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A Remark on the Stability of Peakons for the Degasperis-Procesi Equation

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Abstract

In this paper, we present a new argument (see Lemma 3.4) that allows us to simplify the proof of stability of peakons established in Lin and Liu (2009) (Theorem 1.1).

1 Introduction

In this paper, we consider the Degasperis-Procesi equation (DP)

\[ u_t - u_{txx} + 4u u_x = 3u_x u_{xx} + u u_{xxx}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \]  

(1.1)

with \( u(0) = u_0 \in L^2(\mathbb{R}) \) and \((1- \partial_x^2) u_0 \in \mathcal{M}^+ (\mathbb{R}) \).

The DP equation is completely integrable (see [3]) and has been proved to be physically relevant for water waves (see [1]). It possesses, among others, the following conservation laws

\[ E(u) = \int_{\mathbb{R}} y u = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2), \quad F(u) = \int_{\mathbb{R}} u^3 = \int_{\mathbb{R}} (-v_{xx}^3 + 12v_x v_{xx}^2 - 48v^2 v_x + 64v^3), \]  

(1.2)

where \( y = (1- \partial_x^2) u \) and \( v = (4- \partial_x^2)^{-1} u \). One can notice that the conservation law \( E(\cdot) \) is equivalent to \( \| \cdot \|^2_{L^2(\mathbb{R})} \). Indeed, using integration by parts (we assume that \( u(\pm \infty) = v(\pm \infty) = v_x (\pm \infty) = 0 \)), it holds

\[ \|u\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} (16v^2 + 8v_x^2 + v_{xx}^2) \sim E(u). \]  

(1.3)

In the sequel we will denote

\[ \|u\|_{\mathcal{H}} = \sqrt{E(u)}. \]  

(1.4)

Applying \((1 - \partial_x^2)^{-1}(\cdot) \) to (1.1), we obtain

\[ u_t + \frac{1}{2} \partial_x u_x^2 + \frac{3}{2} (1 - \partial_x^2)^{-1} \partial_x u_x^2 = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \]  

(1.5)

In this form, the DP equation admits explicit solitary waves called peakons (see [3]) that are defined by

\[ u(t, x) = \varphi_c(x - ct) = c \varphi(x - ct) = c e^{-|x-ct|}, \quad c \in \mathbb{R}^*, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \]  

(1.6)
Our goal is to simplify the proof given in [7] of the stability of a single peakon for the DP equation. Recall that the proof of the stability for the Camassa-Holm equation (CH) in [2] follows from two integral relations between two conservation laws of CH, \( \max x u \) and functions related to \( u \). In [7] the proof is more complicated, since all the local maxima and minima of \( v = (4 - \partial_x^2)^{-1}u \) are involved in the relations. In this paper we present a simplification of this proof, where only the maximum of \( v \) is involved in the relations. Our proof is thus closer to the proof for CH in [2]. The main idea is the following: since \( u \) is \( L^2 \)-close to the peakon \( \varphi_c(\cdot - \xi) \), for some \( \xi \in \mathbb{R} \), and \( (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R}) \), it is easy to check that \( u \) is actually \( C^0 \)-close to the peakon, and thus \( v \) is \( C^2 \)-close to the smooth-peakon:

\[
\rho_c(x - \xi) = (4 - \partial_x^2)^{-1} \varphi_c(x - \xi) = \frac{c}{3} e^{-|x - \xi|} - \frac{c}{6} e^{-2|x - \xi|}, \quad x \in \mathbb{R}. \tag{1.7}
\]

First, since \( \rho_c, \rho'_c \) and \( \rho''_c \) are very small with respect to the amplitude \( c \) outside of the interval \( \Theta_0 = [-6.7, 6.7] \), we can restrict ourself to study \( v \) on \( \Theta_\xi = [\xi - 6.7, \xi + 6.7] \). Now we observe that \( \rho''_c \) has strictly negative values in the interval \( \Theta_0 = [-\ln \sqrt{2}, \ln \sqrt{2}] \), with \( \rho'_c \) strictly positive on \([-6.7, -\ln \sqrt{2}] \) and \( \rho'_c \) strictly negative on \([\ln \sqrt{2}, 6.7] \). This forces \( v_x \) to change sign only one time on \( \Theta_\xi \), and thus \( v \) has only one local extremum (which is a maximum) on \( \Theta_\xi \). This fact will considerably simplify the proof of the stability.

