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A general Doob-Meyer-Mertens decomposition for g-supermartingale systems*

Bruno Bouchard† Dylan Possamaï‡ Xiaolu Tan§

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Abstract

We provide a general Doob-Meyer decomposition for g-supermartingale systems, which does not require any right-continuity on the system, nor that the filtration is quasi left-continuous. In particular, it generalizes the Doob-Meyer decomposition of Mertens [35] for classical supermartingales, as well as Peng's [40] version for right-continuous g-supermartingales. As examples of application, we prove an optional decomposition theorem for g-supermartingale systems, and also obtain a general version of the well-known dual formation for BSDEs with constraint on the gains-process, using very simple arguments.

Key words: Doob-Meyer decomposition, Non-linear expectations, Backward stochastic differential equations.

MSC Classification (2010): 60H99.

1 Introduction

The Doob-Meyer decomposition is one of the fundamental result of the general theory of processes, in particular when applied to the theory of optimal control,

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see El Karoui [17]. Recently, it has been pointed out by Peng [40] that it also holds in the semi-linear context of the so-called g-expectations. Namely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a d-dimensional Brownian motion W, as well as the Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, let $g : (t, \omega, y, z) \in \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be some function, progressively measurable in (t, ω) and Lipschitz in (y, z), and $\xi \in \mathbf{L}^2(\mathcal{F}_\tau)$ for some stopping time τ. We define $\mathcal{E}^g_{\cdot,\tau}[\xi] := Y_\cdot$ where (Y, Z) solves the backward stochastic differential equation

$$-dY_t = g_t(Y_t, Z_t)dt - Z_t \cdot dW_t, \text{ on } [0, \tau],$$

with terminal condition $Y_\tau = \xi$. Then, an optional process X is said to be a (strong) $\mathcal{E}^g$-supermartingale if for all stopping times $\sigma \leq \tau$ we have $X_\tau \in \mathbf{L}^2(\mathcal{F}_\tau)$ and $X_\sigma \geq \mathcal{E}^g_{\sigma,\tau}[X_\tau]$ almost surely. When X is right-continuous, it admits a unique decomposition of the form

$$-dX_t = g_t(X_t, Z^X_t)dt - Z^X_t \cdot dW_t + dA^X_t,$$

in which Z^X is a square integrable and predictable process, and A^X is non-decreasing predictable. See [40] and [10, 25, 32]. In particular, when $g \equiv 0$, this is the classical Doob-Meyer decomposition in a Brownian filtration framework.

As fundamental as its classical version, this result was used by many authors in various contexts : backward stochastic differential equation with constraints [2, 29, 41], minimal supersolutions under non-classical conditions on the driver [16, 24], minimal supersolutions under volatility uncertainty [8, 15, 33, 34, 42, 44, 47, 48], backward stochastic differential equations with weak terminal conditions [3], etc.

However, it is limited to right-continuous $\mathcal{E}^g$-supermartingales, while the right-continuity might be very difficult to prove, if even correct. The method generally used by the authors is then to work with the right-limit process, which is automatically right-continuous, but they then face important difficulties in trying to prove that it still shares the dynamic programming principle of the original process. This was sometimes overcome to the price of stringent assumptions, which are often too restrictive, in particular in the context of singular optimal control problems.

In the classical case, $g \equiv 0$, it is well known that we can avoid these technical difficulties by appealing to the version of the Doob-Meyer decomposition for supermartingales with only right and left limits, see El Karoui [17]. It has been established by Mertens [35], Dellacherie and Meyer [12, Vol. II, Appendice 1] provides an alternative proof. Unfortunately, such a result has not been available so far in the semi-linear context.

This paper fills this gap[1] and provides a version *à la* Mertens of the Doob-Meyer decomposition of $\mathcal{E}^g$-supermartingales. By following the arguments of Mertens [35], we first show that a supermartingale associated to a general family of semi-linear (non-expansive) and time consistent expectation operators can be corrected into a right-continuous one by subtracting the sum of the previous jumps on the right. Applying this result to the g-expectation context, together with the decomposition of [40], we then obtain a decomposition for the original $\mathcal{E}^g$-supermartingale, even when it is not right-continuous. The same arguments apply to g-expectations defined on $\mathbf{L}^p$, $p > 1$, and more general filtrations than the Brownian one considered in [40], in particular we shall not assume that the filtration is quasi left-continuous. This is our Theorem 3.1 below. The only additional difficulty is that the decomposition for right-continuous processes has to be extended first. This is done by using the fact that it can naturally be obtained by considering the BSDE reflected from below on the $\mathcal{E}^g$-supermartingale and by using recent technical extensions of the seminal paper El Karoui et al. [19], see Proposition 3.1 below. Then, using classical results of the general theory of stochastic processes, we can even replace the notion of supermartingale by that of supermartingale systems, for which an optional aggregation process can be easily found, see El Karoui [17] for the classical case $g \equiv 0$.

These key statements aim not only at extending already known results to much more general contexts, but also at simplifying many difficult arguments recently encountered in the literature. We provide two illustrative examples. First, we prove a general optional decomposition theorem for g-supermartingales. To the best of our knowledge, such a decomposition was not obtained before. Then, we show how a general duality for the minimal super-solution of a backward stochastic differential equation with constraint on the gains-process can be obtained. This is an hold problem, but we obtain it in a framework that could not be considered in the literature before, compare with [2, 11]. In both cases, these a-priori difficult results turn out to be easy consequences of our main Theorem 3.1, whenever right continuity *per se* is irrelevant.

Notations: (i) In this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Note that we do not assume that the filtration is quasi left-continuous.

[1] After completing this manuscript, we discovered [22] that was issued at the same time. In this paper, the authors prove the existence of reflected BSDEs for barriers with only right-limits, from which they can infer a similar Doob-Meyer-Mertens decomposition as the one proved in the current paper. Their decomposition is less general, in terms of integrability conditions and assumptions on the filtration. On the other hand, we do not provide any comparable existence result for reflected BSDEs with only right-limited barriers (see however our companion paper [4], where a general existence result for reflected BSDEs with càdlàg obstacles is given). Also, our technic of proof is quite different.

(ii) We fix a time horizon $T > 0$ throughout the paper, and denote by $\mathcal{T}$ the set of stopping times a.s. less than T. We shall also make use of the set $\mathcal{T}_\sigma$ of stopping times $\tau \in \mathcal{T}$ a.s. greater than $\sigma \in \mathcal{T}$. For ease of notations, let us say that $(\sigma, \tau) \in \mathcal{T}_2$ if $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}_\sigma$.

(iii) Let $\sigma \in \mathcal{T}$, conditional expectations or probabilities given $\mathcal{F}_\sigma$ are simply denoted by $\mathbb{E}_\sigma$ and $\mathbb{P}_\sigma$. Inequalities between random variable are taken in the a.s. sense unless something else is specified. If $\mathbb{Q}$ is another probability measure on $(\Omega, \mathcal{F})$, which is equivalent to $\mathbb{P}$, we will write $\mathbb{Q} \sim \mathbb{P}$.

(iv) For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, $\mathbf{L}^0(\mathcal{G})$ denotes the set of random variables on $(\Omega, \mathcal{F})$ which are in addition $\mathcal{G}$-measurable. Similarly, for any $p \in (0, \infty]$, and any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we let $\mathbf{L}^p(\mathcal{G}, \mathbb{Q})$ be the collection of real-valued $\mathcal{G}$-measurable random variables with absolute value admitting a p-moment under $\mathbb{Q}$. For ease of notations, we denote $\mathbf{L}^p(\mathcal{G}) := \mathbf{L}^p(\mathcal{G}, \mathbb{P})$ and also $\mathbf{L}^p := \mathbf{L}^p(\mathcal{F})$. These spaces are endowed with their usual norm.

(v) For $p \in (0, \infty]$, we denote by $\mathbf{X}^p$ (resp. $\mathbf{X}^p_r$, $\mathbf{X}^p_{\ell r}$) the collection of all optional processes X such that X_τ lies in $\mathbf{L}^p(\mathcal{F}_\tau)$ for all $\tau \in \mathcal{T}$ (resp. and such that X admits right-limits, and such that X admits right- and left-limits). We denote by $\mathbf{S}^p$ the set of all càdlàg, $\mathbb{F}$-optional processes Y, such that $\sup_{0 \le t \le T} Y_t \in \mathbf{L}^p$, and by $\mathbf{H}^p$ the set of all predictable d-dimensional processes Z such that

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] < +\infty.$$

Finally, we denote by $\mathbf{A}^p$ the set of all non-decreasing predictable processes A such that $A_0 = 0$ and $A_T \in \mathbf{L}^p$.

(vi) For any $d \in \mathbb{N}\backslash\{0\}$, we will denote by $x \cdot y$ the usual inner product of two elements $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. We will also abuse notation and let $|x|$ denote the Euclidean norm of any $x \in \mathbb{R}^d$, as well the associated operator norm of any $d \times d$ matrix with real entries.

2 Stability of $\mathcal{E}$-supermatingales under Mertens's regularization

In this section, we provide an abstract regularization result for supermartingales associated to a family of semi-linear non-expansive and time consistent conditional expectation operators (see below for the exact meaning we give to this, for the moment, vague appellation). It states that we can always modify a supermartingale with right-limits so as to obtain a right-continuous process which is still a supermatingale. This was the starting point of Mertens's proof of the Doob-Meyer decomposition theorem for supermatingales (in the classical sense)

with only right-limits. Our proof actually mimics the one of Mertens [35]. This abstract formulation has the merit to point out the key ingredients that are required for it to go through, in a non-linear context. It will then be applied to g-expectation operators, in the terminology of Peng [39], to obtain our Doob-Meyer type decomposition, which is the main result of this paper.

2.1 Semi-linear time consistent expectation operators

Let $p \in (1, +\infty]$. Throughout the paper, q will denote the conjugate of p (i.e. $p^{-1} + q^{-1} = 1$). Then, we define a non-linear conditional expectation operator as a family $\mathcal{E} = \{\mathcal{E}_{\sigma,\tau},\ (\sigma,\tau) \in \mathcal{T}_2\}$ of maps

$$\mathcal{E}_{\sigma,\tau} : \mathbf{L}^p(\mathcal{F}_\tau) \longmapsto \mathbf{L}^p(\mathcal{F}_\sigma), \quad \text{for } (\sigma,\tau) \in \mathcal{T}_2.$$

One needs it to satisfy certain structural and regularity properties. Let us start with the notions related to time consistency.

Assumption (Tc)**.** *Fix $(\tau_i)_{i\leq 3} \subset \mathcal{T}$ such that $\tau_1 \vee \tau_2 \leq \tau_3$. Then,*

(a) *$\mathcal{E}_{\tau_1,\tau_1}$ is the identity.*

(b) *$\mathcal{E}_{\tau_1,\tau_2} \circ \mathcal{E}_{\tau_2,\tau_3} = \mathcal{E}_{\tau_1,\tau_3}$, if $\tau_1 \leq \tau_2$.*

(c) *$\mathcal{E}_{\tau_1,\tau_3}[\xi] = \mathcal{E}_{\tau_2,\tau_3}[\xi]$* a.s. *on $\{\tau_1 = \tau_2\}$, for all $\xi \in \mathbf{L}^p(\mathcal{F}_{\tau_3})$.*

We also need some regularity with respect to monotone convergence.

