Local Error Estimates of the Finite Element Method for an Elliptic Problem with a Dirac Source Term

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Abstract: The solutions of elliptic problems with a Dirac measure right-hand side are not H^1 and therefore the convergence of the finite element solutions is suboptimal. The use of graded meshes is standard remedy to recover quasi-optimality, namely optimality up to a log-factor, for low order finite elements in the $\mathbb{L}^2$-norm. Optimal (or quasi-optimal for the lowest order case) convergence for Lagrange finite elements has been shown, in the $\mathbb{L}^2$-norm, on a subdomain which excludes the singularity. Here, on such subdomains, we show a quasi-optimal convergence in the H^s-norm, for $s \geqslant 1$, and, in the particular case of Lagrange finite elements, an optimal convergence in H^1-norm, on a family of quasi-uniform meshes in dimension 2. The study of this problem is motivated by the use of the Dirac measure as a reduced model in physical problems, for which high accuracy of the finite element method at the singularity is not required. Our results are obtained using local Nitsche and Schatz-type error estimates, a weak version of Aubin-Nitsche duality lemma and a discrete inf-sup condition. These theoretical results are confirmed by numerical illustrations.

Key words: Dirichlet problem, Dirac measure, Green function, finite element method, local error estimates.

1 Introduction.

This paper deals with the accuracy of the finite element method on elliptic problems with a singular right-hand side. More precisely, let us consider the Dirichlet problem

$$(P_\delta)\left\{\begin{array}{rcll} -\triangle u_\delta & = & \delta_{x_0} & \text{in } \Omega, \\ u_\delta & = & 0 & \text{on } \partial\Omega, \end{array}\right.$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open $\mathcal{C}^\infty$ domain or a square, and δ_{x_0} denotes the Dirac measure concentrated at a point $x_0 \in \Omega$ such that $\text{dist}(x_0, \partial\Omega) > 0$.

Problems of this type occur in many applications from different areas, like in the mathematical modeling of electromagnetic fields [17]. Dirac measures can also be found on the right-hand side of adjoint equations in optimal control of elliptic problems with state constraints [8]. As further examples where such measures play an important role, we mention controllability for elliptic and parabolic equations [9, 10, 21] and parameter identification problems with pointwise measurements [23].

Our interest in (P_δ) is motivated by the modeling of the movement of a thin structure in a viscous fluid, such as cilia involved in the muco-ciliary transport in the lung [15]. In the asymptotic of a zero diameter cilium with an infinite velocity, the cilium is modelled by a line Dirac of force in the source term. In order to make the computations easier, the line Dirac can be approximated by a sum of punctual Dirac forces distributed along the cilium [20]. In this paper, we address a scalar version of this problem: problem (P_δ).

In the regular case, namely the Laplace problem with a regular right-hand side, the finite element solution u^h is well-defined and for $u \in H^{k+1}(\Omega)$, we have, for all $0 \leqslant s \leqslant 1$,

$$\|u - u^h\|_s \leqslant Ch^{k+1-s}\|u\|_{k+1}, \tag{1}$$

where k is the degree of the method [11] and h the mesh size. In dimension 1, the solution u_δ of Problem (P_δ) belongs to $H^1(\Omega)$, but it is not $H^2(\Omega)$. In this case, the numerical solution u_δ^h and the exact solution u_δ can be computed explicitly. If x_0 coincides with a node of the discretization, $u_\delta^h = u_\delta$. Otherwise, this equality holds only on the complementary of the element which contains x_0, and the convergence orders in H^1-norm and $\mathbb{L}^2$-norm are 1/2 and 3/2 respectively. In dimension 2, Problem (P_δ) has no $H^1(\Omega)$-solution, and so, although the finite element solution can be defined, the $H^1(\Omega)$-error makes no sense and the $\mathbb{L}^2(\Omega)$-error estimates cannot be obtained by a straightforward application of the Aubin-Nitsche method.

Let us review the literature on error estimates for problem (P_δ), starting with discretizations on quasi-uniform meshes. Babuška [4] showed an $\mathbb{L}^2(\Omega)$-convergence of order $h^{1-\varepsilon}$, $\varepsilon > 0$, for a two-dimensional smooth domain. Scott proved in [25] an a priori error estimate of order $2 - \frac{d}{2}$, where the dimension d is 2 or 3. The same result has been proved by Casas [7] for general Borel measures on the right-hand side.

To the best of our knowledge, in order to improve the convergence order, Eriksson [14] was the first who studied the influence of locally refined meshes near x_0. Using results from [24], he proved convergence of order k and $k+1$ in the $W^{1,1}(\Omega)$-norm and the $\mathbb{L}^1(\Omega)$-norm respectively, for approximations with a P_k-finite element method. Recently, by Apel and co-authors [2], an $\mathbb{L}^2(\Omega)$-error estimate of order $h^2|\ln h|^{3/2}$ has been proved in dimension 2, using graded meshes. Optimal convergence rates with graded meshes were also recovered by D'Angelo [12] using weighted Sobolev spaces. A posteriori error estimates in weighted spaces have been established by Agnelli and co-authors [1].

These theoretical a priori results are based upon graded meshes, which increase the complexity of the meshing and the computational cost, even if the mesh is refined only locally, especially when the right-hand side includes several Dirac measures, that can be static or moving. Therefore Eriksson [13] developped a numerical method to solve the problem and recovers the optimal convergence rate: the numerical solution is searched in the form $u_0 + v_h$ where u_0 contains the singularity of the solution and v_h is the numerical solution of a smooth problem. This method has been developped in the case of the Stokes problem in [20].

However, in applications, the Dirac measure at x_0 is often a model reduction approach, and a high accuracy at x_0 of the finite element method is not necessary. Thus, it is interesting to study the error on a fixed subdomain which excludes the singularity. Recently, Köppl and Wohlmuth have shown in [19] optimal convergence in $\mathbb{L}^2$-norm for the Lagrange finite elements (the result is quasi-optimal for the P^1-element). In this paper, we consider the problem in dimension 2, and we show :

1. Quasi-optimal convergence in H^s-norm, for $s \geqslant 1$. This result applies to a wide class of finite-element methods and beyond, including Lagrange and Hermite finite elements and wavelets. The $\mathbb{L}^2$-error estimates established in [19] are not used and the proof is based on different arguments.

2. Optimal convergence in H^1-norm for the Lagrange finite elements. This result is obtained by direct use of the optimal $\mathbb{L}^2$-norm convergence result in [19].

3. Optimal convergence in H^1-norm in the particular case of the P^1-Lagrange finite element using different arguments than those used for the previous results.

These results imply that graded meshes are not required to recover optimality far from the singularity and that there are no pollution effects. In addition, by linearity of Problem (P_δ), the result holds in the case of several Dirac masses. The paper is organized as follows. Our main results are presented in Section 2 after recalling the Nitsche and Schatz Theorem, which is an important tool for the proof presented in Section 3. In Section 4 another argument is presented to obtain an optimal estimate in the particular case of the P_1-finite elements. We illustrate in Section 5 our theoretical results by numerical simulations and, in Section 6, we discuss the generalization of our approach to the three-dimensional case.

2 Main results.

In this section, we define all the notations used in this paper, formulate our main results and recall an important tool for the proof, the Nitsche and Schatz Theorem.