2 Preliminaries

In this section, we briefly recall the global well-posedness results for the Cauchy problem of the DP equation (see [5] and [8]), and its consequences. For \( I \) a finite or infinite time interval of \( \mathbb{R}_+ \), we denote by \( \mathcal{X}(I) \) the function space \(^1\)

\[
\mathcal{X}(I) = \left\{ u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})), \; u_x \in L^\infty(I; BV(\mathbb{R})) \right\}. \tag{2.1}
\]

**Theorem 2.1** (Global Weak Solution; See [5] and [8]). Assume that \( u_0 \in L^2(\mathbb{R}) \) with \( y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \). Then the DP equation has a unique global weak solution \( u \in \mathcal{X}(\mathbb{R}_+) \) such that

\[
y(t, \cdot) = (1 - \partial^2_x)u(t, \cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+ \tag{2.2}
\]

and

\[
|u_x(t, x)| \leq u(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.3}
\]

Moreover \( E(\cdot) \) and \( F(\cdot) \) are conserved by the flow.

**Remark 2.1** (Control of \( L^\infty \) Norm by \( L^2 \) Norm). (2.3) and the well-known Sobolev embedding of \( H^1(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \) lead to

\[
\|u\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})}. \tag{2.4}
\]

3 Stability of peakons

In this section, we present our simplification of the proof of stability of peakons for the DP equation.

**Theorem 3.1** (Stability of Peakons). Let \( u \in \mathcal{X}([0, T[) \), with \( 0 < T \leq +\infty \), be a solution of the DP equation and \( \varphi_c \) be the peakon defined in (1.6), traveling to the right at the speed \( c > 0 \). There exist \( C > 0 \) and \( \varepsilon_0 > 0 \) only depending on the speed \( c \), such that if

\[
y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{3.1}
\]

\(^1\)\( W^{1,1}(\mathbb{R}) \) is the space of \( L^1(\mathbb{R}) \) functions with derivatives in \( L^1(\mathbb{R}) \) and \( BV(\mathbb{R}) \) is the space of function with bounded variation.
and
\[ \|u_0 - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2, \quad \text{with} \quad 0 < \varepsilon < \varepsilon_0, \] (3.2)

then
\[ \|u(t, \cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0, T], \] (3.3)

where \( \xi(t) \in \mathbb{R} \) is the only point where the function \( v(t, \cdot) = (4 - \partial_x^2)^{-1}u(t, \cdot) \) attains its maximum.

We first recall that \( E(u) \sim E(\varphi_c) \) and \( F(u) \sim F(\varphi_c) \) in \( \mathbb{R} \), if \( u \sim \varphi_c \) in \( L^2(\mathbb{R}) \), with \( y \in \mathcal{M}^+(\mathbb{R}) \) (see for instance [7] or [6]).

**Lemma 3.1** (Control of Distances Between Energies; See [6]). Let \( u \in L^2(\mathbb{R}) \) with \( y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R}) \). If \( \|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2 \), then
\[ |E(u) - E(\varphi_c)| \leq O(\varepsilon^2) \] (3.4)

and
\[ |F(u) - F(\varphi_c)| \leq O(\varepsilon^2), \] (3.5)

where \( O(\cdot) \) only depends on the speed \( c \).

To prove Theorem 3.1, by the conservation of \( E(\cdot) \), \( F(\cdot) \) and the continuity of the map \( t \mapsto u(t) \) from \([0, T]\) to \( \mathcal{H} \) (since \( \mathcal{H} \simeq L^2 \)), it suffices to prove that for any function \( u \in L^2(\mathbb{R}) \) satisfying \( y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R}) \), (3.4) and (3.5), if
\[ \inf_{z \in \mathbb{R}} \|u - \varphi_c(\cdot - z)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \] (3.6)

then
\[ \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \] (3.7)

where \( \xi \in \mathbb{R} \) is the only point of maximum of \( v \).

