Graphs with Forbidden and Required Vertices

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Sommets interdits et obligatoires

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Une instance du problème est un graphe, un ensemble F de *sommets interdits* et un ensemble R de *sommets obligatoires*. Nous montrons que construire un vertex cover, connexe ou pas, de taille minimale, contenant tous les sommets de R et aucun sommet de F, peut être 2-approché (s'il existe). Nous montrons aussi que décider s'il existe ou pas un ensemble dominant indépendant contenant tout R et aucun sommet de F est $\mathcal{NP}$-complet.

Keywords: Graphs, Vertex Cover, Minimum Independent Dominating Set, Approximation Algorithms.

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1 Introduction

In this paper we consider undirected, unweigted graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is its set of *vertices* and $\mathcal{E}$ its set of *edges*. We also distinguish two subsets of vertices: $F \subseteq \mathcal{V}$ the set of *Forbidden* vertices and $R \subseteq \mathcal{V}$ the set of *Required* vertices.

The generic problem addressed in this paper is to construct a subset $\mathcal{S}$ of vertices of $\mathcal{G}$ having some given properties (for example $\mathcal{S}$ can be a vertex cover of $\mathcal{G}$ or a dominating set of $\mathcal{G}$, etc) and such that no (forbidden) vertex of F is in $\mathcal{S}$ ($F \cap \mathcal{S} = \emptyset$) and every (required) vertex of R must be in $\mathcal{S}$ ($R \subseteq \mathcal{S}$).

An *instance* is given by the triplet $(\mathcal{G}, F, R)$. Once the properties of the solution are defined, the main objectives are of twofold: (1) Decide whether a solution for the instance $(\mathcal{G}, F, R)$ exists; if it is the case (2) try to minimize its size. Of course $R \cap F = \emptyset$ otherwise no solution $\mathcal{S}$ is possible.

Notations. We give here some notations and definitions that will be useful throughout the paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be any undirected, unweighted graph. An edge between vertices u and v of $\mathcal{G}$ is noted uv; in this case u and v are *neighbors*. Let $S \subseteq \mathcal{V}$ a set of vertices of $\mathcal{G}$. We note $\mathcal{G}[S]$ the graph *induced* by S in $\mathcal{G}$: its set of vertices is S and its edges are the ones of $\mathcal{G}$ between vertices of S: $\{uv : u \in S, v \in S, uv \in \mathcal{E}\}$. Set S is a *vertex cover* of $\mathcal{G}$ if each edge $e = uv$ of $\mathcal{G}$ is *covered* by (i.e. contains) at least a vertex of S: $u \in S$ or $v \in S$ or both. S is a *connected vertex cover* of $\mathcal{G}$ if S is a vertex cover of $\mathcal{G}$ and if $\mathcal{G}[S]$ is a connected graph. Set S is an *independent set* of $\mathcal{G}$ if $\mathcal{G}[S]$ contains non edges. S is a *dominating set* of $\mathcal{G}$ if for all $u \in \mathcal{V} - S$, u has at least one neighbor in S: $\forall u \in \mathcal{V} - S, \exists v \in S, uv \in \mathcal{E}$. S is an *independent dominating set* of $\mathcal{G}$ if S is a dominating set of $\mathcal{G}$ and also an independent set of $\mathcal{G}$.

Related works. Recently several papers have been published concerning the construction of structures in graphs under constraints like *conflicts* of edges: if two edges e and e' are in conflict they cannot be part of the same structure (at most one can be in the structure, not both). This has been investigated for paths, trees and Hamiltonian paths in [DPSW11, KLM13a, KLM13b, Sze03]. The same kind of study has been carried out when the conflicts concern pairs of vertices (see [Kov13] for example). All these constraints are conditional: if an edge (or a vertex) is in the structure then the other one cannot be part of it. This can be used in applications to take into account incompatible devices in a network for example.

In this paper, there is no such conditional exclusion. If a vertex is in the forbidden set F then it *cannot be* in the structure. If it is in the required set R then it *must be* in the structure. This changes the nature of the constraints and type of applications.

Motivations for network monitoring. Some monitoring devices or software can be installed on some nodes of a network; these equipped nodes will monitor all their incident links (vertex cover or connected vertex cover) or neighbors (dominating set). However, some nodes are incompatible with the device or software and cannot be equipped (forbidden nodes). On the contrary, some nodes must be in the set because they have particular properties (required nodes).

