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SOME NEW PROPERTIES FOR THE FAREY DIAGRAMS

Daniel Khoshnoudirad

Abstract

Farey diagrams are associated to the \((m, n)\)-cubes. The \((m, n)\)-cubes are the pieces of discrete planes, occurring in Discrete Mathematics, Number Theory, Combinatorics and Graph Theory. The aim of the paper is to prove an optimization of the triangles-quadrangles conjecture related to the combinatorial structure of Farey diagrams of order \((m, n)\). This conjecture claims that the connected components of the Farey diagram of order \((m, n)\) are either triangles or convex quadrangles. Indeed, we prove a result which is stronger: in the induction, the result is still maintained when we add only one Farey line of order \((m + 1, n)\)(this point makes our proof similar to the proof of McIlroy for the discrete segments in [10]). With that result, we can control better the number of \((m, n)\)-cubes generated by a connected component of the Farey diagram of order \((m, n)\). In fact, the bound is independent of the parameters of the plane. Then we establish the relation with discrete segments, and new applications of this result to control the generation of the \((m, n)\)-cubes in Computer Sciences.

1. INTRODUCTION
The geometrical connected component of the Farey diagrams for discrete segments were studied by McIlroy in [10]. This article of McIlroy gave an algorithm to recognize a discrete segment with a powerful log \( n \) algorithm. A new type of Farey diagrams appear when one studies the discrete planes or \((m,n)\)-cubes ([5]). In 2013, a preprint was posted on a website of archives ([11]) proving that: In the Farey diagram for \((m,n)\)-cubes, the connected component are either triangles or quadrangles.

I now give my proof of this conjecture, and indeed of the generalization of this conjecture as announced in the abstract. We show that at each step of the induction, the structure of Farey facets is preserved, whatever be the order of adding the new Farey lines.

2. PRELIMINARIES

Let \([a, b]\) denote the set \(\{a, \ldots, b\}\) of consecutive integers between \(a\) and \(b\).

**Definition 1** (Farey lines of order \((m,n)\)). [5] A Farey line of order \((m,n)\) is a line whose equation is \(u\alpha + v\beta + w = 0\) with \((u,v,w) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\), and which has at least 2 intersection points with the frontier of \([0,1]^2\). \((u,v,w)\) are the coefficients. \((\alpha, \beta)\) are the variables. Let denote the set of Farey lines of order \((m,n)\) by \(FL(m,n)\).

We remind that \(\zeta\) denotes the well-known Riemann Zeta function.

**Proposition 1.** [5] The number of straight Farey lines verifies

\[
|FL(m,n)| \sim \frac{mn(m+n)}{\zeta(3)} \text{ when } m \text{ and } n \text{ go to infinity.}
\]

**Definition 2** (Farey vertex). [5] A Farey vertex of order \((m,n)\) is the intersection of two Farey lines.

We will denote the set of Farey vertices of order \((m,n)\), obtained as intersection points of Farey lines of order \((m,n)\), by \(FV(m,n)\).

**Proposition 2.** [6] The number of Farey vertices verifies:

\[
\exists K > 0, \forall (m,n) \in \mathbb{N}^2, \quad |FV(m,n)| \leq Km^2n^2(m+n)\ln^2(mn)
\]
Definition 3 (Farey diagrams for the pieces of discrete planes of order \((m, n)\) (or \((m, n)\)-cubes)). \([5]\) The Farey diagram for the \((m, n)\)-cubes of order \((m, n)\) is the diagram composed of Farey connected component of order \((m, n)\) defined by Farey lines. It will be denoted by \(FD(m, n)\).

We recall that \(\lfloor \cdot \rfloor\) denotes the integer part, and \(\langle \cdot \rangle\) denotes the fractional part. If \(a\) and \(b\) are two integers, \(a \wedge b\) denotes the greatest common divisor of \(a\) and \(b\), and \(a \lor b\) denotes the least common multiple. 
\(\varphi\) denotes the Euler’s totient function.

Definition 4 (Farey sequences of order \(n\)). \([2]\) The Farey sequence of order \(n\) is the set 

\[ F_n = \{0\} \cup \left\{ \frac{p}{q} \mid 1 \leq p \leq q \leq n, p \wedge q = 1 \right\} \]

We mention \([2]\) as a forthcoming modern reference work on the Farey sequences. Several standard variants of the notion of Farey diagram are mentioned there.