Assumption (S)**.** *Fix $(\sigma,\tau) \in \mathcal{T}_2$.*

(a) *Fix $s \in [0,T)$ and $\xi \in \mathbf{L}^0(\mathcal{F}_s)$. Let $(s_n)_{n\geq 1} \subset [s,T]$ decrease to s and $(\xi_n)_{n\geq 1}$ be such that $\xi_n \in \mathbf{L}^p(\mathcal{F}_{s_n})$ for each n, $(\xi_n^-)_{n\geq 1}$ is bounded in $\mathbf{L}^p$, and $\xi_n \longrightarrow \xi$* a.s. *as $n \longrightarrow \infty$, then*

$$\limsup_{n\to\infty} \mathcal{E}_{s,s_n}[\xi_n] \geq \xi.$$

(b) *Let $(\sigma_n)_{n\geq 1} \subset \mathcal{T}$ be a decreasing sequence which converges* a.s. *to σ and s.t. $\sigma_n \leq \tau$* a.s. *for all $n \geq 1$. Fix $\xi \in \mathbf{L}^p(\mathcal{F}_\tau)$. Then,*

$$\limsup_{n\to\infty} \mathcal{E}_{\sigma_n,\tau}[\xi] \geq \mathcal{E}_{\sigma,\tau}[\xi].$$

(c) *Let $(\xi_n)_{n\geq 1} \subset \mathbf{L}^p(\mathcal{F}_\tau)$ be a non-decreasing sequence which converges* a.s. *to $\xi \in \mathbf{L}^p$. Then,*

$$\limsup_{n\to\infty} \mathcal{E}_{\sigma,\tau}[\xi_n] \geq \mathcal{E}_{\sigma,\tau}[\xi].$$

The idea that $\mathcal{E}$ should be semi-linear and non-expansive is encoded in the following.

Let $\mathbb{Q}^1$, $\mathbb{Q}^2$ be two probability measures on $(\Omega, \mathcal{F})$ and $\tau \in \mathcal{T}$, we define the concatenated probability measure $\mathbb{Q}^1 \otimes_\tau \mathbb{Q}^2$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{E}^{\mathbb{Q}^1 \otimes_\tau \mathbb{Q}^2}\big[\xi\big] \;:=\; \mathbb{E}^{\mathbb{Q}^1}\big[\mathbb{E}^{\mathbb{Q}^2}\big[\xi\big|\mathcal{F}_\tau\big]\big], \text{ for all bounded measurable variable } \xi.$$

Assumption (Sld)**.** *There is a family $\mathcal{Q}$ of $\mathbb{P}$-equivalent probability measures such that:*

- $\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^q + \left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^{1-q}\right] \leq L$ *for all $\mathbb{Q} \in \mathcal{Q}$, for some $L > 1$.*
- $\mathbb{Q}^1 \otimes_\tau \mathbb{Q}^2 \in \mathcal{Q}$*, for all $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ and $\tau \in \mathcal{T}$.*
- *For all $(\sigma, \tau) \in \mathcal{T}_2$ and $(\xi, \xi') \in \mathbf{L}^p(\mathcal{F}_\tau) \times \mathbf{L}^p(\mathcal{F}_\tau)$ there exists $\mathbb{Q} \in \mathcal{Q}$ and a $[L^{-1}, 1]$-valued $\beta \in \mathbf{L}^0(\mathcal{F})$ satisfying*

$$\mathcal{E}_{\sigma,\tau}[\xi] \leq \mathcal{E}_{\sigma,\tau}[\xi'] + \mathbb{E}^{\mathbb{Q}}_\sigma[\beta(\xi - \xi')].$$

Let us comment this last condition. Assume that $(\mathbb{Q}, \beta)$ is the same for (ξ, ξ') and (ξ', ξ). Then, inverting the roles of ξ and ξ', it indeed says that

$$\mathcal{E}_{\sigma,\tau}[\xi] - \mathcal{E}_{\sigma,\tau}[\xi'] = \mathbb{E}^{\mathbb{Q}}_\sigma[\beta(\xi - \xi')].$$

Otherwise stated, in this case, the operator $\mathcal{E}$ can be linearized as each point. However, the linearization, namely $(\mathbb{Q}, \beta)$, depends in general on $(\xi, \xi'), \sigma$ and τ, so that it is not a linear operator. Thus the label semi-linear.

In any case, it is non-expansive in the sense that $\mathcal{E}_{\sigma,\tau}[\xi] - \mathcal{E}_{\sigma,\tau}[\xi'] \leq \mathbb{E}^{\mathbb{Q}}_\sigma[|\xi - \xi'|]$, since $\beta \leq 1$. Moreover, $\mathcal{E}_{\sigma,\tau}[\xi] \leq \mathcal{E}_{\sigma,\tau}[\xi']$ whenever $\xi \leq \xi'$ a.s., and with strict inequality on $\{\mathbb{P}_\sigma[\xi < \xi'] > 0\}$, since $\beta > 0$.

2.2 Stability by regularization on the right

Before stating the main result of this section, one needs to define the notion of $\mathcal{E}$-supermartingales.

We say that X is a $\mathcal{E}$-supermatingale if $X \in \mathbf{X}^p$ and $X_\sigma \geq \mathcal{E}_{\sigma,\tau}[X_\tau]$ a.s. for all $(\sigma, \tau) \in \mathcal{T}_2$. We say that it is a local $\mathcal{E}$-supermatingale if there exists a non-decreasing sequence of stopping times $(\vartheta_n)_{n\geq 1}$ s.t. $X_{\sigma\wedge\vartheta_n} \geq \mathcal{E}_{\sigma\wedge\vartheta_n,\tau\wedge\vartheta_n}[X_{\tau\wedge\vartheta_n}]$ for all $(\sigma, \tau) \in \mathcal{T}_2$ and $n \geq 1$, and $\vartheta_n \uparrow \infty$ a.s. as $n \longrightarrow \infty$.

Lemma 2.1. *Let Assumptions* (Tc)*,* (S) *and* (Sld) *hold. Let $X \in \mathbf{X}^p_\tau$ be a $\mathcal{E}$-supermartingale such that $(X^-_t)_{t\leq T}$ is bounded in $\mathbf{L}^p$. Define the process I by*

$$I_t := \sum_{s<t}(X_s - X_{s+}),\ t \leq T. \tag{1}$$

Then, I is non-decreasing, left-continuous and belongs to $\mathbf{X}^{\frac{1}{p}}$. Moreover, $\overline{X} := X + I$ is a right-continuous local $\mathcal{E}$-supermatingale.

Proof. We split the proof in several steps. As already mentioned, we basically only check that the arguments of Mertens [35] go through under our assumptions.

(a) *$\overline{X}$ is right-continuous.* Indeed, for every $t \in [0, T)$, one has

$$\overline{X}_{t+} \;=\; X_{t+} + I_{t+} \;=\; X_{t+} + I_t + (X_t - X_{t+}) \;=\; X_t + I_t.$$

(b) *Jumps from the right are non-positive, i.e. $X_t \geq X_{t+}$ for each $t \in [0, T)$, so that I is non-decreasing, and $X_{\sigma+} \geq \mathcal{E}_{\sigma,\tau}[X_\tau]$ for all $(\sigma, \tau) \in \mathcal{T}_2$ with $\sigma < \tau$.*

By the $\mathcal{E}$-supermartingale property, $X_t \geq \mathcal{E}_{t,t+h}[X_{t+h}]$ for any $h \in (0, T - t]$ and $t < T$. Since $X_{t+h} \longrightarrow X_{t+}$ as $h \downarrow 0$ and $(X^-_{t+h})_h$ is bounded in $\mathbf{L}^p$, it follows from (S)(a) that $X_t \geq X_{t+}$. Similarly, $X_{\sigma+} \geq \mathcal{E}_{\sigma,\tau}[X_\tau]$ as a consequence of (S)(b).

(c) *Let $k \in \mathbb{N}$, $\varepsilon > 0$, and $(\sigma_i)_{i \leq k} \subset \mathcal{T}$ be the non-decreasing sequence of stopping times which exhausts the first k jumps from the right of X of size bigger than ε (recall that X admits right-limits). Denote*

$$I^{\varepsilon,k}_t := \sum_{i=1}^{k} (X_{\sigma_i} - X_{\sigma_i+}) \mathbf{1}_{\sigma_i < t}, \quad \textit{and} \quad \overline{X}^{\varepsilon,k}_t := X_t + I^{\varepsilon,k}_t, \tag{2}$$

then $\overline{X}^{\varepsilon,k}$ is still a $\mathcal{E}$-supermartingale.

Note that we can always assume that there is a.s. at least k jumps, as we can always add jumps of size 0 at T. We shall use the conventions $\sigma_0 = 0$ and $\sigma_{k+1} = T$. The proof proceeds by induction and requires several steps. For ease of notation, we omit the superscript (ε, k) in $(\overline{X}^{\varepsilon,k}, I^{\varepsilon,k})$ and write $(\overline{X}, I)$ in this part (that is in item (c) only).

(i) Fix $i \leq k$ and $\tau_1, \tau_2 \in \mathcal{T}$ such that $\sigma_i \leq \tau_1 \leq \tau_2 \leq \sigma_{i+1}$ a.s. Let us show that

$$\overline{X}_{\tau_1} \geq \mathcal{E}_{\tau_1,\tau_2}[\overline{X}_{\tau_2}].$$

Indeed, since $X \geq X_+$ and $I \geq 0$ by (b), (Sld) implies that

$$\begin{aligned} \mathcal{E}_{\tau_1,\tau_2}[\overline{X}_{\tau_2}] &= \mathcal{E}_{\tau_1,\tau_2}\left[X_{\tau_2} + I_{\sigma_i} + (X_{\sigma_i} - X_{\sigma_i+}) \mathbf{1}_{\{\sigma_i < \tau_2\}}\right] \\ &\leq \mathcal{E}_{\tau_1,\tau_2}[X_{\tau_2}] + I_{\sigma_i} + (X_{\sigma_i} - X_{\sigma_i+}) \mathbf{1}_{\{\sigma_i < \tau_2\}}. \end{aligned}$$

On the other hand, it follows from (b) that $\mathcal{E}_{\tau_1,\tau_2}[X_{\tau_2}] \leq X_{\tau_1+}$. Hence,

$$\mathcal{E}_{\tau_1,\tau_2}[\overline{X}_{\tau_2}] \leq X_{\tau_1+} + I_{\sigma_i} + (X_{\sigma_i} - X_{\sigma_i+}) \mathbf{1}_{\{\sigma_i < \tau_2\}} = X_{\tau_1} + I_{\tau_1} = \overline{X}_{\tau_1}.$$

(ii) In view of (Tc)(b), the result of (i) implies in particular that $\overline{X}_{\tau_1} \geq \mathcal{E}_{\tau_1,\tau_2}[\overline{X}_{\tau_2}]$ for any $(\tau_1,\tau_2) \in \mathcal{T}_2$ such that $\sigma_i \leq \tau_1 \leq \sigma_{i+1}$ and $\sigma_j \leq \tau_2 \leq \sigma_{j+1}$ a.s., for some $i \leq j \leq k$.

