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UNRESTRICTED VIRTUAL BRAIDS, FUSED LINKS AND OTHER QUOTIENTS OF VIRTUAL BRAID GROUPS

VALERIY G. BARDAKOV, PAOLO BELLINGERI, AND CELESTE DAMIANI

ABSTRACT. We consider the group of unrestricted virtual braids, describe its structure and explore its relations with fused links. Also, we define the groups of flat virtual braids and virtual Gauss braids and study some of their properties, in particular their linearity.

1. INTRODUCTION

Fused links were defined by L. H. Kauffman and S. Lambropoulou in [21]. Afterwards, the same authors introduced their “braided” counterpart, the unrestricted virtual braids, and extended S. Kamada work ([17]) by presenting a version of Alexander and Markov theorems for these objects [22]. In the group of unrestricted virtual braids, which shall be denoted by \( \text{UVB}_n \), we consider braid-like diagrams in which we allow two kinds of crossing (classical and virtual), and where the equivalence relation is given by ambient isotopy and by the following transformations: classical Reidemeister moves (Figure 1), virtual Reidemeister moves (Figure 2), a mixed Reidemeister move (Figure 3), and two moves of type Reidemeister III with two real crossings and one virtual crossing (Figure 4). These two last moves are called forbidden moves.

The group \( \text{UVB}_n \) appears also in [16], where it is called symmetric loop braid group, being it a quotient of the loop braid group \( \text{LB}_n \) studied in [1], usually known as the welded braid group \( \text{WB}_n \).

![Figure 1. Classical Reidemeister moves.](image)

It has been shown that all fused knots are equivalent to the unknot ([18,27]). Moreover, S. Nelson’s proof in [27] of the fact that every virtual knot unknots, when allowing forbidden moves, which is carried on using Gauss diagrams, can be verbatim adapted to links with several components. So, every fused link diagram is fused isotopic to a link diagram where the only crossings (classical or virtual) are the ones involving different components.

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On the other hand, there are non trivial fused links and their classification is not (completely) trivial ([12]): in particular in [11], A. Fish and E. Keyman proved that fused links that have only classical crossings are characterized by their (classical) linking numbers. However, this result does not generalize to links with virtual crossings: as conjectured in [11] it is easy to find non equivalent fused links with the same (classical) linking number (see Section 3).

The first aim of this note is to give a short survey on above knotted objects, describe unrestricted virtual braids and compare more or less known invariants for fused links. In Section 2 we give a description of the structure of the group of unrestricted virtual braids $UVB_n$ (Theorems 2.4 and 2.7), answering a question of Kauffman and Lambropoulou from [22]. In Section 3 we construct a representation for $UVB_n$ in $\text{Aut}(N_n)$, the group of automorphisms of the free 2-step nilpotent group of rank $n$ (Proposition 3.11). Using this representation we define a notion of group of fused links and we compare this invariant to other known invariants (Proposition 3.16). Finally, in Section 4 we describe the structure of other quotients of virtual braid groups: the flat virtual braid group (Proposition 4.1 and Theorem 4.3), the flat welded braid group (Proposition 4.5) and the virtual Gauss braid group (Theorem 4.7). As a corollary we prove that flat virtual braid groups and virtual Gauss braid groups are linear and that have solvable word problem (the fact that unrestricted virtual braid groups are linear and have solvable word problem is a trivial consequence of Theorem 2.7).

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2. UNRESTRICTED VIRTUAL BRAID GROUPS

In this Section, in order to define unrestricted virtual braid groups, we will first introduce virtual and welded braid groups by simply recalling their group presentation; for other definitions, more intrinsic, see for instance [2, 9, 17] for the virtual case and [8, 10, 17] for the welded one.

**Definition 2.1.** The virtual braid group $\text{VB}_n$ is the group defined by the group presentation

$$\langle \{ \sigma_i, \rho_i \mid i = 1, \ldots, n-1 \} \mid R \rangle$$

where $R$ is the set of relations:

1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, \ldots, n-2$;
2. $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i-j| \geq 2$;
3. $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$, for $i = 1, \ldots, n-2$;
4. $\rho_i \rho_j = \rho_j \rho_i$, for $|i-j| \geq 2$;
5. $\rho_i^2 = 1$, for $i = 1, \ldots, n-1$;
6. $\sigma_i \rho_j = \rho_j \sigma_i$, for $|i-j| \geq 2$;
7. $\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}$, for $i = 1, \ldots, n-2$.

We define the virtual pure braid group, denoted $\text{VP}_n$, to be the kernel of the map $\text{VB}_n \to \text{S}_n$ sending, for every $i = 1, 2, \ldots, n-1$, generators $\sigma_i$ and $\rho_i$ to $s_i$, where $s_i$ is the transposition $(i, i+1)$. A presentation for $\text{VP}_n$ is given in [3]; it will be recalled in the proof of Theorem 2.7.

The welded braid group $\text{WB}_n$ can be defined as a quotient of $\text{VB}_n$ via the normal subgroup generated by relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \text{for} \ i = 1, \ldots, n-2. \quad (1)$$

Relations (1) will be referred to as relations of type $F1$.

**Remark 2.2.** We will see in Section 3 that the symmetrical relations

$$\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_{i+1}, \quad \text{for} \ i = 1, \ldots, n-2 \quad (2)$$

called $F2$ relations, do not hold in $\text{WB}_n$. This justifies Definition 2.3.

**Definition 2.3.** We define the group of unrestricted virtual braids $\text{UVB}_n$ as the group defined by the group presentation

$$\langle \{ \sigma_i, \rho_i \mid i = 1, \ldots, n-1 \} \mid R' \rangle$$

where $R'$ is the set of relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for} \ i = 1, \ldots, n-2; \quad (R1)$$
\[ (R2) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } |i - j| \geq 2; \]

\[ (R3) \quad \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \quad \text{for } i = 1, \ldots, n - 2; \]

\[ (R4) \quad \rho_i \rho_j = \rho_j \rho_i, \quad \text{for } |i - j| \geq 2; \]

\[ (R5) \quad \rho_i^2 = 1, \quad \text{for } i = 1, \ldots, n - 1; \]

\[ (R6) \quad \sigma_i \rho_j = \rho_j \sigma_i, \quad \text{for } |i - j| \geq 2; \]

\[ (R7) \quad \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \text{for } i = 1, \ldots, n - 2; \]

\[ (F1) \quad \rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \text{for } i = 1, \ldots, n - 2; \]

\[ (F2) \quad \rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, \quad \text{for } i = 1, \ldots, n - 2. \]

The main result of this section is to prove that \( U VB_n \) can be described as semi-direct product of a right-angled Artin group and the symmetric group \( S_n \): this way we answer a question posed in \([22]\) about the (non trivial) structure of \( U VB_n \).

**Theorem 2.4.** Let \( X_n \) be the right-angled Artin group generated by \( x_{i,j} \) for \( 1 \leq i \neq j \leq n \) where all generators commute except the pairs \( x_{i,j} \) and \( x_{j,i} \) for \( 1 \leq i \neq j \leq n \). The group \( U VB_n \) is isomorphic to \( X_n \rtimes S_n \) where \( S_n \) acts by permutation on the indices of generators of \( X_n \).