2 The Vertex Cover Problem in $(\mathcal{G},F,R)$

Given $(\mathcal{G},F,R)$ a Vertex Cover with Forbidden and Required Vertices (*VCwFaRV*) S is a vertex cover of $\mathcal{G}$ such that $F\cap\mathcal{S}=\emptyset$ and $R\subseteq\mathcal{S}$. We first make the following remark, easy to prove.

Remark 1 *Let $(\mathcal{G},F,R)$ be any instance. If $F\cap R\neq\emptyset$ or if F is not an independent set of $\mathcal{G}$ then $(\mathcal{G},F,R)$ has no* VCwFaRV. *Otherwise $\mathcal{S}=\mathcal{V}-F$ is a* VCwFaRV *of $(\mathcal{G},F,R)$.*

This remark induces the fact that deciding whether there is a *VCwFaRV* in $(\mathcal{G},F,R)$ is polynomial. Optimizing the size of a *VCwFaRV* is hard, since the very particular case $R=F=\emptyset$ is the classical $\mathcal{NP}$-*complete* vertex cover problem.

Theorem 1 *When $(\mathcal{G},F,R)$ has at least a* VCwFaRV *then a minimum size one can be 2-approximated with a polynomial approximation algorithm.*

`Proof.` Let $(\mathcal{G},F,R)$ be any instance such that $F\cap R=\emptyset$ and F is an independent set of $\mathcal{G}$. Let $N(F)$ be the set of neighbors of the vertices of F in $\mathcal{G}$: $N(F)=\{v : uv\in\mathcal{E}, u\in F\}$ (as F is an independent set, at most one extremity of any edge can be in F). Any *VCwFaRV* must contain $N(F)$ otherwise edges incident to F cannot be covered. By definition any *VCwFaRV* must also contain R, thus it must contain $R\cup N(F)$.

Let $\mathcal{V}'=\mathcal{V}-(F\cup R\cup N(F))$ and $\mathcal{G}'=\mathcal{G}[\mathcal{V}']$ be the graph induced by the vertices of $\mathcal{V}'$ in $\mathcal{G}$. Let $\mathcal{S}'$ be any vertex cover of $\mathcal{G}'$ obtained by applying any polynomial 2-approximation algorithm (see [DLP13] for recent new ones) and let $\mathcal{S}=\mathcal{S}'\cup R\cup N(F)$ be the final result. By polynomial construction, $\mathcal{S}$ contains R and contains no vertex of F. $\mathcal{S}$ covers all edges of $\mathcal{E}$: edges incident to F are covered by vertices of $N(F)$, edges incident to R are covered by R (a part of the solution $\mathcal{S}$) and all the other edges are covered by $\mathcal{S}$'. $\mathcal{S}$ is then a *VCwFaRV* of $(\mathcal{G},F,R)$. Let us prove now the approximation ratio.

Let $\mathcal{S}^*$ be a minimum size *VCwFaRV* of $(\mathcal{G},F,R)$ and $OPT=|\mathcal{S}^*|$ be its size. Let $T=\mathcal{S}^*-(R\cup N(F))$ (thus we have $OPT=|T|+|R\cup N(F)|$). We prove that T is an optimal vertex cover of $\mathcal{G}'=\mathcal{G}[\mathcal{V}']$. Suppose that it is false: Thus a vertex cover T_2 of $\mathcal{G}'$ with $|T_2|<|T|$ exists; Hence $\mathcal{S}_2=T_2\cup R\cup N(F)$ is a *VCwFaRV* of size $|\mathcal{S}_2|\leq|T_2|+|R\cup N(F)|<|T|+|R\cup N(F)|=|\mathcal{S}^*|$. This is in contradiction with the optimality of $\mathcal{S}^*$.

