# ISOMONODROMIC DEFORMATIONS OF LOGARITHMIC CONNECTIONS AND STABILITY

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INDRANIL BISWAS, VIKTORIA HEU, AND JACQUES HURTUBISE

Abstract. Let $X_0$ be a compact connected Riemann surface of genus $g$ with $D_0 \subset X_0$ an ordered subset of cardinality $n$, and let $E_G$ be a holomorphic principal $G$–bundle on $X$, where $G$ is a reductive affine algebraic group defined over $\mathbb{C}$, that admits a logarithmic connection $\nabla_0$ with polar divisor $D_0$. Let $(\mathcal{E}_G, \nabla)$ be the universal isomonodromic deformation of $(E_G, \nabla_0)$ over the universal Teichmüller curve $(\mathcal{X}, \mathcal{D}) \longrightarrow \text{Teich}_{g,n}$, where $\text{Teich}_{g,n}$ is the Teichmüller space for genus $g$ Riemann surfaces with $n$–marked points. We prove the following (see Section 5):

(1) Assume that $g \geq 2$ and $n = 0$. Then there is a closed complex analytic subset $\mathcal{Y} \subset \text{Teich}_{(g,n)}$, of codimension at least $g$, such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$, the principal $G$–bundle $\mathcal{E}_G|_{\mathcal{X}_t}$ is semistable, where $\mathcal{X}_t$ is the compact Riemann surface over $t$.
(2) Assume that $g \geq 1$, and if $g = 1$, then $n > 0$. Also, assume that the monodromy representation for $\nabla_0$ does not factor through some proper parabolic subgroup of $G$. Then there is a closed complex analytic subset $\mathcal{Y}' \subset \text{Teich}_{(g,n)}$, of codimension at least $g$, such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$, the principal $G$–bundle $\mathcal{E}_G|_{\mathcal{X}_t}$ is semistable.
(3) Assume that $g \geq 2$. Assume that the monodromy representation for $\nabla_0$ does not factor through some proper parabolic subgroup of $G$. Then there is a closed complex analytic subset $\mathcal{Y}'' \subset \text{Teich}_{(g,n)}$, of codimension at least $g-1$, such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$, the principal $G$–bundle $\mathcal{E}_G|_{\mathcal{X}_t}$ is stable.

In [He1], the second–named author proved the above results for $G = \mathrm{GL}(2, \mathbb{C})$.

## 1. Introduction

Take any triple of the form $(E \longrightarrow X, D, \nabla)$, where $E$ is a holomorphic vector bundle over a smooth connected complex variety $X$, and $\nabla$ is an integrable logarithmic connection on $E$ singular over a simple normal crossing divisor $D \subset X$. The monodromy functor associates to it a representation $\rho_\nabla : \pi_1(X \setminus D, x_0) \longrightarrow \mathrm{GL}(E_{x_0})$. Altering the connection by a holomorphic automorphism of $E$ leads to a conjugated representation. The monodromy functor produces an equivalence between the category of logarithmic connections $(E, \nabla)$ on $(X, D)$ such that the real parts of residues lie in $[0, 1)$ and the category of equivalence classes of linear representations of $\pi_1(X \setminus D, x_0)$. Given a monodromy representation $\rho$, one can consider the set of all logarithmic connections $(E \longrightarrow X, D, \nabla)$ (with no condition on the residues) up to holomorphic isomorphisms that produce the same monodromy representation $\rho = \rho_\nabla$ up to conjugation. All these connections are conjugated to each other by meromorphic gauge transformations with possible poles over $D$ (see for example [Sa]).

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The classical Riemann–Hilbert problem can be formulated as follows: *Given a representation* $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus D_0, x) \longrightarrow \mathrm{GL}(r, \mathbb{C})$*, is there a logarithmic connection* $(E \longrightarrow X, D_0, \nabla)$ *such that* $\rho = \rho_\nabla$ *and* $E$ *is holomorphically trivial?* The answer to this problem is

(1) positive if rank $r = 2$ [Pl], [De],
(2) negative in general ($r \geq 3$) [AB],
(3) positive if $\rho$ is irreducible [Bol1], [Ko].

On the other hand, the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus D_0, x)$ depends only on the topological and not the complex structure of $\mathbb{P}^1_{\mathbb{C}} \setminus D_0$. So given an integrable connection, one can consider variations of the complex structure without changing the monodromy representation. More precisely, consider the Teichmüller space $\mathrm{Teich}_{0,n}$ of the $n$–pointed Riemann sphere together with its universal Teichmüller curve

$$\tau : (\mathcal{X} = \mathbb{P}^1_{\mathbb{C}} \times \mathrm{Teich}_{0,n}, \mathcal{D}) \longrightarrow \mathrm{Teich}_{0,n},$$

where $n = \deg(D_0)$. Since $\mathrm{Teich}_{0,n}$ is contractible, the inclusion

$$(\mathbb{P}^1_{\mathbb{C}}, D_t) := \tau^{-1}(t) \hookrightarrow (\mathcal{X}, \mathcal{D}), \quad t \in \mathrm{Teich}_{0,n},$$

induces an isomorphism $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus D_t, x_t) \simeq \pi_1(\mathcal{X} \setminus \mathcal{D}, x_t)$. Hence by the Riemann–Hilbert correspondence, we can associate to any logarithmic connection $(E_0, \nabla_0)$ on $\mathbb{P}^1_{\mathbb{C}}$, with polar divisor $D_0$, its *universal isomonodromic deformation*: a flat logarithmic connection $(\mathcal{E}, \nabla)$ over $\mathcal{X}$ with polar divisor $\mathcal{D}$ that extends the connection $(E_0, \nabla_0)$. The monodromy representation for $\nabla|_{\tau^{-1}(t)}$ does not change as $t$ moves over $\mathrm{Teich}_{0,n}$ (see for example [He2]).