Let \( \nu : U VB_n \to S_n \) be the map defined as follows:

\[ \nu(\sigma_i) = \nu(\rho_i) = s_i, \quad i = 1, 2, \ldots, n - 1, \]

where \( s_i \) is the transposition \( (i, i+1) \). We will call the kernel of \( \nu \) unrestricted virtual pure braid group and we will denote it by \( U VP_n \). Since \( \nu \) admits a natural section, we have that \( U VB_n = U VP_n \rtimes S_n \).

Let us define some elements of \( U VB_n \). For \( i = 1, \ldots, n - 1 \):

\[ \lambda_{i,i+1} = \rho_i \sigma_i^{-1}, \]

\[ \lambda_{i+1,i} = \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i. \]

For \( 1 \leq i < j - 1 \leq n - 1 \):

\[ \lambda_{ij} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \]

\[ \lambda_{ji} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}. \]

The next lemma was proved in \([3]\) for the corresponding elements in \( VB_n \) and therefore is also true in the quotient \( U VB_n \).

**Lemma 2.5.** The following conjugating rules are fulfilled in \( U VB_n \):

1) for \( 1 \leq i < j \leq n \) and \( k < i - 1 \) or \( i < k < j - 1 \) or \( k > j \):

\[ \rho_k \lambda_{ij} \rho_k = \lambda_{ij}, \quad \text{and} \quad \rho_k \lambda_{ji} \rho_k = \lambda_{ji}; \]

2) for \( 1 \leq i < j \leq n \):

\[ \rho_{i-1} \lambda_{ij} \rho_{i-1} = \lambda_{i-1,j}, \quad \text{and} \quad \rho_{i-1} \lambda_{ji} \rho_{i-1} = \lambda_{j,i-1}; \]
3) for $1 \leq i < j - 1 \leq n$:
\[
\rho_i \lambda_{i,i+1} \rho_i = \lambda_{i+1,i}, \quad \rho_i \lambda_{ij} \rho_i = \lambda_{i+1,j}, \\
\rho_i \lambda_{i+1,i} \rho_i = \lambda_{i,i+1}, \quad \rho_i \lambda_{ji} \rho_i = \lambda_{j,i+1};
\]
4) for $1 \leq i + 1 < j \leq n$:
\[
\rho_{j-1} \lambda_{ij} \rho_{j-1} = \lambda_{i,j-1}, \quad \text{and} \quad \rho_{j-1} \lambda_{ji} \rho_{j-1} = \lambda_{j-1,i};
\]
5) for $1 \leq i < j \leq n$:
\[
\rho_j \lambda_{ij} \rho_j = \lambda_{i,j+1}, \quad \text{and} \quad \rho_j \lambda_{ji} \rho_j = \lambda_{j+1,i}.
\]

**Corollary 2.6.** The group $S_n$ acts by conjugation on the set $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$. This action is transitive.

In particular, the group $S_n$ acts by permutation on the set $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$.

We prove that the group generated by $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$ coincides with $UVP_n$, and then we will find the defining relations. This will show that $UVP_n$ is a right-angled Artin group.

Let $m_{kl} = \rho_{k-1} \rho_{k-2} \cdots \rho_1$ for $l < k$ and $m_{kl} = 1$ in other cases. Then the set
\[
\Lambda_n = \left\{ \prod_{k=2}^n m_{k,j} \mid 1 \leq j_k \leq k \right\}
\]
is a Schreier set of coset representatives of $UVP_n$ in $UVB_n$.

**Theorem 2.7.** The group $UVP_n$ admits a presentation with generators $\lambda_{kl}$ for $1 \leq k \neq l \leq n$, and defining relations: $\lambda_{ij}$ commute with $\lambda_{kl}$ if and only if $k \neq j$ or $l \neq i$.

**Proof.** The proof is a straightforward application of Reidemeister–Schreier method (see, for example, [24, Ch. 2.2]); most part of relations were already proven in [3] in the case of the virtual pure braid group $VP_n$.

Define the map $-: UVB_n \to \Lambda_n$ which takes an element $w \in UVB_n$ to the representative $\overline{w}$ in $\Lambda_n$. In this case the element $w \overline{\gamma}^{-1}$ belongs to $UVP_n$. By Theorem 2.7 from [24] the group $UVP_n$ is generated by

\[
s_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},
\]
where $\lambda$ runs over the set $\Lambda_n$ and $a$ runs over the set of generators of $UVB_n$.

It is easy to establish that $s_{\lambda,\rho_i} = e$ for all representatives $\lambda$ and generators $\rho_i$.

Consider the generators
\[
s_{\lambda,\sigma_i} = \lambda \sigma_i \cdot (\overline{\lambda \rho_i})^{-1}.
\]
For $\lambda = e$ we get $s_{\epsilon,\sigma_i} = \sigma_i \rho_i = \lambda_{i,i+1}^{-1}$. Note that $\lambda \rho_i$ is equal to $\overline{\lambda \rho_i}$ in $S_n$. Therefore,
\[
s_{\lambda,\sigma_i} = \lambda (\sigma_i \rho_i) \lambda^{-1}.
\]
From Lemma 2.5 it follows that each generator $s_{\lambda,\sigma_i}$ is equal to some $\lambda_{kl}$, $1 \leq k \neq l \leq n$.

By Corollary 2.6, the inverse statement is also true, i.e., each element $\lambda_{kl}$ is equal to some generator $s_{\lambda,\sigma_i}$. The first part of the theorem is proven.
To find defining relations of $UVP_n$, we define a rewriting process $\tau$. It allows us to rewrite a word which is written in the generators of $UVB_n$ and presents an element in $UVP_n$ as a word in the generators of $UVP_n$. Let us associate to the reduced word

$$u = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \ldots a_{\nu}^{\varepsilon_{\nu}}, \quad \varepsilon_i = \pm 1, \quad a_i \in \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \rho_1, \rho_2, \ldots, \rho_{n-1}\},$$

the word

$$\tau(u) = s_{k_1,a_1}^{\varepsilon_1} s_{k_2,a_2}^{\varepsilon_2} \ldots s_{k_{\nu},a_{\nu}}^{\varepsilon_{\nu}}$$

in the generators of $UVP_n$, where $k_j$ is a representative of the $(j - 1)$th initial segment of the word $u$ if $\varepsilon_j = 1$ and $k_j$ is a representative of the $j$-th initial segment of the word $u$ if $\varepsilon_j = -1$.

By [24, Theorem 2.9], the group $UVP_n$ is defined by relations

$$r_{\mu,\lambda} = \tau(\lambda r_{\mu,\lambda}^{-1}), \quad \lambda \in \Lambda_n,$$

where $r_{\mu}$ is the defining relation of $UVB_n$.