(iii) Given $\tau \in \mathcal{T}$, we next show by induction that

$$\overline{X}_{\sigma_i} \geq \mathcal{E}_{\sigma_i,\tau}[\overline{X}_\tau] \ \text{ on } \{\sigma_i \leq \tau\},\ \forall\, i \leq k.$$

For $i = k$, this follows from (Tc)(c) and (i). Assume that it is true for $1 \leq i+1 \leq k$. Then, on $\{\sigma_i \leq \tau\}$,

$$\overline{X}_{\sigma_i} \geq \mathcal{E}_{\sigma_i,\tau\wedge\sigma_{i+1}}\left[\overline{X}_{\tau\wedge\sigma_{i+1}}\right] = \mathcal{E}_{\sigma_i,\tau\wedge\sigma_{i+1}}\left[\overline{X}_\tau \mathbf{1}_{\tau\leq\sigma_{i+1}} + \overline{X}_{\sigma_{i+1}}\mathbf{1}_{\tau>\sigma_{i+1}}\right].$$

But, by (a) and (c) of (Tc) and the induction hypothesis, we deduce immediately

$$\overline{X}_\tau \mathbf{1}_{\tau\leq\sigma_{i+1}} = \mathcal{E}_{\tau\wedge\sigma_{i+1},\tau}\left[\overline{X}_\tau\right]\mathbf{1}_{\tau\leq\sigma_{i+1}} \text{ and } \overline{X}_{\sigma_{i+1}}\mathbf{1}_{\tau>\sigma_{i+1}} \geq \mathcal{E}_{\tau\wedge\sigma_{i+1},\tau}\left[\overline{X}_\tau\right]\mathbf{1}_{\tau>\sigma_{i+1}}.$$

It remains to appeal to (Sld) to deduce that $\overline{X}_{\sigma_i} \geq \mathcal{E}_{\sigma_i,\tau\wedge\sigma_{i+1}} \circ \mathcal{E}_{\tau\wedge\sigma_{i+1},\tau}[\overline{X}_\tau]$, on $\{\sigma_i \leq \tau\}$, and to conclude by (Tc)(b).

(iv) We are in position to conclude this step. Fix $(\tau_1,\tau_2) \in \mathcal{T}_2$. Set $\tilde{\tau}_1^i := (\tau_1 \vee \sigma_i)\wedge\sigma_{i+1}$. Then, (Tc)(c) implies that $\mathcal{E}_{\tau_1,\sigma_{i+1}\wedge\tau_2}[\overline{X}_{\tau_2}] = \mathcal{E}_{\tilde{\tau}_1^i,\sigma_{i+1}\wedge\tau_2}[\overline{X}_{\sigma_{i+1}\wedge\tau_2}]$ on $\{\sigma_i \leq \tau_1 \leq \sigma_{i+1}\}$. But it follows from (iii), and the same arguments as above, that

$$\begin{aligned}
\mathcal{E}_{\tilde{\tau}_1^i,\sigma_{i+1}\wedge\tau_2}\left[\overline{X}_{\sigma_{i+1}\wedge\tau_2}\right] &= \mathcal{E}_{\tilde{\tau}_1^i,\sigma_{i+1}\wedge\tau_2}\left[\overline{X}_{\tau_2}\mathbf{1}_{\tau_2\leq\sigma_{i+1}} + \overline{X}_{\sigma_{i+1}}\mathbf{1}_{\tau_2>\sigma_{i+1}}\right] \\
&\geq \mathcal{E}_{\tilde{\tau}_1^i,\sigma_{i+1}\wedge\tau_2}\left[\mathcal{E}_{\tau_2\wedge\sigma_{i+1},\tau_2}\left[\overline{X}_{\tau_2}\right]\right] \\
&= \mathcal{E}_{\tilde{\tau}_1^i,\tau_2}\left[\overline{X}_{\tau_2}\right].
\end{aligned}$$

Recalling the result of (i), we conclude that, on $\{\sigma_i \leq \tau_1 \leq \sigma_{i+1}\}$,

$$\overline{X}_{\tau_1} \geq \mathcal{E}_{\tau_1,\sigma_{i+1}\wedge\tau_2}\left[\overline{X}_{\sigma_{i+1}\wedge\tau_2}\right] \geq \mathcal{E}_{\tau_1,\tau_2}\left[\overline{X}_{\tau_2}\right].$$

Since $\cup_{i=0}^k \{\sigma_i \leq \tau_1 \leq \sigma_{i+1}\} = \Omega$, this concludes the proof of this step.

(d) *We now provide a bound on $I_T^{\varepsilon,k}$ defined by* (2).

Let $(\sigma_i)_{i\leq k}$ be as in (c) associated to the parameter (ε,k). We first prove by induction that

$$\mathcal{E}_{\sigma_i,T}[I_T^{\varepsilon,k}] \ \leq\ I_{\sigma_i} + X_{\sigma_i} + \mathbb{E}_{\sigma_i}^{\mathbb{Q}_i}[X_T^-],\ i \leq k+1,$$

in which $\mathbb{Q}_i \in \mathcal{Q}$. The result is true for $i = k+1$, recall our convention $\sigma_{k+1} = T$ and (Tc)(a). Let us assume that it holds for some $i+1 \leq k+1$. Then, by

(Tc)(a)-(b) and (Sld) combined with (b),

$$\begin{aligned}\mathcal{E}_{\sigma_i,T}\left[I_T^{\varepsilon,k}\right] &= \mathcal{E}_{\sigma_i,\sigma_{i+1}} \circ \mathcal{E}_{\sigma_{i+1},T}\left[I_T^{\varepsilon,k}\right] \\ &\le \mathcal{E}_{\sigma_i,\sigma_{i+1}}\left[I_{\sigma_{i+1}}^{\varepsilon,k} + X_{\sigma_{i+1}} + \mathbb{E}^{\mathbb{Q}_{i+1}}_{\sigma_{i+1}}\left[X_T^-\right]\right] \\ &= \mathcal{E}_{\sigma_i,\sigma_{i+1}}\left[I_{\sigma_i}^{\varepsilon,k} + X_{\sigma_i} - X_{\sigma_i+} + X_{\sigma_{i+1}} + \mathbb{E}^{\mathbb{Q}_{i+1}}_{\sigma_{i+1}}[X_T^-]\right] \\ &\le I_{\sigma_i}^{\varepsilon,k} + X_{\sigma_i} - X_{\sigma_i+} + \mathcal{E}_{\sigma_i,\sigma_{i+1}}\left[X_{\sigma_{i+1}}\right] + \mathbb{E}^{\tilde{\mathbb{Q}}_i}_{\sigma_i}\left[\mathbb{E}^{\mathbb{Q}_{i+1}}_{\sigma_{i+1}}[X_T^-]\right] \\ &\le I_{\sigma_i}^{\varepsilon,k} + X_{\sigma_i} + \mathbb{E}^{\tilde{\mathbb{Q}}_i}_{\sigma_i}\left[\mathbb{E}^{\mathbb{Q}_{i+1}}_{\sigma_{i+1}}[X_T^-]\right],\end{aligned}$$

in which $\tilde{\mathbb{Q}}_i \in \mathcal{Q}$. Then, our induction claim follows for i by composing $\tilde{\mathbb{Q}}_i$ and $\mathbb{Q}_{i+1}$ in an obvious way. Recalling our convention $\sigma_0 = 0$, this implies that $\mathcal{E}_{0,T}[I_T^{\varepsilon,k}] \le X_0 + \mathbb{E}^{\mathbb{Q}_0}[X_T^-]$, from which (Sld) provides the estimate

$$L^{-1}\mathbb{E}^{\mathbb{Q}}\left[I_T^{\varepsilon,k}\right] \;\le\; \mathcal{E}_{0,T}\left[I_T^{\varepsilon,k}\right] - \mathcal{E}_{0,T}[0] \;\le\; X_0 + \mathbb{E}^{\mathbb{Q}_0}[X_T^-] - \mathcal{E}_{0,T}[0],$$

in which $\mathbb{Q} \sim \mathbb{P}$ is such that $\mathbb{E}^{\mathbb{Q}}[|d\mathbb{P}/d\mathbb{Q}|^q] \le L$. Since p and q are conjugate, it remains to use Hölder's inequality to deduce that

$$\mathbb{E}\left[\left(I_T^{\varepsilon,k}\right)^{\frac{1}{p}}\right]^p \le C_L\left(1 + |X_0| + \mathbb{E}[(X_T^-)^p]^{\frac{1}{p}} + |\mathcal{E}_{0,T}[0]|\right), \tag{3}$$

for some $C_L > 0$ which only depends on L.

(e) *We now extend the bound* (3) *to the general case.*

Notice that the r.h.s. of (3) does not depend on ε nor k, so we can first send k to ∞ and then ε to 0 and apply the monotone convergence theorem, to obtain that

$$\mathbb{E}\left[\left(I_T\right)^{\frac{1}{p}}\right]^p \le C_L\left(1 + |X_0| + \mathbb{E}[(X_T^-)^p]^{\frac{1}{p}} + |\mathcal{E}_{0,T}[0]|\right).$$

(f) *It remains to show that* $\overline{X} := X + I$ *is a local* $\mathcal{E}$*-supermartingale.*

Recall that I is defined in (1), and $(I^{\varepsilon,k}, \overline{X}^{\varepsilon,k})$ are defined in (2). Let ϑ_n be the first time when $I \ge n$. Note that $(\vartheta_n)_{n\ge1}$ is a.s. increasing and converges to ∞, this follows from (e). We know from (c) that $\overline{X}^{\varepsilon,k}$ is a $\mathcal{E}$-supermartingale. Hence, for $(\sigma,\tau) \in \mathcal{T}_2$, we have

$$\overline{X}^{\varepsilon,k}_{\sigma\wedge\vartheta_n} \ge \mathcal{E}_{\sigma\wedge\vartheta_n,\tau\wedge\vartheta_n}\left[\overline{X}^{\varepsilon,k}_{\tau\wedge\vartheta_n}\right].$$

But $\overline{X}^{\varepsilon,k}_{\vartheta} \uparrow \overline{X}_{\vartheta}$ a.s. for any stopping time ϑ, when one let k first go to ∞ and then ε to 0. Since $\overline{X}_{\tau\wedge\vartheta_n} \in \mathbf{L}^p(\mathcal{F}_\tau)$, by definition of $(\vartheta_n)_{n\ge1}$ and the fact that $X \in \mathbf{X}^p_r$, (S)(c) implies that

$$\overline{X}_{\sigma\wedge\vartheta_n} \ge \mathcal{E}_{\sigma\wedge\vartheta_n,\tau\wedge\vartheta_n}\left[\overline{X}_{\tau\wedge\vartheta_n}\right],$$

which concludes the proof. □

3 Doob-Meyer-Mertens decomposition of g-supermartingale systems

We now specialize to the context of g-expectations introduced by Peng [39] (notice however that we consider a slightly more general version). The object is to provide a Doob-Meyer-Mertens decomposition of g-supermartingale systems without càdlàg conditions. This is our Theorem 3.1 below.