We are led to another Riemann–Hilbert problem: *Given a logarithmic connection* $(E_0, \nabla_0)$ *on* $\mathbb{P}^1_{\mathbb{C}}$ *with polar divisor* $D_0$ *of degree* $n$*, is there a point* $t \in \mathrm{Teich}_{0,n}$ *such that the vector bundle* $E_t = \mathcal{E}|_{\mathbb{P}^1_{\mathbb{C}} \times \{t\}}$ *underlying the universal isomonodromic deformation* $(\mathcal{E}, \nabla)$ *is trivial?*

A partial answer to this question is given by the following theorem of Bolibruch:

**Theorem 1.1** ([Bol2])**.** *Let* $(E_0, \nabla_0)$ *be an irreducible trace–free logarithmic rank two connection with* $n \geq 4$ *poles on* $\mathbb{P}^1_{\mathbb{C}}$ *such that each singularity is resonant. There is a proper closed complex analytic subset* $\mathcal{Y} \subset \mathrm{Teich}_{0,n}$ *such that for all* $t \in \mathrm{Teich}_{0,n} \setminus \mathcal{Y}$*, the vector bundle* $E_t = \mathcal{E}|_{\mathbb{P}^1_{\mathbb{C}} \times \{t\}}$ *underlying the universal isomonodromic deformation* $(\mathcal{E}, \nabla)$ *of* $(E_0, \nabla_0)$ *is trivial.*

It should be mentioned that the condition in Theorem 1.1, that each singularity is resonant, can actually be removed [He1].

From the Birkhoff–Grothendieck classification of holomorphic vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ it follows immediately that the only semistable holomorphic vector bundle of degree 0 and rank $r$ on $\mathbb{P}^1_{\mathbb{C}}$ is the trivial bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}^{\oplus r}$. This leads to the following more general question: *Given a representation* $\rho : \pi_1(X \setminus D, x) \longrightarrow \mathrm{GL}_r(\mathbb{C})$*, where* $X$ *is a compact connected Riemann surface, is there a logarithmic connection* $(E \longrightarrow X, D, \nabla)$ *such that* $\rho = \rho_\nabla$ *and* $E$ *is semistable of degree zero?* The answer to this problem is

(1) negative in general [EH],

(2) positive if $\rho$ is irreducible [EV].

On the other hand, we can ask the following: Let $\tau : (\mathcal{X}, \mathcal{D}) \longrightarrow \text{Teich}_{g,n}$ be the universal Teichmüller curve. *Given a logarithmic connection* $(E_0, \nabla_0)$*, with polar divisor* $D_0$ *of degree* $n$ *on a compact connected Riemann surface* $X_0$ *of genus* $g$*, is there an element* $t \in \text{Teich}_{(g,n)}$ *such that the vector bundle* $E_t = \mathcal{E}|_{\mathcal{X}_t} \longrightarrow \mathcal{X}_t = \tau^{-1}(t)$ *underlying the universal isomonodromic deformation* $(\mathcal{E}, \nabla)$ *of* $(E_0, \nabla_0)$ *is semistable?*

Note that we necessarily have $\text{degree}(E_t) = \text{degree}(E_0)$. Again, Theorem 1.1 can be generalized:

**Theorem 1.2** ([He1])**.** *Let* $(E_0, \nabla_0)$ *be an irreducible logarithmic rank* 2 *connection with polar divisor* $D_0$ *of degree* $n$ *on a compact connected Riemann surface* $X_0$ *of genus* $g$ *such that* $3g - 3 + n > 0$*. Consider its universal isomonodromic deformation* $(\mathcal{E}, \nabla)$ *over* $\tau : (\mathcal{X}, \mathcal{D}) \longrightarrow \text{Teich}_{(g,n)}$*. There is a closed complex analytic subset* $\mathcal{Y} \subset \text{Teich}_{(g,n)}$ *of codimension at least* $g$ *(respectively,* $g-1$*) such that for any* $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$*, the vector bundle* $E_t = \mathcal{E}|_{\mathcal{X}_t}$*, where* $(\mathcal{X}_t, D_t) = \tau^{-1}(t)$*, is semistable (respectively, stable).*

Our aim here is to prove an analog of Theorem 1.2 in the more general context of logarithmic connections on principal $G$–bundles over a compact connected Riemann surface (see [Boa] for logarithmic connections on principal $G$–bundles).

Let $X_0$ be a compact connected Riemann surface of genus $g$, and let $D_0 \subset X_0$ be an ordered subset of cardinality $n$. Let $G$ be a reductive affine algebraic group over $\mathbb{C}$. Let $E_G$ be a holomorphic principal $G$–bundle on $X$ and $\nabla_0$ a logarithmic connection on $E_G$ with polar divisor $D_0$. Let $(\mathcal{E}_G, \nabla)$ be the universal isomonodromic deformation of $(E_G, \nabla_0)$ over the universal Teichmüller curve $\tau : (\mathcal{X}, \mathcal{D}) \longrightarrow \text{Teich}_{g,n}$. For any point $t \in \text{Teich}_{g,n}$, the restriction $\mathcal{E}_G|_{\tau^{-1}(t)} \longrightarrow \mathcal{X}_t := \tau^{-1}(t)$ will be denoted by $\mathcal{E}^t_G$.

We prove the following (see Section 5):

**Theorem 1.3.**

(1) *Assume that* $g \geq 2$ *and* $n = 0$*. There is a closed complex analytic subset* $\mathcal{Y} \subset \text{Teich}_{(g,n)}$ *of codimension at least* $g$ *such that for any* $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$*, the principal* $G$*–bundle* $\mathcal{E}^t_G \longrightarrow \mathcal{X}_t$ *is semistable.*
(2) *Assume that* $g \geq 1$*, and if* $g = 1$*, then* $n > 0$*. Also, assume that the monodromy representation for* $\nabla_0$ *does not factor through some proper parabolic subgroup of* $G$*. There is a closed complex analytic subset* $\mathcal{Y}' \subset \text{Teich}_{(g,n)}$ *of codimension at least* $g$ *such that for any* $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$*, the principal* $G$*–bundle* $\mathcal{E}^t_G$ *is semistable.*
(3) *Assume that* $g \geq 2$*. Assume that the monodromy representation for* $\nabla_0$ *does not factor through some proper parabolic subgroup of* $G$*. There is a closed complex analytic subset* $\mathcal{Y}'' \subset \text{Teich}_{(g,n)}$ *of codimension at least* $g-1$ *such that for any* $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$*, the principal* $G$*–bundle* $\mathcal{E}^t_G$ *is stable.*

It is known that if a holomorphic principal $G$–bundle $E_G$ over an elliptic curve admits a holomorphic connection, then $E_G$ is semistable. Therefore, a stronger version of Theorem 1.3(1) is valid when $g = 1$.