In [3, Theorem 1] was proven that relations $(R1) - (R7)$ imply the following set of relations:

\begin{align*}
(RS1) & \quad \lambda_{ij} \lambda_{kl} = \lambda_{kl} \lambda_{ij} \\
(RS2) & \quad \lambda_{ki}(\lambda_{kj} \lambda_{ij}) = (\lambda_{ij} \lambda_{kj}) \lambda_{ki},
\end{align*}

where distinct letters stand for distinct indices (this a complete set of relations for the virtual pure braid group $VP_n$). Consider now relation $(F1)$. We get the element

$$(f_1) := \rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}. $$

Using the rewriting process, we obtain

$$r_{f_1,e} = \tau(f_1) = s_{e,\rho_i} s_{\sigma_{i+1} \sigma_i} s_{\rho_{i+1} \sigma_{i+1}}^{-1} s_{\rho_i \sigma_{i+1} \sigma_i} s_{\rho_{i+1} \sigma_{i+1}}^{-1} s_{\sigma_{i+1} \sigma_i} s_{(\lambda_{ij})^{\sigma_{i+1}} \sigma_i}^{-1}.$$ 

Using the conjugating rules from Lemma 2.5, we get

$$r_{f_1,e} = \lambda_{i,i+2}^{-1} \lambda_{i+1,i+2}^{-1} \lambda_{i,i+2} \lambda_{i+1,i+2}. $$

Therefore, the following relation

$$\lambda_{i,i+2} \lambda_{i+1,i+2} = \lambda_{i+1,i+2} \lambda_{i,i+2}$$

is fulfilled in $UVP_n$. The remaining relations $r_{f_1,\lambda}$, $\lambda \in \Lambda_n$, can be obtained from this relation using conjugation by $-1$. By the formulas from Lemma 2.5, we obtain the set of relations $\lambda_{i,j} \lambda_{k,j} = \lambda_{k,j} \lambda_{i,j}$, where $i, j, k$ are distinct letters. On the other hand rewriting relations of type (F2) we get the set of relations $\lambda_{i,j} \lambda_{i,k} = \lambda_{i,k} \lambda_{i,j}$, where $i, j, k$ are distinct letters. Using the proven relations $\lambda_{ki} \lambda_{kj} = \lambda_{kj} \lambda_{ki}$ and $\lambda_{ij} \lambda_{kj} = \lambda_{kj} \lambda_{ij}$ we can rewrite relation $(RS2)$ in the form

$$\lambda_{kj}(\lambda_{ki} \lambda_{ij}) = \lambda_{kj}(\lambda_{ij} \lambda_{ki}). $$

After cancelation we have relations $[\lambda_{ki}, \lambda_{ij}] = [\lambda_{ij}, \lambda_{ki}] = 1$. This complete the proof. □
Proof of Theorem 2.4. The group $X_n$ is evidently isomorphic to $UVP_n$ (sending any $x_{i,j}$ into the corresponding $\lambda_{i,j}$). Recall that $UVP_n$ is the kernel of the map $\nu: UVB_n \rightarrow S_n$ defined as $\nu(\sigma_i) = \nu(\rho_i) = s_i$ for $i = 1, \ldots, n - 1$. Recall also that $\nu$ has a natural section $s: S_n \rightarrow UVB_n$, defined as $s(s_i) = \rho_i$ for $i = 1, \ldots, n - 1$. Therefore $UVB_n$ is isomorphic to $UVP_n \times S_n$ where $S_n$ acts by permutation on the indices of generators of $UVP_n$ (see Corollary 2.6).

We recall that the pure braid group $P_n$ is the kernel of the homomorphism from $B_n$ to the symmetric group $S_n$ sending every generator $\sigma_i$ to the permutation $(i, i + 1)$. It is generated by the set $\{a_{ij} \mid 1 \leq i < j \leq n\}$, where

\[ a_{i,i+1} = \sigma_i^2, \]
\[ a_{i,j} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} \cdots \sigma_{j-1}^{-1}, \quad \text{for } i + 1 < j \leq n. \]

Corollary 2.8. Let $p: P_n \rightarrow UVP_n$ be the canonical projection of the pure braid group $P_n$ in $UVP_n$. Then $p(P_n)$ is the abelianization of $P_n$.

Proof. As remarked in ([3, page 6]), generators $a_{i,j}$ of $P_n$ can be rewritten as

\[ a_{i,i+1} = \lambda_{i,i+1}^{-1} \lambda_{i+1,i}^{-1}, \quad \text{for } i = 1, \ldots, n - 1; \]
\[ a_{i,j} = \lambda_{j-1,i} \lambda_{j-2,i} \cdots \lambda_{i+1,i} (\lambda_{i,j}^{-1} \lambda_{j,i}) \lambda_{i+1,j} \cdots \lambda_{j-2,j} \lambda_{j-1,j}, \quad \text{for } 2 \leq i + 1 < j \leq n. \]

According to Theorem 2.7, $UVP_n$ is the cartesian product of the free groups of rank 2 $F_{i,j} = \langle \lambda_{i,j}, \lambda_{j,i} \rangle$ for $1 \leq i < j \leq n$.

For every generator $a_{i,j}$ for $1 \leq i < j \leq n$ of $P_n$ we have that its image is in $F_{i,j}$ and it is not trivial. In fact, $p(a_{i,j}) = \lambda_{i,j}^{-1} \lambda_{j,i}^{-1}$. So $p(P_n)$ is isomorphic to $\mathbb{Z}^{n(n-1)/2}$. The statement therefore follows readily since the abelianized of $P_n$ is $\mathbb{Z}^{n(n-1)/2}$. □

3. UNRESTRICTED VIRTUAL BRAIDS AND FUSED LINKS

Definition 3.1. A virtual link diagram is a closed oriented 1-manifold $D$ immersed in $\mathbb{R}^2$ such that all multiple points are transverse double point, and each double point is provided with an information of being positive, negative or virtual as in Figure 5. We assume that virtual link diagrams are the same if they are isotopic in $\mathbb{R}^2$. Positive and negative crossings will also be called classical crossings.

![Figure 5](image)

Figure 5. a) Positive crossing, b) Negative crossing, c) Virtual crossing.

Definition 3.2. Fused isotopy is the equivalence relation on the set of virtual link diagrams given by classical Reidemeister moves, virtual Reidemeister moves, and the forbidden moves $F1$ and $F2$. 
Remark 3.3. These moves are the same moves pictured in Figure 1, 2, 3, and 4, with the addition of Reidemeister moves of type I, both classical and virtual, see Figure 6.

![Figure 6. Reidemeister moves of type I.](image)

**Definition 3.4.** A **fused link** is an equivalence class of virtual link diagrams with respect to fused isotopy.

The classical Alexander Theorem generalizes to virtual braids and links, and it directly implies that every oriented welded (resp. fused) link can be represented by a welded (resp. unrestricted virtual) braid, whose Alexander closure is isotopic to the original link. Two braiding algorithms are given in [17] and [21].

