}{n}} \to \hat{w}_\infty, \tag{4.12}$$

where

$$w = 2 - \frac{4}{e^u + 1} \tag{4.13}$$

and where $\hat{w}_\infty$ is a strictly positive function, due to [?, Lemma 3.1].

The proof of Theorem 4.1 is very similar to the proof of a corresponding positive result in this direction by the second author. In [?] he proved that due to a strong decay of the traceless second fundamental form along the IMCF in $\mathbb{R}^{n+1}$ we indeed obtain spherical roundness in this case without rescaling. The idea how to obtain a negative result in the hyperbolic space is that if we could improve the spherical closeness, then we could mimic the proof in [?] to deduce a roundness result in $\mathbb{H}^{n+1}$, which is not possible in view of Hung's and Wang's paper.

The idea of the proof of Theorem 4.1 goes as follows: The estimate in (4.12) provides closeness of the flow hypersurfaces to the sphere of radius 2 in the ball model. The order of the closeness is $e^{-\frac{t}{n}}$. The traceless second fundamental form decays correspondingly, as we will point in more detail later in the proof. But if we had this additional exponent α in the spherical closeness estimate, we could even deduce better spherical closeness (to a sphere different from S_2) than we have in (4.12) and then we would be able to translate this to a spherical closeness in the hyperbolic space. This would in turn yield a contradiction to Hung's and Wang's result. Now let us prove Theorem 4.1 in detail. First we need some helpful notation and an auxillary result.

4.2. **Definition.** (i) Let N be either the Euclidean space, the hyperbolic space or an open hemisphere. For a starshaped hypersurface $M \hookrightarrow N$, let M^* be the set of points in N, with respect to which M is starshaped.

(ii) For a starshaped hypersurface $M \hookrightarrow N$ let $p \in M^*$. Then for the graph representation

$$M = \{(r, x^i) \colon r = u(x^i), (x^i) \in \mathcal{S}_p\}, \tag{4.14}$$

by

$$\operatorname{osc}_p u = \max_{x \in \mathcal{S}_p} u(x) - \min_{x \in \mathcal{S}_p} u(x) \tag{4.15}$$

we denote the oscillation of the geodesic distance of the point (u, x^i) to the point p. Here $\mathcal{S}_p$ denotes a geodesic sphere around p.

By a simple argument we obtain the following alternative for a general expanding sequence of hypersurfaces with controlled oscillation.

4.3. **Lemma.** *Let N be as in Definition 4.2 and $M_t \hookrightarrow N$, $0 \leq t \in \mathbb{R}$, be a family of starshaped hypersurfaces such that*

$$M_t^* \subset M_s^* \quad \forall s \geq t \tag{4.16}$$

and such that for all $t_0 \in \mathbb{N}$ and $p \in M_{t_0}^*$ there exists a constant c independent of t_0, such that

$$\operatorname{osc}_p u_t \leq c \operatorname{osc}_p u_{t_0} \quad \forall t \geq t_0. \tag{4.17}$$

Then for fixed p, $\operatorname{osc}_p u_t$ does not have zero as a limit value for $t \to \infty$ unless

$$\operatorname{osc}_p u_t \to 0, \quad t \to \infty. \tag{4.18}$$

Proof. For given $\epsilon > 0$, if zero is a limit point, we may choose t_0, such that

$$\operatorname{osc}_p u_{t_0} \leq \frac{\epsilon}{c}, \tag{4.19}$$

then

$$\operatorname{osc}_p u_t \leq c \operatorname{osc}_p u_{t_0} \leq \epsilon \quad \forall t \geq t_0. \tag{4.20}$$

□

Now we can prove Theorem 4.1.

Proof. Assume the contrary, i.e. that there exist $\alpha > 1$ and $k \in \mathbb{N}$, such that for all uniformly convex hypersurfaces $\tilde{M} \hookrightarrow \mathbb{R}^{n+1}$ with

$$\max(|\tilde{M}|, \|\tilde{A}\|_\infty) \leq C \tag{4.21}$$

we have that

$$\|\mathring{\tilde{A}}\|_\infty < \frac{1}{k} \tag{4.22}$$

implies

$$\tilde{d}_{\mathcal{H}}(\tilde{M}, \tilde{S}) \leq k \|\mathring{\tilde{A}}\|_\infty^\alpha \tag{4.23}$$

for some suitable sphere $\tilde{S} \subset \mathbb{R}^{n+1}$. According to [**?**, Thm. 1] for $n = 2$ and [**?**, Sec. 4] for $n \geq 3$ there exists a starshaped and mean-convex hypersurface $M_0 \hookrightarrow \mathbb{H}^{n+1}$, such that for no graph representation

$$M_t = \text{graph } u \tag{4.24}$$

the rescaled IMCF flow hypersurfaces

$$\hat{M}_t = \text{graph}\left(u - \frac{t}{n}\right) \equiv \text{graph } \hat{u} \tag{4.25}$$

converge to a geodesic sphere. However, for each graph representation, we obtain smooth convergence of

$$\hat{u} \to \hat{u}_\infty. \tag{4.26}$$

In [**?**, Thm. 1.2 (2)] it is deduced that

$$\|\mathring{A}\|_\infty \leq ce^{-\frac{2t}{n}}, \tag{4.27}$$

where $c = c(n, M_0)$. Now fix a graph representation around $p \in M_0^*$. From (3.4) and (3.5) we obtain that the corresponding Euclidean traceless part decays like