## 2. Infinitesimal deformations

We first recall some classical results in deformation theory, thereby setting up our notation.

2.1. **Deformations of a $n$–pointed curve.** Let $X_0$ be an irreducible smooth complex projective curve of genus $g$, with $g > 0$, and let

$$D_0 := \{x_1, \cdots, x_n\} \subset X_0$$

be an ordered subset of cardinality $n$ (it may be zero). We assume that $n > 0$ if $g = 1$. This condition implies that the pair $(X_0, D_0)$ does not have any infinitesimal automorphism, equivalently, the automorphism group of $(X_0, D_0)$ is finite.

Let $B := \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ be the spectrum of the local ring. An *infinitesimal deformation of* $(X_0, D_0)$ is given by a quadruple

$$(\mathcal{X}, q, \mathcal{D}, f), \tag{2.1}$$

where

- $q : \mathcal{X} \longrightarrow B$ is a smooth proper algebraic morphism of relative dimension one,
- $\mathcal{D} = (\mathcal{D}_1, \cdots, \mathcal{D}_n)$ is a collection of $n$ ordered disjoint sections of $q$, and
- $f : X_0 \longrightarrow \mathcal{X}$ is an algebraic morphism such that
$$f(X_0) \subset \mathcal{X}_0 := q^{-1}(0) \quad \text{with} \quad x_i = \mathcal{D}_i(0) \ \forall \ 1 \leq i \leq n,$$
and the morphism $X_0 \xrightarrow{f} \mathcal{X}_0$ is an isomorphism.

The divisor $\sum_{i=1}^n \mathcal{D}_i(B)$ on $\mathcal{X}$ will also be denoted by $\mathcal{D}$. For a vector bundle $\mathcal{V}$ on $\mathcal{X}$, the vector bundle $\mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ will be denoted by $\mathcal{V}(-\mathcal{D})$.

The differential of $f$

$$\mathrm{d}f : \mathrm{T}X_0 \longrightarrow f^*\mathrm{T}\mathcal{X}$$

produces a homomorphism $\mathrm{T}X_0(-D_0) := \mathrm{T}X_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0}(-D_0) \longrightarrow f^*\mathrm{T}\mathcal{X}(-\mathcal{D})$ which will also be denoted by $\mathrm{d}f$. Consider the following short exact sequence of sheaves on $X_0$:

$$0 \longrightarrow \mathrm{T}X_0(-D_0) \xrightarrow{\mathrm{d}f} f^*\mathrm{T}\mathcal{X}(-\mathcal{D}) \xrightarrow{h} \mathcal{O}_{X_0} \longrightarrow 0; \tag{2.2}$$

note that the normal bundle to $\mathcal{X}_0 \subset \mathcal{X}$ is the pullback of $\mathrm{T}_0B$ by $q|_{\mathcal{X}_0}$ and hence this normal bundle is identified with $\mathcal{O}_{X_0}$. Consider the connecting homomorphism

$$\mathbb{C} = \mathrm{H}^0(X_0, \mathcal{O}_{X_0}) \xrightarrow{\phi} \mathrm{H}^1(X_0, \mathrm{T}X_0(-D_0)) \tag{2.3}$$

in the long exact sequence of cohomologies associated to the short exact sequence in (2.2). Let $1_{X_0}$ denote the constant function 1 on $X_0$. The cohomology element

$$\phi(1_{X_0}) \in \mathrm{H}^1(X_0, \mathrm{T}X_0(-D_0)), \tag{2.4}$$

where $\phi$ is the homomorphism in (2.3), is the *Kodaira–Spencer infinitesimal deformation class* for the family in (2.1). If this infinitesimal deformation class is zero, then the family $(\mathcal{X}, \mathcal{D}) \longrightarrow B$ is isomorphic to the trivial family $(X_0 \times B, D_0 \times B) \longrightarrow B$.

2.2. **Deformations of a curve together with a principal bundle.** Take $(X_0\,,D_0)$ as before. Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Let

$$p\,:\,E_G\,\longrightarrow\,X_0 \tag{2.5}$$

be a principal $G$–bundle on $X_0$. The infinitesimal deformations of the triple

$$(X_0\,,D_0\,,E_G) \tag{2.6}$$

are guided by the Atiyah bundle $\mathrm{At}(E_G)\,\longrightarrow\,X_0$, the construction of which we shall briefly recall (see [At] for a more detailed exposition). Consider the direct image $p_*\mathrm{T}E_G\,\longrightarrow\,X_0$, where $\mathrm{T}E_G$ is the holomorphic tangent bundle of $E_G$, and $p$ is the projection in (2.5). It is a quasicoherent sheaf equipped with an action of $G$ given by the action of $G$ on $E_G$. The invariant part

$$\mathrm{At}(E_G)\,:=\,(p_*\mathrm{T}E_G)^G\,\subset\,(p_*\mathrm{T}E_G)$$

is a vector bundle on $X_0$ of rank $1+\dim G$ which is known as the Atiyah bundle of $E_G$. Consequently, we have $\mathrm{At}(E_G)\,=\,(TE_G)/G$. Let

$$\mathrm{ad}(E_G)\,:=\,E_G\times^G\mathfrak{g}\,\longrightarrow\,X_0$$

be the adjoint vector bundle associated to $E_G$ for the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. Let