Similarly we have the following version of Markov Theorem ([22]):

**Theorem 3.5 ([22]).** Two oriented fused links are isotopic if and only if any two corresponding unrestricted virtual braids differ by moves defined by braid relations in $\text{UVB}_\infty$ (braid moves) and a finite sequence of the following moves (extended Markov moves):

- Virtual and real conjugation: $\rho_i \beta \rho_i^{-1} \beta \sim \beta \sim \sigma_i^{-1} \beta \sigma_i^{-1}$;
- Right virtual and real stabilization: $\beta \rho_n \sim \beta \sim \beta \sigma_n^{\pm 1}$,

where $\text{UVB}_\infty = \bigcup_{n=2}^{\infty} \text{UVB}_n$, $\beta$ is a braid in $\text{UVB}_n$, $\sigma_i, \rho_i$ generators of $\text{UVB}_n$ and $\sigma_n, \rho_n \in \text{UVB}_{n+1}$.

A. Fish and E. Keyman proved the following result about fused links.

**Theorem 3.6 ([11]).** A fused link with only classical crossings $L$ with $n$-components is completely determined by the linking numbers of each pair of components under fused isotopy.

The proof in [11] is quite technical, it involves several computations on generators of the pure braid group and their images in $\text{UVP}_n$.

In order to be self contained let us give an easier proof which uses the structure of $\text{UVB}_n$ described in Section 2: the advantage is that no preliminary lemma on the properties of the pure braid group generators is necessary. The main point of the proof is that if $L$ is a fused link with $n$ component that has only classical crossings, then there is a braid $\beta \in \text{UVP}_n$ such that $L$ is equivalent to its closure $\hat{\beta}$.

**Proof.** The case $n = 1$ is trivial because then $L$ is fused isotopic to the unknot ([18, 27]). So let us consider $n > 1$. Let $U_j$ be the subgroup of $P_n$ generated by $\{a_{i,j} \mid 1 \leq i < j\}$, and $B_{i,j} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i$, for $i < j$, and $B_{i,i} = 1$.

We consider a fused link with only classical crossings $L$ with $n$ components, and $\alpha$ in $B_m$, with $n \leq m$, such that $\hat{\alpha} \sim L$.

Even though $B_n$ and $P_n$ are not subgroups of $\text{UVB}_n$, since $L$ has only classical crossings, we can consider $B_n$ and $P_n$’s images in $\text{UVB}_n$. 
The braid $\alpha$ can be written in the form $\alpha = x_2B_{k_2,2} \cdots x_mB_{k_m,m}$ where $x_i \in U_i < P_m$, and $1 \leq k_i$ (we refer to [7] for a complete proof).

If $B_{k_i,i} = 1$ for $i = 2, \ldots, m$, then $\alpha$ is a pure braid, so $m = n$. So we will assume that $B_{k_s,s} \neq 1$ for some $s$, and that if $i > s$ then $B_{k_i,i} = 1$. The permutation induced by $\alpha$ is the identity on the strands $s + 1, \ldots, m$ so each of these form a component of $\hat{\alpha}$.

Conjugating $\alpha$ by $B_{1,m}$ we obtain a braid $\alpha_1 = B_{1,m}^{-m} \alpha B_{1,m}^{-m}$ and the $s$-strand of $\alpha$ is the $m$-strand of $\alpha_1$ (whose closure is isotopic to $\alpha$'s).

We remark that until now we only used classical Alexander theorem and classical isotopy.

Now we can write $\alpha_1 = y_2B_{k_2,2} \cdots y_mB_{k_m,m}$ with $y_i \in U_i$. Considering the projection $P_n \to UP_n$, we rewrite generators $a_{i,j}$ in terms of $\lambda_{i,j}$: then each $y_i$ is in $D_{i-1}$, the subset of $UP_n$ generated by $\{\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,i-1}, \lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i-1,i}\}$. From Theorem 2.4 we deduce that: $D_{i-1} = (\lambda_{i,1}, \lambda_{i,1}) \times \cdots \times (\lambda_{i,i-1}, \lambda_{i,i-1})$. We can order the words so that $y_m$ is of the form $w_1 \cdots w_{m-1}$ where $w_i \in (\lambda_{m,i}, \lambda_{i,m})$. In addition $B_{t,m} = \sigma_{m-1}B_{t,m-1}$.

Let $\gamma$ be $y_2B_{k_2,2} \cdots y_mB_{k_m,m-1}$. Then $\alpha_1 = \gamma w_1 \cdots w_{m-1} \sigma_{m-1} B_{t,m-1}$, where $w_1 \cdots w_{m-1}$ is a pure braid, and $\gamma$ does not involve the $m$-strand.

The $m$-strand and the other strand involved in the occurrence of $\sigma_{m-1}$ before $B_{t,m-1}$, and hence in all the crossings of $w_{m-1}$, belong to the same component of $L_1 = \hat{\alpha_1}$ (see Figure 7).

![Figure 7. Braid $\alpha_1$.](image)

We virtualize all classical crossings of $w_{m-1}\sigma_{m-1}$ using Kanenobu’s technique ([18, Proof of Theorem 1]): since $w_{m-1}$ has a even total number of generators $\sigma_{m-1}$ and $\rho_{m-1}$, after virtualizing $w_{m-1}\sigma_{m-1}$ becomes a word composed by an odd number of $\rho_{m-1}$.

Applying the relation associated with the virtual Reidemeister move of type 2 we obtain a new link $L_2$, fused isotopic to $L$, associated to $\alpha_2 = \gamma w_1 \cdots w_{m-2} \rho_{m-1} B_{t,m-1}$.

We remark that for all $\lambda_{i,m}$ and $\lambda_{m,i}$ with $i < m - 1$ we have:

$\lambda_{i,m}\rho_{m-1} = \rho_{m-1}\lambda_{i,m-1}$, and $\lambda_{m,i}\rho_{m-1} = \rho_{m-1}\lambda_{m,i-1}$.

Then $\alpha_2 = \gamma \rho_{m-1} w'_1 \cdots w'_{m-2} B_{t,m-1}$, where $w'_i$ is a word in $\langle \lambda_{m-1,i}, \lambda_{i,m-1}\rangle$.

In $\alpha_2$ there is only one (virtual) crossing on the $m$-strand, so, using Markov moves (conjugation and virtual stabilisation) we obtain a new braid $\alpha_3$ whose closure is again fused isotopic to $L$ and has $(m - 1)$ strands.

If we continue this process, eventually we will get to a braid $\beta$ in $B_n$ whose closure is fused isotopic to $L$. At this point, each strand of $\beta$ corresponds to a different component of $L$, so $\beta$ must be a pure braid.
At this point, we can rewrite the pure braid in terms of $a_{i,j}$ generators, and conclude as Fish and Keyman do, defining a group homomorphism $\delta_{i,j} : P_n \to \mathbb{Z}$ by

$$a_{s,t} \mapsto \begin{cases} 
1 & \text{if } s = i \text{ and } t = j; \\
0 & \text{otherwise}
\end{cases}$$

which is the classical linking number $lk_{i,j}$ of $L$'s $i$-th and $j$-th components. Any fused link with only classical crossings $L$ with $n$ components can be obtained as a closure of a pure braid $\beta = x_2 \cdots x_n$ where each $x_i$ can be written in the form $x_i = a_{1,i}^{\delta_{i,1}} \cdots a_{i-1,i}^{\delta_{i-1,i}}$ (Corollary 2.8). This shows that $\beta$ only depends on the linking number of the components. \hfill $\square$

In [14, Section 1] a virtual version of the linking number is defined in the following way: to a 2-component link we associate a couple of integers $(vlk_{1,2}, vlk_{2,1})$ where $vlk_{1,2}$ is the sum of signs of real crossings where the first component passes over the second one, while $vlk_{2,1}$ is computed by exchanging the components in the definition of $vlk_{1,2}$. Clearly the classical linking number $lk_{1,2}$ is equal to the sum of $vlk_{1,2}$ and $vlk_{2,1}$.

