HAL open science

An asymptotic shape theorem for random linear growth models

Aurelia Deshayes

► To cite this version:

Aurelia Deshayes. An asymptotic shape theorem for random linear growth models. 2015. hal-01153313

HAL Id: hal-01153313
https://hal.science/hal-01153313

Preprint submitted on 19 May 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AN ASYMPTOTIC SHAPE THEOREM FOR RANDOM LINEAR GROWTH MODELS

AURELIA DESHAYES

Abstract. In this note, we generalize the asymptotic shape theorem proved in [Des14a] for a class of random growth models whose growth is at least and at most linear. In this way, we obtain asymptotic shape theorems conjectured for several models: the contact process in a randomly evolving environment [SW08], the oriented percolation with hostile immigration [GM12b] and the bounded modified contact process [DS00].

1. Introduction

In 1974, Harris [Har74] introduced the *classical contact process* as an interacting particle system modeling the spread of a population through the grid $\mathbb{Z}^d$. This process has been well studied and it satisfies in particular a shape theorem. Since then, a lot of extensions appeared in the literature: *two stage contact process* by Krone [Kro99], *boundary modified contact process* by Durett and Schinazi [DS00], *contact process in randomly evolving environment* by Broman [Bro07], Remenik [Rem08], Steif and Warfheimer [SW08], *contact process with aging* by the author [Des14a] *etc.* All these processes are linear random growth models in the sense that we can exhibit a quantity of interest A_t taking values in $\mathcal{P}(\mathbb{Z}^d)$ and deterministic compact sets A_- and A_+ such that, for t large enough,

$$tA_- \subset A_t \subset tA_+.$$

The first inclusion traduces the at least linear growth and the second one the at most linear growth. These models have also others characteristics like attractivity, phase transition phenomenon *etc.* Here, we want to show that they satisfy asymptotic shape results *i.e.* there exists a deterministic set B such that $\frac{A_t}{t}$ converges to B in a sense to precise and we want to highlight their common properties which lead to a general asymptotic shape theorem. In order to do that we introduce a general class of random growth models including the above mentioned contact process extensions and a related good quantity, the *essential hitting time* σ, which allows us to use the same techniques that the ones involved in [Des14a] to prove an asymptotic shape theorem. Indeed, as the standard contact process itself, its extensions are non permanent models: the extinction is possible and the hitting times can be infinite so that the standard integrability conditions are not satisfied. On the other hand, if we condition the model to survive, then stationarity and subadditivity properties can be lost. The quantity σ overcome such lacks: it turns out that this function σ satisfies adequate stationarity properties as well as the almost-subaddivity conditions involved in Kesten and Hammersley's theorem (see [Kes73] and

2000 *Mathematics Subject Classification.* 60K35, 82B43.

Key words and phrases. interacting particles system, asymptotic shape theorem, essential hitting time, subadditivity, contact processes.

[Ham74]), a well-known extension of Kingman's seminal result. We will define precisely all the quantities involved but we will refer to [Des14a] for proofs working in the same way.

We obtain a general asymptotic shape theorem for a class of random linear growth models and deduce asymptotic shape theorems conjectured for several models: the contact process in a randomly evolving environment [SW08], the oriented percolation with hostile immigration [GM12b] and the bounded modified contact process [DS00].

2. A class of random linear growth models

2.1. Notations and definitions.

2.1.1. *State space.* We will work on interacting particle systems on $\mathbb{Z}^d$. For x and y in $\mathbb{Z}^d$, we say that x and y are neighbors, and we denote by $x \sim y$, if $\sum_{i=1}^d |x_i - y_i| = 1$. We denote by T_x the spatial translation operator of vector $x \in \mathbb{Z}^d$.

Let S be a finite totally ordered set which will represent the possible state (or type) of a particle $x \in \mathbb{Z}^d$. We denote by $S^{\mathbb{Z}^d}$ the set of mappings $\xi : \mathbb{Z}^d \mapsto S$ which can also be seen as the set of partitions of $\mathbb{Z}^d$ in $|S|$ subsets:

$$\xi \in S^{\mathbb{Z}^d} \Leftrightarrow \left\{ \begin{array}{lccc} \xi : & \mathbb{Z}^d & \to & S \\ & x & \to & \xi(x); \end{array} \right.$$

and we denote by $S_f^{\mathbb{Z}^d}$ the subset of functions with finitely many coordinates non equal to $\min S$ (often 0). Let $\mathcal{C}(S^{\mathbb{Z}^d})$ be the set of continuous functions from $S^{\mathbb{Z}^d}$ to $\mathbb{R}$ and $\mathcal{C}_0(S^{\mathbb{Z}^d})$ the subset of functions depending on only finitely many coordinates of ξ. The set $S^{\mathbb{Z}^d}$ will be our state space of configurations.