Definition 5 (Farey edge). A Farey edge of order \((m, n)\) is an edge of the Farey diagram of order \((m, n)\). We denote the set of Farey edges by \(FE(m, n)\).

Definition 6 (Farey graph). \([5]\) The Farey graph of order \((m, n)\) is the graph \(GF(m, n) = (FV(m, n), FE(m, n))\).

Definition 7 (Farey facet). A Farey facet of order \((m, n)\) is a facet of the Farey graph of order \((m, n)\). We will denote the set of Farey facets of order \((m, n)\) by \(FF(m, n)\).

Definition 8 (Farey triangle, Farey quadrangle). A Farey triangle (resp. a Farey quadrangle) of order \((m, n)\) is a triangle (resp. a convex quadrangle) of the Farey diagram of order \((m, n)\).

Definition 9 (\(\tau\)-Farey quadrangle). A \(\tau\)-Farey quadrangle of order \((m, n)\) is a Farey quadrangle of order \((m, n)\), such that two consecutives Farey edges of this quadrangle, are on Farey lines with strict and opposite growing. (for example strictly ascending and strictly descending).

Definition 10 (Strictly crossing a side or a Farey edge). We will say that a Farey line \(D\) strictly crosses through a side \([A, B]\) (or a Farey edge) of a Farey facet, if there exists a unique point \(P\) such that \(D \cap \]A, B[ = \{P\} \).
3. THE PROOF BY DOUBLE INDUCTION

As we did in [5] and [6], we can always suppose that in the equation of a Farey line (of the type: $u\alpha + v\beta + w = 0$, with $(u, v, w) \in [−m, m] \times [−n, n] \times \mathbb{Z}$),

\[ \begin{cases} 
\{ & \text{the non-redundancy condition can be expressed by: } u \wedge v \wedge w = 1. \\
& v \geq 0 \text{ (even consider the equation } (−u)\alpha + (−v)\beta + (−w) = 0). \Rightarrow v \in [0, n] 
\end{cases} \]

In particular, in order to explain the definition of a $\tau$-Farey quadrangle:

**Definition 11** (Details for the definition of a $\tau$-Farey quadrangle). A $\tau$-Farey quadrangle of order $(m, n)$ is a Farey quadrangle of order $(m, n)$, such that two consecutives Farey edges of this quadrangle, are on Farey lines with strict and opposite growing. If the equations of these two Farey lines are $u_1\alpha + v_1\beta + w_1 = 0$
and $u_2\alpha + v_2\beta + w_2 = 0$, we have, for example in the case where these two lines are strictly ascending and strictly descending:

\[
\begin{align*}
-\frac{u_1}{v_1} > 0 &\iff u_1 < 0 \\
-\frac{u_2}{v_2} < 0 &\iff u_2 > 0 
\end{align*}
\]

Proposition 3. For all $n \in \mathbb{N}^*$, the Farey facets of order $(1, n)$, (in $FD(1, n)$), are either triangles or quadrangles. A quadrangle is a $\tau$-Farey quadrangle.

**Proof.** By symmetry, it is equivalent to study $FD(n, 1)$. We prove the property by induction:

1. Initialization: for $n = 1$, there are four triangles.

2. Induction hypothesis: $FD(n, 1)$ has for facets, triangles and convex quadrangles and a quadrangle is a $\tau$-Farey quadrangle.

We fix one of the Farey facets of order $(n, 1)$. Let us denote this facet by $K_{n,1}$. We are going to prove by induction on the number of Farey lines of order $(n+1, 1)$, that the new Farey facets generated inside $K_{n,1}$ are still triangles and convex quadrangles, and that a quadrangle is a $\tau$-Farey quadrangle.

Let us denote $NFL_{n,1}$ the number of Farey lines of order $(n+1, 1)$ passing through $K_{n,1}$. Let us assume that we drew any $k$ ($k < NFL_{n,1}$) Farey lines of order $(n+1, 1)$ passing through $K_{n,1}$, and let us assume that only triangles and convex quadrangles are generated. And a quadrangle is a $\tau$-Farey quadrangle.