We assume that the space $(\Omega, \mathcal{F}, \mathbb{P})$ carries a d-dimensional Brownian motion W, adapted to the filtration $\mathbb{F}$, which may be strictly larger than the natural (completed) filtration of W. Recall that $\mathbb{F}$ satisfies the usual conditions.

3.1 g-expectation and Doob-Meyer decomposition

Fix some $p > 1$. Let $g : (\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longmapsto g_t(\omega, y, z) \in \mathbb{R}$ be such that $(g_t(\cdot, y, z))_{t \leq T}$ is $\mathbb{F}$-progressively measurable for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and

$$\begin{aligned} |g_t(\omega, y, z) - g_t(\omega, y', z')| &\leq L_g(|y - y'| + |z - z'|), \\ \forall\, (y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d, \ \text{ for } dt \times d\mathbb{P} - \text{a.e. } (t, \omega) \in [0,T] \times \Omega, \end{aligned} \tag{4}$$

for some constant number $L_g > 0$. We also assume that $(g_t(\omega, 0, 0))_{t \leq T}$ satisfies the following integrability condition

$$\mathbb{E}\left[\int_0^T |g_t(0,0)|^p dt\right] < \infty. \tag{5}$$

In the following, we most of the time omit the argument ω in g.

Given $(\sigma, \tau) \in \mathcal{T}_2$ and $\xi \in \mathbf{L}^p(\mathcal{F}_\tau)$, we set $\mathcal{E}^g_{\sigma,\tau}[\xi] := Y_\sigma$ in which (Y, Z, N) is the unique solution of

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} g_s(Y_s, Z_s) ds - \int_{t\wedge\tau}^{\tau} Z_s \cdot dW_s - \int_{t\wedge\tau}^{\tau} dN_s, \ t \leq T, \tag{6}$$

such that $(Y, Z) \in \mathbf{S}^p \times \mathbf{H}^p$ and N is a càdlàg $\mathbb{F}$-martingale orthogonal to W in the sense that the bracket $[W, N]$ is null, $\mathbb{P}$-a.s., and such that

$$\mathbb{E}\left[[N]_T^{\frac{p}{2}}\right] < +\infty.$$

The wellposedness of this equation follows from [4, Thm 4.1], see also [30, Thm 2] and [29] or [43, Prop. A.1] for the case $p = 2$. We also remind the reader that the introduction of the orthogonal martingale N in the definition of the solution is necessary, since the martingale predictable representation property may not

hold with a general filtration $\mathbb{F}$. The map $\mathcal{E}^g$ is usually called the g-expectation operator.

We define $\mathcal{E}^g$-supermartingales, also called g-supermatingales, as in the previous section, for $\mathcal{E} = \mathcal{E}^g$, i.e. X is a $\mathcal{E}^g$-supermatingale iff $X \in \mathbf{X}^p$ and $X_\sigma \geq \mathcal{E}^g_{\sigma,\tau}[X_\tau]$ a.s. for all $(\sigma, \tau) \in \mathcal{T}_2$. For càdlàg $\mathcal{E}^g$-supermartingales, we have the following classical Doob-Meyer decomposition, which is a consequence of the well-posedness of a corresponding reflected backward stochastic differential equation. Its proof is provided in the Appendix (see also Peng [40, Thm. 3.3] in the case of a Brownian filtration).[2]

Proposition 3.1. *Let $X \in \mathbf{X}^p$ be a càdlàg $\mathcal{E}^g$-supermartingale. Then there exists $Z \in \mathbf{H}^p$, a càdlàg process $A \in \mathbf{A}^p$ and a càdlàg martingale N, orthogonal to W, satisfying $\mathbb{E}[[N]_T^{p/2}] < \infty$, such that*

$$X_\sigma = X_\tau + \int_\sigma^\tau g_s(X_s, Z_s)ds + A_\tau - A_\sigma - \int_\sigma^\tau Z_s \cdot dW_s - \int_\sigma^\tau dN_s,$$

for all $(\sigma, \tau) \in \mathcal{T}_2$. Moreover, this decomposition is unique.

Proof. See Appendix A.1. □

3.2 Time consistence and regularity of g-expectations

We now verify that the conditions of Lemma 2.1 apply to $\mathcal{E}^g$.

Proposition 3.2. *Assume that $y \longmapsto g_t(\omega, y, z)$ is non-increasing for all $z \in \mathbb{R}$, for $dt \times d\mathbb{P} -$ a.e. $(t, \omega) \in [0, T] \times \Omega$. Then, Assumptions* (Tc), (S) *and* (Sld) *hold for $\mathcal{E}^g$.*

Proof. First, notice that since W is actually continuous, we not only have $[W, N] = 0$, a.s., but also

$$\langle W, N \rangle = \langle W, N^c \rangle = \langle W, N^d \rangle = 0, \text{ a.s.},$$

where N^c (resp. N^d) is the continuous (resp. purely discontinuous) martingale part of N. Then (Tc) follows from the definition of $\mathcal{E}^g$ and the uniqueness of a solution. The stability properties (S)(b) and (c) follow from the path continuity of the Y component of the solution of (6) and the standard estimates given in

[2] We emphasize that we work with a filtration which is not assumed to be quasi-left continuous, a case which, as far as we know, has never been covered in the literature. The main technical results needed to establish Proposition 3.1 are given in our companion paper [4].

[4, Thm 2.1 and Thm 4.1], see also [30, Prop. 3] for the case where the filtration is quasi left-continuous[3].

The fact that (Sld) holds is a consequence of the usual linearization argument. Let (Y, Z, N) and (Y', Z', N') be the solutions of (6) with terminal conditions ξ and ξ'. Then, since g is uniformly Lipschitz continuous, there exist two processes λ and η, which are $\mathbb{F}$-progressively measurable, such that

$$g_s(Y_s, Z_s) - g_s(Y'_s, Z'_s) = \lambda_s \left(Y_s - Y'_s\right) + \eta_s \cdot \left(Z_s - Z'_s\right), \ ds \times d\mathbb{P} - \text{a.e.}$$

These two processes are bounded by L_g for $dt \times d\mathbb{P} -$ a.e. $(t, \omega) \in [0, T] \times \Omega$, as a consequence of (4). Moreover, $\lambda \leq 0$ since g is non-increasing in y.

Then, for any $0 \leq t \leq s \leq T$, let us define the following continuous, positive and $\mathbb{F}$-progressively measurable process

$$A_{t,s} := \exp\left(\int_t^s \lambda_u du - \int_t^s \eta_u \cdot dW_u - \frac{1}{2}\int_t^s |\eta_u|^2 du\right).$$

By applying Itô's formula, we deduce classically (see [30, Lem. 9]) that

$$Y_\sigma - Y'_\sigma = \mathbb{E}_\sigma\left[A_{\sigma,\tau}(\xi - \xi')\right],$$

which is nothing else but Assumption (Sld) by Girsanov's theorem (recall that $\lambda \leq 0$ and that λ and η are bounded by L_g, i.e. it suffices to consider $\mathcal{Q}$ as the collection of measures with density with respect to $\mathbb{P}$ given by an exponential of Doléans-Dade of the above form with η bounded by L_g).

Finally, the condition (S)(a) follows from a similar linearization argument. Let $s \in [0, T)$ and $\xi \in \mathbf{L}^0(\mathcal{F}_s)$, $s_n \searrow s$ and $(\xi_n)_{n\geq 1}$ be such that $\xi_n \in \mathbf{L}^p(\mathcal{F}_{s_n})$ for each n, $(\xi_n^-)_{n\geq 1}$ is bounded in $\mathbf{L}^p$ and $\xi_n \to \xi$ as $n \to \infty$. One has

$$\mathcal{E}^g_{s,s_n}[\xi_n] \geq \mathbb{E}_s\left[A_n\left(\xi_n - C\int_s^{s_n} |g_s(0,0)|ds\right)\right],$$

for a sequence $(A_n)_{n\geq 1}$ bounded in any $\mathbf{L}^{p'}$, $p' \geq 1$, which converges a.s. to 1, and some $C \geq 1$ independent on n. Since $\left(\xi_n^-, \int_s^{s_n} |g_s(0,0)|ds\right)_{n\geq 1}$ is bounded in $\mathbf{L}^p$, and $p > 1$, the negative part of term in the above expectation is uniformly integrable, and we can apply Fatou's Lemma to conclude the proof. □

Remark 3.1. One easily checks that $X_{\sigma+} \geq \mathcal{E}^g_{\sigma,\tau}[X_{\tau+}]$ for $(\sigma, \tau) \in \mathcal{T}_2$, whenever X is a $\mathcal{E}^g$-supermartingale. Again, this follows from the path continuity of the Y component of the solution of (6) and the estimates of [4, Rem 4.1].

[3]Notice that these estimates can be readily extended to the difference of two solutions of BSDEs, since, as pointed out in the proof of [5, Thm 4.2], such a difference is itself the solution to a BSDE.

Corollary 3.1. *Assume that $y \longmapsto g_t(\omega, y, z)$ is non-increasing for all $z \in \mathbb{R}^d$, for $dt \times d\mathbb{P} -$ a.e. $(t, \omega) \in [0, T] \times \Omega$. Let $X \in \mathbf{X}_r^p$ be an $\mathcal{E}^g$-supermartingale. Define the process I by*

$$I_t := \sum_{s<t} (X_s - X_{s+}),\ t \leq T. \tag{7}$$

Then, I is a non-decreasing and left-continuous process satisfying $I_T \in \mathbf{L}^{\frac{1}{p}}$. Moreover, $\overline{X} := X + I$ is a right-continuous local $\mathcal{E}^g$-supermatingale.

Proof. This is an immediate consequence of Lemma 2.1 and Proposition 3.2 if $(X_t^-)_{t \leq T}$ is bounded in $\mathbf{L}^p$. But this follows from the fact that $X^- \leq \mathcal{E}^g_{\cdot,T}[X_T]^- \in \mathbf{S}^p$. □

For later use, let us a provide another version in which the monotonicity of g in y is not used anymore. The price to pay is that the I process defined below may not be non-decreasing anymore.

Corollary 3.2. *Let $X \in \mathbf{X}_r^p$ be an $\mathcal{E}^g$-supermartingale. Then, $X_t \geq X_{t+}$ for all $t \in [0, T)$. Define the process I by*

$$I_t := \sum_{s<t} e^{L_g(s-t)} (X_s - X_{s+}),\ t \leq T. \tag{8}$$

Then, $I_T \in \mathbf{L}^{\frac{1}{p}}$, I is left-continuous. Moreover, $\overline{X} := X + I$ is a right-continuous local $\mathcal{E}^g$-supermatingale.