2.1.2. *Markov process and property of interest.* We work with a time space $\mathcal{T}$ which can be discrete ($\mathcal{T} = \mathbb{N}$) or continuous ($\mathcal{T} = \mathbb{R}_+$). Let $(\xi_t)_{t\in\mathcal{T}}$ be a stationary Markov process taking values in $S^{\mathbb{Z}^d}$ and let (P) be a *property of interest* about elements of S. We define the process (A_t) taking values in $\mathcal{P}(\mathbb{Z}^d)$ by, for $t \in \mathcal{T}$,

$$A_t = \{x \in \mathbb{Z}^d, \xi_t(x) \text{ satisfies } (P)\};$$

the quantity A_t represents the set of points which satisfies the property (P) at time t.

We can also write, for $t \in \mathcal{T}$ and $x \in \mathbb{Z}^d$, $A_t(x) = g(\xi_t(x))$ where g is the indicator function corresponding to the property (P); we extend $A_t = g(\xi_t)$ in the natural way. One should notice that the process (A_t) is not necessarily Markovian.

Let $\delta_{\min}$ be the minimal non trivial configuration of the process (ξ_t) such that $A_0 = g(\delta_{\min})$ is not empty (*i.e.* someone in $\delta_{\min}$ satisfies (P)). A such configuration $\delta_{\min}$ exists because S is totally ordered and it is unique modulo translation. For every $x \in \mathbb{Z}^d$, we denote by

$$\tau = \inf\{t > 0, A_t^{\delta_{\min}} = \emptyset\},$$
$$t(x) = \inf\{t > 0, x \in A_t^{\delta_{\min}}\}.$$

Starting from $\delta_{\min}$, τ is the first time when no one satisfies (P) and $t(x)$ is the first time when x satisfies (P).

For $x \in \mathbb{Z}^d$ and $t \in \mathcal{T}$ we denote by T_x the spatial translation and by θ_t the temporal one. For $t \in \mathcal{T}$, let

$$H_t = \bigcup_{s \leq t} A_t = \{x \in \mathbb{Z}^d, t(x) \leq t\}$$

be the set of points which have satisfied (P) before time t.

2.1.3. *Interacting particle system.*

Definition. *Let (ξ_t) be a Markov process taking values in $S^{\mathbb{Z}^d}$ and (P) a property of interest. We say that (ξ_t, P) belongs to the class $\mathcal{C}$ if it satisfies the following conditions:*

(1) The generator $\mathcal{A}$ of (ξ_t) satisfies, for every $f \in \mathcal{C}_0(S^{\mathbb{Z}^d})$ and every configuration $\xi \in S^{\mathbb{Z}^d}$,

$$\mathcal{A}(f)(\xi) = \sum_{x \in \mathbb{Z}^d} \sum_{s \in S} c(x, \xi, s) \left(f(\xi_{(x,s)}) - f(\xi) \right)$$

where, for all $x \in \mathbb{Z}^d$ and ξ, $\xi_{(x,s)}$ is the configuration ξ where $\xi(x)$ is replaced by s. $c(x, \xi, s) \geq 0$ is the intensity for a jump $\xi \to \xi_{(x,s)}$ and depends on ξ only through $\{\xi(y), y \sim x\}$ and $\xi(x)$ and we take $c(x, \xi, s) = 0$ if $s = \xi(x)$.

(2) The process (ξ_t) is additive in the following sense: for all $f, g \in S^{\mathbb{Z}^d}$, $\xi_t^{f \vee g} = \xi_t^f \vee \xi_t^g$; moreover, $\{A_t = \emptyset\}$ in an absorbing state.

The first condition implies that (ξ_t) is a nearest neighbor interacting particle system; this definition was introduced by Harris in [Har74] but today the term NNI represents a larger class of interacting particle systems (including for example the exclusion process which does not satisfy this first condition). We can find the general definition of NNI particle systems and the proof of their existence in [Lig05].

Harris [Har78] introduced a graphical representation of additive processes which make possible to work with a percolation view on a diagram of Poisson processes impacts (see [Des14a] for an example of such construction).