We consider a new Farey line of order $(n+1, 1)$ passing through $K_{n,1}$. There are two possibilities of type for this new line:

Let us denote by

\[ D^+_{v,w} : \left\{ (\alpha, \beta) \mid (n+1)\alpha + v\beta + w = 0 \right\} \text{ with } (v, w) \in [0, 1] \times \mathbb{Z} \]

the first type of Farey line of order $(n+1, 1)$. $D^+_{v,w}$ will be denoted by $D^+$ for the sake of simplicity.

\[ D^-_{v,w} : \left\{ (\alpha, \beta) \mid -(n+1)\alpha + v\beta + w = 0 \right\} \text{ with } (v, w) \in [0, 1] \times \mathbb{Z} \]

the second type of Farey line of order $(n+1, 1)$. $D^-_{v,w}$ will be denoted by $D^-$ for the sake of simplicity. We also note that we took $v \in [0, 1]$ for $D^+$ (resp.
$D^-$), because if $v \in \mathbb{[-1,0]}$, we multiply the equation by $-1$ to obtain an equation of the type $D^-$ (resp. $D^+$).

We consider each one of the subpolygons inside $K_{n,1}$ crossed through by $D^+$. For each subpolygons of $K_{n,1}$, which is strictly crossed through by $D^+$, the only case being able to produce a polygon with strictly more than 4 sides is the case of a quadrangle, with the Farey line $D^+$ (or $D^-$) strictly crossing through two consecutive sides (A triangle which is crossed by a line can generate polygons with at most 4 sides). We prove that this case does not occur. By contradiction, if it could occur, both $u_1$ associated to both consecutive Farey edges, would verify $u_1 \leq 0$ according to the lemma 1. However, in a $\tau$-Farey quadrangle, one of the $u_1$ verifies $u_1 > 0$ by definition. It gives a contradiction.

- Either $D^+$ strictly crosses two opposite sides. In this case, it gives two $\tau$-Farey quadrangles.
- or $D^+$ crosses a vertex and strictly crosses a side. It produces a triangle and a $\tau$-Farey quadrangle.
- or $D^+$ crosses two opposite vertices. It produces two triangles.

•

Proposition 4. For all $n \in \mathbb{N^*}$, \[ \text{for all } m \in \mathbb{N^*}, \text{ the Farey facets of order } (m,n), \]

(in $FD(m,n)$) are either triangles or convex quadrangles. A quadrangle is a $\tau$-Farey quadrangle.

Proof. For all $n \in \mathbb{N^*}$, we prove by induction that

\[ \forall m \in \mathbb{N^*}, \text{ the Farey diagram of order } (m,n) \text{ has for facets, triangles and quadrangles.} \]

1. Initialization: for $m = 1$, we have to prove that $FD(1,n)$ verifies the property.

It is the object of the previous proposition.

2. Induction hypothesis: $FD(m,n)$ has for facets, triangles and convex quadrangles and a quadrangle is a $\tau$-Farey quadrangle.

We fix one of the Farey facets of order $(m,n)$. Let us denote this facet by $K_{m,n}$. We are going to prove by induction on the number of Farey lines of order $(n+1,m)$, that the new Farey facets generated inside $K_{m,n}$ are still triangles and convex quadrangles, and that a quadrangle is a $\tau$-Farey quadrangle.

Let us denote $NFL_{m,n}$ the number of Farey lines of order $(m+1,n)$ passing through $K_{m,n}$. Let us assume that we drew any $k$ ($k < NFL_{m,n}$) Farey lines of order $(m+1,n)$ passing through $K_{m,n}$, and let us assume that only triangles and convex quadrangles are generated. And a quadrangle is a $\tau$-Farey quadrangle.
We consider a new Farey line of order \((m + 1, n)\) passing through \(K_{m,n}\). There is two possibilities of type for this new line:

Let us denote by

\[
D^+_{v,w} : \{(\alpha, \beta) \mid (m + 1)\alpha + v\beta + w = 0\} \text{ with } (v, w) \in \mathbb{Z}_{0,n} \times \mathbb{Z}
\]

the first type of Farey line of order \((m + 1, n)\). \(D^+_{v,w}\) will be denoted by \(D^+\) for the sake of simplicity.