2.1 Notations.

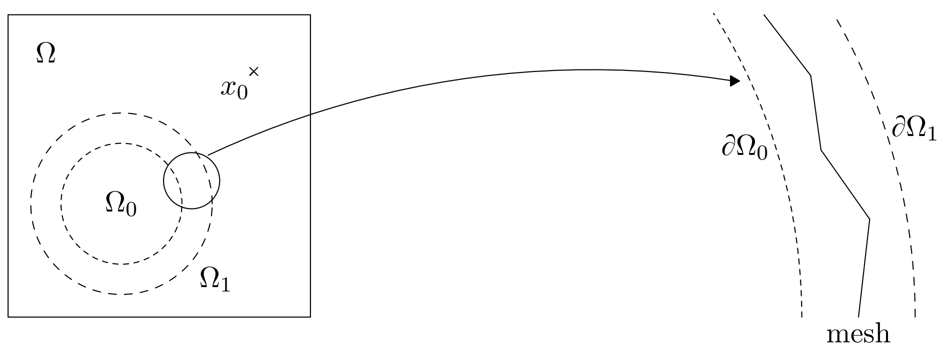

Figure 1: Domains Ω_0 and Ω_1.

For a domain D, we will denote by $\|\cdot\|_{s,p,D}$ (respectively $|\cdot|_{s,p,D}$) the norm (respectively the semi-norm) of the Sobolev space $W^{s,p}(D)$, while $\|\cdot\|_{s,D}$ (respectively $|\cdot|_{s,D}$) will stand for the norm (respectively the semi-norm) of the Sobolev space $H^s(D)$.

For the numerical solution, let us introduce a family of quasi-uniform simplicial triangulations $\mathcal{T}_h$ of Ω and an order k finite element space $V_h^k \subset H_0^1(\Omega)$. To ensure that the numerical solution is well-defined, the space V_h^k is assumed to contain only continuous functions. The finite element solution $u_\delta^h \in V_h^k$ of problem (P_δ) is defined by

$$\int_\Omega \nabla u_\delta^h \cdot \nabla v_h = v_h(x_0), \ \forall v_h \in V_h^k. \tag{2}$$

For $s \geqslant 2$, we will also evaluate the H^s-norm of the error on a subdomain of Ω which does not contain the singularity, and, whenever we do so, we will of course assume the finite elements to be H^s-conforming. We fix two subdomains Ω_0 and Ω_1 of Ω, such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and $x_0 \notin \overline{\Omega_1}$ (see Figure 1). We consider a mesh which satisfies the following condition:

Assumption 1. *For some h_0, we have for all $0 < h \leqslant h_0$ (see Figure 1),*

$$\overline{\Omega}_0^m \bigcap \Omega_1^c = \varnothing, \text{ where } \overline{\Omega}_0^m = \bigcup_{\substack{T \in \mathcal{T}_h \\ T \bigcap \Omega_0 \neq \varnothing}} T,$$

and Ω_1^c is the complement of Ω_1 in Ω.

2.2 Regularity of the solution u_δ.

In this subsection, we focus on the singularity of the solution, which is the main difficulty in the study of this kind of problems. In dimension 2, problem (P_δ) has a unique variational solution $u_\delta \in W_0^{1,p}(\Omega)$ for all $p \in [1, 2[$ (see for instance [3]). Indeed, denoting by G the Green function, G is defined by

$$G(x) = -\frac{1}{2\pi} \log(|x|).$$

This function G satisfies $-\triangle G = \delta_0$, so that $G(\cdot - x_0)$ contains the singular part of u_δ. As it is done in [3], the solution u_δ can be built by adding to $G(\cdot - x_0)$ a corrector term $\omega \in H^1(\Omega)$, solution of the Laplace(problem

$$\begin{cases} -\triangle\omega &= \quad 0 \quad \text{in } \Omega, \\ \quad \omega &= -G(\cdot - x_0) \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

Then, the solution is given by

$$u_\delta(x) = G(x - x_0) + \omega(x) = -\frac{1}{2\pi} \log(|x - x_0|) + \omega(x).$$

It is easy to verify that $u_\delta \notin H_0^1(\Omega)$. Actually, we can specify how the quantity $\|u_\delta\|_{1,p,\Omega}$ goes to infinity when p goes to 2, with $p < 2$. According to the foregoing, if we write $u_\delta = G + \omega$, since $\omega \in H^1(\Omega)$, estimating the behavior of $\|u_\delta\|_{1,p,\Omega}$ as p converges to 2 from below (which will be denoted by $p \nearrow 2$) is reduced to estimating the behavior $\|G\|_{1,p,B}$, where $B = B(0, 1)$ where B is the ball of center 0 and radius 1. $G \in \mathbb{L}^p(\Omega)$ for all $1 \leqslant p < \infty$, and using polar coordinates, we get, for $p < 2$,

$$|G|_{1,p,B}^p = \int_B |\nabla G(x)|^p \mathrm{d}x = \int_0^1 \int_0^{2\pi} \left(\frac{1}{2\pi} \frac{1}{r}\right)^p r \mathrm{d}\theta \mathrm{d}r = (2\pi)^{1-p} \int_0^1 r^{1-p} \mathrm{d}r = \frac{(2\pi)^{1-p}}{2 - p}.$$

Finally, when $p \nearrow 2$,

$$\|u_\delta\|_{1,p,\Omega} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2 - p}}. \tag{4}$$

2.3 The Nitsche and Schatz Theorem.

Before stating the Nitsche and Schatz Theorem, let us introduce some known properties of the finite element spaces V_h^k.

Assumption 2. *Given two fixed concentric spheres B_0 and B with $B_0 \subset\subset B \subset\subset \Omega$, there exists an h_0 such that for all $0 < h \leqslant h_0$, we have for some $R \geqslant 1$ and $M > 1$:*

B1 *For any $0 \leqslant s \leqslant R$ and $s \leqslant \ell \leqslant M$, for each $u \in H^\ell(B)$, there exists $\eta \in V_h^k$ such that*

$$\|u - \eta\|_{s,B} \leqslant C h^{\ell - s} \|u\|_{\ell,B}.$$

Moreover, if $u \in H_0^1(B_0)$ then η can be chosen to satisfy $\eta \in H_0^1(B)$.

B2 *Let $\varphi \in \mathscr{C}_0^\infty(B_0)$ and $u_h \in V_h^k$, then there exists $\eta \in V_h^k \cap H_0^1(B)$ such that*

$$\|\varphi u_h - \eta\|_{1,B} \leqslant C(\varphi, B, B_0) h \|u_h\|_{1,B}.$$

B3 *For each $h \leqslant h_0$ there exists a domain B_h with $B_0 \subset\subset B_h \subset\subset B$ such that if $0 \leqslant s \leqslant \ell \leqslant R$ then for all $u_h \in V_h^k$ we have*

$$\|u_h\|_{\ell,B_h} \leqslant C h^{s-\ell} \|u_h\|_{s,B_h}.$$

We now state the following theorem, a key tool in the forthcoming proof of Theorem 1.