2.1.4. *Random linear growth models.* We define the class of models with which we are going to work:

Definition. *We say that (ξ_t, P) belongs to the class $\mathcal{C}_L$ if it satisfies the following conditions:*

(1) we have $\mathbb{P}(\tau = +\infty) > 0$

(2) there exist $C_1, C_2, M_1, M_2 > 0$ such that for all $t > 0$ and $x \in \mathbb{Z}^d$

$$\text{(AML)} \qquad \mathbb{P}\left(\exists x \in \mathbb{Z}^d : t(x) \leq t \text{ and } \|x\|_1 \geq M_1 t\right) \leq C_1 \exp(-C_2 t),$$

$$\text{(SC)} \qquad \mathbb{P}\left(t < \tau < \infty\right) \leq C_1 \exp(-C_2 t),$$

$$\text{(ALL)} \qquad \mathbb{P}\left(t(x) \geq M_2 \|x\| + t, \tau = \infty\right) \leq C_1 \exp(-C_2 t).$$

The second condition means that the growth of the set of points satisfying (P) is at most linear (AML), at least linear (ALL) and, if the extinction time τ is finite, then it is small (SC for small clusters in analogy with percolation vocabulary). Since $\mathbb{P}(\tau = +\infty) > 0$, we will work with the probability conditioned to survive: $\overline{\mathbb{P}} = \mathbb{P}(.|\tau = +\infty)$.

2.2. **Examples.**

2.2.1. *Classical contact process.* For $x \in \mathbb{Z}^d$, $\xi \in \{0,1\}^{\mathbb{Z}^d}$ and $k \in \mathbb{N}$:

$$
\begin{aligned}
c(x,\xi,0) &= \mathbb{1}_{\xi(x)=1} \\
c(x,\xi,1) &= \mathbb{1}_{\xi(x)=0} \times \lambda \sum_{y \sim x} \mathbb{1}_{\xi(y)=1}.
\end{aligned}
$$

We consider P the property 'to be alive' and we get back to the already known asymptotic shape theorem for $A_t = \xi_t$. In the following we denote by λ_c the critical value for classical contact process.

Next examples are extensions of the classical contact process from the literature; we give here definitions with the notations of our context. They belong to class $\mathcal{C}$ by construction. Several authors have worked hard to show that they belong to the class $\mathcal{C}_L$.

2.2.2. *Contact process in a randomly evolving environment (CPREE).* The contact process in a randomly evolving environment was introduced by Broman [Bro07] and studied in particular by Steif and Warfheimer [SW08]. Take $\mathcal{T} = \mathbb{R}_+$, $S = \{(0,0),(0,1),(1,0),(1,1)\}$. An element of S represents a pair (type, stage) where type can be favorable (1) or unfavorable (0) and the stage can be alive (1) or dead (0). We can write the jump intensities:

	$\xi(x) \to s$	$c(x,\xi,s)$
birth	$(0,0) \to (0,1)$	$\sum_{x \sim y} \xi_t(y)$
(type independent)	$(1,0) \to (1,1)$	$\sum_{x \sim y} \xi_t(y)$
death	$(0,1) \to (0,0)$	δ_0
(type dependent)	$(1,1) \to (1,0)$	δ_1
	$(0,0) \to (1,0)$	γp
evolution	$(0,1) \to (1,1)$	γp
of type	$(1,0) \to (0,0)$	$\gamma(1-p)$
	$(1,1) \to (0,1)$	$\gamma(1-p)$

with $0 \le \delta_1 < \delta_0$, $p \in [0,1]$ and $\gamma \ge 0$ four parameters of the system.

Let (P) ='to be alive'. The process (A_t) represents the set of alive points regardless of their type (favorable or unfavorable) and corresponds to the process (C_t) in Steif and Warfheimer's article [SW08]. Here, $\delta_{\min}$ is the configuration where 0 is in state $(0,1)$ and all the others are in state $(0,0)$. Using the Steif-Warfheimer construction and a restart argument like the one in section 5.3 in [Des14a], it is easy to see that the controls (AML), (SC) and (ALL) are satisfied, so (ξ_t, P) belongs to $\mathcal{C} \cap \mathcal{C}_L$.

2.2.3. *Contact process with aging (CPA).* We introduced the contact process with aging in [Des14a] where the dynamics are detailed; in this article we made a precise construction which allows us to consider a general case where a particle has an integer age. In the present context we have to suppose S finite so we can consider an aging contact process with a maximal age N. Then take $\mathcal{T} = \mathbb{R}_+$ and $S = \{0,\dots,N\}$.

Let (P) ='to be alive'. The process (A_t) represents the set of alive points regardless of their age. Here, $\delta_{\min} = \delta_0$ is the configuration where all particles are dead except 0 which is alive with age 1. From the results obtained in [Des14a], (ξ_t, P) belongs to $\mathcal{C} \cap \mathcal{C}_L$.