Fish and Keyman in [11] suggest that their theorem cannot be extended to links with virtual crossings between different components. They consider the unlink on two components $U_2$ and $L = \hat{\alpha}$, where $\alpha = \sigma_1 \rho_1 \sigma_1^{-1} \rho_1$, they remark that their classical linking number is 0 but they conjecture that these two links are not fused isotopic. In fact, considering the virtual linking number we can see that $(vlk_{1,2}, vlk_{2,1})(U_2) = (0, 0)$, while $(vlk_{1,2}, vlk_{2,1})(L) = (-1, 1)$.

Using this definition of virtual linking number, we could be tempted to extend Fish and Keyman results, claiming that a fused link $L$ is completely determined by the virtual linking numbers of each pair of components under fused isotopy.

However for the unrestricted case the previous argument cannot be straightforward applied: the virtual linking number is able to distinguish $\lambda_{i,j}$ from $\lambda_{j,i}$, but it is still an application from $UVP_n$ to $(\mathbb{Z}^2)^{n(n-1)/2} = \mathbb{Z}^{n(n-1)}$ that counts the exponents (i.e, the number of appearances) of each generator. Since $UVP_n$ isn’t abelian, this is not sufficient to completely determine the braid.

3.1. A representation for the unrestricted virtual braid group. Let us recall that the braid group $B_n$ may be represented as a subgroup of $\text{Aut}(F_n)$ by associating to any generator $\sigma_i$, for $i = 1, 2, \ldots, n - 1$, of $B_n$ the following automorphism of $F_n$:

$$\sigma_i : \begin{cases} 
 x_i \mapsto x_i x_{i+1} x_i^{-1}, \\
x_{i+1} \mapsto x_i, \\
x_l \mapsto x_l, & l \neq i, i + 1.
\end{cases}$$

Moreover Artin provided (see for instance [15, Theorem 5.1]) a characterization of braids as automorphisms of free groups: an automorphism $\beta \in \text{Aut}(F_n)$ lies in $B_n$ if and only if $\beta$ satisfies the following conditions:

i) $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$, $1 \leq i \leq n$

ii) $\beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n$

where $\pi \in S_n$ and $a_i \in F_n$. 


Remark 3.7. The group \( PC_n \) admits also other equivalent definitions in terms of mapping classes and configuration spaces: it appears often in the literature with different names and notations, such as group of flying rings [2, 8], McCool group [6], motions group [13] and loop braid group \([1]\).

Remark 3.8. Kamada remarks in \([17]\) that, through the canonical epimorphism \( VB_n \to W B_n \), the classical braid group \( B_n \) embeds in \( V B_n \). It can be seen via an argument in \([10]\) that \( B_n \) is isomorphic to the subgroup of \( V B_n \) generated by \( \{ \sigma_1, \ldots, \sigma_n \} \).

Remark 3.9. As a consequence of the isomorphism between \( W B_n \) and \( PC_n \), we can show that relation \( F2 \) does not hold in \( W B_n \). In fact applying \( \rho_{i+1}\sigma_i \sigma_{i+1} \) one gets

\[
\rho_{i+1}\sigma_i \sigma_{i+1} : \begin{cases} 
  x_i \mapsto x_i \\
  x_{i+1} \mapsto x_{i+1}x_i x_i^{-1} \\
  x_{i+2} \mapsto x_{i+2}x_{i+1} x_i^{-1}
\end{cases}
\]

while applying \( \sigma_i \sigma_{i+1}\rho_i \) one gets

\[
\sigma_i \sigma_{i+1}\rho_i : \begin{cases} 
  x_i \mapsto x_i x_{i+1} x_i^{-1} \\
  x_{i+1} \mapsto x_i x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1} \\
  x_{i+2} \mapsto x_i x_{i+2} x_{i+1}
\end{cases}
\]

Since \( x_i x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1} \neq x_i x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1} \) in \( F_n \) we deduce that relation \( F2 \) does not hold in \( W B_n \).

Our aim is to find a representation for unrestricted virtual braids as automorphisms of a group \( G \). Since the map \( \psi : W B_n \to Aut(F_n) \) does not factor to the quotient \( UV B_n \) (Remark 3.9) we need to find a representation in the group of automorphisms of a quotient of \( F_n \) in which relation \( F2 \) is preserved.

Remark 3.10. In \([16]\) the authors look for representations of the braid group \( B_n \) that can be extended to the loop braid group \( W B_n \) but do not factor over \( U V B_n \), which is its quotient via relations of type \( F2 \), while we look for a representation that does factor.

Let \( F_n = \gamma_1 F_n \supseteq \gamma_2 F_n \supseteq \cdots \) be the lower central series of \( F_n \), the free group of rank \( n \), where \( \gamma_{i+1} F_n = [F_n, \gamma_i F_n] \). Let us consider its third term, \( \gamma_3 F_n = [F_n, [F_n, F_n]] \); the free 2-step nilpotent group \( N_n \) of rank \( n \) is defined to be the quotient \( F_n/\gamma_3 F_n \).
There is an epimorphism from $F_n$ to $N_n$ that induces an epimorphism from $\text{Aut}(F_n)$ to $\text{Aut}(N_n)$. Then, let $\phi: \text{UVB}_n \to \text{Aut}(N_n)$ be the composition of $\varphi: \text{UVB}_n \to \text{Aut}(F_n)$ and $\text{Aut}(F_n) \to \text{Aut}(N_n)$.

**Proposition 3.11.** The map $\phi: \text{UVB}_n \to \text{Aut}(N_n)$ is a representation for $\text{UVB}_n$.

*Proof.* In $N_n$ we have that $[x_i, x_{i+1}, x_{i+2}] = 1$, for $i = 1, \ldots, n - 2$, meaning that $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} = x_{i+1} x_i x_{i+2} x_i^{-1}$, i.e., relation $F2$ is preserved. \qed

**Proposition 3.12.** The image of the representation $\phi: \text{UP}_n \to \text{Aut}(N_n)$ is a free abelian group of rank $n(n-1)$.

*Proof.* From Theorem 2.4 we have that the only generators that do not commute in $\text{UP}_n$ are $\lambda_i$ and $\lambda_j$ with $1 \leq i \neq j \leq n$.