$$\mathrm{d}p\,:\,\mathrm{T}E_G\,\longrightarrow\,p^*\mathrm{T}X_0$$

be the differential of the map $p$ in (2.5). Being $G$–equivariant it produces a homomorphism $\mathrm{At}(E_G)\,\longrightarrow\,\mathrm{T}X_0$ which will also be denoted by $\mathrm{d}p$. Now the action of $G$ on $E_G$ produces an isomorphism $E_G\times\mathfrak{g}\,\longrightarrow\,\mathrm{kernel}(\mathrm{d}p)$. Therefore, we have $\mathrm{kernel}(\mathrm{d}p)/G\,=\,\mathrm{ad}(E_G)$. In other words, the above isomorphism descends to an isomorphism

$$\mathrm{ad}(E_G)\,\xrightarrow{\sim}\,(p_*(\mathrm{kernel}(\mathrm{d}p)))^G$$

that preserves the Lie algebra structure on the fibers of the two vector bundles (the Lie algebra structure on the fibers of $(p_*(\mathrm{kernel}(\mathrm{d}p)))^G$ is given by the Lie bracket of $G$–invariant vertical vector fields). Therefore, $\mathrm{At}(E_G)$ fits in the following short exact sequence of vector bundles on $X_0$

$$0\,\longrightarrow\,\mathrm{ad}(E_G)\,\longrightarrow\,\mathrm{At}(E_G)\,\xrightarrow{\mathrm{d}p}\,\mathrm{T}X_0\,\longrightarrow\,0\,, \tag{2.7}$$

which is known as the Atiyah exact sequence for $E_G$. The logarithmic Atiyah bundle $\mathrm{At}(E_G\,,D_0)$ is defined by

$$\mathrm{At}(E_G\,,D_0)\,:=\,(\mathrm{d}p)^{-1}(\mathrm{T}X_0(-D_0))\,\subset\,\mathrm{At}(E_G)\,.$$

From (2.7) we have the short exact sequence

$$0\,\longrightarrow\,\mathrm{ad}(E_G)\,\longrightarrow\,\mathrm{At}(E_G\,,D_0)\,\xrightarrow{\sigma}\,\mathrm{T}X_0(-D_0)\,\longrightarrow\,0\,, \tag{2.8}$$

which is called the logarithmic Atiyah exact sequence. The above homomorphism $\sigma$ is the restriction of $\mathrm{d}p$ to $\mathrm{At}(E_G\,,D_0)\,\subset\,\mathrm{At}(E_G)$.

An *infinitesimal deformation* of the triple $(X_0\,,D_0\,,E_G)$ in (2.6) is a 6–tuple

$$(\mathcal{X}\,,q\,,\mathcal{D}\,,f\,,\mathcal{E}_G\,,\psi)\,, \tag{2.9}$$

where

- $(\mathcal{X}\,,q\,,\mathcal{D}\,,f)$ in an infinitesimal deformation of the $n$–pointed curve $(X_0\,,D_0)$ as in (2.1),
- $\mathcal{E}_G\,\longrightarrow\,\mathcal{X}$ is a principal $G$–bundle, and
- $\psi$ is an isomorphism
$$\psi\,:\,E_G\,\longrightarrow\,f^*\mathcal{E}_G \tag{2.10}$$
of principal $G$–bundles.

The logarithmic Atiyah bundle
$$\mathrm{At}(\mathcal{E}_G\,,\mathcal{D})\,\longrightarrow\,\mathcal{X}$$
for $(\mathcal{E}_G\,,\mathcal{D})$ is the inverse image, in $\mathrm{At}(\mathcal{E}_G)$, of the subsheaf $\mathrm{T}\mathcal{X}(-\mathcal{D}))\,\subset\,\mathrm{T}\mathcal{X}$. We have the following short exact sequence of sheaves on $X_0$:
$$0\,\longrightarrow\,\mathrm{At}(E_G\,,D_0)\,\longrightarrow\,f^*\mathrm{At}(\mathcal{E}_G\,,\mathcal{D})\,\longrightarrow\,\mathcal{O}_{X_0}\,\longrightarrow\,0 \tag{2.11}$$
given by $\psi$ in (2.10). Let
$$\mathbb{C}\,=\,\mathrm{H}^0(X_0,\,\mathcal{O}_{X_0})\,\xrightarrow{\widetilde{\phi}}\,\mathrm{H}^1(X_0,\,\mathrm{At}(E_G\,,D_0))$$
be the connecting homomorphism in the long exact sequence of cohomologies associated to (2.11). The cohomology element
$$\widetilde{\phi}(1_{X_0})\,\in\,\mathrm{H}^1(X_0,\,\mathrm{At}(E_G\,,D_0)) \tag{2.12}$$
is the *infinitesimal deformation class* of the triple $(X_0\,,D_0\,,E_G)$ given by (2.9). Let
$$\sigma_*\,:\,\mathrm{H}^1(X_0,\,\mathrm{At}(E_G\,,D_0))\,\longrightarrow\,\mathrm{H}^1(X_0,\,\mathrm{T}X_0(-D_0))$$
be the homomorphism given by the projection $\sigma$ in (2.8). From the commutativity of the diagram
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{At}(E_G\,,D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_G\,,\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0\\
 & & \Big\downarrow & & \Big\downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{T}X_0(-D_0) & \longrightarrow & f^*\mathrm{T}\mathcal{X}(-\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0
\end{array}$$
where the top and bottom rows are as in (2.11) and (2.2) respectively, it follows that
$$\sigma_*(\widetilde{\phi}(1_{X_0}))\,=\,\phi(1_{X_0})\,,$$
where $\widetilde{\phi}(1_{X_0})$ and $\phi(1_{X_0})$ are constructed in (2.12) and (2.4) respectively. We note that $\sigma_*$ is the forgetful map that sends an infinitesimal deformation of $(X_0\,,D_0\,,E_G)$ to the underlying infinitesimal deformation of $(X_0\,,D_0)$.

## 3. Obstruction to an extension of a reduction of structure group

Given a reduction of structure group of $E_G$ to a parabolic subgroup of $G$, our aim in this section is to compute the obstruction for it to extend to a reduction of an infinitesimal deformation $\mathcal{E}_G\,\longrightarrow\,\mathcal{X}$ as in (2.9).