Proof. It follows from Corollary 3.1 that the result holds if g is non-increasing in its y-variable. On the other hand, it is immediate to check that ζ is an $\mathcal{E}^g$-supermartingale if and only if $\tilde{\zeta}$ is an $\mathcal{E}^{\tilde{g}}$-supermartingale, with

$$\tilde{\zeta} := e^{L_g T} \zeta \ \text{ and } \ \tilde{g}_t(y, z) := e^{L_g t} g(y e^{-L_g t}, z e^{-L_g t}) - L_g y.$$

The map $\tilde{g}$ is now non-increasing in its y-component as a consequence of (4). Moreover, $\tilde{g}$ still satisfies (4), with the same constant L_g, and, by (5), $\tilde{g}(0, 0)$ satisfies the integrability condition needed to define the corresponding BSDE. Hence $\tilde{X} + \tilde{I}$ is a right-continuous $\mathcal{E}^{\tilde{g}}$-supermartingale, $\tilde{I}$ is non-decreasing and $\tilde{I}_T \in \mathbf{L}^{\frac{1}{p}}$, where we have defined

$$\tilde{I}_t := \sum_{s<t} (\tilde{X}_s - \tilde{X}_{s+}) = \sum_{s<t} e^{L_g s} (X_s - X_{s+}).$$

Hence, $X + I = e^{-L_g \cdot}(\tilde{X} + \tilde{I})$ is a $\mathcal{E}^g$-supermartingale, and $I_T \in \mathbf{L}^{\frac{1}{p}}$ since $\tilde{I}_T \in \mathbf{L}^{\frac{1}{p}}$. □

3.3 The Doob-Meyer-Mertens's decomposition for $\mathcal{E}^g$-supermartingales

We are now in position to state the main result of this paper.

Let $S = \{S(\tau),\ \tau \in \mathcal{T}\}$ be a $\mathcal{T}$-system in the sense that for all $\tau, \tau' \in \mathcal{T}$

(i) $S(\tau) \in \mathbf{L}^0(\mathcal{F}_\tau)$,

(ii) $S(\tau) = S(\tau')$ a.s. on $\{\tau = \tau'\}$.

If $S(\tau) \in \mathbf{L}^p(\mathcal{F}_\tau)$ for every $\tau \in \mathcal{T}$ and $S(\sigma) \geq \mathcal{E}^g_{\sigma,\tau}[S(\tau)]$ for all $(\sigma, \tau) \in \mathcal{T}_2$, then we say that it is a $\mathcal{E}^g$-supermartingale system.

Theorem 3.1 (Mertens's decomposition)**.** *Let S be a $\mathcal{E}^g$-supermatingale system s.t. $\{S(\tau),\ \tau \in \mathcal{T}\}$ is uniformly integrable, then there exists $X \in \mathbf{X}^p_{\ell r}$ such that $S(\sigma) = X_\sigma$ for all $\sigma \in \mathcal{T}$. If in addition,* esssup$\{S(\tau)\ \tau \in \mathcal{T}\} \in \mathbf{L}^p$*, then there exists $(Z, A) \in \mathbf{H}^p \times \mathbf{A}^p$ and a càdlàg martingale N, orthogonal to W, satisfying $\mathbb{E}[[N]_T^{p/2}] < \infty$, such that*

$$S(\sigma) = X_\sigma = X_\tau + \int_\sigma^\tau g_s(X_s, Z_s)ds + A_\tau - A_\sigma - \int_\sigma^\tau Z_s \cdot dW_s - \int_\sigma^\tau dN_s,$$

for all $(\sigma, \tau) \in \mathcal{T}_2$. This decomposition is unique.

Proof. (a) Let us first prove that there exists an optional process $X \in \mathbf{X}^p$ such that $S(\sigma) = X_\sigma$ a.s. for all $\sigma \in \mathcal{T}$. Since S is uniformly integrable, [13, Thm. 6 and Rem. 7 c)] imply that it suffices to show that

$$\mathbb{E}[S(\sigma)] \geq \limsup_{n\to\infty} \mathbb{E}[S(\sigma_n)],$$

for all non-increasing sequence $(\sigma_n)_{n\geq 1} \in \mathcal{T}_\sigma$ such that $\sigma_n \longrightarrow \sigma \in \mathcal{T}$, a.s. By using a similar linearization argument as the one used in the proof of Proposition 3.2, we can find $\mathbb{F}$-progressively measurable processes λ^n and η^n that are bounded by $L_g\ dt \times d\mathbb{P}$-a.e. and such that

$$S(\sigma) \geq \mathbb{E}_\sigma \left[H^n_{\sigma_n} \left(e^{\int_\sigma^{\sigma_n} \lambda^n_s ds} S(\sigma_n) + \int_\sigma^{\sigma_n} e^{\int_\sigma^s \lambda^n_s ds} g_s(0,0)ds \right) \right]$$

where

$$H^n := \exp\left(-\frac{1}{2} \int_\sigma^{\cdot\vee\sigma} |\eta^n_s|^2 ds + \int_\sigma^{\cdot\vee\sigma} \eta^n_s \cdot dW_s \right).$$

Then,

$$\begin{aligned} \mathbb{E}[S(\sigma)] \geq\ & \mathbb{E}[S(\sigma_n)] + \mathbb{E}\left[e^{\int_\sigma^{\sigma_n} \lambda^n_s ds} (H^n_{\sigma_n} - 1) S(\sigma_n) \right] \\ & + \mathbb{E}\left[H^n_{\sigma_n} \int_\sigma^{\sigma_n} e^{\int_\sigma^s \lambda^n_s ds} g_s(0,0)ds \right] \\ & + \mathbb{E}\left[\left(e^{\int_\sigma^{\sigma_n} \lambda^n_s ds} - 1 \right) S(\sigma_n) \right]. \end{aligned} \tag{9}$$

Note that

$$(H^n_{\sigma_n} - 1)S(\sigma_n) \geq -(S(\sigma_n))^+ - H^n_{\sigma_n}(S(\sigma_n))^-.$$

Since S is uniformly integrable, so is S^+. Besides, we have by definition

$$S(\sigma_n) \geq \mathcal{E}^g_{\sigma_n,T}[S(T)].$$

But, once more it is clear that $\mathcal{E}^g_{\sigma_n,T}[S(T)]$ is bounded in $\mathbf{L}^p$, uniformly in n, see [4, Thm 4.1]. Since $H^n_{\sigma_n}$ has bounded (uniformly in n) moments of any order, de la Vallée-Poussin criterion ensures that $H^n S^-$ is also uniformly integrable. Therefore, $\{[(H^n_{\sigma_n} - 1)S(\sigma_n)]^-, n \geq 1\}$ is uniformly integrable. Using the fact that $(\lambda^n, \eta^n)_n$ is uniformly bounded by L_g, as well as (5), we can use Fatou's lemma in (9) to obtain that the second and the third terms on the right-hand side converges to 0 as $n \longrightarrow \infty$.

(b) The fact that X has right- and left-limits, up to an evanescent set, follows from Lemma A.2 stated below, since X is an $\mathcal{E}^g$-supermartingale.

(c) Let I be defined as in Corollary 3.2 for X. Since $X + I$ is right-continuous, we can apply the Doob-Meyer decomposition of Proposition 3.1 to $\overline{X}^n := (X + I)_{\cdot\wedge\vartheta_n}$ where ϑ_n is the first time when $I \geq n$. There exists $(Z^n, \bar{A}^n) \in \mathbf{H}^p \times \mathbf{A}^p$ and a càdlàg martingale N^n, orthogonal to W, such that, for $(\sigma, \tau) \in \mathcal{T}_2$,

$$\begin{aligned}
\overline{X}^n_\sigma &= \overline{X}^n_\tau + \int_{\vartheta_n\wedge\sigma}^{\vartheta_n\wedge\tau} g_s(\overline{X}^n_s, Z^n_s)ds + \bar{A}^n_\tau - \bar{A}^n_\sigma - \int_{\vartheta_n\wedge\sigma}^{\vartheta_n\wedge\tau} Z^n_s \cdot dW_s - \int_{\vartheta_n\wedge\sigma}^{\vartheta_n\wedge\tau} dN^n_s \\
&= \overline{X}^n_\tau + \int_{\vartheta_n\wedge\sigma}^{\vartheta_n\wedge\tau} \{g_s(X_s, Z^n_s) + \eta_s I_s\}ds + \bar{A}^n_\tau - \bar{A}^n_\sigma - \int_{\vartheta_n\wedge\sigma}^{\vartheta_n\wedge\tau} Z^n_s \cdot dW_s \\
&\quad - \int_{\vartheta_n\wedge\sigma}^{\vartheta_n\wedge\tau} dN^n_s,
\end{aligned}$$

in which η is a progressively measurable process bounded by L_g, $dt \times d\mathbb{P}$-a.e., as a consequence of (4). Set

$$A^n := I_{\cdot\wedge\vartheta_n} + \bar{A}^n + \int_0^{\cdot\wedge\vartheta_n} \eta_s I_s ds, \tag{10}$$

and observe that $(A^n, Z^n, N^n) = (A^k, Z^k, N^k)$ on $[\![0, \vartheta_k]\!]$ for $n \leq k$, by uniqueness of the decomposition in Proposition 3.1. We can then define

$$(A, Z, N) := \mathbf{1}_{[\![0,\vartheta_1]\!]}(A^1, Z^1, N^1) + \sum_{n\geq 1} \mathbf{1}_{]\!]\vartheta_n,\vartheta_{n+1}]\!]}(A^{n+1}, Z^{n+1}, N^{n+1}), \tag{11}$$

so that

$$X_\sigma = X_\tau + \int_\sigma^\tau g_s(X_s, Z_s)ds + A_\tau - A_\sigma - \int_\sigma^\tau Z_s \cdot dW_s - \int_\sigma^\tau dN_s. \tag{12}$$

We claim that A is non-decreasing and that the above decomposition is unique. The fact that $(Z, A, [N]_T) \in \mathbf{H}^p \times \mathbf{A}^p \times \mathbf{L}^{p/2}$ then follows from [4, Proposition 2.1].