Theorem (Nitsche and Schatz [22])**.** *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and let V_h^k satisfy Assumption 2. Let $u \in H^\ell(\Omega_1)$, let $u_h \in V_h^k$ and let q be a nonnegative integer, arbitrary but fixed. Let us suppose that $u - u_h$ satisfies*

$$\int_\Omega \nabla(u-u_h)\cdot\nabla v_h = 0, \ \forall v_h \in V_h^k \cap H_0^1(\Omega_1).$$

Then there exists h_1 such that if $h \leqslant h_1$ we have

(i) for $s = 0, 1$ and $1 \leqslant \ell \leqslant M$,

$$\|u-u_h\|_{s,\Omega_0} \leqslant C\left(h^{\ell-s}\|u\|_{\ell,\Omega_1} + \|u-u_h\|_{-q,\Omega_1}\right),$$

(ii) for $2 \leqslant s \leqslant \ell \leqslant M$ and $s \leqslant k < R$,

$$\|u-u_h\|_{s,\Omega_0} \leqslant C\left(h^{\ell-s}\|u\|_{\ell,\Omega_1} + h^{1-s}\|u-u_h\|_{-q,\Omega_1}\right).$$

In this paper, we will actually need a more general version of the assumptions on the approximation space V_h^k:

Assumption 3. *Given $B \subset \Omega$, consider $p' \geqslant 2$, there exists an h_0 such that for all $0 < h \leqslant h_0$, we have for some $R \geqslant 1$ and $M > 1$:*

$\widetilde{\mathbf{B}}\mathbf{1}$ *For any $0 \leqslant s \leqslant R$ and $s \leqslant \ell \leqslant M$, for each $u \in H^\ell(B)$, there exists $\eta \in V_h^k$ such that, for any finite element $T \subset B$,*

$$|u-\eta|_{s,p',T} \leqslant C h^{d(1/p'-1/2)} h^{\ell-s} |u|_{\ell,2,T}.$$

$\widetilde{\mathbf{B}}\mathbf{3}$ *For $0 \leqslant s \leqslant \ell \leqslant R$, for all $u_h \in V_h^k$, for any finite element T in the family $\mathcal{T}_h$, we have*

$$\|u_h\|_{\ell,p',T} \leqslant C h^{d(1/p'-1/2)} h^{s-\ell} \|u_h\|_{s,2,T}.$$

Assumptions $\widetilde{\mathbf{B}}\mathbf{1}$ and $\widetilde{\mathbf{B}}\mathbf{3}$ are generalizations of assumptions **B1** and **B3**. They are quite standard and satisfied by a wide variety of approximation spaces, including all finite element spaces defined on quasi-uniform meshes [11]. The parameters R and M play respectively the role of the regularity and order of approximation of the approximation space V_h^k. For example, in the case of P_1-finite elements, we have $R = 3/2 - \varepsilon$ and $M = 2$. Assumption **B2** is less common but also satisfied by a wide class of approximation spaces. Actually, for Lagrange and Hermite finite elements, a stronger property than assumption **B2** is shown in [5]: let $0 \leqslant s \leqslant \ell \leqslant k$, $\varphi \in \mathscr{C}_0^\infty(B)$ and $u_h \in V_h^k$, then there exists $\eta \in V_h^k$ such that

$$\|\varphi u_h - \eta\|_{s,B} \leqslant C(\varphi) h^{\ell-s+1} \|u_h\|_{\ell,B}. \tag{5}$$

Applied for $s = \ell = 1$, inequality (5) gives assumption **B2**.

2.4 Statement of our main results.

Our main results are Theorems 1, 2 and 3. The rest of the paper is mostly concerned by the proof and the numerical illustration.

Theorem 1. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ satisfy Assumption 1, $1 \leqslant s \leqslant k$. Let u_δ be the solution of problem (P_δ) and u_δ^h its Galerkin projection onto V_h^k, satisfying (2). Under Assumptions 2 and 3, there exists h_1 such that if $0 < h \leqslant h_1$, we have,*

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega) h^k \sqrt{|\ln h|}. \tag{6}$$

In addition, for $s \geqslant 2$, if the finite elements are supposed H^s-conforming, we have

$$\|u_\delta - u_\delta^h\|_{s,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega) h^{k+1-s} \sqrt{|\ln h|}. \tag{7}$$

Remark 1. *The main tool in proving Theorem 1 is the Nitsche and Schatz Theorem, and the result holds for all the spaces verifying Assumptions 2 and 3. The class of such spaces includes spaces beyond finite elements, including, for instance, wavelets.*

Section 3 will be dedicated to the proof of Theorem 1.

In the particular case of Lagrange finite elements, Köppl and Wohlmuth [19] showed, in the $\mathbb{L}^2$-norm of a subdomain which does not contain x_0, quasi-optimality for the lowest order case, and optimal a priori estimates for higher order. The proof is based on Wahlbin-type arguments, which are similar to the Nitsche and Schatz Theorem (see [27, 28]), and different arguments from the ones presented in this paper, like the use of an operator of Scott and Zhang type [26]. Using this result it is possible to prove quite easily optimal convergence in H^1-norm for Lagrange finite elements. This result reads as follows:

Theorem 2. *Consider a domain Ω_2 such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, $x_0 \notin \overline{\Omega_2}$, and satisfying Assumption 1. Let u_δ be the solution of problem (P_δ) and u_δ^h its Galerkin projection onto the space of Lagrange finite elements of order $k+1$. There exists h_1 such that if $0 < h \leqslant h_1$, we have*

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C(\Omega_1, \Omega_2, \Omega) h^k. \tag{8}$$

Remark 2. *This result is optimal and thus slightly stronger than inequality (6), but it is limited to Lagrange finite elements and to the H^1-norm, due to the use of an operator of Scott-Zhang type. Theorem 1 is more general: it holds for a wide class of finite elements and it allows to estimate the error in H^s-norm, for any $s \geqslant 1$.*

Proof of Theorem 2. In the particular case of Lagrange finite elements, Köppl and Wohlmuth proved in [19] the following convergence in the $\mathbb{L}^2$-norm of a subdomain which does not contain x_0:

$$\|u_\delta - u_\delta^h\|_{0,\Omega_1} \leqslant C(\Omega_1, \Omega_2, \Omega) \begin{cases} h^2|\ln(h)| & \text{if } k = 1, \\ h^{k+1} & \text{if } k > 1. \end{cases} \tag{9}$$

Let us apply the Nitsche and Schatz Theorem on Ω_0 and Ω_1 for $l = k+1$ and $q = 0$,

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C \left(h^k \|u_\delta\|_{2,\Omega_1} + \|u_\delta - u_\delta^h\|_{0,\Omega_1} \right).$$

Using (9), we get

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C h^k.$$

For the particular P_1-Lagrange finite elements, we prove the optimal convergence in H^1-norm using completely different arguments. This proof involves a technical assumption on the mesh, namely Assumption 4 in Section 4.2: the distance of the Dirac mass to the edges of the mesh triangles is assumed to be at least of the same order as the mesh size h. The result reads as follows:

Theorem 3. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ satisfy Assumption 1 and consider a mesh such that there exists a domain B_ε satisfying Assumption 4 with ϵ of the same order as the mesh size. The P_1-finite element method converges with order 1 for the $H^1(\Omega_0)$-norm. More precisely:*

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega)h.$$

The proof of this result is detailed in Section 4.

3 Proof of Theorem 1.

This section is devoted to the proof of Theorem 1. We first show a weak version of the Aubin-Nitsche duality lemma (Lemma 1) and establish a discrete inf-sup condition (Lemma 2). Then, we use these results to prove Theorem 1.

3.1 Aubin-Nitsche duality lemma with a singular right-hand side.

The proof of Theorem 1 is based on the Nitsche and Schatz Theorem of Section 2.3. In order to estimate the quantity $\|u_\delta - u_\delta^h\|_{-q,\Omega_1}$, we will first show a weak version of Aubin-Nitsche Lemma, for the case of Poisson Problem with a singular right-hand side.