Let us present some important properties of smooth-peakons, defined in (1.7), which will play a crucial role in the proof of Theorem 3.1. The smooth-peakon \( \rho_c \) belongs to \( H^3(\mathbb{R}) \rightarrow C^2(\mathbb{R}) \) (by the Sobolev embedding) since \( \varphi_c \) belongs to \( H^1(\mathbb{R}) \) (defined in (1.6)). It is a positive even function, which admits a single maximum \( c/6 \) at point 0, and decays at infinity to 0 (see Fig. 1a). Its derivative \( \rho'_c \) belongs to \( H^2(\mathbb{R}) \rightarrow C^1(\mathbb{R}) \), it is an odd function, which vanishes only at the origin and has negative values on \([0, +\infty[\). It admits a a single minimum \(-c/12\) at point \( \ln2 \) and tends at infinity to 0 (see Fig. 1b). Its second derivative \( \rho''_c \) belongs to \( H^1(\mathbb{R}) \rightarrow C^0(\mathbb{R}) \), it is an even function, which vanishes at \( \pm\ln2 \), takes positive values on \([-\infty, -\ln2[\cup\ln2, +\infty[\) and negative values on \([-\ln2, \ln2[\). It admits a single minimum \(-c/3\) at point 0 and two maxima \(c/24\) at points \( \pm\ln4 \), and decays at infinity to 0 (see Fig. 1c).

Next, we will need the following estimates.

**Lemma 3.2** (\( C^0, C^1 \) and \( C^2 \) Approximations). Let \( u \in L^2(\mathbb{R}) \) with \( y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R}) \). If \( \|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^{1/4} \), then
\[ \|u - \varphi_c\|_{C^0(\mathbb{R})} + \|v - \rho_c\|_{C^2(\mathbb{R})} \leq O(\varepsilon^{1/8}) \] (3.8)

and
\[ \|v - \rho_c\|_{C^1(\mathbb{R})} \leq O(\varepsilon^{1/4}). \] (3.9)

**Proof.** Let us begin with the second estimate. From the definition of \( E(\cdot) \) and \( \mathcal{H} \) (see respectively (1.2) and (1.4)), one can see that \( \|u\|_{\mathcal{H}} \) is equivalent to \( \|v\|_{H^2(\mathbb{R})} \), since \( \|v\|_{H^2(\mathbb{R})} \leq \|u\|_{\mathcal{H}} \leq 5\|v\|_{H^2(\mathbb{R})} \). Then, assumption \( u \) is \( \mathcal{H} \)-close to \( \varphi_c \) implies that \( v \) is \( H^2 \)-close to \( \rho_c \). Now, using the Sobolev embedding of \( H^2(\mathbb{R}) \) into \( C^1(\mathbb{R}) \), we deduce (3.9).

For the first estimate, note that the assumption \( y = (1 - \partial_x^2)u \geq 0 \) implies that \( u = (1 - \partial_x^2)^{-1}y \geq 0 \) and satisfies \( |\varphi'_c| = \varphi_c \) on \( \mathbb{R} \) (see (2.3)). Then, applying triangular inequality, and using that \( |\varphi'_c| = \varphi_c \) on
$\mathbb{R}$ and (2.4), we have
\[
\|u - \varphi_c\|_{H^1(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} + \|\varphi_c\|_{H^1(\mathbb{R})}
\leq 2\|u\|_{L^2(\mathbb{R})} + 2\|\varphi_c\|_{L^2(\mathbb{R})}
\leq 2\|u - \varphi_c\|_{L^2(\mathbb{R})} + 4\|\varphi_c\|_{L^2(\mathbb{R})}
\leq O(\varepsilon^{1/4}) + O(1),
\]
where $\|\varphi_c\|_{L^2(\mathbb{R})} = c$. Therefore, applying the Gagliardo-Nirenberg inequality and using that $\|u - \varphi_c\|_H \leq \varepsilon^{1/4}$ (with $H \simeq L^2$), we obtain
\[
\|u - \varphi_c\|_{C^0(\mathbb{R})} \leq \|u - \varphi_c\|_{L^2(\mathbb{R})}^{1/2} \|u - \varphi_c\|_{H^1(\mathbb{R})}^{1/2}
\leq O(\varepsilon^{1/8}) \left( O(\varepsilon^{1/8}) + O(1) \right)
\leq O(\varepsilon^{1/8}).
\]