Recalling the expressions of $\lambda_i$ and $\lambda_j$ in terms of generators $\sigma_i$ and $\rho_i$, we see that the automorphisms associated to $\lambda_i$ and $\lambda_j$ are

\[
\phi(\lambda_{ij}) : \begin{cases}
  x_i \mapsto x_j^{-1} x_i x_j = x_i [x_i, x_j] \\
  x_k \mapsto x_k, \text{ for } k \neq i;
\end{cases}
\]

\[
\phi(\lambda_{ji}) : \begin{cases}
  x_j \mapsto x_i^{-1} x_j x_i = x_j [x_j, x_i] = x_j [x_i, x_j]^{-1}; \\
  x_k \mapsto x_k \text{ for } k \neq i.
\end{cases}
\]

It is then easy to check that the automorphisms associated to $\lambda_{ij}\lambda_{ji}$ and to $\lambda_{ji}\lambda_{ij}$ coincide:

\[
\phi(\lambda_{ij}\lambda_{ji}) = \phi(\lambda_{ji}\lambda_{ij}) : \begin{cases}
  x_i \mapsto x_i [x_i, x_j] \\
  x_j \mapsto x_j [x_i, x_j]^{-1}.
\end{cases}
\]

\qed

**Remark 3.13.** As a consequence of the previous calculation we have that the homomorphism $\phi$ coincides on $\text{UP}_n$ with the abelianization map.

As a consequence of Proposition 3.12, representation $\phi$ is not faithful. However, according to previous characterization of $\text{WB}_n$ as subgroup of $\text{Aut}(F_n)$ it is natural to ask if we can give a characterization of automorphisms of $\text{Aut}(N_n)$ that belong to $\phi(\text{UVB}_n)$.

**Proposition 3.14.** Let $\beta$ be an element of $\text{Aut}(N_n)$, then $\beta \in \phi(\text{UVB}_n)$ if and only if $\beta$ satisfies the condition $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$ with $1 \leq i \leq n$, where $\pi \in S_n$ and $a_i \in N_n$.

*Proof.* Let us denote with $\text{UVB}(N_n)$ the subgroup of $\text{Aut}(N_n)$ such that any element $\beta \in \text{UVB}(N_n)$ has the form $\beta(x_i) = g_i^{-1} x_{\pi(i)} g_i$, denoted $x_{\pi(i)}^{g_i}$, with $1 \leq i \leq n$, where $\pi \in S_n$ and $g_i \in N_n$. We need to prove that $\phi: \text{UVB}_n \to \text{UVB}(N_n)$ is an epimorphism. Let $\beta$ be an element of $\text{UVB}(N_n)$. Since $S_n$ is both isomorphic to the subgroup of $\text{UVB}_n$ generated by the $\rho_i$, generators, and to the subgroup of $\text{UVB}(N_n)$ generated by the permutation automorphisms, we can assume that for $\beta$ the permutation is trivial, i.e., $\beta(x_i) = x_i^{g_i}$. We define $\varepsilon_{ij}$ to be $\phi(\lambda_{ij})$ as in Proposition 3.12, and we prove that $\beta$ is a product of such automorphisms. We recall that every element of $N_n$, can be written in the form

\[
x_1^{\alpha_1} \cdots x_n^{\alpha_n} [x_1, x_2]^{\beta_{1,2}} [x_1, x_3]^{\beta_{1,3}} \cdots [x_{n-1}, x_n]^{\beta_{n-1,n}}
\]
with $\alpha, \beta_{j,k} \in \mathbb{Z}$, so $\beta(x_i)$ can be expressed as
\[
\beta(x_i) = x_i^{\alpha_1 \cdots x_n^{a_n}[x_1,x_2]^\beta \cdots [x_{n-1},x_n]^{\beta_n}}.
\]
In addition $[x_i,x_j]$ for $1 \leq i < j \leq n$ is a basis for $\gamma_2 F_n / \gamma_3 F_n$, that is central in $N_n$, so commutators commute among themselves and with generators $x_i$. Using this fact, and the property of commutators that allows us to express $x_i^\beta x_j^\beta [x_i,x_j]^{-\alpha\beta}$, we can rewrite
\[
\beta(x_i) = x_i^{\alpha_1 \cdots x_n^\alpha}.
\]
where $\alpha_i = 0$.
In particular we can assume that
\[
\beta(x_i) = x_i^{\alpha_2 \cdots x_n^\alpha}.
\]
We define a new automorphism $\beta_1$ multiplying $\beta$ for $\varepsilon_{12} \cdots \varepsilon_{1n}$.
We have that $\beta_1(x_1) = x_1$, and $\beta_1(x_2) = x_2^{b_1} x_2^{b_2} \cdots x_2^{b_n}$, with $b_2 = 0$. Then again we define a new automorphism $\beta_2 = \beta_1 \varepsilon_{21} \varepsilon_{23} \cdots \varepsilon_{2n} = \beta \varepsilon_{12} \cdots \varepsilon_{1n} \varepsilon_{21} \varepsilon_{23} \cdots \varepsilon_{2n}$ that fixes $x_1$ and $x_2$.
Carrying on in this way for $n$ steps we get to an automorphism
\[
\beta_n = \beta_{n-1} \varepsilon_{1n}^z \cdots \varepsilon_{n,n-1} = \beta \prod_{j=1}^n \varepsilon_{1j}^{-a_j} \prod_{j=1}^n \varepsilon_{2j}^{-b_j} \cdots \prod_{j=1}^n \varepsilon_{nj}^{-e_j}
\]
setting $\varepsilon_{ii} = 1$. The automorphism $\beta_n$ is the identity automorphism: then $\beta$ is a product of $\varepsilon_{ij}$ automorphisms, hence it has a pre-image in $UVB_n$.

### 3.2. The knot group.
Let $L$ be a fused link. Then there exists a unrestricted virtual braid $\beta$ such that its closure $\hat{\beta}$ is equivalent to $L$.

**Definition 3.15.** The fused link group $G(L)$ is the group given by the presentation
\[
\left\langle x_1, \ldots, x_n \begin{array}{r}
\phi(\hat{\beta})(x_i) = x_i \quad \text{for } i \in \{1, \ldots, n\}, \\
[x_i, [x_k, x_l]] = 1 \quad \text{for } i, k, l \text{ not necessarily distinct}
\end{array} \right\rangle
\]
where $\phi : UVB_n \to \text{Aut}(N_n)$ is the map from Proposition 3.11.

**Proposition 3.16.** The fused link group is invariant under fused isotopy.

**Proof.** According to [22] two unrestricted virtual braids have fused isotopic closures if and only if they are related by braid moves and extended Markov moves. We should check that under these moves the fused link group $G(L)$ of a fused link $L$ does not change. This is the case. However a quicker strategy to verify the invariance of this group is to remark that it is a projection of the welded link group defined in [5, Section 5]. This last one being an invariant for welded links, we only have to do the verification for the second forbidden braid move, coming from relation $F2$. This invariance is guaranteed by the fact that $\phi$ preserves relation $F2$ as seen in Proposition 3.11. \qed
The group distinguishes the unlink \( U_2 \) from the Hopf link \( H \), but does not distinguish the Hopf link with two classical crossings \( H_1 \) from the one with a classical and a virtual crossing \( H_1 \). In fact:

\[ G(U_2) = \mathbb{Z}_2, \]
\[ G(H) = G(H_1) = \mathbb{Z}^2. \]

We remark however that \( H \) and \( H_1 \) are distinguished by the virtual linking number.