Lemma 1. *Let $f \in W^{-1,p}(\Omega) = (W_0^{1,p'}(\Omega))'$, $1 < p < 2$, and $u \in W_0^{1,p}(\Omega)$ be the unique solution of*

$$\begin{cases} -\triangle u &= f \quad \textit{in } \Omega, \\ u &= 0 \quad \textit{on } \partial\Omega. \end{cases}$$

Let $u_h \in V_h^k$ be the Galerkin projection of u. For finite elements of order k, letting $e = u - u_h$, we have for all $0 \leqslant q \leqslant k-1$,

$$\|e\|_{-q,\Omega} \leqslant Ch^{q+1}h^{2(1/p'-1/2)}|e|_{1,p,\Omega}. \tag{10}$$

Proof. We aim at estimating, for $q \geqslant 0$, the H^{-q}-norm of the error e:

$$\|e\|_{-q,\Omega} = \sup_{\phi \in \mathscr{C}_0^\infty(\Omega)} \frac{|\int_\Omega e\phi|}{\|\phi\|_{q,\Omega}}. \tag{11}$$

The error $e \in W_0^{1,p}$ satisfies

$$\int_\Omega \nabla e \cdot \nabla v_h = 0, \ \forall v_h \in V_h^k.$$

Consider $\phi \in \mathscr{C}_0^\infty(\Omega)$ and let $w^\phi \in H^{q+2}$ be the solution of

$$\begin{cases} -\triangle w^\phi &= \phi \quad \text{in } \Omega, \\ w^\phi &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

In dimension 2, by the Sobolev injections established for instance in [6], $H^{q+2}(\Omega) \subset W^{1,p'}(\Omega)$ for all p' in $[2, +\infty[$. Thus, for any $w_h \in V_h^k$,

$$\left|\int_\Omega e\phi\right| = \left|\int_\Omega e\triangle w^\phi\right| = \left|\int_\Omega \nabla e \cdot \nabla w^\phi\right| = \left|\int_\Omega \nabla e \cdot \nabla (w^\phi - w_h)\right| \leqslant |w^\phi - w_h|_{1,p',\Omega}|e|_{1,p,\Omega}.$$

We have to estimate $|w^\phi - w_h|_{1,p',\Omega}$. It holds

$$|w^\phi - w_h|_{1,p',\Omega}^{p'} = \sum_T |w^\phi - w_h|_{1,p',T}^{p'}.$$

For all $0 \leqslant q \leqslant k-1$ and for all element T in $\mathcal{T}_h$, thanks to Assumption $\widetilde{\mathbf{B1}}$ applied for $s = 1$, $\ell = q+2$, there exists $w_h \in V_h^k$ such as

$$|w^\phi - w_h|_{1,p',T} \leqslant Ch^{2(1/p'-1/2)}h^{q+1}|w^\phi|_{q+2,2,T}. \tag{12}$$

We number the triangles of the mesh $\{T_i, i = 1, \cdots, N\}$ and we set

$$a = (a_i)_i \text{ and } b = (b_i)_i, \text{ where } a_i = |w^\phi - w_h|_{1,p',T_i} \text{ and } b_i = |w^\phi|_{q+2,2,T_i}.$$

By (12), we have, for all i in $[\![1, N]\!]$,

$$a_i \leqslant Ch^{2(1/p'-1/2)}h^{q+1}b_i.$$

We recall the norm equivalence in $\mathbb{R}^N$ for $0 < r < s$,

$$\|x\|_{\ell^s} \leqslant \|x\|_{\ell^r} \leqslant N^{1/r-1/s}\|x\|_{\ell^s}.$$

Remark that here $N \sim Ch^{-2}$. As $2 < p'$, we have $\|b\|_{\ell^{p'}} \leqslant \|b\|_{\ell^2}$. Then, we can write

$$\begin{aligned} |w^\phi - w_h|_{1,p',\Omega} = \|a\|_{\ell^{p'}} &\leqslant Ch^{q+1}h^{2(1/p'-1/2)}\|b\|_{\ell^{p'}} \\ &\leqslant Ch^{q+1}h^{2(1/p'-1/2)}\|b\|_{\ell^2} \\ &\leqslant Ch^{q+1}h^{2(1/p'-1/2)}|w^\phi|_{q+2,2,\Omega} \\ &\leqslant Ch^{q+1}h^{2(1/p'-1/2)}\|\phi\|_{q,\Omega}. \end{aligned}$$

Finally, using this estimate in (11), we obtain, for $q \leqslant k-1$,

$$\|e\|_{-q,\Omega} \leqslant Ch^{q+1}h^{2(1/p'-1/2)}|e|_{1,p,\Omega}.$$

□

Corollary 1. *For finite elements of order k, for any $0 < \varepsilon < 1$,*

$$\|u_\delta - u_\delta^h\|_{-k+1,\Omega} \leqslant Ch^k h^{-\varepsilon}|u_\delta - u_\delta^h|_{1,p,\Omega}, \tag{13}$$

where $p \in]1, 2[$ is defined by

$$p = \frac{2}{1+\varepsilon}\left(\text{and so } p' = \frac{2}{1-\varepsilon}\right). \tag{14}$$

Proof. We will apply Lemma 1 to estimate $\|u_\delta - u_\delta^h\|_{-q,\Omega}$ for (p, p') defined in (14). In inequality (10):

$$2\left(\frac{1}{p'} - \frac{1}{2}\right) = 2\left(\frac{1-\varepsilon}{2} - \frac{1}{2}\right) = -\varepsilon. \tag{15}$$

Finally, for finite elements of order k,

$$\|u_\delta - u_\delta^h\|_{-k+1,\Omega} \leqslant Ch^k h^{-\varepsilon}|u_\delta - u_\delta^h|_{1,p,\Omega}.$$

□

3.2 Estimate of $|u_\delta - u_\delta^h|_{1,p,\Omega}$.

It remains to estimate the quantity $|u_\delta - u_\delta^h|_{1,p,\Omega}$ by bounding $|u_\delta^h|_{1,p,\Omega}$ in terms of $|u_\delta|_{1,p,\Omega}$ (equality (17)). To achieve this, we will need the following discrete inf-sup condition.

Lemma 2. *For $0 < \varepsilon < 1$, p and p' defined in (14), we have the discrete inf-sup condition*

$$\inf_{u_h \in V_h^k} \sup_{v_h \in V_h^k} \frac{\int_\Omega \nabla u_h \cdot \nabla v_h}{\|u_h\|_{1,p,\Omega}\|v_h\|_{1,p',\Omega}} \geqslant C h^\varepsilon.$$

Proof. The continuous inf-sup condition

$$\inf_{u \in W_0^{1,p}} \sup_{v \in W_0^{1,p'}} \frac{\int_\Omega \nabla u \cdot \nabla v}{\|u\|_{1,p}\|v\|_{1,p'}} \geqslant \beta > 0$$

holds for β independent of p and p' (and thus independent of ε). It is a consequence of the duality of the two spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,p'}(\Omega)$, see [18]. For $v \in W_0^{1,p'}(\Omega)$, let $\Pi_h v$ denote the H_0^1-Galerkin projection of v onto V_h^k. This is well defined since $W_0^{1,p'}(\Omega) \subset H_0^1(\Omega)$. We apply Assumption $\widetilde{\mathbf{B}}\mathbf{3}$ to $\Pi_h v$ for $\ell = s = 1$, and get

$$\|\Pi_h v\|_{1,p',\Omega} \leqslant C h^{-2(1/2-1/p')}\|\Pi_h v\|_{1,2,\Omega} \leqslant C h^{-2(1/2-1/p')}\|v\|_{1,2,\Omega} \leqslant C h^{-2(1/2-1/p')}\|v\|_{1,p',\Omega}.$$