\[
D^-_{v,w} : \{(\alpha, \beta) \mid -(m + 1)\alpha + v\beta + w = 0\} \text{ with } (v, w) \in \mathbb{Z}_{0,n} \times \mathbb{Z}
\]

the second type of Farey line of order \((m + 1, n)\). \(D^-_{v,w}\) will be denoted by \(D^-\) for the sake of simplicity. We also note that we took \(v \in [0, n]\) for \(D^+\) (resp. \(D^-\)), because if \(v \in [-n, 0]\), we multiply the equation by \(-1\) to obtain an equation of the type \(D^-\) (resp. \(D^+\)).

We consider each one of the subpolygons inside \(K_{m,n}\) crossed through by \(D^+\). For each subpolygons of \(K_{m,n}\), which is strictly crossed through by \(D^+\), the only case being able to produce a polygon with strictly more than 4 sides is the case of a quadrangle, with the Farey line \(D^+\) (or \(D^-\)) strictly crossing through two consecutive sides. (A triangle which is crossed by a line can generate polygons with at most 4 sides). We prove that this case does not occur. By contradiction, if it could occur, both \(u_1\) associated to both consecutive Farey edges, would verify \(u_1 \leq 0\) according to the lemma 1. However, in a \(\tau\)-Farey quadrangle, one of the \(u_1\) verifies \(u_1 > 0\) by definition. It gives a contradiction.

- Either \(D^+\) strictly crosses two opposite sides. In this case, it gives two \(\tau\)-Farey quadrangles.
- or \(D^+\) crosses a vertex and strictly crosses a side. It produces a triangle and a \(\tau\)-Farey quadrangle.
- or \(D^+\) crosses two opposite vertices. It produces two triangles.

It yields the result. \(\square\)

**Lemma 1.** • If \(D^+\) strictly crosses a Farey edge of order \((m + 1, n)\), and by considering the equation

\[
D_1 : u_1\alpha + v_1\beta + w_1 = 0
\]

of the line of the crossed Farey edge (with \((u_1, v_1, w_1) \in [(m + 1), m + 1] \times [0, n] \times \mathbb{Z}\)), we have \(u_1 \leq 0\)
If \( D^- \) strictly crosses a Farey edge of order \((m + 1, n)\), and by considering the equation
\[
D_1 : u_1 \alpha + v_1 \beta + w_1 = 0
\]
of the line of the crossed Farey edge (with \((u_1, v_1, w_1) \in [-m+1, m+1] \times [0, n] \times \mathbb{Z})\), we have \( u_1 \geq 0 \)

**Proof.** If we consider
\[
D^+ - D_1 : \{(\alpha, \beta) \mid ((m+1) - u_1)\alpha + (v - v_1)\beta + (w - w_1) = 0\}
\]
By contradiction, 
\[u_1 > 0 \Rightarrow D^+ - D_1 \in FL(m, n).\]
However, we know that \( D^+ - D_1 \) is in the pencil of lines generated by \( D^+ \) and \( D_1 \).
So, it strictly crosses \( K_{m,n} \), what is impossible.
The idea of the proof is the same for \( D^- \). \qed

4. APPLICATIONS IN INFORMATION TECHNOLOGY AND DISCRETE GEOMETRY

McIlroy [10] used the Farey diagrams, to understand the combinatorics of discrete segments. And he could give a characterization of a discrete segment by what he called preimage [10]. The proof widely uses the Farey sequences. In our paper, the purpose is to study further the properties of Farey diagrams for the pieces of discrete planes. After a new result on Farey lines was given in [5], and on Farey vertices ([6]), we now focused on Farey facets, and we show that the structure of Farey facets is very simple: triangles or quadrangles. So it gives new informations on the facets of the Farey Graph. In addition to it, on the algorithmic point of view, this property controls the random generation of discrete pieces of planes: for a Farey facet, the number of \((m,n)\)-cubes generated, can not exceed \(4mn\).

But we have more than this result: we shew that, whatever be the order of adding the new Farey lines in the induction for switching from the Farey diagram of order \((m, n)\) to the Farey diagram of order \((m + 1, n)\), the result is still preserved, which is a generalization.

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REFERENCES