Let us now prove our claim. Define $\tilde{X}$ and $\tilde{g}$ by

$$\tilde{X} := e^{L_g \cdot} X \text{ and } \tilde{g}_t(y,z) := e^{L_g t} g(ye^{-L_g t}, ze^{-t}) - L_g y.$$

Then, $\tilde{X}$ is a $\mathcal{E}^{\tilde{g}}$-supermartingale, and so is its right-limits process $\tilde{X}^+ := \tilde{X}_+$, as a consequence of Remark 3.1, recall Lemma A.2 below. Applying Proposition 3.1, we can find a right-continuous non-decreasing process $\tilde{A} \in \mathbf{A}^p$, $\tilde{Z} \in \mathbf{H}^p$ and a càdlàg martingale $\tilde{N}$, orthogonal to W, such that

$$\tilde{X}^+_\sigma = \tilde{X}^+_\tau + \int_\sigma^\tau \tilde{g}_s(\tilde{X}^+_s, \tilde{Z}_s)ds + \tilde{A}_\tau - \tilde{A}_\sigma - \int_\sigma^\tau \tilde{Z}_s \cdot dW_s - \int_\sigma^\tau d\tilde{N}_s$$

for all $(\sigma, \tau) \in \mathcal{T}_2$. This decomposition is unique. On the other hand, (12) implies that

$$\tilde{X}^+_\sigma = \tilde{X}^+_\tau + \int_\sigma^\tau \tilde{g}_s(\tilde{X}^+_s, e^{L_g s} Z_s)ds + \tilde{B}_\tau - \tilde{B}_\sigma - \int_\sigma^\tau e^{L_g s} Z_s \cdot dW_s - \int_\sigma^\tau e^{L_g s} dN_s,$$

in which

$$\begin{aligned}\tilde{B}_\tau - \tilde{B}_\sigma := & \int_\sigma^\tau (\tilde{g}_s(\tilde{X}_s, e^{L_g s} Z_s) - \tilde{g}_s(\tilde{X}^+_s, e^{L_g s} Z_s))ds \\ & + \int_\sigma^\tau e^{L_g s} dA_s + e^{L_g \tau}(A_{\tau+} - A_\tau) - e^{L_g \sigma}(A_{\sigma+} - A_\sigma).\end{aligned}$$

Hence, $\tilde{B} = \tilde{A}$ is non-decreasing. But, since $(\tilde{g}(\tilde{X}, e^{L_g \cdot} Z) - \tilde{g}(\tilde{X}^+, e^{L_g \cdot} Z)) \leq 0$ as a consequence of Corollary 3.2 (namely $\tilde{X} \geq \tilde{X}^+$) and the fact that $\tilde{g}$ is non-increasing in its first component, we must have that the continuous part of $\int_0^\cdot e^{L_g s} dA_s$ is non-decreasing, and so must be the continuous part of A. We now deduce from the definition of I in (8) and (10)-(11) that A can only decrease in a continuous manner, recall that $\bar{A}_n$ is non-decreasing. Hence, A is non-decreasing. The fact that the decomposition is unique comes from the uniqueness of the decomposition for $\tilde{X}^+$. □

3.4 Remarks

The framework of this section corresponds to the case where the BSDEs are driven by a continuous martingale M, whose quadratic variation is absolutely continuous with respect to the Lebesgue measure, and with an invertible density. Extensions to the context of [6], see also [18], [29] or [9], would be of interest. Similarly, one could certainly consider BSDEs with jumps, generators with quadratic growth,

obstacles, stochastic Lipschitz conditions, etc. We have chosen to work in a simpler setting so as not to drown our arguments with unneeded technicalities, and to focus on the novelty of our approach.

However, the case $p = 1$ can not be treated by the same technics, in particular we can not appeal to the classical linearization procedure. It would also require a reinforcement of the condition (4), see [5].

4 Applications

We now consider two problems studied in the recent literature, which are solved with sophisticated arguments under technical conditions. Using Theorem 3.1, we can solve these problems in a very general context with quite simple arguments.

4.1 Optional decomposition of g-supermartingale systems

We are still in the context of the previous section, with the slight modification that, instead of the Brownian motion W, we consider a continuous $(\mathbb{P}, \mathbb{F})$-martingale M of the form: $M_t = \int_0^t \alpha_s^\top dW_s$, in which α is a $\mathbb{R}^{d\times d}$-bounded predictable process with bounded inverse. Recall that $\mathbb{F}$ satisfies the usual conditions.

Let $S = \{S(\tau),\ \tau \in \mathcal{T}\}$ be a $\mathcal{T}$-system, g be as in Section 3 such that (4) and (5) hold. Let $\mathcal{M}_0$ denote the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are equivalent to $\mathbb{P}$ and such that M is a $(\mathbb{Q}, \mathbb{F})$-martingale. We then say that a $\mathcal{T}$-system S is a $\mathcal{E}^g$-supermartingale system under some $\mathbb{Q} \in \mathcal{M}_0$ if $S(\tau) \in \mathbf{L}^p(\mathbb{Q})$ for all $\tau \in \mathcal{T}$ and $S(\sigma) \geq \mathcal{E}^{\mathbb{Q},g}_{\sigma,\tau}[S(\tau)]$ for all $(\sigma, \tau) \in \mathcal{T}_2$, where, with $(\sigma, \tau) \in \mathcal{T}_2$ and $\xi \in \mathbf{L}^p(\mathcal{F}_\tau, \mathbb{Q})$, we set $\mathcal{E}^{\mathbb{Q},g}_{\sigma,\tau}[\xi] := Y_\sigma$, with (Y, Z, N) the unique solution of

$$\begin{aligned} Y_t &= \xi + \int_{t\wedge\tau}^{\tau} g_s(Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s \cdot dM_s - \int_{t\wedge\tau}^{\tau} dN_s, \\ &= \xi + \int_{t\wedge\tau}^{\tau} g_s(Y_s, \alpha_s^{-1}\alpha_s Z_s)ds - \int_{t\wedge\tau}^{\tau} \alpha_s Z_s \cdot dW_s - \int_{t\wedge\tau}^{\tau} dN_s, \quad t \leq T, \end{aligned}$$

such that $Y \in \mathbf{S}^p(\mathbb{Q})$, Z belongs to $\mathbf{H}^p(\mathbb{Q})$ and N is a càdlàg $(\mathbb{F}, \mathbb{Q})$-martingale orthogonal to M, and such that

$$\mathbb{E}^{\mathbb{Q}}\left[[N]_T^{p/2}\right] < +\infty.$$

The spaces $\mathbf{S}^p(\mathbb{Q})$ and $\mathbf{H}^p(\mathbb{Q})$ are defined as $\mathbf{S}^p$ and $\mathbf{H}^p$, but with $\mathbb{Q}$ instead of $\mathbb{P}$.

The main result of this section is the following optional type decomposition (see e.g. [20, 26, 21]).

Theorem 4.1 (Optional decomposition)**.** *If for any $\mathbb{Q} \in \mathcal{M}_0$, S is a $\mathcal{E}^{\mathbb{Q},g}$-supermartingale system which is $\mathbb{Q}$-uniformly integrable s.t.* esssup$\{|S(\tau)|,\ \tau \in \mathcal{T}\} \in \mathbf{L}^p(\mathbb{Q})$*, then there exists $(X, Z) \in \mathbf{X}^p_{\ell r} \times \mathbf{H}^p$ such that $S(\sigma) = X_\sigma$ for all $\sigma \in \mathcal{T}$, and*

$$X_\cdot + \int_0^\cdot g_s(X_s, Z_s)ds - \int_0^\cdot Z_s \cdot dM_s \text{ is non-increasing, a.s.}$$

Proof. The existence of the process $X \in \mathbf{X}^p_{\ell r}$ such that $S(\sigma) = X_\sigma$ for all $\sigma \in \mathcal{T}$ follows from Theorem 3.1. Fix then some $\mathbb{Q} \in \mathcal{M}_0$. Using Theorem 3.1, we deduce the existence of $(Z^{\mathbb{Q}}, A^{\mathbb{Q}}) \in \mathbf{H}^p(\mathbb{Q}) \times \mathbf{A}^p(\mathbb{Q})$ and of a $\mathbb{Q}$-martingale $N^{\mathbb{Q}}$ orthogonal to M such that $\mathbb{P}-$a.s. (or $\mathbb{Q}-$a.s.)

$$X_\sigma = X_\tau + \int_\sigma^\tau g_s(X_s, Z^{\mathbb{Q}}_s)ds + A^{\mathbb{Q}}_\tau - A^{\mathbb{Q}}_\sigma - \int_\sigma^\tau Z^{\mathbb{Q}}_s \cdot dM_s - \int_t^T dN^{\mathbb{Q}}_s,$$

for $(\sigma, \tau) \in \mathcal{T}_2$. Recall the definition of I in Corollary 3.2 and that $X + I$ is right-continuous. Then,

$$[X + I, M]_\cdot = \int_0^\cdot \alpha_s^\top \alpha_s Z^{\mathbb{Q}}_s ds, \tag{13}$$

and the family $(Z^{\mathbb{Q}})_{\mathbb{Q}\in\mathcal{M}_0}$ can actually be aggregated into a universal predictable process Z, since α is invertible. Hence, we deduce that $X + \int_0^\cdot g_s(X_s, Z_s)ds$ is actually a supermartingale under any $\mathbb{Q} \in \mathcal{M}_0$, and we can apply the classical optional decomposition theorem ([21, Thm.1]) together with the classical Mertens's decomposition ([35, T2 Lemme]) to deduce the existence of an $\mathbb{F}$-predictable process $\widetilde{Z}$ such that

$$X_\cdot + \int_0^\cdot g_s(X_s, Z_s)ds - \int_0^\cdot \widetilde{Z}_s \cdot dM_s \text{ is non-increasing, } \mathbb{P}-\text{a.s.}$$

Next, using (13), we obtain $Z = \widetilde{Z}\ dt \times d\mathbb{P}$-a.e., which ends the proof. □

4.2 Dual formulation for minimal super-solutions of BSDEs with constraints on the gains process

In this section, we provide an application to the dual representation for BSDEs with constraints. We specialize to the situation where Ω is the canonical space of $\mathbb{R}^d$-valued continuous functions on $[0, T]$, starting at 0, endowed with the Wiener measure $\mathbb{P}$. We let $\mathbb{F}^\circ = (\mathcal{F}^\circ_t)_{t\le T}$ denote the raw filtration of the canonical process $\omega \longmapsto W(\omega) = \omega$, while $\mathbb{F}$ denotes its $\mathbb{P}$-augmentation. We also fix $p' > p > 1$.

We let g be as in Section 3 such that (4) and (5) hold for p' and fix $\xi \in \mathbf{L}^{p'}$. Further, let $\mathcal{O} = (\mathcal{O}_t(\omega))_{(t,\omega)\in[0,T]\times\Omega}$ be a family of non-empty closed convex

random subsets of $\mathbb{R}^d$, such that $\mathcal{O}$ is $\mathbb{F}^\circ$-progressively measurable in the sense of random sets (see e.g. Rockafellar [45]) i.e. $\{(s,\omega) \in [0,t] \times \Omega \ : \mathcal{O}_s(\omega) \cap O \neq \emptyset\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}$ for all $t \in [0,T]$ and all closed $O \subseteq \mathbb{R}^d$. In particular, it admits a Castaing representation, see e.g. [45], which in turn ensures that the support function defined by

$$\delta_t(\omega,\cdot) : u \in \mathbb{R}^d \longmapsto \delta_t(\omega,u) := \sup\{u \cdot z,\ z \in \mathcal{O}_t(\omega)\}$$

is $\mathcal{F}_t^\circ \otimes \mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d \cup \{\infty\})$-measurable, for each $t \in [0,T]$.