Moreover, for any $u_h \in V_h^k \subset W^{1,p}(\Omega)$,

$$\begin{aligned}\|u_h\|_{1,p,\Omega} &\leqslant C \sup_{v \in W_0^{1,p'}} \frac{\int_\Omega \nabla u_h \cdot \nabla v}{\|v\|_{1,p',\Omega}} = C \sup_{v \in W_0^{1,p'}} \frac{\int_\Omega \nabla u_h \cdot \nabla \Pi_h v}{\|v\|_{1,p',\Omega}} \\ &\leqslant C h^{-2(1/2-1/p')} \sup_{v \in W_0^{1,p'}} \frac{\int_\Omega \nabla u_h \cdot \nabla \Pi_h v}{\|\Pi_h v\|_{1,p',\Omega}} \\ &\leqslant C h^{-2(1/2-1/p')} \sup_{v_h \in V_h^k} \frac{\int_\Omega \nabla u_h \cdot \nabla v_h}{\|v_h\|_{1,p',\Omega}}.\end{aligned}$$

Finally, thanks to Poincaré inequality, and to inequality (15),

$$\inf_{u_h \in V_h^k} \sup_{v_h \in V_h^k} \frac{\int_\Omega \nabla u_h \cdot \nabla v_h}{\|u_h\|_{1,p,\Omega}\|v_h\|_{1,p',\Omega}} \geqslant C h^\varepsilon.$$

□

Then, we can estimate $|u_\delta - u_\delta^h|_{1,p,\Omega}$:

Lemma 3. *With p and p' defined in (14),*

$$|u_\delta - u_\delta^h|_{1,p,\Omega} \leqslant C \frac{h^{-\varepsilon}}{\sqrt{\varepsilon}}. \tag{16}$$

Proof. According to Lemma 2, it exists $v_h \in V_h^k$, with $\|v_h\|_{1,p',\Omega} = 1$, such that

$$h^{2(1/2-1/p')}\|u_\delta^h\|_{1,p,\Omega} \leqslant C \int_\Omega \nabla u_\delta^h \cdot \nabla v_h = C \int_\Omega \nabla u_\delta \cdot \nabla v_h \leqslant C \|u_\delta\|_{1,p,\Omega}.$$

So we have

$$|u_\delta - u_\delta^h|_{1,p,\Omega} \leqslant |u_\delta|_{1,p,\Omega} + |u_\delta^h|_{1,p,\Omega} \leqslant C h^{-2(1/2-1/p')}\|u_\delta\|_{1,p,\Omega}. \tag{17}$$

All that remains is to substitute $\|u_\delta\|_{1,p,\Omega}$ for the expression established in (4). For p defined as in (14),

$$\|u_\delta\|_{1,p,\Omega} \leqslant \frac{C}{\sqrt{2-p}} \leqslant \frac{C}{\sqrt{\varepsilon}}.$$

Finally, with (15) and (17), we get

$$|u_\delta - u_\delta^h|_{1,p,\Omega} \leqslant C\frac{h^{-\varepsilon}}{\sqrt{\varepsilon}}.$$

□

3.3 Proof of Theorem 1.

We can now prove Theorem 1.

Proof. The function u_δ is analytic on $\overline{\Omega}_1$, therefore the quantity $\|u_\delta\|_{k+1,\Omega_1}$ is bounded. If we suppose $s=1$, Nitsche and Schatz Theorem gives, for $\ell = k+1$ and $q = k-1$,

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C\left(h^k + \|u_\delta - u_\delta^h\|_{-k+1,\Omega_1}\right).$$

Thanks to (13) and (16),

$$\|u_\delta - u_\delta^h\|_{-k+1,\Omega} \leqslant Ch^k\frac{h^{-2\varepsilon}}{\sqrt{\varepsilon}},$$

therefore, taking $\varepsilon = |\ln h|^{-1}$,

$$\|u_\delta - u_\delta^h\|_{-k+1,\Omega} \leqslant Ch^k\sqrt{|\ln h|}. \tag{18}$$

Finally, we get the result of Theorem 1 for $s=1$ (inequality (6)):

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant Ch^k\sqrt{|\ln h|}.$$

Now, let us fix $2 \leqslant s \leqslant k$, Nitsche and Schatz Theorem gives, for $\ell = k+1$ and $q = k-1$,

$$\|u_\delta - u_\delta^h\|_{s,\Omega_0} \leqslant C\left(h^{k+1-s} + h^{1-s}\|u_\delta - u_\delta^h\|_{-k+1,\Omega_1}\right).$$

So, thanks to (18), we get the second result of Theorem 1 (inequality (7)),

$$\|u_\delta - u_\delta^h\|_{s,\Omega_0} \leqslant Ch^{k+1-s}\sqrt{|\ln h|}.$$

which ends the proof of Theorem 1. □

4 Proof of Theorem 3.

To prove this theorem, we first regularize the right-hand side, and prove that in our case the solution u_δ of (P_δ) and the solution of the regularized problem coincide on the complementary of a neighborhood of the singularity (Theorem 4). The proof of Theorem 3 is based, once again, on the Nitsche and Schatz Theorem and on the observation that the discrete right-hand sides of problem (P_δ) and of the regularized problem are exactely the same, so that the numerical solutions are the same too (Lemma 5).

4.1 Direct problem and regularized problem.

The results presented in this section are valid in any dimension $d \geqslant 1$. However they will only be applied in dimension 2 in Section 4.3 in order to prove Theorem 2. Let $\varepsilon > 0$, and f_ε be defined on Ω by

$$f_\varepsilon = \frac{d}{\sigma(\mathbb{S}_{d-1})\varepsilon^d} \mathbb{1}_{B_\varepsilon}, \tag{19}$$

where $B_\varepsilon = B(x_0, \varepsilon)$ and $\sigma(\mathbb{S}_{d-1})$ is the Lebesgue measure of the unit sphere in dimension d. The parameter ε is supposed to be small enough so that $\overline{B_\varepsilon} \subset\subset \Omega$. The function f_ε is a regularization of the Dirac distribution δ_{x_0}. Let us consider the following problem:

$$(P_\varepsilon) \left\{ \begin{array}{rcll} -\triangle u_\varepsilon & = & f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

Since $f_\varepsilon \in \mathbb{L}^2(\Omega)$, it is possible to show that problem (P_ε) has a unique variational solution u_ε in $H_0^1(\Omega) \bigcap H^2(\Omega)$ [16]. We will show the following result:

Theorem 4. *The solution u_δ of (P_δ) and the solution u_ε of (P_ε) coincide on $\widetilde{\Omega} = \overline{\Omega} \backslash B_\varepsilon$, ie,*

$$u_\delta|_{\widetilde{\Omega}} = u_\varepsilon|_{\widetilde{\Omega}}.$$

The proof is based on the following lemma.