Finally to estimate the second term of the left-hand side of (3.8), we first notice that the continuity of $(4 - \partial_x^2)^{-1} (\cdot)$ from $H^s(\mathbb{R})$ to $H^{s+2}(\mathbb{R})$ and the above estimates ensure that $\|v - \rho_c\|_{H^1} = O(1)$ and $\|v - \rho_c\|_{H^2} = O(\varepsilon^{1/4})$. These last estimates combined with the Gagliardo-Nirenberg inequality yield the result as above. $\square$

The following lemma specifies the distance to minimize for stability.

**Lemma 3.3** (Quadratic Identity; See [7]). For any $u \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, it holds
\[
E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - \xi)\|_H^2 + 4c \left( v(\xi) - \frac{c}{6} \right),
\tag{3.10}
\]
where $v = (4 - \partial_x^2)^{-1} u$ and $\rho_c(0) = c/6$.

**Proof.** We follow the idea of Constantin and Strauss with the CH equation (see [2], Lemma 1). We compute
\[
E(u - \varphi_c(\cdot - \xi)) = E(u) + E(\varphi_c) - 2 \langle (1 - \partial_x^2)\varphi_c(\cdot - \xi), (4 - \partial_x^2)^{-1} u \rangle_{H^{-1},H^1}
\]
\[
= E(u) + E(\varphi_c) - 2 \langle (1 - \partial_x^2)\varphi_c(\cdot - \xi), v \rangle_{H^{-1},H^1},
\tag{3.11}
\]
where $\langle \cdot, \cdot \rangle_{H^{-1},H^1}$ denotes the duality bracket $H^{-1}(\mathbb{R}), H^1(\mathbb{R})$. Now, using the definition of $\varphi_c(\cdot - \xi)$ and integration by parts, we have
\[
\langle (1 - \partial_x^2)\varphi_c(\cdot - \xi), v \rangle_{H^{-1},H^1} = \int_\mathbb{R} v\varphi_c(\cdot - \xi) + \int_\mathbb{R} v_x\varphi'_c(\cdot - \xi)
\]
\[
= \int_\mathbb{R} v\varphi_c(\cdot - \xi) + \int_\mathbb{R} v_x\varphi_c(\cdot - \xi) - \int_\mathbb{R} v_x\varphi_c(\cdot - \xi)
\]
\[
= 2cv(\xi). \tag{3.12}
\]
Recalling that the energy of peakons is given by
\[
E(\varphi_c) = \langle (1 - \partial_x^2)\varphi_c, (4 - \partial_x^2)^{-1}\varphi_c \rangle_{H^{-1},H^1} = \int_\mathbb{R} \rho_c\varphi_c + \int_\mathbb{R} \rho'_c\varphi'_c
\]
\[
= \int_\mathbb{R} \rho_c\varphi_c + \int_\mathbb{R} (\varphi'_c)^2 - \int_\mathbb{R} \rho'_c\varphi_c = 2c\rho_c(0) = \frac{c^2}{3}, \tag{3.13}
\]
we obtain the lemma. $\square$

Now we will study carefully the local extrema of $v = (4 - \partial_x^2)^{-1} u$. Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, and assume that (3.6) holds for some $z \in \mathbb{R}$. We consider the interval in which the mass of
smooth-peaks is concentrated, and the interval in which the mass of second derivative of smooth-peaks is strictly negative. In the sequel of this paper, the notation $\alpha \simeq \beta$ means that $0.9 \times \beta \leq \alpha \leq 1.1 \times \beta$. We set, for any $z \in \mathbb{R}$,

$$\Theta_z = [z - 6.7, z + 6.7],$$

where $6.7 \simeq \ln\left(\frac{20}{20 - \sqrt{399}}\right)$, \hspace{1cm} (3.14)

and

$$\mathcal{V}_z = \left[z - \ln\sqrt{2}, z + \ln\sqrt{2}\right].$$

(3.15)