We consider solutions $(Y,Z,A) \in \mathbf{X}^p_{\ell r} \times \mathbf{H}^p \times \mathbf{A}^p$ of

$$Y = \xi + \int_\cdot^T g_s(Y_s,Z_s)ds + A_T - A - \int_\cdot^T Z_s \cdot dW_s, \tag{14}$$

under the constraint

$$Z \in \mathcal{O},\ dt \times d\mathbb{P} - \text{a.e.} \tag{15}$$

We say that a solution $(Y,Z,A) \in \mathbf{X}^p_{\ell r} \times \mathbf{H}^p \times \mathbf{A}^p$ of (14)-(15) is minimal if any other solution $(Y',Z',A') \in \mathbf{X}^p_{\ell r} \times \mathbf{H}^p \times \mathbf{A}^p$ is such that $Y_\tau \leq Y'_\tau$ a.s., for any $\tau \in \mathcal{T}$.

The dual characterization relies on the following construction.

Let us also define $\mathcal{U}$ as the class of $\mathbb{R}^d$-valued, progressively measurable processes such that $|\nu| + |\delta(\nu)| \leq c$, $dt \times d\mathbb{P}$-a.e., for some $c \in \mathbb{R}$. Given $\nu \in \mathcal{U}$, we let $\mathbb{P}^\nu$ be the probability measure whose density with respect to $\mathbb{P}$ is given by the Doléans-Dade exponential of $\int_0^\cdot \nu_s \cdot dW_s$, and denote by $W^\nu := W - \int_0^\cdot \nu_s ds$ the corresponding $\mathbb{P}^\nu$-Brownian motion. Then, given $\xi' \in \mathbf{L}^p(\mathcal{F}_\tau, \mathbb{P}^\nu)$, $\tau \in \mathcal{T}$, we define $\mathcal{E}^\nu_{\cdot,\tau}[\xi']$ as the Y^ν-component of the solution $(Y^\nu, Z^\nu) \in \mathbf{S}^p(\mathbb{P}^\nu) \times \mathbf{H}^p(\mathbb{P}^\nu)$ of the BSDE

$$Y^\nu = \xi' + \int_\cdot^\tau \left(g_s(Y^\nu_s, Z^\nu_s) - \delta_s(\nu_s)\right) ds - \int_\cdot^\tau Z^\nu_s \cdot dW^\nu_s.$$

In the above, $\mathbf{S}^p(\mathbb{P}^\nu)$ and $\mathbf{H}^p(\mathbb{P}^\nu)$ are defined as $\mathbf{S}^p$ and $\mathbf{H}^p$ but with respect to $\mathbb{P}^\nu$ in place of $\mathbb{P}$.

Theorem 4.2. *Define*

$$S(\tau) := \operatorname{esssup}\left\{\mathcal{E}^\nu_{\tau,T}[\xi],\ \nu \in \mathcal{U}\right\},\ \ \tau \in \mathcal{T}. \tag{16}$$

Assume that $\operatorname{esssup}\{|S(\tau)|,\ \tau \in \mathcal{T}\} \in \mathbf{L}^{p'}$ *for some* $p' > p$. *Then, there exists* $X \in \mathbf{X}^p_{\ell r}$ *such that* $X_\tau = S(\tau)$ *for all* $\tau \in \mathcal{T}$, *and* $(Z,A) \in \mathbf{H}^p \times \mathbf{A}^p$ *such that* (X,Z,A) *is the minimal solution of* (14)-(15).

Before providing the proof of this result, let us comment it. This formulation is known since [11], however it was proven only under strong assumptions. Although it should essentially be a consequence of the Doob-Meyer decomposition for g-supermatingales, the main difficulty comes from the fact that the family of controls in $\mathcal{U}$ is not uniformly bounded. Hence, (16) is a singular control problem for which the right-continuity of $\tau \longmapsto S(\tau)$ is very difficult to establish, *a priori*, see [2] for a restrictive Markovian setting. This fact prevents us to apply the result of [40]. Theorem 3.1 allows us to bypass this issue and provides a very simple proof.

Proof of Theorem 4.2. Let $(Y, Z, A) \in \mathbf{X}^p_{\ell r} \times \mathbf{H}^p \times \mathbf{A}^p$ be a solution of (14)-(15). Then, for $(\sigma, \tau) \in \mathcal{T}_2$,

$$\begin{aligned} Y_\sigma &= Y_\tau + \int_\sigma^\tau g_s(Y_s, Z_s)ds + A_\tau - A_\sigma - \int_\sigma^\tau Z_s \cdot dW_s \\ &= Y_\tau + \int_\sigma^\tau (g_s(Y_s, Z_s) - \nu_s \cdot Z_s)ds + A_\tau - A_\sigma - \int_\sigma^\tau Z_s \cdot dW^\nu_s \\ &= Y_\tau + \int_\sigma^\tau (g_s(Y_s, Z_s) - \delta_s(\nu_s))ds + A_\tau - A_\sigma + \int_\sigma^\tau (\delta_s(\nu_s) - \nu_s \cdot Z_s)ds \\ &\quad - \int_\sigma^\tau Z_s \cdot dW^\nu_s. \end{aligned}$$

Notice that $Z \in \mathcal{O}$, $dt \times d\mathbb{P}$-a.e. and hence $\delta(\nu) - \nu \cdot Z \geq 0$, $dt \times d\mathbb{P}$-a.e. Then, it follows by comparison that

$$Y_\sigma \geq \mathcal{E}^\nu_{\sigma,T}[\xi], \text{ for all } \nu \in \mathcal{U} \text{ and } \sigma \in \mathcal{T}. \tag{17}$$

Conversely, it is not difficult to deduce from the definition of S that it satisfies a dynamic programming principle:

$$S(\sigma) = \operatorname{esssup} \left\{ \mathcal{E}^\nu_{\sigma,\tau}[S(\tau)],\ \nu \in \mathcal{U} \right\},\ \forall\ (\sigma, \tau) \in \mathcal{T}_2,$$

see e.g. [2]. Taking $\nu \equiv 0$, we deduce that S is a $\mathcal{E}^0$-supermartingale system. The existence of the aggregating process $X \in \mathbf{X}^p_{\ell r}$ then follows from Theorem 3.1. Since it is also a $\mathcal{E}^\nu$-supermartingale system for $\nu \in \mathcal{U}$, the same theorem implies that we can find $(Z^\nu, A^\nu) \in \mathbf{H}^p(\mathbb{P}^\nu) \times \mathbf{A}^p(\mathbb{P}^\nu)$ such that

$$X_\sigma = \xi + \int_\sigma^T (g_s(X_s, Z^\nu_s) - \delta_s(\nu_s))ds + A^\nu_T - A^\nu_\sigma - \int_\sigma^T Z^\nu_s \cdot dW^\nu_s,\ \ \sigma \in \mathcal{T}.$$

Identifying the quadratic variation terms implies that $Z^\nu = Z^0 =: Z$. Thus for all $\nu \in \mathcal{U}$,

$$e(\nu) := \int_0^T (\nu_s Z_s - \delta_s(\nu_s))ds \leq \int_0^T (\nu_s Z_s - \delta(\nu_s))ds + A^\nu_T - A^\nu_0 = A^0_T - A^0_0.$$

We claim that if $N := \{(\omega,t) : Z_t(\omega) \notin \mathcal{O}_{t,\omega}\}$ has a non-zero measure w.r.t $d\mathbb{P} \times dt$, then we can find $\hat{\nu} \in \mathcal{U}$ such that $e(\hat{\nu}) \geq 0$ and $\mathbb{P}[e(\hat{\nu}) > 0] > 0$. However, for any real $\lambda > 0$, one has $\lambda\hat{\nu} \in \mathcal{U}$ and $e(\lambda\hat{\nu}) = \lambda e(\hat{\nu}) \leq A^0_T - A^0_0$, by the above, which is a contradiction since $A^0_T - A^0_0$ is independent of λ. Hence, (15) holds for $Z = Z^0$ and

$$X_\sigma = \xi + \int_\sigma^T g_s(X_s, Z_s)ds + A^0_T - A^0_\sigma - \int_\sigma^T Z_s \cdot dW_s, \;\; \sigma \in \mathcal{T}.$$

By (17), it is clear that (X, Z, A^0) is the minimal solution of (14)-(15).

It remains to prove the above claim. Assume that N has non-zero measure. Then, it follows from [46, Thm. 13.1] that $\{(\omega,t) : \bar{F}_t(\omega) := \sup\{F_t(\omega,u),\ |u| = 1\} \geq 2\iota\}$ has non-zero measure, for some $\iota > 0$, in which

$$F_t(\omega,u) := u \cdot Z_t(\omega) - \delta_t(\omega,u).$$

After possibly passing to another version (in the $dt \times d\mathbb{P}$-sense), we can assume that Z is $\mathbb{F}^\circ$-progressively measurable. Since δ is $\mathcal{F}^\circ_T \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, $(\omega,t,u) \in \Omega \times [0,T] \times \mathbb{R}^d \longmapsto F_t(\omega,u)$ is Borel-measurable. By [1, Prop. 7.50 and Lem. 7.27], we can find a Borel map $(t,\omega) \longmapsto \hat{u}(t,\omega)$ such that $|\hat{u}| = 1$ and $F_t(\omega,\hat{u}(t,\omega)) \geq \bar{F}_t(\omega) - \iota$ $dt \times d\mathbb{P}$-a.e. Then, $\tilde{u}(t,\omega) := \hat{u}(t,\omega)\mathbf{1}_N(\omega,t)$ is Borel and satisfies $F_t(\omega,\hat{u}(t,\omega)) \geq \iota\mathbf{1}_N(\omega,t)$ $dt \times d\mathbb{P}$-a.e. Since $F_t(\omega,\cdot)$ depends on ω only though $\omega_{\cdot\wedge t}$, the same holds for $(t,\omega) \longmapsto \hat{u}(t,\omega_{\cdot\wedge t})$, which is progressively measurable. We conclude by setting

$$\hat{\nu}_t(\omega) := \hat{u}(t,\omega_{\cdot\wedge t})/(1 + |\delta_t(\omega,\hat{u}(t,\omega_{\cdot\wedge t}))|).$$

□

A Appendix

A.1 Doob-Meyer decomposition for right-continuous supermartingales

We complete here the proof of Proposition 3.1, based on a personal communication with Nicole El Karoui.

Proof of Proposition 3.1. Let us start by considering the following reflected BSDEs with lower obstacle X on $[0,\tau]$

$$\begin{cases} Y = Y_\tau + \int_\cdot^\tau g_s(Y_s,Z_s)ds - \int_\cdot^\tau Z_s \cdot dW_s - \int_\cdot^\tau dN_s - \int_\cdot^\tau dK_s, \\ Y \geq X \;\text{ on } [0,\tau], \\ \int_0^\tau (Y_{s-} - X_{s-})dK_s = 0, \end{cases} \tag{18}$$

where N is again a càdlàg martingale orthogonal to W, and K is a càdlàg non-decreasing and predictable process. Since the obstacle X is assumed to be càdlàg, the wellposedness of such an equation is guaranteed by [4, Theorem 3.1].