**Remark 3.17.** This invariant does not distinguish the Hopf link with two classical crossings from the one with one virtual crossing (Figure 8). More generally, the knot group does not distinguish the closures of the following braids: let us consider \( \lambda_{1,2}^\alpha \lambda_{2,1}^\beta \) and \( \lambda_{1,2}^\gamma \), where \( \gamma \) is the greatest common divisor of \( \alpha \) and \( \beta \). The automorphisms associated to them are

\[
\phi(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) : \begin{cases} 
  x_1 &\mapsto x_1 [x_1, x_2]^\beta x_1, \ x_2 &\mapsto x_2 [x_1, x_2]^\beta x_1
\end{cases}
\]
\[
\phi(\lambda_{1,2}^\gamma) : \begin{cases} 
  x_1 &\mapsto x_1 [x_1, x_2]^\gamma x_1, \ x_2 &\mapsto x_2 x_2
\end{cases}
\]

Then

\[
G(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) = G(\lambda_{1,2}^\gamma) = \langle x_1, x_2 | [x_1, x_2]^\gamma = 1, [x_1, [x_k, x_l]] = 1 \text{ for } i, k, l \in \{1, 2\} \rangle.
\]

However it distinguishes \( L = \hat{\beta} \) and \( U_2 \), whose associated group is \( \mathbb{Z}_2 \) and \( U_2 \), whose associated group is \( N_2 \). On the other hand, as we saw above, \( L = \hat{\beta} \) and \( U_2 \) have the same classical linking number.

### 4. Other Quotients

Several other quotients of virtual braid groups have been studied in the literature: we end this paper with a short survey on them, giving the structure of the corresponding pure subgroups and some results on their linearity.

#### 4.1. Flat Virtual Braids

The study of flat virtual knots and links was initiated by Kauffman [19] and their braided counterpart was introduced in [20]. The category of flat virtual knots is identical in structure to what are called virtual strings by V. Turaev in [28] (remark that every virtual string is the closure of a flat virtual braid).

The flat virtual braids were introduced in [20] as quotients obtained from \( VB_n \) adding relations

\[ \sigma_i^2 = 1, \text{ for } i = 1, \ldots, n - 1. \]
It is evident that $FVB_n$ is a quotient of the free product $S_n \ast S_n$. In addition to relations coming from the two copies of $S_n$, in $FVB_n$ we have mixed relations

\[(8)\quad \rho_i \rho_{i+1}s_i = s_{i+1}\rho_i \rho_{i+1}, \quad \text{for } i = 1, \ldots, n - 1.\]

Let us consider natural map

- $\pi_V : VB_n \rightarrow FVB_n$ defined by $\sigma_i \mapsto s_i$ and $\rho_i \mapsto \rho_i$, setting $s_i$ (for $i = 1, \ldots, n - 1$) to be the image of $\sigma_i$ in the corresponding quotient.

We call flat virtual pure braid group $FVP_n$ the kernel of this map. If we consider the natural projection $f : VB_n \rightarrow FVB_n$, we have that $f(VP_n) = FVP_n$.

**Proposition 4.1.** Let $VP_n^+$ the subgroup of $VP_n$ defined as

\[VP_n^+ = \langle \lambda_{ij} \mid 1 \leq i < j \leq n \rangle.\]

There is an isomorphism $FVP_n \cong VP_n^+$.

**Proof.** Recall that $VP_n$ is generated by elements $\lambda_{ij}$ as defined in Eq. (3) and (4). and with the following complete set of relations

(i) $\lambda_{ij}\lambda_{kl} = \lambda_{kl}\lambda_{ij}$,

(ii) $\lambda_{ki}(\lambda_{kj}\lambda_{ij}) = (\lambda_{ij}\lambda_{kj})\lambda_{ki}$.

Hence, $FVP_n$ is generated by the images of these elements under $f$. Then

\[f(\lambda_{i,i+1}) = \rho_i s_i, \quad f(\lambda_{i+1,i}) = s_i \rho_i\]

and we can check that

\[f(\lambda_{i+1,i}) = (f(\lambda_{i,i+1}))^{-1}, \quad \text{for } i = 1, \ldots, n - 1.\]

In the general case

\[f(\lambda_{j,i}) = (f(\lambda_{i,j}))^{-1}, \quad \text{for } 1 \leq i < j - 1 \leq n - 1.\]

This means that $FVP_n$ is generated by elements

\[f(\lambda_{i,j}), \quad \text{for } 1 \leq i < j \leq n.\]

To find the set of defining relations we must replace elements $\lambda_{ji}$ with elements $\lambda_{ij}^{-1}$ whenever $i < j$, in all of $FVP_n$’s defining relations. The resulting relations are consequences of relations in $VP_n^+$. Hence, the map $FPV_n \rightarrow VP_n^+$, which is defined by the rule

\[f(\lambda_{i,j}) \mapsto \lambda_{i,j}, \quad \text{for } 1 \leq i < j \leq n\]

is an isomorphism. \(\square\)

**Remark 4.2.** For $n = 3$, the group

\[FVP_3 = \langle \lambda_{12}, \lambda_{13}, \lambda_{23} \mid \lambda_{12}^{-1}(\lambda_{23}\lambda_{13})\lambda_{12} = \lambda_{13}\lambda_{23} \rangle\]

is the HNN-extension of the free group $\langle \lambda_{13}, \lambda_{23} \rangle$ of rank 2 with stable element $\lambda_{12}$ and with associated subgroups $A = \langle \lambda_{23}\lambda_{13} \rangle$ and $B = \langle \lambda_{13}\lambda_{23} \rangle$, which are isomorphic to the infinite cyclic group. Moreover, the group $FVP_3$ is isomorphic to the free product $\mathbb{Z}^2 \ast \mathbb{Z}$. 
Let us recall that there is another remarkable surjection of the virtual braid group $VB_n$ in the symmetric group $S_n$, which sends $\sigma_i$ into 1 and $\rho_i$ into $\rho_i$: the kernel of this map is denoted by $H_n$ in [4]. In the same way we can define the group $FH_n$ as the kernel of the homomorphism $\mu : FVB_n \to S_n$, which is defined as follows:

$$\mu(s_i) = 1, \quad \mu(\rho_i) = \rho_i, \quad i = 1, 2, \ldots, n - 1.$$ 

Now let us define, for $i = 1, \ldots, n - 1$:

\begin{align*}
  y_{i,i+1} &= s_i, \\
  y_{i+1,i} &= \rho_i s_i \rho_i.
\end{align*}

For $1 \leq i < j - 1 \leq n - 1$:

\begin{align*}
  y_{i,j} &= \rho_{j-1} \cdots \rho_{i+1}s_i \rho_{i+1} \cdots \rho_{j-1}, \\
  y_{j,i} &= \rho_{j-1} \cdots \rho_{i+1}s_i \rho_i \rho_{i+1} \cdots \rho_{j-1}.
\end{align*}

(10)

It is not difficult to prove that these elements belong $FH_n$ and that:

**Theorem 4.3.** The group $FH_n$ admits a presentation with the generators $y_{k,l}$, $1 \leq k \neq l \leq n$, and the defining relations:

\begin{align*}
  y_{k,l}^2 &= 1, \\
  (y_{k,l} y_{k,l})^2 &= 1, \\
  (y_{k,l} y_{k,l})^3 &= 1,
\end{align*}

where distinct letters stand for distinct indices.