One can clearly see that $\mathcal{V}_0$ is a subset of $\Theta_0$ (since $20/(20 - \sqrt{399}) > \sqrt{2}$). We chose the values $\pm 6.7$ such that $\rho_c(\pm 6.7) \simeq c/2400 \simeq 4.1 \times 10^{-4}c$ as in [6]. Also, we have $\rho'_c(-6.7) = -\rho'_c(6.7) \simeq 4.1 \times 10^{-4}c$ and $\rho''_c(\pm 6.7) \simeq 4.1 \times 10^{-4}c$. Then $\rho_c$, $\rho'_c$ and $\rho''_c$ are very small with respect to the amplitude $c$ on $\mathbb{R} \setminus \Theta_0$.

We claim the following result.

**Lemma 3.4** (Uniqueness of the Local Maximum). Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in M^+(\mathbb{R})$, that satisfies (3.6) for some $z \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ only depending on the speed $c$, such that if $0 < \varepsilon < \varepsilon_0$, then the only solution of the Cauchy problem

$$u(x, t) = \frac{1}{4}e^{-|x|} - \frac{1}{4}e^{-2|x|}$$

profile.
then the function \( v = (4 - \partial_z^2)^{-1} u \) admits a unique local extremum on \( \Theta_z \). This extremum is a maximum, and it holds

\[
v(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta_z, \quad (3.16)
\]

\[
u(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta_z. \quad (3.17)
\]

**Proof.** The key is to study the impact of the assumption \( y \in \mathcal{M}^+(\mathbb{R}) \) on \( v \). First, let us show that \(|v_x| \leq 2v\) on \( \mathbb{R} \). We recall that from the assumption \( y \geq 0 \), we have \( u \geq 0 \) and \( v \geq 0 \) on \( \mathbb{R} \). According to the definition of \( v \), we have for all \( x \in \mathbb{R} \),

\[
v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^{x} e^{2x'} u(x') dx' + \frac{e^{2x}}{4} \int_{x}^{+\infty} e^{-2x'} u(x') dx'
\]

and

\[
v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^{x} e^{2x'} u(x') dx' + \frac{e^{2x}}{2} \int_{x}^{+\infty} e^{-2x'} u(x') dx',
\]

which yields

\[
|v_x(x)| \leq 2v(x), \quad \forall x \in \mathbb{R}. \quad (3.18)
\]

Second, let us show that \( u \leq 6v \) on \( \mathbb{R} \). Using the Fourier transform, one can check that

\[
(1 - \partial_z^2)^{-1}(4 - \partial_z^2)^{-1} \theta = F^{-1}\left[ \frac{1}{3(1 + \omega^2)} - \frac{1}{3(4 + \omega^2)} \right] \theta = \frac{1}{3}(1 - \partial_z^2)^{-1} \theta - \frac{1}{3}(4 - \partial_z^2)^{-1} \theta, \quad (3.19)
\]

and one can rewrite \( v \) as

\[
v = (4 - \partial_z^2)^{-1}(1 - \partial_z^2)^{-1} y = \frac{1}{3}(1 - \partial_z^2)^{-1} y - \frac{1}{3}(4 - \partial_z^2)^{-1} y. \quad (3.20)
\]

Then for all \( x \in \mathbb{R} \),

\[
u(x) - 6v(x) = -(1 - \partial_z^2)^{-1} y(x) + 2(4 - \partial_z^2)^{-1} y(x)
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}} e^{-|x-x'|} y(x') dx' + \frac{1}{2} \int_{\mathbb{R}} e^{-2|x-x'|} y(x') dx' \leq 0,
\]

\[
(3.21)
\]

since \( e^{-2|x|} \leq e^{-|x|} \) on \( \mathbb{R} \).