As $\mathcal{S}'$ is a 2-approximation of an optimal vertex cover of $\mathcal{G}'$ we obtain: $|\mathcal{S}'|\leq 2|T|$. Moreover, as T is only a part of $\mathcal{S}^*$ we also have: $|T|\leq OPT$. Combining all these points we get: $|\mathcal{S}|\leq|\mathcal{S}'|+|R\cup N(F)|\leq 2|T|+|R\cup N(F)|\leq OPT+|T|+|R\cup N(F)|=2OPT$. □

3 The Connected Vertex Cover Problem in $(\mathcal{G},F,R)$

Given $(\mathcal{G},F,R)$ a Connected Vertex Cover with Forbidden and Required Vertices (*CVCwFaRV*) $\mathcal{S}$ is a vertex cover of $\mathcal{G}$ such that $\mathcal{G}[\mathcal{S}]$ is connected, $F\cap\mathcal{S}=\emptyset$ and $R\subseteq\mathcal{S}$. We suppose that $\mathcal{G}$ is connected.

Proposition 1 *Let $\mathcal{G}$ be a connected graph. $(\mathcal{G},F,R)$ contains a* CVCwFaRV *if and only if F is an independent set of $\mathcal{G}$, $F\cap R=\emptyset$ and $\mathcal{G}[\mathcal{V}-F]$ is connected.*

`Proof.` If $(\mathcal{G},F,R)$ contains a *CVCwFaRV* noted $\mathcal{S}$, this means that $F\cap R=\emptyset$ and F is an independent set of $\mathcal{G}$ (otherwise edges of $\mathcal{G}[F]$ could not be covered). Moreover, as $\mathcal{S}$ is a vertex cover of $\mathcal{G}$, any edge uv of $\mathcal{G}$ has at least one extremity in $\mathcal{S}$. Let us consider any vertex $u\in\mathcal{V}-F$: u is in $\mathcal{S}$ or has a neighbor in $\mathcal{S}$ (since $\mathcal{G}$ is connected, u has at least a neighbor) or both. Thus for any $u,v\in\mathcal{V}-F$ there is a path between u and v in $\mathcal{G}[\mathcal{V}-F]$ (through the connected graph $\mathcal{G}[\mathcal{S}]$) which is so connected.

Now suppose that $\mathcal{G}[\mathcal{V}-F]$ is connected, that $F\cap R=\emptyset$ and that F is an independent set of $\mathcal{G}$. This means that $\mathcal{S}=\mathcal{V}-F$ contains no vertex of F and contains all vertices of R. Moreover, let uv be any edge of $\mathcal{G}$. As F is an independent set of $\mathcal{G}$, $u \in \mathcal{S}$ or $v \in \mathcal{S}$ (or both). This means that $\mathcal{S}$ is a vertex cover of $\mathcal{G}$. In conclusion, $\mathcal{S}$ is a *CVCwFaRV* of $(\mathcal{G}, F, R)$. □

By the previous result it is polynomial to decide whether there exists a *CVCwFaRV* of $(\mathcal{G}, F, R)$ and, if it is the case, to construct one (namely $\mathcal{V} - F$). Like in Section 2, if $F = R = \emptyset$, we get the classic connected vertex cover $\mathcal{NP}$*-complete* problem.

Theorem 2 *When* $(\mathcal{G}, F, R)$ *has at least a* CVCwFaRV *then a minimum size one can be* 2*-approximated with a polynomial algorithm.*

`Proof.` Proposition 1 gives conditions that can be checked in polynomial time under which $(\mathcal{G}, F, R)$ contains a *CVCwFaRV*. We suppose now that these conditions are verified.

Let us consider the problem where $F = \emptyset$ (no forbidden vertices). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph and R be the set of required vertices. Let us construct $\mathcal{G}_+ = (\mathcal{V}_+, \mathcal{E}_+)$ the graph $\mathcal{G}$ in which we add for each vertex r of R a new proper neighbor r_+ (of degree 1, that is a *leaf* of $\mathcal{G}$), and the associated new edge rr_+. $\mathcal{G}_+$ is also connected. Now, use any polynomial approximation algorithm of ratio 2 to construct a 2-approximation connected vertex cover of $\mathcal{G}_+$ (the most widely known is based on DFS [Sav82]). Any new edge rr_+ is covered by this solution but if r is not in the solution and r_+ is in the solution, then just reverse: put r into the solution and extract the leaf r_+ from it. Let us note $\mathcal{S}_+$ this new solution that is always a connected vertex cover of $\mathcal{G}_+$, with the same size and that verifies now $R \subseteq \mathcal{S}_+$. We construct the final solution, returned by our algorithm: $\mathcal{S} = \mathcal{S}_+ \cap \mathcal{V}$; $\mathcal{S}$ is then $\mathcal{S}_+$ except the new vertices of $\mathcal{G}_+$ that could be in $\mathcal{S}_+$. The set $\mathcal{S}$ always contains R. Moreover, $\mathcal{S}$ is a vertex cover of $\mathcal{G}$. In addition, $\mathcal{G}[\mathcal{S}]$ is connected since $\mathcal{G}_+[\mathcal{S}_+]$ is connected and the only potential deleted vertices are of degree one, useless to ensure connectivity for the remaining vertices. $\mathcal{S}$ is then a *CVCwFaRV* of $(\mathcal{G}, \emptyset, R)$ that can be constructed with a polynomial algorithm.