**Proof.** We know that $FVB_n$ is the quotient of $VB_n$ by the set of relations

$$\sigma_i^2 = 1, \quad i = 1, 2, \ldots, n - 1.$$ 

Hence, in $FH_n$ relation 12 holds. All other relations follow from the homomorphism $H_n \to FH_n$ and the group presentation for $H_n$ provided in [4].

**Corollary 4.4.** The group $FVB_n$ is linear and it has solvable word problem.

**Proof.** From the decomposition $VB_n = VP_n \rtimes S_n$ we have that $FVB_n = FH_n \rtimes S_n$, where $FH_n$ is a finitely generated Coxeter group. The linearity and the solvability of the word problem therefore follow from the fact that all finitely generated Coxeter groups have these properties and that these properties pass to finite extensions.

**4.2. Flat welded braids.** In a similar way we can define the flat welded braid group $FWB_n$ as the quotient obtained from $WB_n$ adding relations

$$\sigma_i^2 = 1, \quad i = 1, \ldots, n - 1,$$

and we can consider the map

- $\pi_W : WB_n \to FWB_n$ defined by $\sigma_i \mapsto s_i$ and $\rho_i \mapsto \rho_i$;
setting $s_i$ (for $i = 1, \ldots, n - 1$) to be the image of $\sigma_i$ in the corresponding quotient.

In $FWB_n$, in addition to relations (8) and (9), we also have relations coming from relations of type $F1$, i.e.,

\begin{equation}
(16) \quad s_{i+1}s_i\rho_{i+1} = \rho_i s_{i+1}s_i, \quad \text{for } i = 1, \ldots, n - 1.
\end{equation}

In $FWB_n$ relations (15) and (16) imply that also relations of type $F2$ hold, since from $\rho_is_{i+1}s_i = s_{i+1}s_i\rho_{i+1}$ one gets $s_is_{i+1}\rho_i = \rho_{i+1}s_is_{i+1}$.

Adapting Theorem 2.7 one can easily verify that $FWP_n$ is isomorphic to $Z^{n(n-1)}/2$. As a straightforward consequence of Theorem 2.4, we can describe the structure of $FWB_n$.

**Proposition 4.5.** Let $Z^{n(n-1)}/2$ be the free abelian group of rank $n(n-1)/2$. Let us denote by $x_{i,j}$ for $1 \leq i \neq j \leq n$ a possible set of generators. The group $FWB_n$ is isomorphic to $Z^{n(n-1)}/2 \rtimes S_n$, where $S_n$ acts by permutation on the indices of generators of $Z^{n(n-1)}/2$ (setting $x_{j,i} := x_{i,j}$ for $1 \leq i \neq j \leq n$).

**Proof.** Let us recall how elements $\lambda_{i,j}$ in $UVB_n$ were defined.

For $i = 1, \ldots, n - 1$:

\[
\lambda_{i,i+1} = \rho_i \sigma_i^{-1}, \\
\lambda_{i+1,i} = \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i.
\]

For $1 \leq i < j - 1 \leq n - 1$:

\[
\lambda_{ij} = \rho_j \rho_{j-1} \rho_{j-2} \ldots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \ldots \rho_{j-2} \rho_{j-1}, \\
\lambda_{ji} = \rho_j \rho_{j-1} \rho_{j-2} \ldots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \ldots \rho_{j-2} \rho_{j-1}.
\]

Relations (15) are therefore equivalent to relations $\lambda_{i,j}\lambda_{j,i} = 1$. Adding these relations and following verbatim the proof of Theorem 2.7 we get the statement. \hfill \Box

4.3. **Virtual Gauss braids.** From the notion of flat virtual knot we can get the notion of Gauss virtual knot or simply Gauss knot. Turaev [29] introduced these knots under the name of “homotopy classes of Gauss words”, while Manturov [25] used the name “free knots”.

The “braided” analogs of Gauss knots, called free virtual braid group on $n$ strands, was introduced in [26]. From now on we will be calling it virtual Gauss braid group and will denote by $GVB_n$.

The group of virtual Gauss braids $GVB_n$ is the quotient of $FVB_n$ by relations

\[ s_i\rho_i = \rho_is_i, \quad \text{for } i = 1, \ldots, n - 1. \]

Note also that the virtual Gauss braid group is a natural quotient of the twisted virtual braid group, studied for instance in [23].

Once again we can consider the homomorphism from $GVB_n$ to $S_n$ that sends each generator $s_i$ and $\rho_i$ in $\rho_i$. The virtual Gauss pure braid group $GVP_n$ is defined to be the kernel of this map. Since this map admits a natural section $GVB_n$ is isomorphic to $GVP_n \rtimes S_n$.

Adapting the proof of Theorem 2.7, we get the following.
Proposition 4.6. The group $GVP_n$ admits a presentation with generators $\lambda_{kl}$ for $1 \leq k < l \leq n$ and the defining relations of $FVP_n$ plus relations

$$\lambda_{ij}^2 = 1, \quad \text{for} \quad 1 \leq i < j \leq n.$$ 

Moreover as in the case of $FVB_n$ also in the case of $GVB_n$ we can consider the map $\mu : GVB_n \rightarrow S_n$, defined as follows:

$$\mu(s_i) = 1, \quad \mu(\rho_i) = \rho_i, \quad \text{for} \quad i = 1, 2, \ldots, n - 1.$$ 

Let $GH_n$ the kernel of $\mu : GVB_n \rightarrow S_n$ and $y_{k,l}$, defined in subsection 4.1: we can prove the following result.

Theorem 4.7. The group $GH_n$ admits a presentation with generators $y_{k,l}$, $1 \leq k < l \leq n$, and defining relations:

\begin{align*}
(17) & \quad y_{k,l}^2 = 1, \\
(18) & \quad (y_{i,j}y_{k,l})^2 = 1, \\
(19) & \quad (y_{i,k}y_{k,j})^3 = (y_{i,j}y_{k,j})^3 = (y_{i,k}y_{i,j})^3 = 1,
\end{align*}

where distinct letters stand for distinct indices.

Proof. We know that $GVB_n$ is the quotient of $FVB_n$ by the set of relations

$$s_i\rho_i = \rho_i s_i, \quad i = 1, 2, \ldots, n - 1.$$ 

One can easily verify that it implies that $y_{i,j} = y_{j,i}$, $1 \leq i < j \leq n$. Hence, $GH_n$ is generated by elements $y_{k,l}$, $1 \leq k < l \leq n$. If we rewrite the set of relations of $FH_n$ in these generators we get the set of relations given in the statement. \hfill $\Box$

As corollary, we have:

Corollary 4.8. The group $GVB_n$ is linear and it has solvable word problem.

References