Fix a parabolic subgroup $P\,\subset\,G$. The Lie algebra of $P$ will be denoted by $\mathfrak{p}$. Let
$$E_P\,\subset\,E_G \tag{3.1}$$

be a holomorphic reduction of structure group of $E_G$ to the subgroup $P \subset G$. Let

$$\mathrm{ad}(E_P) := E_P \times^P \mathfrak{p} \longrightarrow X_0$$

be the adjoint vector bundle associated to $E_P$ for the adjoint action of $P$ on its Lie algebra $\mathfrak{p}$. The vector bundle over $X_0$ associated to the principal $P$–bundle $E_P$ for the adjoint action of $P$ on the quotient $\mathfrak{g}/\mathfrak{p}$ will be denoted by $E_P(\mathfrak{g}/\mathfrak{p})$. The logarithmic Atiyah bundle for $(E_P\, , D_0)$ will be denoted by $\mathrm{At}(E_P, D_0)$. We have

$$\mathrm{ad}(E_P) \subset \mathrm{ad}(E_G) \quad \text{and} \quad \mathrm{At}(E_P, D_0) \subset \mathrm{At}(E_G\, , D_0)\,;$$

both the quotient bundles above are identified with $E_P(\mathfrak{g}/\mathfrak{p})$. In other words, we have the following commutative diagram of vector bundles on $X_0$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}(E_P\, , D_0) & \xrightarrow{\beta} & \mathrm{T}X_0(-D_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \xi & & \| & & \\
0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G\, , D_0) & \xrightarrow{\gamma} & \mathrm{T}X_0(-D_0) & \longrightarrow & 0 \\
 & & \downarrow \mu_1 & & \downarrow \mu & & & & \\
 & & E_P(\mathfrak{g}/\mathfrak{p}) & = & E_P(\mathfrak{g}/\mathfrak{p}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.2}$$

Let

$$\widetilde{\xi} : \mathrm{H}^1(X_0,\, \mathrm{At}(E_P\, , D_0)) \longrightarrow \mathrm{H}^1(X_0,\, \mathrm{At}(E_G\, , D_0)) \tag{3.3}$$

be the homomorphism induced by the canonical injection $\xi$ in (3.2).

Take $(\mathcal{X}\, , q\, , \mathcal{D}\, , f\, , \mathcal{E}_G\, , \psi)$ as in (2.9). Assume that the reduction $E_P \subset E_G$ in (3.1) extends to a reduction of structure group

$$\mathcal{E}_P \subset \mathcal{E}_G$$

to $P \subset G$ on $\mathcal{X}$. Consider the short exact sequence on $X_0$

$$0 \longrightarrow \mathrm{At}(E_P\, , D_0) \longrightarrow f^*\mathrm{At}(\mathcal{E}_P\, , \mathcal{D}) \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0\,, \tag{3.4}$$

where $\mathrm{At}(\mathcal{E}_P\, , \mathcal{D}) \longrightarrow \mathcal{X}$ is the logarithmic Atiyah bundle associated to the principal $P$–bundle $\mathcal{E}_P$, and $f$ is the map in (2.1). Let

$$\theta \in \mathrm{H}^1(X_0,\, \mathrm{At}(E_P\, , D_0)) \tag{3.5}$$

be the image of the constant function $1_{X_0}$ by the homomorphism

$$\mathrm{H}^0(X_0,\, \mathcal{O}_{X_0}) \longrightarrow \mathrm{H}^1(X_0,\, \mathrm{At}(E_P\, , D_0))$$

in the long exact sequence of cohomologies associated to (3.4).

**Lemma 3.1.** *The cohomology class $\theta$ in* (3.5) *satisfies the equation*

$$\widetilde{\xi}(\theta) = \widetilde{\phi}(1_{X_0})\,,$$

*where $\widetilde{\xi}$ and $\widetilde{\phi}(1_{X_0})$ are constructed in* (3.3) *and* (2.12) *respectively.*

*Proof.* Consider the commutative diagram of vector bundles on $X_0$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{At}(E_P\,,D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_P\,,\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \\ & & \Big\downarrow \xi & & \Big\downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{At}(E_G\,,D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_G\,,\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \end{array} \tag{3.6}$$

where the top and bottom rows are as in (3.4) and (2.11) respectively, and $\xi$ is the homomorphism in (3.2); the above homomorphism $f^*\mathrm{At}(\mathcal{E}_P\,,\mathcal{D}) \longrightarrow f^*\mathrm{At}(\mathcal{E}_G\,,\mathcal{D})$ is the pullback of the natural homomorphism $\mathrm{At}(\mathcal{E}_P\,,\mathcal{D}) \longrightarrow \mathrm{At}(\mathcal{E}_G\,,\mathcal{D})$. In view of (3.6), the lemma follows by comparing the constructions of $\theta$ and $\widetilde{\phi}(1_{X_0})$. $\square$

## 4. Logarithmic connections and the second fundamental form

In this section we characterize those infinitesimal deformations of the principal bundle $E_G$ on the $n$–pointed curve that arise from the isomonodromic deformations.

### 4.1. Canonical extension of a logarithmic connection.

As before, let $p\,:\,E_G \longrightarrow X_0$ be a principal $G$–bundle. A *logarithmic connection* on $E_G$ with polar divisor $D_0$ is a holomorphic splitting of the logarithmic Atiyah exact sequence in (2.8). In other words, a logarithmic connection is a homomorphism

$$\delta\,:\,\mathrm{T}X_0(-D_0) \longrightarrow \mathrm{At}(E_G\,,D_0) \tag{4.1}$$

such that $\sigma\circ\delta \,=\, \mathrm{Id}_{\mathrm{T}X_0(-D_0)}$, where $\sigma$ is the homomorphism in (2.8). Note that given such a $\delta$, there is a unique homomorphism

$$\delta''\,:\,\mathrm{At}(E_G\,,D_0) \longrightarrow \mathrm{ad}(E_G) \tag{4.2}$$

such that $\delta''\circ\delta \,=\, 0$, and the composition

$$\mathrm{ad}(E_G) \hookrightarrow \mathrm{At}(E_G\,,D_0) \stackrel{\delta''}{\longrightarrow} \mathrm{ad}(E_G)$$

(see (2.8)) is the identity map of $\mathrm{ad}(E_G)$. As there are no nonzero $(2\,,0)$–forms on $X_0$, all logarithmic connections on $E_G$ are automatically integrable.