Lemma 4. *Let $d \in \mathbb{N}\backslash\{0\}$, $\varepsilon > 0$, $x \in \mathbb{R}^d$, v a function defined on $\mathbb{R}^d$, harmonic on $\overline{B}(x, \varepsilon)$, and $f \in L^1(\mathbb{R}^d)$ such that*

- *f is radial and positive,*
- *$supp(f) \subset B(0, \varepsilon)$, $\varepsilon > 0$,*
- *$\displaystyle\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = 1$.*

Then, $f * v(x) = \displaystyle\int_{\mathbb{R}^d} f(y)v(x-y)\,\mathrm{d}y = v(x)$.

Proof. As $supp(f) \subset B(0, \varepsilon)$, using spherical coordinates, we have:

$$f * v(x) = \int_0^\varepsilon \int_{\mathbb{S}^{d-1}} f(r)v(x - r\omega)r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}r = \int_0^\varepsilon r^{d-1} f(r) \left(\int_{\mathbb{S}^{d-1}} v(x - r\omega)\mathrm{d}\omega \right) \mathrm{d}r.$$

Besides, v is harmonic on $\overline{B}(x, \varepsilon)$, so that the mean value property gives, for $0 < r \leqslant \varepsilon$,

$$v(x) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} v(y)\,\mathrm{d}y = \frac{r^{d-1}}{\sigma(\partial B(x,r))} \int_{\mathbb{S}^{d-1}} v(x - r\omega)\,\mathrm{d}\omega,$$

thus

$$f * v(x) = \int_0^\varepsilon f(r)v(x)\sigma(\partial B(x,r))\,\mathrm{d}r = v(x) \int_0^\varepsilon \int_{\mathbb{S}^{d-1}} f(r)r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}r = v(x) \int_{B(0,\varepsilon)} f(y)\,\mathrm{d}y = v(x).$$

$\square$

Now, let us prove Theorem 4.

Proof. First, let us leave out boundary conditions and consider the following problem

$$-\triangle u = f_\varepsilon \text{ in } \mathscr{D}'(\mathbb{R}^d). \tag{20}$$

As $-\triangle G = \delta_0$ in $\mathscr{D}'(\mathbb{R}^d)$, we can build a function u satisfying (20) as:

$$u(x) = f_\varepsilon * G(x) = \int_{\mathbb{R}^d} f_\varepsilon(y)G(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f_\varepsilon(x_0+y)G(x-x_0-y)\,\mathrm{d}y = \Big(f_\varepsilon(x_0+\cdot)*G\Big)(x-x_0).$$

Moreover, for all $x \in \Omega\backslash\overline{B_\varepsilon}$, G is harmonic on $\overline{B}(x-x_0,\varepsilon)$, and $f_\varepsilon(\cdot+x_0)$ satisfies the assumptions of Lemma 4, so that $u(x) = \Big(f_\varepsilon(x_0+\cdot)*G\Big)(x-x_0) = G(x-x_0)$. We conclude that u and $G(\cdot-x_0)$ have the same trace on $\partial\Omega$, and so $u+\omega$, where ω is the solution of the Poisson problem (3), is a solution of the problem (P_ε). By the uniqueness of the solution, we have $u_\varepsilon = u+\omega$. Finally, for all $x \in \Omega\backslash\overline{B_\varepsilon}$, $u_\varepsilon(x) = u_\delta(x)$. Since these functions are continuous on $\widetilde{\Omega} = \Omega\backslash B_\varepsilon$, this equality is true on the closure of $\widetilde{\Omega}$, which ends the proof of Theorem 4. ☐

Remark 3. *Theorem 4 holds for any radial positive function $f \in \mathbb{L}^1(\mathbb{R}^d)\bigcap\mathbb{L}^2(\mathbb{R}^d)$ such that*

$$supp(f) \subset B(0,\varepsilon) \text{ and } \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = 1,$$

taking $f_\varepsilon = f(\cdot - x_0)$. It is a direct consequence of Lemma 4.

Remark 4. *Theorem 4 is true in dimension 1, taking $f_\varepsilon = \dfrac{1}{2\varepsilon}\mathbb{1}_{I_\varepsilon}$, where $I_\varepsilon = [x_0-\varepsilon, x_0+\varepsilon] \subset\,]a,b[= I$. In this case, we can easily write down the solutions u_δ and u_ε explicitly,*

$$u_\delta(x) = \begin{cases} \dfrac{b-x_0}{b-a}x - a\dfrac{b-x_0}{b-a} & \text{if } x \in [a,x_0], \\[2ex] -\dfrac{x_0-a}{b-a}x + b\dfrac{x_0-a}{b-a} & \text{if } x \in [x_0,b]. \end{cases}$$

$$u_\varepsilon(x) = \begin{cases} \dfrac{b-x_0}{b-a}x - a\dfrac{b-x_0}{b-a} & \text{if } x \in [a,x_0-\varepsilon], \\[3ex] \begin{aligned} &-\dfrac{x^2}{4\varepsilon} + \left(\dfrac{x_0}{2\varepsilon} + \dfrac{a+b-2x_0}{2(b-a)}\right)x \\ &+\dfrac{a(x_0-b)+b(x_0-a)}{2(b-a)} - \dfrac{x_0^2+\varepsilon^2}{4\varepsilon} \end{aligned} & \text{if } x \in [x_0-\varepsilon, x_0+\varepsilon], \\[3ex] -\dfrac{x_0-a}{b-a}x + b\dfrac{x_0-a}{b-a} & \text{if } x \in [x_0+\varepsilon, b]. \end{cases}$$

and observe, as shown in Figure 2, that u_δ and u_ε coincide outside I_ε.

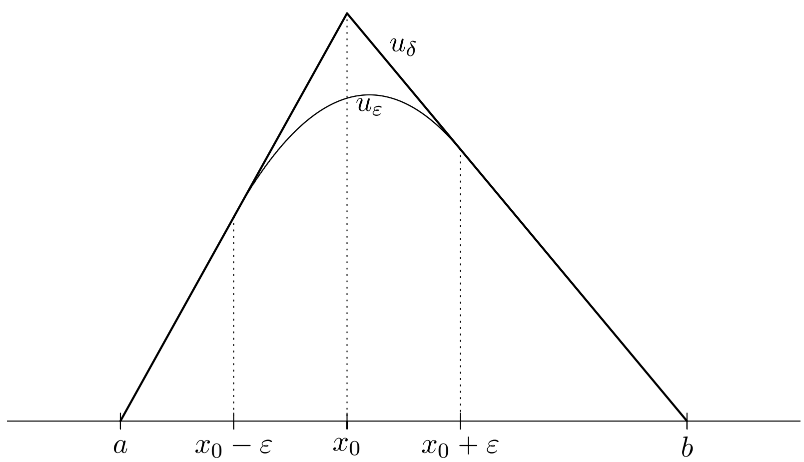

Figure 2: Illustration of Theorem 4 in 1D.

4.2 Discretizations of the right-hand sides.

At this point, we introduce a technical assumption on B_ε and the mesh.

Assumption 4. *The domain of definition B_ε of the function f_ε is supposed to satisfy*

$$B_\varepsilon \subset T_0,$$

where T_0 denotes the triangle of the mesh which contains the point x_0 (Figure 3).

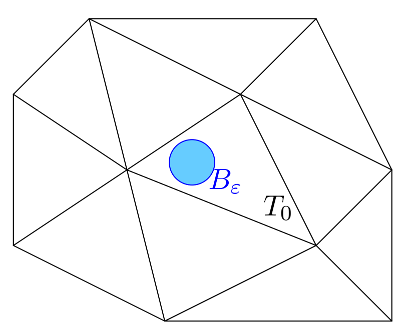

Figure 3: Assumption on B_ε.