2.2.4. *Dependent oriented percolation (DOP).* In [GM12b], Garet and Marchand introduced a dependent oriented percolation model where fertile bacterium (represented by a type 1 particle) is submerged in a population of immune cells (type 2 particle) that are going to impede its development. The immune cells are not very fertile but benefit from a constant immigration process. Take $\mathcal{T} = \mathbb{N}$ and $S = \{0, 1, 2\}$. The system is described by a discrete time Markov chain depending on 3 parameters $p, q, \alpha \in (0, 1)$. Firstly, between time n and time $n + 1/2$, each particle attempts to infect its neighbors: it succeeds with probability p if it is a type 1 particle, and with probability q if it is a type 2 particle (squeezing a type 1 particle if there is one). Secondly, between time $n + 1/2$ and time $n + 1$, there is an immigration of type 2 particles on each site with probability $\alpha > 0$ (again eventually squeezing a type 1 particle). Let (P) ='to be of type 1'. The process (A_t), denote by $(\eta_{1,t})$ in [GM12b], represents alive fertile particles. Here, $\delta_{\min} = \delta_0$ where all particles are dead except 0 which is alive and fertile (of type 1). Theorem 1.2 of [GM12b] assure that the controls (AML), (SC) and (ALL) are satisfied if the process survives (that is if α is smaller than a non trivial value $\alpha_c(p, q)$). So, (ξ_t, P) belongs to $\mathcal{C} \cap \mathcal{C}_L$.

2.2.5. *Boundary modified contact process (BMCP).* Durrett and Schinazi [DS00] introduced a Boundary Modified Contact Process where particles can be infected, susceptible that has never been infected (state -1) and susceptible that as been previously infected. Take $\mathcal{T} = \mathbb{R}_+$, $S = \{-1, 0, 1\}$ and the jump intensities as follows:

$$c(x, \eta, 1) = \begin{cases} \lambda_e N_1(x, \eta) & \text{if } \eta(x) = -1, \\ \lambda_i N_1(x, \eta) & \text{if } \eta(x) = 0, \end{cases} \qquad \text{and } c(x, \eta, 0) = 1 \quad \text{if } \eta(x) = 1.$$

The quantities λ_e and λ_i are two non negative parameters of the system. The process (A_t) represents infected particles at time t. Here, $\delta_{\min}$ is the la configuration where all site are susceptible never infected (*i.e.* in state -1) except site 0 which is infected (state 1). We think that the renormalization work done by Durrett and Schinazi in [DS00] makes possible to obtain the controls (AML), (SC and (ALL) thanks to restart techniques (like in section 5.3 of [Des14a]) for $\lambda_i > \max(\lambda_c, \lambda_e)$.

For contact processes (with aging, boundary modified, in a randomly evolving environment), $t(x)$ is the first time when the particle x is alive regardless its age, type or memory. For dependent oriented percolation, $t(x)$ is the first time when the particle x is one of type 1 alive

2.3. **Results.** In this note, we give the elements to show the following general asymptotic shape theorem.

Theorem 1. *Let (ξ_t) be a Markov process taking values in $S^{\mathbb{Z}^d}$ and (P) an associated property of interest. If (ξ_t, P) belongs to classes $\mathcal{C}$ and $\mathcal{C}_L$ then there exists a norm μ on $\mathbb{R}^d$ such that for all $\varepsilon > 0$, $\mathbb{P}(.|\tau = +\infty)$ almost surely, for all t large enough,*

$$(1 - \varepsilon)B_\mu \subset \frac{\tilde{G}_t}{t} \subset (1 + \varepsilon)B_\mu,$$

with $\tilde{G}_t = \{x \in \mathbb{Z}^d, t(x) \le t\} + [0,1]^d$ *and* B_μ *the unit ball associated to* μ*.*

We can immediately deduce asymptotic shape theorems for the above examples. In each statement, we denote by B_μ the unit ball associated to the norm μ and $\overline{\mathbb{P}}$ the probability conditioned to the survival of the property (P) starting from the configuration $\delta_{\min}$.

First we recall theorems already proved.

Theorem 2 (Asymptotic shape theorem for CPA)**.** *Let* (ξ_t) *a contact process with aging. If* (ξ_t) *is supercritical, then there exists a norm* μ *on* $\mathbb{R}^d$ *such that for all* $\varepsilon > 0$, $\overline{\mathbb{P}}$ *almost surely, and for all* t *large enough*

$$(1-\varepsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1+\varepsilon)B_\mu,$$

where $\tilde{H}_t = H_t + [0,1]^d$ *with* H_t *the set of points born before* t*, regardless of their age.*

We deduce from Theorem 2 an asymptotic shape theorem for Krone's model [Kro99] which is a particular case of the contact process with aging.

Now we announce theorems previously conjectured.