At the level of first order deformations, given a principal $G$–bundle $\mathcal{E}_G \longrightarrow \mathcal{X} \stackrel{q}{\longrightarrow} B$, a logarithmic connection on $\mathcal{E}_G$ with polar divisor $\mathcal{D}$ is a homomorphism $\mathrm{At}(\mathcal{E}_G\,,\mathcal{D}) \longrightarrow \mathrm{ad}(\mathcal{E}_G)$ that splits the logarithmic Atiyah exact sequence for $\mathcal{E}_G$. We note that a connection on $\mathcal{E}_G$ need not be integrable, as we have added an (infinitesimal) extra dimension. However, the Riemann–Hilbert correspondence for principal $G$–bundles yields the following:

**Lemma 4.1.** *Let $(\mathcal{X}\,,q\,,\mathcal{D}\,,f)$ be an infinitesimal deformation of $(X_0\,,D_0)$ as in* (2.1)*. Let $\delta$ be a logarithmic connection on a principal $G$–bundle $E_G$ on $X_0$ with polar divisor $D_0$. Then there exists a unique pair $(\mathcal{E}_G\,,\nabla)$, where*

- *$\mathcal{E}_G$ is a principal $G$–bundle on $\mathcal{X}$, and*
- *$\nabla$ is an integrable logarithmic connection on $\mathcal{E}_G$ with polar divisor $\mathcal{D}$,*

*such that $(f^*\mathcal{E}_G\,,f^*\nabla) \,=\, (E_G\,,\delta)$.*

Let us recall a few elements of the proof of this (classical) result. Choose a covering $\mathfrak{U}$ of $X_0\backslash D_0$ by complex discs and a small neighborhood $U_i$ for each $x_i \in D_0$. Since $\delta$ is integrable, we can choose local charts for $E_G$ over $\mathfrak{U}$ such that all transition functions are constants. Now if the curve fits into an analytic family $\mathcal{X} \longrightarrow \mathcal{B}$, one can, restricting $\mathcal{B}$ if necessary, extend all the $U_i$ into tubular neighborhoods of $\mathcal{D}_i$ and extend the open subsets in $\mathfrak{U}$ such that $\mathfrak{U}\cup U_1\cup\cdots\cup U_n$ is a covering of $\mathcal{X}$. The isomonodromic deformation is then given by simply extending the transition maps by keeping them to be constant in deformation parameters.

The logarithmic connection $\delta$ gives a splitting of the logarithmic Atiyah bundle

$$\mathrm{At}(E_G\,,D_0)\;=\;\mathrm{ad}(E_G)\oplus\mathrm{T}X_0(-D_0)\,.$$

The corresponding cohomological decomposition

$$\mathrm{H}^1(X_0,\mathrm{At}(E_G\,,D_0))\;=\;\mathrm{H}^1(X_0,\mathrm{ad}(E_G))\oplus\mathrm{H}^1(X_0,\mathrm{T}X_0(-D_0))$$

given a splitting of the infinitesimal deformations of $(X_0\,,D_0\,,E_G)$ into infinitesimal the deformations of $(X_0\,,D_0)$ and the infinitesimal deformations $E_G$ (keeping $(X_0\,,D_0)$ fixed). In other words, let

$$\delta'\,:\,\mathrm{H}^1(X_0,\,\mathrm{T}X_0(-D_0))\;\longrightarrow\;\mathrm{H}^1(X_0,\,\mathrm{At}(E_G\,,D_0))$$

be the homomorphism induced by the homomorphism $\delta\,:\,\mathrm{T}X_0(-D_0)\longrightarrow\mathrm{At}(E_G\,,D_0)$ in Lemma 4.1 defining the logarithmic connection on $E_G$. Given an infinitesimal deformation $(\mathcal{X}\,,q\,,\mathcal{D}\,,f)$ of $(X_0\,,D_0)$, the above homomorphism $\delta'$ produces an infinitesimal deformation $(\mathcal{X}\,,q\,,\mathcal{D}\,,f\,,\mathcal{E}_G\,,\psi)$ of $(X_0\,,D_0\,,E_G)$. As explained above, this principal $G$–bundle $\mathcal{E}_G$ on $\mathcal{X}$ coincides with the principal $G$–bundle on $\mathcal{X}$ produced by the isomonodromic deformation in Lemma 4.1.

We will now construct the exact sequence in (2.11) corresponding to the above infinitesimal deformation $(\mathcal{X}\,,q\,,\mathcal{D}\,,f\,,\mathcal{E}_G\,,\psi)$. Consider the injective homomorphism

$$\mathrm{T}X_0(-D_0)\;\longrightarrow\;\mathrm{At}(E_G\,,D_0)\oplus f^*\mathrm{T}\mathcal{X}(-\mathcal{D})\,,\quad v\;\longmapsto\;(\delta(v)\,,-(\mathrm{d}f)(v))\,,$$

where $\mathrm{d}f$ is the differential in (2.2). The corresponding cokernel

$$\mathrm{At}^\delta(E_G\,,D_0)\;:=\;(\mathrm{At}(E_G\,,D_0)\oplus f^*\mathrm{T}\mathcal{X}(-\mathcal{D}))/(\mathrm{T}X_0(-D_0))$$

possesses a canonical projection

$$\widehat{\delta}\,:\,\mathrm{At}^\delta(E_G\,,D_0)\;\longrightarrow\;\mathcal{O}_{X_0}\,,\quad(v\,,w)\;\longmapsto\;h(w)\,,\tag{4.3}$$