Let us study its approximation ratio. Let $\mathcal{S}^*$ be an optimal *CVCwFaRV* of $(\mathcal{G}, R)$ and OPT be its size. Let OPT_+ be the size of an optimal connected vertex cover of $\mathcal{G}_+$. By definition, $\mathcal{S}^*$ contains all R, covers every edge of $\mathcal{G}$ and $\mathcal{G}[\mathcal{S}^*]$ is connected. Hence, $\mathcal{S}^*$ is a connected vertex cover of $\mathcal{G}_+$ and its size is then larger than the optimal one: $OPT_+ \leq OPT$. Now, by construction of $\mathcal{S}$ and with previous properties we have: $|\mathcal{S}| \leq |\mathcal{S}_+| \leq 2OPT_+ \leq 2OPT$.

The last part of this proof is just to take into account a set F of forbidden vertices. We have now an instance $(\mathcal{G}, F, R)$. We suppose that it satisfies the conditions of Proposition 1 that ensures the existence of at least a solution. Consider $N(F)$ the set of neighbors of vertices of F in $\mathcal{G}$. All the vertices of $N(F)$ must be in any solution to cover the edges incident to F. So now consider $R' = R \cup N(F)$ as the new set of required vertices and solve the problem as previously in the graph $\mathcal{G}[\mathcal{V} - F]$ by considering that the set of forbidden vertices is empty. This is a polynomial 2-approximation algorithm for the instance $(\mathcal{G}, F, R)$. □

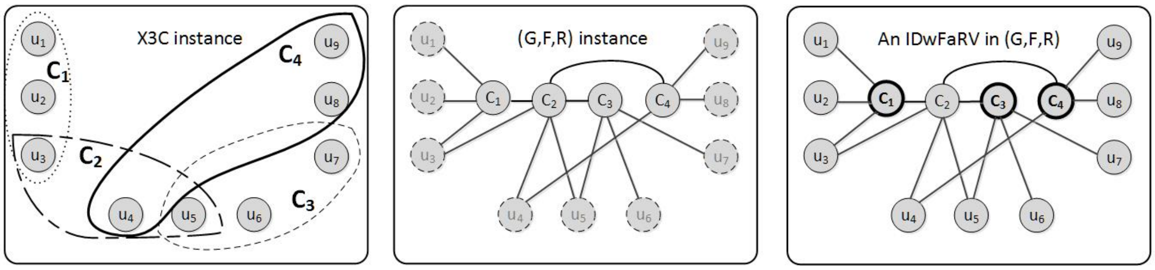
4 Independent Dominating Set in $(\mathcal{G}, F, R)$

An Independent Dominating Set with Forbidden and Required Vertices (*IDwFaRV*) in $(\mathcal{G}, F, R)$ is an independent dominating set of $\mathcal{G}$ containing all vertices of R and no vertices of F. Note that when $R = F = \emptyset$ the problem is polynomial.

Theorem 3 *Given* $(\mathcal{G}, F, R)$*, deciding whether an* IDwFaRV *exists is* $\mathcal{NP}$-complete*, even if* $R = \emptyset$*.*