Theorem 3 (Asymptotic shape theorem for CPREE)**.** *Let* (ξ_t) *a contact process in a randomly evolving environment. If* $p > p_c$ *(critical value for bond percolation on* $\mathbb{Z}^d$*), then there exists a norm* μ *on* $\mathbb{R}^d$ *such that for all* $\varepsilon > 0$, $\mathbb{P}\big(\ .\ |\tau^{\emptyset,\{0\}} = +\infty\big)$ *almost surely, for all* t *large enough,*

$$(1-\varepsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1+\varepsilon)B_\mu,$$

where $\tilde{H}_t = H_t + [0,1]^d$ *with* H_t *the set of points born before* t*, regardless of their type.*

Theorem 4 (Asymptotic shape theorem for DOP)**.** *Let* (η_t) *a dependent oriented percolation. If* $p > \overrightarrow{p_c}$ *(critical value for oriented bond percolation on* $\mathbb{Z}^d \times \mathbb{N}$*),* $q < \overrightarrow{p_c}$ *and* $\alpha \in (0, \alpha_c(p,q))$*, then there exists a norm* μ *on* $\mathbb{R}^d$ *such that for all* $\varepsilon > 0$, $\mathbb{P}(\ .\ |\tau_1^{0,\emptyset} = +\infty)$ *almost surely, for all* t *large enough,*

$$(1-\varepsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1+\varepsilon)B_\mu,$$

where $\tilde{H}_t = H_t + [0,1]^d$ *with* H_t *the set of sites where particles of type* 1 *are before* t*.*

Theorem 5 (Asymptotic shape theorem for BMCP)**.** *Let* (η_t) *a contact process boundary modified. If* $\lambda_i > \max(\lambda_c, \lambda_e)$ *and subject to that there exist* $C_1, C_2, M_1, M_2 > 0$ *such that for all* $t > 0$ *and* $x \in \mathbb{Z}^d$

$$\mathbb{P}\left(\exists x \in \mathbb{Z}^d : t(x) \le t \text{ and } \|x\|_1 \ge M_1 t\right) \le C_1 \exp(-C_2 t),$$
$$\mathbb{P}\left(t < \tau < \infty\right) \le C_1 \exp(-C_2 t),$$
$$\mathbb{P}\left(t(x) \ge M_2\|x\| + t, \tau = \infty\right) \le C_1 \exp(-C_2 t),$$

où $\tau = \inf\{t > 0, \forall x \in \mathbb{Z}^d, \eta_t(x) \le 0\}$ *et* $t(x) = \inf\{t > 0, \eta_t(x) = 1\}$*, then there exists a norm* μ *on* $\mathbb{R}^d$ *such that for all* $\varepsilon > 0$, $\mathbb{P}(\ .\ |\tau = +\infty)$ *almost surely, for all* t *large* *enough,*

$$(1-\varepsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1+\varepsilon)B_\mu,$$

where $\tilde{H}_t = H_t + [0,1]^d$ *with* H_t *the set of particles of type 1 born before* t*.*

From now, (ξ_t, P) belongs to $\mathcal{C}\cap\mathcal{C}_L$. In Section 3 we introduce the essential hitting time, a quantity that has good properties regarding to the dynamical system. In Section 4 we prove the Theorem 1 thanks subadditive ergodic theorems applied to the essential hitting time.

3. Essential hitting time

3.1. **Definition.** With non permanent models like contact process extensions, the hitting times can be infinite (because extinction is possible) and if we condition on the survival, we can lose independence, stationarity and even subadditivity properties required by Kingman theory.

We are inspired by the construction of Garet and Marchand [GM12a] for contact process in random environment to build the *essential hitting time* for a model which belongs to $\mathcal{C}\cap\mathcal{C}_L$.

We set $u_0(x) = v_0(x) = 0$ and we define by induction two sequences of stopping times $(u_n(x))_n$ and $(v_n(x))_n$ as follows.

- Suppose that $v_k(x)$ is defined. We set
$$u_{k+1}(x) = \inf\{t \geq v_k(x) : x \in A_t^{\delta_{\min}}\}.$$
If $v_k(x)$ is finite, then $u_{k+1}(x)$ is the first time after $v_k(x)$ where the site x satisfies (P); otherwise, $u_{k+1}(x) = +\infty$.
- Suppose that $u_k(x)$ is defined, with $k \geq 1$. We set $v_k(x) = u_k(x) + \tau^x \circ \theta_{u_k(x)}$. If $u_k(x)$ is finite, then the time $\tau^x \circ \theta_{u_k(x)}$ is the (possibly infinite) extinction time of (P) starting at time $u_k(x)$ from the configuration $\delta_{\min} \circ T_x$; otherwise, $v_k(x) = +\infty$.