where $h$ is the homomorphism in (2.2); note that the above homomorphism $\widehat{\delta}$ is well–defined because $h$ vanishes on the image of $\mathrm{T}X_0(-D_0)$ in $\mathrm{At}(E_G\,,D_0)\oplus f^*\mathrm{T}\mathcal{X}(-\mathcal{D})$. The kernel of $\widehat{\delta}$ is identified with $\mathrm{At}(E_G\,,D_0)$ by sending any $z\in\mathrm{At}(E_G\,,D_0)$ to the image in $\mathrm{At}^\delta(E_G\,,D_0)$ of $(z\,,0)\in\mathrm{At}(E_G\,,D_0)\oplus f^*\mathrm{T}\mathcal{X}(-\mathcal{D})$. Therefore, we obtain the following exact sequence of vector bundles over $X_0$:

$$0\;\longrightarrow\;\mathrm{At}(E_G\,,D_0)\;\longrightarrow\;\mathrm{At}^\delta(E_G\,,D_0)\;\xrightarrow{\widehat{\delta}}\;\mathcal{O}_{X_0}\;\longrightarrow\;0\,,$$

This exact sequence coincides with the one in (2.11).

Consider the projection

$$\mathrm{At}(E_G\,,D_0)\oplus f^*\mathrm{T}\mathcal{X}(-\mathcal{D}) \longrightarrow \mathrm{ad}(E_G)\,, \quad (z_1\,,z_2)\longmapsto \delta''(z_1)\,,$$

where $\delta''$ is constructed in (4.2) from $\delta$. It vanishes on the image of $\mathrm{T}X_0(-D_0)$, yielding a projection

$$\lambda\,:\,\mathrm{At}^\delta(E_G\,,D_0)\longrightarrow \mathrm{ad}(E_G)\,. \tag{4.4}$$

Let $\nabla''\,:\,\mathrm{At}(\mathcal{E}_G\,,\mathcal{D})\longrightarrow \mathrm{ad}(\mathcal{E}_G)$ be the homomorphism given by the logarithmic connection $\nabla$ in Lemma 4.1. The homomorphism in (4.4) fits in the commutative diagram

$$\begin{array}{ccc} \mathrm{At}^\delta(E_G\,,D_0) & \xrightarrow{\lambda} & \mathrm{ad}(E_G) \\ \| & & \| \\ f^*\mathrm{At}(\mathcal{E}_G\,,\mathcal{D}) & \xrightarrow{f^*\nabla''} & f^*\mathrm{ad}(\mathcal{E}_G) \end{array} \tag{4.5}$$

(the vertical identifications are evident).

We summarize the above constructions in the following lemma:

**Lemma 4.2.** *Given* $(\mathcal{X}\,,q\,,\mathcal{D}\,,f)$ *as in* (2.1)*, and also given a logarithmic connection* $\delta$ *on a principal* $G$*–bundle* $E_G\longrightarrow X_0$*, the exact sequence in* (2.11) *corresponding to the infinitesimal deformation of* $(X_0\,,D_0\,,E_G)$ *in Lemma 4.1 is*

$$0\longrightarrow \mathrm{At}(E_G\,,D_0)\longrightarrow \mathrm{At}^\delta(E_G\,,D_0)\xrightarrow{\widehat{\delta}}\mathcal{O}_{X_0}\longrightarrow 0\,,$$

*where* $\widehat{\delta}$ *is constructed in* (4.3).

4.2. **The second fundamental form.** Fix a logarithmic connection $\delta$ on $(E_G\,,D_0)$ as in (4.1). Take a reduction of structure group $E_P\subset E_G$ to $P$ as in (3.1). The composition

$$S(\delta)\,:=\,\mu\circ\delta\,:\,\mathrm{T}X_0(-D_0)\longrightarrow E_P(\mathfrak{g}/\mathfrak{p})\,, \tag{4.6}$$

where $\mu$ is constructed in (3.2), is called the *second fundamental form* of $E_P$ for the connection $\delta$. We note that $\delta$ is induced by a logarithmic connection on the principal $P$–bundle $E_P$ if and only if we have $S(\delta)\,=\,0$.

Assume that $E_P$ satisfies the condition that $S(\delta)\,\neq\,0$. Let

$$\mathcal{L}\,\subset\,E_P(\mathfrak{g}/\mathfrak{p}) \tag{4.7}$$

be the holomorphic line subbundle generated by the image of the homomorphism $S(\delta)$ in (4.6). More precisely, $\mathcal{L}$ is the inverse image, in $E_P(\mathfrak{g}/\mathfrak{p})$, of the torsion part of the quotient $E_P(\mathfrak{g}/\mathfrak{p})/(S(\delta)(\mathrm{T}X_0(-D_0)))$. Now consider the homomorphism

$$S(\delta)'\,:\,\mathrm{T}X_0(-D_0)\longrightarrow \mathcal{L} \tag{4.8}$$

given by the second fundamental form $S(\delta)$. Let

$$S'\,:\,\mathrm{H}^1(X_0,\,\mathrm{T}X_0(-D_0))\longrightarrow \mathrm{H}^1(X_0,\,\mathcal{L}) \tag{4.9}$$

be the homomorphism induced by $S(\delta)'$ in (4.8).

**Proposition 4.3.** *As before, let $\delta$ be a logarithmic connection on $E_G \longrightarrow X_0$ with polar divisor $D_0$, and let $(\mathcal{X}\,, q\,, \mathcal{D}\,, f)$ be an infinitesimal deformation of $(X_0, D_0)$. Let $\mathcal{E}_G \longrightarrow \mathcal{X}$ be the isomonodromic deformation of $\delta$ along $(\mathcal{X}\,, \mathcal{D})$ obtained in Lemma 4.1. Let $E_P \subset E_G$ be a reduction of structure group to $P$ over $X_0$ that extends to a reduction of structure group $\mathcal{E}_P \subset \mathcal{E}_G$ to $P$ over $\mathcal{X}$. Then*

$$S'(\phi(1_{X_0})) \,=\, 0\,,$$

*where $\phi(1_{X_0})$ is the corresponding cohomology class constructed in* (2.4)*, and $S'$ is constructed in* (4.9)*.*