Remark 5. *The parameter ε will be chosen to be $h/10$, so it remains to fix a "good" triangle T_0 and to build the mesh accordingly, so that Assumption 4 is satisfied. Remark that it is always possible to locally modify any given mesh so that it satisfies this assumption.*

Lemma 5. *Under Assumption 4,*

$$u_\varepsilon^h = u_\delta^h,$$

where u_δ^h and u_ε^h are respectively the P_1-finite element solutions of problems (P_δ) and (P_ε).

Proof. Let us write down explicitly the discretized right-hand side F_ε^h associated to the function f_ε: for all node i and associated test function $v_i \in V_h^1$,

$$\left(F_\varepsilon^h\right)_i = \int_\Omega \frac{1}{\sigma(B_\varepsilon)} \mathbb{1}_{B_\varepsilon}(x) v_i(x)\,\mathrm{d}x = \int_{B_\varepsilon \subset T_0} \frac{1}{\sigma(B_\varepsilon)} v_i(x)\,\mathrm{d}x,$$

and v_i is affine (and so harmonic) on T_0, therefore

$$\left(F_\varepsilon^h\right)_i = \begin{cases} v_i(x_0) & \text{if } i \text{ is a node of the triangle } T_0, \\ 0 & \textit{otherwise.} \end{cases}$$

We note that $F_\varepsilon^h = D^h$, where D^h is the discretized right-hand side vector associated to the Dirac mass. That is why, with A_h the Laplacian matrix,

$$u_\varepsilon^h - u_\delta^h = \sum_{i \text{ node}} \left[A_h^{-1}\left(F_\varepsilon^h - D^h\right)\right]_i v_i = 0.$$

□

Remark 6. *$F_\varepsilon^h = D^h$ holds as long as $B_\varepsilon \subset T_0$. Otherwise, we still have $u_\delta{}_{|\Omega_0} = u_\varepsilon{}_{|\Omega_0}$ (Theorem 4), but $F_\varepsilon^h \neq D^h$, and so $u_\delta^h{}_{|\Omega} \neq u_\varepsilon^h{}_{|\Omega}$.*

4.3 Proof of Theorem 3.

Theorem 3 can now be proved.

Proof. First, by triangular inequality, we can write, for $s \in \{0, 1\}$:

$$\|u_\delta - u_\delta^h\|_{s,\Omega_0} \leqslant \|u_\delta - u_\varepsilon\|_{s,\Omega_0} + \|u_\varepsilon - u_\varepsilon^h\|_{s,\Omega_0} + \|u_\varepsilon^h - u_\delta^h\|_{s,\Omega_0}.$$

Besides, thanks to Theorem 4, we have

$$\|u_\delta - u_\varepsilon\|_{s,\Omega_0} = 0, \tag{21}$$

and thanks to Lemma 5, we have

$$\|u_\delta^h - u_\varepsilon^h\|_{s,\Omega_0} = 0.$$

Finally we get

$$\|u_\delta - u_\delta^h\|_{s,\Omega_0} \leqslant \|u_\varepsilon - u_\varepsilon^h\|_{s,\Omega_0}. \tag{22}$$

We will apply the Nitsche and Schatz Theorem to $e = u_\varepsilon - u_\varepsilon^h$. With $\ell = 2$, $s = 1$, and $p = 0$,

$$\|e\|_{1,\Omega_0} \leqslant C\left(h\|u_\varepsilon\|_{2,\Omega_1} + \|e\|_{0,\Omega_1}\right). \tag{23}$$

The domain Ω is smooth and $f_\varepsilon \in \mathbb{L}^2(\Omega)$, so $u_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$, and then, thanks to inequality (1),

$$\|e\|_{0,\Omega_1} \leqslant \|e\|_{0,\Omega} \leqslant Ch^2\|u_\varepsilon\|_{2,\Omega} \leqslant Ch^2\|f_\varepsilon\|_{0,\Omega}.$$

As $\|f_\varepsilon\|_{0,\Omega}$ can be computed exactely,

$$\|f_\varepsilon\|_{0,\Omega} = \left(\int_\Omega \left(\frac{1}{\pi\varepsilon^2}\mathbb{1}_{B_\varepsilon}(y)\right)^2 \mathrm{d}y\right)^{1/2} = \frac{1}{\varepsilon\sqrt{\pi}},$$

for $\varepsilon \sim h/10$ (in order to satisfy the assumption on B_ε), we get

$$\|e\|_{0,\Omega_1} \leqslant Ch. \tag{24}$$

Finally, according to Theorem 4, ${u_\delta}_{|\Omega_1} = {u_\varepsilon}_{|\Omega_1}$, therefore combining (23) and (24), we get

$$\|u_\varepsilon - u_\varepsilon^h\|_{1,\Omega_0} = \|e\|_{1,\Omega_0} \leqslant Ch. \tag{25}$$

At last, using inequalities (22) and (25) we obtain the expected error estimate, that is

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant Ch.$$

□

5 Numerical illustrations.

In this section, we illustrate our theorical results by numerical examples.

Concentration of the error around the singularity. First, we present one of the computations which drew our attention to the fact that the convergence could be better far from the singularity. For this example, we define Ω as the unit disk,

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_2 < 1\},$$

Ω_0 as the portion of Ω

$$\Omega_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0.2 < \|x\|_2 < 1\},$$

and finally $x_0 = (0, 0)$ the origin. In this case, the exact solution u_δ of problem (P_δ) is given by

$$u(x) = -\frac{1}{4\pi} \log\left(x_1^2 + x_2^2\right).$$

When problem (P_δ) is solved by the P_1-finite element method, the numerical solution u_δ^h converges to the exact solution u_δ with order 1 in the $\mathbb{L}^2$-norm on the entire domain Ω (see [25]). This example shows that the convergence far from the singularity is faster, since the order of convergence in this case is 2 (see [19]). The difference between the convergence rates for the $\mathbb{L}^2$-norms on Ω and Ω_0, led us to make the conjecture that the preponderant part of the error is concentrated around the singularity, as can be seen in Figures 4, 5, 6, and 7, which show the distribution of the error for $1/h \simeq 10, 15, 20$ and 30.

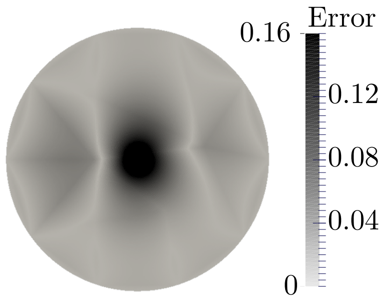

Figure 4: Error for $1/h \simeq 10$.

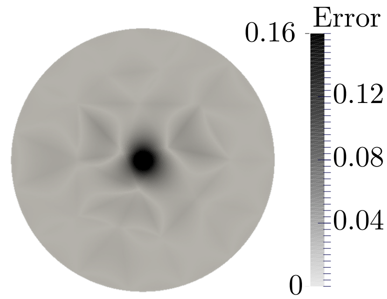

Figure 5: Error for $1/h \simeq 15$.