We have $u_0(x) = v_0(x) \leq u_1(x) \leq v_1(x) \leq \ldots \leq u_i(x) \leq v_i(x) \ldots$ We then define $K(x)$ to be the first step when v_k or u_{k+1} becomes infinite:

$$K(x) = \min\{k \geq 0 : v_k(x) = +\infty \text{ or } u_{k+1}(x) = +\infty\}.$$

$K(x)$ has a sub-geometric tail. In particular, $K(x)$ is almost surely finite. It allows us to define the following quantities.

Definition. *For* $x \in \mathbb{Z}^d$*, we call essential hitting time of* x *by the property* (P) *the quantity* $\sigma(x) = u_{K(x)}$*. We define the operator* $\tilde{\theta}_x$ *on* Ω *by setting:*

$$\tilde{\theta}_x = \begin{cases} T_x \circ \theta_{\sigma(x)} & \text{if } \sigma(x) < +\infty, \\ T_x & \text{otherwise.} \end{cases} \tag{1}$$

Thanks to controls (AML), (SC) and (ALL), we can obtain that, for $x \in \mathbb{Z}^d \setminus \{0\}$, $\overline{\mathbb{P}}$ is invariant under the action of $\tilde{\theta}_x$. Besides, under $\overline{\mathbb{P}}$, $\sigma(y) \circ \tilde{\theta}_x$ is independent from $\sigma(x)$ and its law is the same as the law of $\sigma(y)$. We refer to [Des14a] for proofs of the previous properties about $K(x)$ and $\tilde{\theta}_x$ (and to chapter 7 of [Des14b] for more details).

3.2. **Bad growth points.** In order to apply the techniques used in Section 7 of [Des14a] to control σ, we introduce the general box of bad growth point for a model in $\mathcal{C} \cap \mathcal{C}_L$. For every $x \in \mathbb{Z}^d$, $L > 0$ and $t > 0$, we introduce the event

$$E^y(x,t) = \{H_t^y \not\subset y + B_{M_1 t}\} \cup \{t/2 < \tau^y < +\infty\} \cup \{\tau^y = +\infty, \inf\{s \geq 2t : x \in A_s^y\} > \kappa t\}.$$

traducing the fact that a point y has bad growth with respect to the spatio-temporal point (x,t). Let ν_y the counting measure of all events occurring at y during $[0, L]$. Thanks to Harris representation, ν_y can be expressed with Poisson processes. This measure allows us to count the number of bad growth points by

$$N_L(x,t) = \sum_{y \in x + B_{Mt+2}} \int_0^L \mathbb{1}_{E^y(x,t)} \circ \theta_s \, d\nu_y(s).$$

For technical reasons, we add 0 in the measure ν and we modify $E^y(x,t)$ by

$$\tilde{E}^y(x,t) = E^y(x,t) \cup \{\nu_y[0, t/2] = 0\} \tag{2}$$

Going back to our examples:

- For DOP (which is a discrete model), ν is the counting measure on $\mathbb{N}$.
- For CPA we consider:

$$\nu_y = \omega_y^1 + \omega_y^\gamma + \sum_{e \in \mathbb{E}^d : y \in e} \omega_e^\infty + \delta_0$$

  where ω_y^1, ω_y^γ and ω_e^∞ are the Poisson processes respectively giving the possible death times at y, the possible maturations and the possible birth times through e.
- For CPREE, we consider:

$$\nu_y = \omega_y^0 + \omega_y^1 + \sum_{e \in \mathbb{E}^d : y \in e} \omega_e + \delta_0,$$

  where ω_y^0 is the Poisson process associated to type 0 deaths, ω_y^1 the one associated to type 1 deaths and ω_e the one associated to births.

With this definitions we can show the following properties.

Lemma 6. *Let $x \in \mathbb{Z}^d$ and $t \geq 2$. If L, s are positive integers such that $N_L(x,t) \circ \theta_s = 0$ and $u_i(x) \in [s+t, s+L]$, then $v_i(x) = +\infty$ ou $u_{i+1}(x) - u_i(x) \leq \kappa t$.*
Besides, for any $t > 0$, $s \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, the following inclusion holds:

$$\begin{aligned}
&\{\tau = +\infty\} \\
&\cap \{\exists u \in x + B_{M_1 t+2},\ \tau^u \circ \theta_s = +\infty,\ u \in A_s\} && (3)\\
&\cap \left\{N_{K(x)\kappa t}(x,t) \circ \theta_s = 0\right\} && (4)\\
&\cap \bigcap_{1 \leq i < K(x)} \{v_i(x) - u_i(x) < t\} && (5)\\
&\qquad \subset \{\tau = +\infty\} \cap \{\sigma(x) \leq s + K(x)\kappa t\}.
\end{aligned}$$

Then, following the same proof lines that [Des14a], and choosing $\kappa = 3M_1(1 + M_2)$, where M_1 and M_2 are the constants respectively given in (AML) and (ALL), we control the probability that a space-time box contains no bad growth point.