$$\|\mathring{\tilde{A}}\|_\infty = \|e^\psi \mathring{A}\|_\infty \leq e^\psi_{\max} e^{-\frac{2t}{n}}, \tag{4.28}$$

where

$$e^\psi_{\max} = \frac{1}{\left(1 - \frac{1}{4}w^2_{\max}\right)} \tag{4.29}$$

with w as in (4.13) and

$$w_{\max} = \max_{x \in \mathcal{S}_p} w(x). \tag{4.30}$$

Due to (4.12) we obtain

$$\|\mathring{\tilde{A}}\|_\infty \leq ce^{-\frac{t}{n}} \tag{4.31}$$

and due to the C^∞-convergence of $w \to 2$, we are in the situation to apply our assumption (4.23), whenever t is large enough. We obtain a sequence of spheres $\tilde{S}_{\tilde{R}_t} \subset \mathbb{R}^{n+1}$, such that

$$\tilde{d}_{\mathcal{H}}(\tilde{M}_t, \tilde{S}_{\tilde{R}_t}) \leq ce^{-\frac{\alpha}{n}t}, \tag{4.32}$$

where the Hausdorff distance is measured with respect to the Euclidean metric. Due to (4.12) we even have

$$\tilde{S}_{\tilde{R}_t} \subset B_2(0), \tag{4.33}$$

for large times t.

Now let us switch back to the hyperbolic space. The spheres $\tilde{S}_{\tilde{R}_t}$ are geodesic spheres in $\mathbb{H}^{n+1}$ as well since total umbilicity is preserved under a conformal transformation and in the Euclidean space as well as in the hyperbolic space for closed and embedded hypersurfaces total umbilicity is tantamount to being a geodesic sphere. We denote these spheres in $\mathbb{H}^{n+1}$ by S_{R_t}. For the corresponding hyperbolic Hausdorff distance we deduce

$$d_{\mathcal{H}}(M_t, S_{R_t}) \leq e^\psi_{\max}\tilde{d}_{\mathcal{H}}(\tilde{M}_t, \tilde{S}_{\tilde{R}_t}) \leq ce^{\frac{1-\alpha}{n}t}, \tag{4.34}$$

which converges to 0 as $t \to \infty$.

Since the inradius of the M_t converges to infinity and for large t the M_t are strictly convex, for each $\delta > 0$ we find $t_0 > 0$, such that

$$\bar{B}_\delta(p) \subset M^*_{t_0} \subset M^*_t \quad \forall t \geq t_0, \tag{4.35}$$

where the latter inclusion is due to the fact that starshapedness around a given point is preserved. According to [**?**, Prop. 3.2, Lemma 3.5], there holds for the oscillation of u that for all t_0 and all $q \in M^*_{t_0}$ we have

$$\operatorname{osc}_q u(t,\cdot) \leq c \operatorname{osc}_q u(t_0,\cdot) \quad \forall t \geq t_0. \tag{4.36}$$

So in particular, if we choose

$$\delta = c \operatorname{osc}_p u(0,\cdot), \tag{4.37}$$

we find that the oscillation of each M_t is minimized within the set $\bar{B}_\delta(p)$:

$$\operatorname*{argmin}_{q\in M^*_t} \operatorname{osc}_q u(t,\cdot) \in \bar{B}_\delta(p) \quad \forall t \geq t_0, \tag{4.38}$$

because outside $\bar{B}_\delta(p)$ the oscillation is already larger than it is with respect to p.

Due to (4.34) we obtain

$$\operatorname{osc}_{q_t} u(t,\cdot) = \min_{q\in\bar{B}_\delta(p)} \operatorname{osc}_q u(t,\cdot) \leq c e^{\frac{1-\alpha}{n}t} \quad \forall t \geq t_0. \tag{4.39}$$

Let t_k be a sequence of times with $t_k \to \infty$. Due to the compactness of $\bar{B}_\delta(p)$ a subsequence of center points converges,

$$q_k \to q \in \bar{B}_\delta(p), \tag{4.40}$$

where we did not rename the index of the sequence. Since

$$|\operatorname{osc}_{q_k} u(t_k,\cdot) - \operatorname{osc}_q u(t_k,\cdot)| \leq 2\operatorname{dist}(q_k, q) \quad \forall k \in \mathbb{N}, \tag{4.41}$$

we obtain

$$\operatorname{osc}_q u(t_k,\cdot) \to 0, \quad k \to \infty. \tag{4.42}$$

In view of (4.36) and the preservation of starshapedness along IMCF the assumptions of Lemma 4.3 are fulfilled. Applying Lemma 4.3, we obtain that

$$\operatorname{osc}_q(t,\cdot) \to 0, \tag{4.43}$$

in contradiction to the choice of the initial hypersurface. □

4.4. *Remark.* Note that in turn of the proof we even have shown that for given $\alpha > 1$ and $k \in \mathbb{N}$ as in Theorem 4.1, such a counterexample M_k satisfying (4.3) and (4.4) must actually occur along the inverse mean curvature flow in the conformally flat version of the IMCF in $\mathbb{H}^{n+1}$. We only used our contrary assumption within this class of flow hypersurfaces.

5. Concluding remark

We would like to point out that the techniques in section 4 might be useful in other situations. Whenever one would like to estimate the closeness to a sphere in comparison with another geometric quantity, e.g. in comparison with eigenvalue pinching of the Laplacian or also in almost-Schur/almost-CMC type estimates, one could determine how this particular geometric quantity behaves along the IMCF and then determine the best possible roundness estimate using the IMCF in $\mathbb{H}^{n+1}$. It should often be quite straightforward to derive the best possible decay estimate.

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