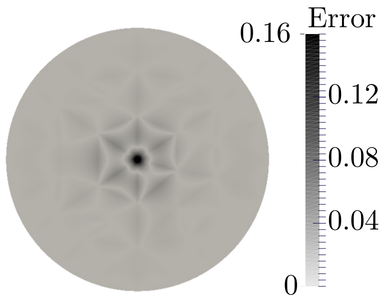

Figure 6: Error for $1/h \simeq 20$.

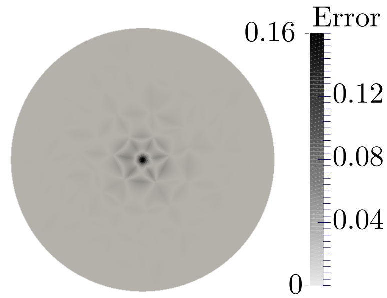

Figure 7: Error for $1/h \simeq 30$.

Estimated orders of convergence. Figure 8 shows the estimated order of convergence for the $H^1(\Omega_0)$-norm for the P_k-finite element method, where $k = 1, 2, 3$ and 4, in dimension 2. The convergence far from the singularity (i.e. excluding a neighborhood of the point x_0) is the same as in the regular case: the P_k-finite element method converges at the order k on Ω_0 for the H^1-norm, as proved in this paper with a $\sqrt{|\ln(h)|}$ multiplier.

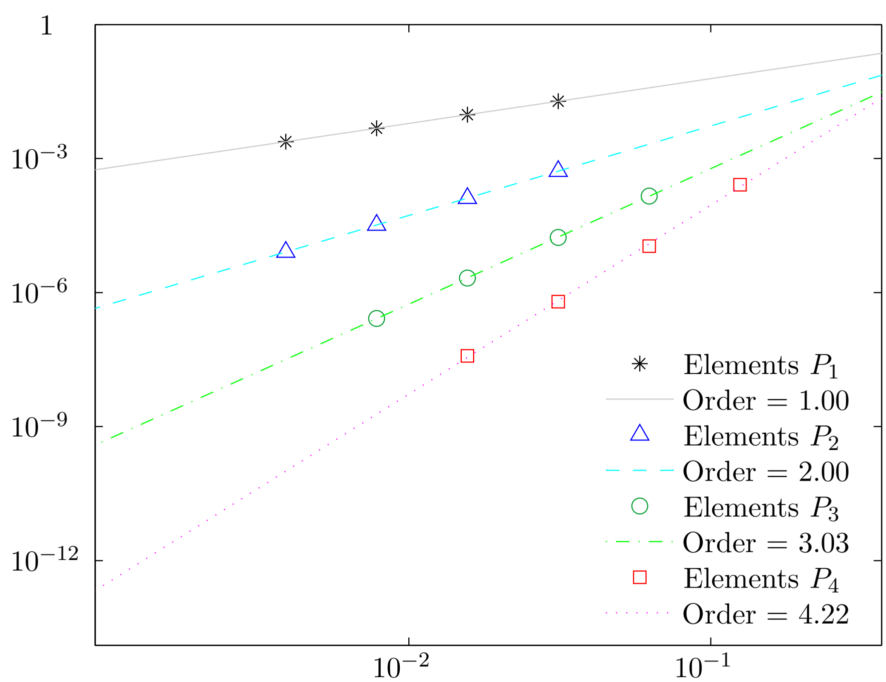

Figure 8: Estimated order of convergence for $H^1(\Omega_0)$-norm for the finite element method P_k, $k = 1, 2, 3, 4$.

6 Discussion

6.1 The three-dimensional case

Dirac mass. The approach presented in this paper can be extended to the three-dimensional case but straighforward adaptations of the proofs lead to a suboptimal result. In the case of Theorem 1, the solution u_δ belongs to $W_0^{1,p}(\Omega)$ for all p in $[1, 3/2[$. As a consequence the couple (p, p') defined in (14) has to be taken near from $(3/2, 3)$. For instance,

$$p = \frac{3}{2+\varepsilon} \text{ and } p' = \frac{3}{1-\varepsilon},$$

so that, with the same notations, the result of Corollary 1 becomes

$$\|u_\delta - u_\delta^h\|_{-k+1,\Omega} \leqslant Ch^k h^{-\varepsilon-1/2} |u_\delta - u_\delta^h|_{1,p,\Omega}.$$

Moreover, the discrete inf-sup condition in dimension 3 is

$$\inf_{u_h \in V_h^k} \sup_{v_h \in V_h^k} \frac{\int_\Omega \nabla u_h \cdot \nabla v_h}{\|u_h\|_{1,p,\Omega} \|v_h\|_{1,p',\Omega}} \geqslant Ch^{\varepsilon+1/2}.$$

Thus when dealing with the estimaye for $|u_\delta - u_\delta^h|_{1,p,\Omega}$, we get

$$|u_\delta - u_\delta^h|_{1,p,\Omega} \leqslant Ch^{-\varepsilon-1/2} |u_\delta|_{1,p,\Omega}.$$

Finally, with the asymptotics in 3d

$$\|u_\delta\|_{1,p,\Omega} \sim \frac{1}{\sqrt[3]{4\pi}} \frac{1}{\sqrt[3]{3-2p}^2},$$

we get the estimate

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega) h^{k-1} \sqrt[3]{|\ln h|}^2.$$

which is clearly suboptimal.

Theorem 3 is also suboptimal in 3d, even if better. Indeed, in 2d or in 3d, the proof readily adapts until the computation of $\|f_\varepsilon\|_{0,\Omega}$, which is in 3d

$$\|f_\varepsilon\|_{0,\Omega} = \frac{1}{2}\sqrt{\frac{3}{\pi}} \frac{1}{\varepsilon\sqrt{\varepsilon}},$$

so that we get

$$\|u_\delta - u_\delta^h\|_{1,\Omega_0} \leqslant C\sqrt{h}.$$

Line Dirac along a curve. In 3-dimension, a line Dirac δ_Γ along a curve $\Gamma \subset\subset \Omega$ belongs to $H^{-1-\eta}$ for all $\eta > 0$, so that the solution u_Γ of the Poisson Problem with the line Dirac δ_Γ belongs to $H^{1-\eta}$. Actually, we have $u_\Gamma \in W^{1,p}(\Omega)$ for all $p \in [1, 2[$. In this case, with the same notations and assumptions as in Theorem 1, we have the following estimate for u_Γ and its Galerkin projection u_Γ^h,

$$\|u_\Gamma - u_\Gamma^h\|_{1,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega) h^k \sqrt{|\ln h|},$$

which is quasi-optimal. This result is shown using the same arguments as the ones presented in Section 3, but cannot be obtained with the tools given in the proof detailed in [19].

6.2 Dirac mass near the boundary

Theorem 3 excludes some critical cases: Dirac mass should not be closer and closer to the border of the domain Ω. Indeed, for example in the case $\mathrm{d}(x_0, \partial\Omega) \sim h^2$, Assumption 4 cannot be satisfied with $\varepsilon \sim h/10$, but only with $\varepsilon \sim h^2/10$. Nevertheless, this small value of ε implies

$$\|u - u_h\|_{1,\Omega_0} \leqslant C,$$

so that our method does not even prove the convergence of the approximate solution in this case. Actually, if the distance $\mathrm{d}(x_0, \partial\Omega)$ tends to 0, the norm $\|u\|_{1,p,\Omega}$, for a fixed $1 \leqslant p < 2$, tends to $+\infty$, so that the problem becomes more and more singular. But this question is a completely different problem and should be treated in a different way.

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