Lemma 7. *There exist $A, B > 0$ such that for all $L > 0$, $x \in \mathbb{Z}^d$ and $t > 0$ one has*

$$\mathbb{P}\left(N_L(x,t) \geq 1\right) \leq A\exp(-Bt).$$

3.3. **Subadditivity and difference between σ and t.** In the same way, the following controls are still available.

Proposition 8. *(I) There exists $A, B > 0$ such that for all $x, y \in \mathbb{Z}^d$,*

$$\forall t > 0 \qquad \overline{\mathbb{P}}\left(\sigma(x+y) - \left(\sigma(x) + \sigma(y)\circ\tilde{\theta}_x\right) \geq t\right) \leq A\exp\left(-B\sqrt{t}\right).$$

Moreover, for $p \geq 1$, there exists $M_p > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}[(\sigma(x+y) - (\sigma(x) + \sigma(y)\circ\tilde{\theta}_x))_+^p] \leq M_p.$$

(II) $\overline{\mathbb{P}}$ almost surely, it holds that $\displaystyle\lim_{\|x\|\to+\infty} \frac{|\sigma(x) - t(x)|}{\|x\|} = 0.$

(III) There exist $A, B, C > 0$ such that

$$\forall x \in \mathbb{Z}^d, \forall t > 0,\ \overline{\mathbb{P}}\left(\sigma(x) \geq C\|x\| + t\right) \leq A\exp(B\sqrt{t}).$$

Moreover, for $p \geq 1$, there exists a constant $C(p) > 0$ such that

$$\forall x \in \mathbb{Z}^d,\ \overline{\mathbb{E}}[\sigma(x)^p] \leq C(p)(1 + \|x\|)^p.$$

(IV) For every $\epsilon > 0$, $\overline{\mathbb{P}}$-a.s., there exists $R > 0$ such that

$$\forall x, y \in \mathbb{Z}^d,\ (\|x\| \geq R \text{ and } \|x - y\| \leq \epsilon\|x\|) \Longrightarrow (|\sigma(x) - \sigma(y)| \leq C\epsilon\|x\|).$$

4. Proof of the asymptotic shape theorem

Thanks to the almost subadditive theorem of Kesten and Hammersley (see [Ham74] and [Kes73]), we obtained a general asymptotic shape theorem on some random variables $(\sigma(x))_{x\in\mathbb{Z}^d}$.

Theorem 9 (Theorem 39 of [Des14a]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\sigma(x))_{x\in\mathbb{Z}^d}$ be random variables with finite second moments and such that, for every $x \in \mathbb{Z}^d$, $\sigma(x)$ and $\sigma(-x)$ have the same distribution. Let $(s(y))_{y\in\mathbb{Z}^d}$ and $(r(x,y))_{x,y\in\mathbb{Z}^d}$ be collections of random variables such that:*

Hyp 1: $\forall x, y \in \mathbb{Z}^d,\ \sigma(x+y) \leq \sigma(x) + s(y) + r(x,y)$ *with $s(y)$ having the same law as $\sigma(y)$, and being independent from $\sigma(x)$,*

Hyp 2: $\forall x, y \in \mathbb{Z}^d,\ \exists C_{x,y}$ *and* $\alpha_{x,y} < 2$ *such that* $\forall n, p, \mathbb{E}[r(nx, py)^2] \leq C_{x,y}(n+p)^{\alpha_{x,y}}$,

Hyp 3: $\exists C > 0$ *such that* $\forall x \in \mathbb{Z}^d,\ \mathbb{P}(\sigma(nx) > Cn\|x\|) \xrightarrow{n\to\infty} 0$,

Hyp 4: $\exists K > 0$ *such that* $\forall \epsilon > 0, \mathbb{P} - p.s\ \exists M$ *such that* $(\|x\| \geq M$ *and* $\|x - y\| \leq K\|x\|) \Rightarrow \|\sigma(x) - \sigma(y)\| \leq \epsilon\|x\|$,

Hyp 5: $\exists c > 0$ *such that* $\forall x \in \mathbb{Z}^d,\ \mathbb{P}(\sigma(nx) < cn\|x\|) \xrightarrow{n\to\infty} 0.$

Then there exists $\mu : \mathbb{Z}^d \to \mathbb{R}^+$ such that

$$\lim_{\|x\|\to\infty} \frac{\sigma(x) - \mu(x)}{\|x\|} = 0 \text{ a.s.}$$

Moreover, μ can be extended to a norm on $\mathbb{R}^d$ and we have the following asymptotic shape theorem: for all $\epsilon > 0$, $\mathbb{P}$ almost surely, for all t large enough,

$$(1-\epsilon)B_\mu \subset \frac{\tilde{G}_t}{t} \subset (1+\epsilon)B_\mu,$$

where $\tilde{G}_t = \{x \in \mathbb{Z}^d : \sigma(x) \leq t\} + [0,1]^d$ and B_μ is the unit ball for μ.