`Proof.` The problem is clearly in $\mathcal{NP}$. We reduce it to the *X3C* (Exact Cover by Sets of size 3) $\mathcal{NP}$*-complete* problem that we recall now. Let $\mathcal{X} = \{u_1, u_2, \ldots, u_{3q}\}$ a set of $3q$ *elements* and $\mathcal{F}$ a family of k sets $C_i \subseteq \mathcal{X}$ such that $|C_i| = 3$ ($i \in \{1, \ldots, k\}$) and $\cup_{i=1}^{k} C_i = \mathcal{X}$. Given the instance $(\mathcal{X}, \mathcal{F})$, the *X3C* problem is to determine whether an *exact cover* $\mathcal{S}_{\mathcal{F}} \subseteq \mathcal{F}$ of $\mathcal{X}$ exists: $\forall C_i \in \mathcal{S}_{\mathcal{F}}, \forall C_j \in \mathcal{S}_{\mathcal{F}} - \{C_i\}, C_i \cap C_j = \emptyset$ and $\cup_{C_i \in \mathcal{S}_{\mathcal{F}}} C_i = \mathcal{X}$. Let $(\mathcal{X}, \mathcal{F})$ be any *X3C* instance. We construct an instance $(\mathcal{G}, F, R)$ as follows. Each $u_j \in \mathcal{X}$ becomes a vertex (also noted u_j) of $\mathcal{G}$. Each $C_i \in \mathcal{F}$ becomes a vertex (also noted C_i) of $\mathcal{G}$. In $\mathcal{G}$ each vertex C_i is connected to the 3 vertices u_a, u_b, u_c if, in $\mathcal{F}$, $C_i = \{u_a, u_b, u_c\}$. Two distinct vertices C_i and C_j are connected in $\mathcal{G}$ if, in $\mathcal{F}$, the two sets are non-disjoint: $C_i \cap C_j \neq \emptyset$. The set F of forbidden vertices is $\mathcal{X}$:

$F = \mathcal{X}$. The set R of required vertices is empty: $R = \emptyset$. This construction can be done in polynomial time. For any *X3C* instance $(\mathcal{X}, \mathcal{F})$ we note $(\mathcal{G}, F, R)$ the associated instance of our problem.



Suppose first that there is a solution $\mathcal{S}_{\mathcal{F}}$ for the *X3C* problem. Let $\mathcal{S}$ be the set composed of all the corresponding vertices of $\mathcal{S}_{\mathcal{F}}$. $\mathcal{S}$ is an independent set of $\mathcal{G}$: indeed, as $\mathcal{S}_{\mathcal{F}}$ is an exact cover of $\mathcal{X}$, all the sets $C_i \in \mathcal{S}_{\mathcal{F}}$ are pairwise disjoint and then the corresponding vertices of $\mathcal{S}$ are non connected in $\mathcal{G}$. $\mathcal{S}$ is also a dominating set of $\mathcal{G}$ because each vertex u_i is dominated by exactly one $C_j \in \mathcal{S}_{\mathcal{F}}$ (the one containing it) and each C_j that is not in $\mathcal{S}$ is dominated by at least a vertex in $\mathcal{S}$; Indeed, as $\mathcal{S}_{\mathcal{F}}$ is a cover of $\mathcal{X}$, the 3 elements of set C_j are covered by sets of $\mathcal{S}_{\mathcal{F}}$ meaning that there is $C_i \in \mathcal{S}_{\mathcal{F}}$ such that $C_i \cap C_j \neq \emptyset$; vertex $C_j \notin \mathcal{S}$ is then dominated by vertex $C_i \in \mathcal{S}$ in $\mathcal{G}$ by the edge C_iC_j. To finish, as $\mathcal{S}$ contains no forbidden vertices (because $F = \mathcal{X}$), $\mathcal{S}$ is a *IDwFaRV* of $(\mathcal{G}, F, R)$.

Suppose now that there is an *IDwFaRV* noted $\mathcal{S}$, of $(\mathcal{G}, F, R)$. In this case, $\mathcal{S}$ contains no vertex of $\mathcal{X}$ (because these vertices are forbidden, $F = \mathcal{X}$). Hence $\mathcal{S}$ must contain some vertices C_i. Note $\mathcal{S}_{\mathcal{F}}$ the family of sets corresponding to the vertices of $\mathcal{S}$. As $\mathcal{S}$ is a dominating set of $\mathcal{G}$, $\cup_{C_i \in \mathcal{S}_{\mathcal{F}}} C_i = \mathcal{X}$. Moreover, as $\mathcal{S}$ is an independent set of $\mathcal{G}$, for all $C_i \in \mathcal{S}_{\mathcal{F}}$ and all $C_j \in \mathcal{S}_{\mathcal{F}}$, there is no edge connecting them, i.e. $C_i \cap C_j = \emptyset$. Thus $\mathcal{S}_{\mathcal{F}}$ is a solution for the *X3C* problem. □

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