Let us now prove that we have $Y_t = X_t$, a.s., for any $t \in [0, \tau]$. Let us argue by contradiction and suppose that this equality does not hold. Without loss of generality, we can assume that $Y_0 > X_0$ (otherwise, we replace 0 by the first time when $Y > X + \iota$ for some $\iota > 0$). Fix then some $\varepsilon > 0$ and consider the following stopping time

$$\tau^\varepsilon := \inf \{t \geq 0,\ Y_t \leq X_t + \varepsilon\} \wedge \tau.$$

Since Y is strictly above X before τ^ε, we know that K is identically 0 on $[\![0, \tau^\varepsilon]\!]$, which implies that

$$Y_t = Y_{\tau^\varepsilon} + \int_t^{\tau^\varepsilon} g_s(Y_s, Z_s)ds - \int_t^{\tau^\varepsilon} Z_s \cdot dW_s - \int_t^{\tau^\varepsilon} dN_s.$$

Consider now the following BSDE on $[\![0, \tau^\varepsilon]\!]$

$$y_t = X_{\tau^\varepsilon} + \int_t^{\tau^\varepsilon} g_s(y_s, z_s)ds - \int_t^{\tau^\varepsilon} z_s \cdot dW_s - \int_t^{\tau^\varepsilon} dn_s.$$

By standard a priori estimates (see for instance [4, Rem. 4.1]), we can find a constant $C > 0$ independent of $\varepsilon > 0$ s.t.

$$Y_0 \leq y_0 + C\mathbb{E}\left[|X_{\tau^\varepsilon} - Y_{\tau^\varepsilon}|^p\right]^{\frac{1}{p}} \leq y_0 + C\varepsilon.$$

But remember that X is an $\mathcal{E}^g$-supermartingale, so that we must have $y_0 \leq X_0$. Hence, we have obtained $Y_0 \leq X_0 + C\varepsilon$, which implies a contradiction by arbitrariness of $\varepsilon > 0$.

The uniqueness of the decomposition is then clear by identification of the local martingale part and the finite variation part of a semimartingale. □

A.2 Down-crossing lemma of $\mathcal{E}^g$-supermartingale

We provide here a down-crossing lemma for $\mathcal{E}^g$-supermartingales (defined in Section 3 with g satisfying (4) and (5) for some $p > 1$), which is an extension of Chen and Peng [7, Thm 6] (see also Coquet et al. [10, Prop. 2.6]). For completeness, we will also provide a proof. As in the classical case, $g \equiv 0$, it ensures the existence of right- and left-limits for $\mathcal{E}^g$-supermartingales, see Lemma A.2 below.

For any $m > 0$, we denote by $\mathcal{E}^{\pm m}_{\sigma,\tau}$ the non-linear expectation operator associated to the generator $(t, \omega, y, z) \longmapsto \pm m|z|$ and stopping times $(\sigma, \tau) \in \mathcal{T}_2$. Let $J := (\tau_n)_{n\in\mathbb{N}}$ be a countable family of stopping times taking values in $[0, T]$, which

are ordered, i.e. for any $i, j \in \mathbb{N}$, one has $\tau_i \leq \tau_j$, a.s., or $\tau_i \geq \tau_j$, a.s. Let $a < b$, X be some process and $J_n \subseteq J$ be a finite subset ($J_n = \{0 \leq \tau_1 \leq \cdots \leq \tau_n \leq T\}$). We denote by $D_a^b(X, J_n)$ the number of down-crossing of the process $(X_{\tau_k})_{1 \leq k \leq n}$ from b to a. We then define

$$D_a^b(X, J) := \sup \big\{ D_a^b(X, J_n) \ : J_n \subseteq J, \text{ and } J_n \text{ is a finite set} \big\}.$$

Lemma A.1 (Down-crossing)**.** *Suppose that the generator g satisfies* (4) *with Lipschitz constant L in y and μ in z, and* (5) *with $p > 1$. Let $X \in \mathbf{X}^p$ be a $\mathcal{E}^g$-supermartingale, $J := (\tau_n)_{n \in \mathbb{N}}$ be a countable family of stopping times taking values in $[0, T]$, which are ordered. Then, for all $a < b$,*

$$\mathcal{E}_{0,T}^{-\mu}\Big[D_a^b(X, J)\Big] \leq \frac{e^{LT}}{b-a} \mathcal{E}_{0,T}^{\mu}\Big[e^{LT}(X_0 \wedge b - a) - e^{-LT}(X_T \wedge b - a)^+ + e^{LT}(X_T \wedge b - a)^- + e^{LT} \int_0^T |g_s(a, 0)| ds\Big]. \quad (19)$$

Proof. First, without loss of generality, we can always suppose that $\tau^0 \equiv 0$ and $\tau^1 \equiv T$ belong to J, and also that $b > a = 0$. Indeed, whenever $b > a \neq 0$, we can consider the barrier constants $(0, b-a)$, and the $\mathcal{E}^{\bar{g}}$-supermartingale $X - a$, with generator $\bar{g}_t(y, z) := g_t(y + a, z)$, which reduces the problem to the case $b > a = 0$.

Now, suppose that $J_n = \{\tau_0, \tau_1, \cdots, \tau_n\}$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$. We consider the following BSDE

$$\begin{aligned} y_t^i :&= X_{\tau_i} + \int_t^{\tau_i} g_s(y_s^i, z_s^i) ds - \int_t^{\tau_i} z_s^i \cdot dW_s - \int_t^{\tau_i} dN_s^i \\ &= X_{\tau_i} + \int_t^{\tau_i} \big(g_s(0,0) + \lambda_s^i y_s^i + \eta_s^i z_s^i\big) ds - \int_t^{\tau_i} z_s^i \cdot dW_s - \int_t^{\tau_i} dN_s^i, \end{aligned}$$

where λ^i and η^i are progressively measurable, coming from the linearization of g. In particular, we have $|\lambda^i| \leq L$ and $|\eta^i| \leq \mu$. Let us now consider another linear BSDE

$$\begin{aligned} \bar{y}_t^i = X_{\tau_i} &+ \int_t^{\tau_i} \big(- |g_s(0,0)| + \lambda_s^i \bar{y}_s^i + \eta_s^i \bar{z}_s^i\big) ds - \int_t^{\tau_i} \bar{z}_s^i \cdot dW_s \\ &- \int_t^{\tau_i} d\overline{N}_s^i. \end{aligned} \quad (20)$$

By the comparison principle for BSDEs (see [30, Prop. 4]), and since X is an $\mathcal{E}^g$-supermartingale, it is clear that

$$\bar{y}_{\tau_{i-1}}^i \leq y_{\tau_{i-1}}^i \leq X_{\tau_{i-1}}.$$

Solving the above linear BSDE (20), it follows that

$$\bar{y}^i_{\tau_{i-1}} = \mathbb{E}^{\mathbb{Q}}\left[X_{\tau_i} e^{\int_{\tau_{i-1}}^{\tau_i} \lambda^i_r dr} - \int_{\tau_{i-1}}^{\tau_i} e^{\int_{\tau_{i-1}}^{s} \lambda^i_r dr} |g_s(0,0)| ds \middle| \mathcal{F}_{\tau_{i-1}} \right],$$

where $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^T |\eta_s|^2 ds + \int_0^T \eta_s \cdot dW_s}, \quad \text{with} \quad \eta_s := \sum_{i=1}^n \eta^i_s \mathbf{1}_{[\tau_{i-1},\tau_i)}(s).$$

Let $\lambda_s := \sum_{i=1}^n \lambda^i_s \mathbf{1}_{[\tau_{i-1},\tau_i)}(s)$, it follows that the discrete process $(Y_{\tau_i})_{0\le i\le n}$ defined by

$$Y_{\tau_i} := X_{\tau_i} e^{\int_0^{\tau_i} \lambda_r dr} - \int_0^{\tau_i} e^{\int_0^s \lambda_r dr} |g_s(0,0)| ds$$

is a $\mathbb{Q}$-supermartingale. Define further

$$\overline{Y}_{\tau_i} := Y_{\tau_i} \wedge \left(b e^{LT} - \int_0^{\tau_i} e^{\int_0^s \lambda_r d\langle M\rangle_r} |g_s(0,0)| ds \right),$$

which is clearly also a $\mathbb{Q}$-supermartingale. Let

$$u_t := b e^{\int_0^t \lambda_r dr} - \int_0^t e^{\int_0^s \lambda_r dr} |g_s(0,0)| ds,$$

and

$$l_t := -\int_0^t e^{\int_0^s \lambda_r dr} |g_s(0,0)| ds.$$

Denote then by $D^u_l(Y,J)$ (resp. $D^u_l(\overline{Y},J)$) the number of down-crossing of the process Y (resp. $\overline{Y}$) from the upper boundary u to lower boundary l. It is clear that $D^u_l(Y,J) = D^u_l(\overline{Y},J)$. Notice that l_t is decreasing in t, so that we can apply the classical down-crossing theorem for supermartingales (see e.g. Doob [14, p.446]) to $\overline{Y}$, and obtain that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[D^b_0(X,J) \right] \\
&\le \mathbb{E}^{\mathbb{Q}}\left[D^u_l(\overline{Y},J) \right] \\
&\le \frac{e^{LT}}{b} \mathbb{E}^{\mathbb{Q}}\left[(\overline{Y}_0 - \overline{Y}_T) - (u_T - \overline{Y}_T) \wedge 0 \right] \\
&\le \frac{e^{LT}}{b} \mathbb{E}^{\mathbb{Q}}\left[X_0 \wedge (b e^{LT}) - e^{\int_0^T \lambda_s ds}(X_T \wedge b) + e^{LT} \int_0^T |g_s(0,0)| ds \right].
\end{aligned}$$

Notice that $|\lambda_s| \le L$, $|\eta_s| \le \mu$ and $(X_T \wedge b) = (X_T \wedge b)^+ - (X_T \wedge b)^-$. Therefore, we have proved (19) for the case $b > a = 0$, from which we conclude the proof, by our earlier discussion. $\square$

Lemma A.2. *Let $X \in \mathbf{X}^p$ be a $\mathcal{E}^g$-supermartingale of class (D). Then, it admits right- and left-limits outside an evanescent set.*

Proof. We follow well-known arguments for (classical) supermartingales. Let $(\vartheta_n)_n \subset \mathcal{T}$ be a non-increasing sequence of stopping times. Then, $(X_{\vartheta_n})_{n\geq 1}$ converges a.s. This is an immediate consequence of the down-crossing inequality of Lemma A.1, see e.g. [12, Proof of Thm V-28]. Set $\bar{X} := X/(1+|X|)$. Then, [12, Thm VI-48] implies that, up to an evanescent set, $\bar{X}$ admits right-limits. Since $a/(1+|a|) = b/(1+|b|)$ implies $a = b$, for all $a, b \in \mathbb{R}$, this shows that X admits right-limits, up to an evanescent set. The existence of left-limits is proved similarly by considering non-decreasing sequences of stopping times. □

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