We now deduce the expected asymptotic shape theorem for the hitting time t:

Proposition 10. *There exists a norm μ on $\mathbb{R}^d$ such that almost surely under $\overline{\mathbb{P}}$,*

$$\lim_{\|x\| \to +\infty} \frac{t(x) - \mu(x)}{\|x\|} = 0,$$

and for every $\epsilon > 0$, $\overline{\mathbb{P}}$-a.s., for every large t,

$$(1-\epsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1+\epsilon)B_\mu$$

where $\tilde{H}_t = \{x \in \mathbb{Z}^d / t(x) \leq t\} + [0,1]^d$ and B_μ is the unit ball for μ.

Theorem 1 is contained is the previous result.

Proof. First, we use Theorem 9 to show that σ satisfies an asymptotic shape theorem. We check the hypotheses of Theorem 9 using the controls of Proposition 8. Thanks to (III), σ has finite second moment required. We take $s(y) = \sigma(y) \circ \tilde{\theta}_x$. The hypotheses 1 and 2 are satisfied thanks to properties of $\tilde{\theta}_x$ and (I). The hypothesis 3 is the at least linear growth (III) and the hypothesis 5 is immediately checked thanks to the at most linear growth (AML): $\overline{\mathbb{P}}(\sigma(nx) < M_2 n\|x\|) \leq \overline{\mathbb{P}}(t(nx) < M_2 n\|x\|) \leq \frac{A}{\rho}\exp(-BM_2 n\|x\|)$. Finally, the hypothesis 4 is the control (IV). We deduce the result for t from the result for σ thanks to (II). $\square$

References

[Bro07] Erik I. Broman. Stochastic domination for a hidden Markov chain with applications to the contact process in a randomly evolving environment. *Ann. Probab.*, 35(6):2263–2293, 2007.

[Des14a] Aurelia Deshayes The contact process with aginig *ALEA Lat. Am. J. Probab. Math. Stat.*, 11 (2), 845–883, 2014.

[Des14b] Aurelia Deshayes Modèles de croissance et théorèmes de forme asymptotique: les processus de contact *Probability, Université de Lorraine*. French, 2014. <tel-01099793>.

[DS00] Rick Durrett and Rinaldo B. Schinazi. Boundary modified contact processes. *J. Theoret. Probab.*, 13(2):575–594, 2000.

[GM12a] Olivier Garet and Régine Marchand. Asymptotic shape for the contact process in random environment. *The Annals of Applied Probability*, 22(4):1362–1410, 2012.

[GM12b] Olivier Garet and Régine Marchand. Growth of a population in a dynamical hostile environment. *Advances in Applied Probability*, to appear, 2012.

[Ham74] J. M. Hammersley. Postulates for subadditive processes. *Ann. Probability*, 2:652–680, 1974.

[Har74] T. E. Harris. Contact interactions on a lattice. *Ann. Probability*, 2:969–988, 1974.

[Har78] T. E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probability*, 6(3):355–378, 1978.

[Kes73] H. Kesten. Discussion on Professor Kingman's paper. *Ann. Probab.*, 1:903, 1973.

[Kin73] J. F. C. Kingman. Subadditive ergodic theory. *Ann. Probability*, 1:883–909, 1973. With discussion by D. L. Burkholder, Daryl Daley, H. Kesten, P. Ney, Frank Spitzer and J. M. Hammersley, and a reply by the author.

[Kro99] Stephen M. Krone. The two-stage contact process. *Ann. Appl. Probab.*, 9(2):331–351, 1999.

[Lig05] Thomas M. Liggett. *Interacting particle systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Reprint of the 1985 original.

[Lig99] Thomas M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Rem08] Daniel Remenik. The contact process in a dynamic random environment. *Ann. Appl. Probab.*, 18(6):2392–2420, 2008.

[SW08] Jeffrey E. Steif and Marcus Warfheimer. The critical contact process in a randomly evolving environment dies out. *ALEA Lat. Am. J. Probab. Math. Stat.*, 4:337–357, 2008.

Institut Élie Cartan Nancy (mathématiques), Université de Lorraine, Campus Scientifique, BP 239, 54506 Vandoeuvre-lès-Nancy Cedex France,

E-mail address: Aurelia.Deshayes@univ-lorraine.fr