We are now ready to prove the uniqueness of local maxima in \( \Theta_z \). Let us first study the sign of \( v_{xx} \) on \( \mathcal{V}_z \). One can easily check that for all \( x \in \mathcal{V}_0 \),

\[
\rho_x''(x) \leq \frac{\sqrt{2} - 2}{6} c. \quad (3.22)
\]

Then, combining (3.8) and (3.22), taking \( 0 < \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 \ll 1 \), we have for all \( x \in \mathcal{V}_z \),

\[
v_{xx}(x) \leq \frac{\sqrt{2} - 2}{6} c + O(\varepsilon^{1/4}) \leq \frac{\sqrt{2} - 2}{600} c < 0,
\]

which implies that \( v_z \) is strictly decreasing on \( \mathcal{V}_z \). Let us study the sign of \( v_z \) on \( \Theta_z \setminus \mathcal{V}_z \). One can easily check that

\[
\rho_z''(-\ln \sqrt{2}) = \frac{\sqrt{2} - 1}{6} c \quad \text{and} \quad \rho_z''(\ln \sqrt{2}) = -\frac{\sqrt{2} - 1}{6} c.
\]

\[
(3.23)
\]
and that \( \rho'_c(x) \geq 10^{-4}c \) for all \( x \in [-6.7, -\ln 2] \). Then using (3.9) and taking \( 0 < \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 < 1 \), we have \( v_c(x) \geq 4 \times 10^{-5}c > 0 \) for all \( x \in [z - 6.7, z - \ln 2] \). Proceeding in the same way, we obtain \( v(x) \leq -4 \times 10^{-5}c < 0 \) for all \( x \in [z + \ln 2, z + 6.7] \). Since \( v_x \) is strictly decreasing on \( V_z \) and changes sign, \( v_x \) vanishes once on \( V_z \) and thus on \( \Theta_z \). Hence, \( v \) admits a single local extremum on \( \Theta_z \), which is a maximum since \( v_{xx} < 0 \) on \( V_z \).

Now, using that \( \rho_c \) is increasing on \( \mathbb{R}^- \), (3.9) and taking \( 0 < \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 < 1 \), it holds for all \( x \in ]-\infty, z - 6.7[ \),

\[
v(x) = \rho_c(x - z) + O(\varepsilon^{1/4}) \leq \frac{c}{2400} + O(\varepsilon^{1/4}) \leq \frac{c}{300}.
\]

Proceeding in the same way for \( x \in ]z + 6.7, +\infty[ \), we obtain (3.16).

Combining (3.8), (3.21) and proceeding as for the estimate (3.16), we get (3.17). Note that \( \varphi_c(\pm 6.7) \simeq 1.2 \times 10^{-3}c \). This completes the proof of the lemma.

Under the assumptions of Lemma 3.4, \( v \) has got a unique point of global maximum on \( \mathbb{R} \). In the sequel of this section, we will denote by \( \xi \) this point of global maximum and we set \( M = v(\xi) = \max_{x \in \mathbb{R}} v(x) \).

The next two lemmas can be directly deduced from the similar lemmas established in [7] (see also [6]).

**Lemma 3.5** (Connection Between \( E(\cdot) \) and \( M^2 \); See [7]). Let \( u \in L^2(\mathbb{R}) \) and \( v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R}) \).

Define the function \( g \) by

\[
g(x) = \begin{cases} 
2v + v_{xx} - 3v_x, & x < \xi, \\
2v + v_{xx} + 3v_x, & x > \xi.
\end{cases}
\]

Then it holds

\[
\int_{\mathbb{R}} g^2(x)dx = E(u) - 12M^2.
\]

**Lemma 3.6** (Connection Between \( F(\cdot) \) and \( M^3 \); See [7]). Let \( u \in L^2(\mathbb{R}) \) and \( v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R}) \).

Define the function \( h \) by

\[
h(x) = \begin{cases} 
-v_{xx} - 6v_x + 16v, & x < \xi, \\
-v_{xx} + 6v_x + 16v, & x > \xi.
\end{cases}
\]

Then it holds

\[
\int_{\mathbb{R}} h(x)g^2(x)dx = F(u) - 144M^3.
\]

**Sketch of proof.** The proof of Lemmas 3.5-3.6 follows by direct computation, using integration by parts, with \( v_x(\xi) = 0 \) and \( v(\pm \infty) = v_x(\pm \infty) = v_{xx}(\pm \infty) = 0 \). See [7] (also [6]) to understand the technique.

We can now connect the conservation laws.

**Lemma 3.7** (Connection Between \( E(\cdot) \) and \( F(\cdot) \)). Let \( u \in L^2(\mathbb{R}) \), with \( y = (1 - \partial_x^2)u \in M^+(\mathbb{R}) \), that satisfies (3.6) for some \( z \in \mathbb{R} \). There exists \( \varepsilon_0 > 0 \) only depending on the speed \( c \), such that if \( 0 < \varepsilon < \varepsilon_0 \), then it holds

\[
M^3 - \frac{1}{4}E(u)M + \frac{1}{72}F(u) \leq 0.
\]

**Proof.** The key is to show that \( h \leq 18M \) on \( \mathbb{R} \). Note that by (3.9) we know that \( 18M \geq c/4 \) and that Lemma 3.4 ensures that \( \xi \in \Theta_z \) for \( \varepsilon_0 \) small enough. Let us set \( \lambda = z - 6.7 \), \( \mu = z + 6.7 \), and rewrite the function \( h \) as

\[
h(x) = \begin{cases} 
-v_{xx} - 6v_x + 16v, & x < \lambda, \\
u - 6v_x + 12v, & \lambda < x < \xi, \\
u + 6v_x + 12v, & \xi < x < \mu, \\
-v_{xx} + 6v_x + 16v, & x > \mu.
\end{cases}
\]
If $x \in \mathbb{R} \setminus \Theta_z$, using that $v_{xx} = 4v - u$, (3.16) and (3.17), it holds

$$h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c}{9} \leq 18M.$$ 

If $\lambda < x < \xi$, then $v_x \geq 0$, and using that $u \leq 6v$ on $\mathbb{R}$, we have

$$h = u - 6v_x + 12v \leq 18v.$$ 

If $\xi < x < \mu$, then $v_x \leq 0$, and similarly using that $u \leq 6v$ on $\mathbb{R}$, we get

$$h = u + 6v_x + 12v \leq 18v.$$ 

Therefore, it holds

$$h(x) \leq 18 \max_{x \in \mathbb{R}} v(x) = 18M, \quad \forall x \in \mathbb{R}.$$  (3.29)

Now, combining (3.25), (3.27) and (3.29), we get

$$F(u) - 144M^3 = \int_{\mathbb{R}} h(x)g^2(x)dx \leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x)dx \leq 18M(E(u) - 12M^2),$$

and we obtain the lemma. \hfill \Box

**Proof of Theorem 3.1.** We argue as El Dika and Molinet in [4]. As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that $u \in L^2(\mathbb{R})$ satisfies (3.1), (3.2) and (3.6). We recall that $M = v(\xi) = \max_{x \in \mathbb{R}} v(x)$ and we set $\delta = c/6 - M$. We first remark that if $\delta \leq 0$, combining (3.4) and (3.10), it holds

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_c)|^{1/2} \leq O(\varepsilon),$$

that yields the desired result. Now suppose that $\delta > 0$, that is the maximum of the function $v$ is less than the maximum of $\rho_c$. Combining (3.4), (3.5) and (3.28), we get

$$M^3 - \frac{1}{4}E(\varphi_c)M + \frac{1}{72}F(\varphi_c) \leq O(\varepsilon^2).$$

Using that $E(\varphi_c) = c^2/3$ and $F(\varphi_c) = 2c^3/3$, our inequality becomes

$$\left(\frac{M - \frac{c}{6}}{\frac{c}{3}}\right)^2 \left(M + \frac{c}{3}\right) \leq O(\varepsilon^2).$$

Next, substituting $M$ by $c/6 - \delta$ and using that $[M + c/3]^{-1} < 3/c$, we obtain

$$\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon).$$ (3.30)

Finally, combining (3.4), (3.10) and (3.30), we infer that

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},$$

where $C > 0$ only depends on the speed $c$. This completes the proof of the stability of peakons.

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References