*Proof.* Consider the inverse images

$$\mathrm{At}_P(E_G\,, D_0) \,:=\, \mu^{-1}(\mathcal{L}) \,\subset\, \mathrm{At}(E_G\,, D_0) \quad \text{and} \quad \mathrm{ad}_P(E_G) \,:=\, \mu_1^{-1}(\mathcal{L}) \,\subset\, \mathrm{ad}(E_G)\,,$$

where $\mu$ and $\mu_1$ are the quotient maps in (3.2), and $\mathcal{L}$ is constructed in (4.7). These two vector bundles fit in the following commutative diagram produced from (3.2):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}(E_P\,, D_0) & \xrightarrow{\beta} & \mathrm{T}X_0(-D_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \xi & & \| & & \\
0 & \longrightarrow & \mathrm{ad}_P(E_G) & \longrightarrow & \mathrm{At}_P(E_G\,, D_0) & \xrightarrow{\gamma} & \mathrm{T}X_0(-D_0) & \longrightarrow & 0 \\
 & & \downarrow \mu_1 & & \downarrow \mu & & & & \\
 & & \mathcal{L} & = & \mathcal{L} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4.10}$$

By the construction of $\mathrm{At}_P(E_G\,, D_0)$, the connection homomorphism $\delta$ in (4.1) factors through a homomorphism

$$\delta^1\,:\, \mathrm{T}X_0(-D_0) \,\longrightarrow\, \mathrm{At}_P(E_G\,, D_0)\,.$$

Consider the homomorphism

$$\delta^1_*\,:\, \mathrm{H}^1(X_0,\, \mathrm{T}X_0(-D_0)) \,\longrightarrow\, \mathrm{H}^1(X_0,\, \mathrm{At}_P(E_G\,, D_0))$$

induced by the above homomorphism $\delta^1$, and let

$$\Phi \,:=\, \delta^1_*(\phi(1_{X_0})) \,\in\, \mathrm{H}^1(X_0,\, \mathrm{At}_P(E_G\,, D_0)) \tag{4.11}$$

be the image of the cohomology class $\phi(1_{X_0})$ characterizing the deformation $(\mathcal{X}\,, \mathcal{D})$ as in (2.4).

As in the statement of the proposition, let $\mathcal{E}_P \longrightarrow \mathcal{X}$ be an extension of the reduction $E_P$. Note that from (3.6), (3.2) and Lemma 4.2 we have

$$\begin{aligned}
\mathrm{At}(E_G\,, D_0)/\mathrm{At}(E_P\,, D_0) \,&=\, E_P(\mathfrak{g}/\mathfrak{p}) \,=\, (f^*\mathrm{At}(\mathcal{E}_G\,, \mathcal{D}))/(f^*\mathrm{At}(\mathcal{E}_P\,, \mathcal{D})) \\
&=\, \mathrm{At}^\delta(E_G\,, D_0)/f^*\mathrm{At}(\mathcal{E}_P\,, \mathcal{D})\,.
\end{aligned}$$

Let $\mu_2\,:\, \mathrm{At}^\delta(E_G\,, D_0) \,\longrightarrow\, E_P(\mathfrak{g}/\mathfrak{p})$ be the above quotient map. Define

$$\mathrm{At}^\delta_P(E_G\,, D_0) \,:=\, \mu_2^{-1}(\mathcal{L}) \,\subset\, \mathrm{At}^\delta(E_G\,, D_0)\,,$$

where $\mathcal{L}$ is constructed in (4.7).

Now we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}X_0(-D_0) & \longrightarrow & f^*\mathrm{T}\mathcal{X}(-\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \\ & & \Big\downarrow \delta & & \Big\downarrow f^*\nabla & & \| & & \\ 0 & \longrightarrow & \mathrm{At}_P(E_G\,,D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_G\,,\mathcal{D}) \,=\, \mathrm{At}^{\delta}_P(E_G\,,D_0) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \end{array} \tag{4.12}$$

where the bottom exact sequence is obtained from (3.6), and the top exact sequence is as in (2.2) (see also (4.5)). Let

$$\nu\,:\,\mathrm{H}^0(X_0,\,\mathcal{O}_{X_0})\,\longrightarrow\,\mathrm{H}^1(X_0,\,\mathrm{At}_P(E_G\,,D_0)$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to the bottom exact sequence in (4.12). From the commutativity of (4.12) and the construction of $\phi(1_{X_0})$ (see (2.4)) it follows that

$$\Phi\,=\,\nu(1_{X_0})\,=\,\delta_*(\phi(1_{X_0}))\,, \tag{4.13}$$

where $\delta_*\,:\,\mathrm{H}^1(X_0,\,\mathrm{T}X_0(-D_0))\,\longrightarrow\,\mathrm{H}^1(X_0,\,\mathrm{At}_P(E_G\,,D_0))$ is the homomorphism induced by $\delta$, and $\Phi$ is the cohomology class in (4.11).

The diagram in (3.6) produces the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \Big\downarrow & & \Big\downarrow & & & & \\ 0 & \longrightarrow & \mathrm{At}(E_P\,,D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_P\,,D_0) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \\ & & \Big\downarrow \xi & & \Big\downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{At}_P(E_G\,,D_0) & \longrightarrow & \mathrm{At}^{\delta}_P(E_G\,,D_0) & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \\ & & \Big\downarrow \mu & & \Big\downarrow & & & & \\ & & \mathcal{L} & = & \mathcal{L} & & & & \\ & & \Big\downarrow & & \Big\downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{4.14}$$

where $\xi$ and $\mu$ are the homomorphisms in (4.10). Using this diagram we can check that

$$\Phi\,=\,\xi_*(\theta)\,, \tag{4.15}$$

where $\theta$ is the cohomology classes in (3.5), and

$$\xi_*\,:\,\mathrm{H}^1(X_0,\,\mathrm{At}(E_P\,,D_0))\,\longrightarrow\,\mathrm{H}^1(X_0,\,\mathrm{At}_P(E_G\,,D_0))$$

is the homomorphism induced by $\xi$ in (4.14). Indeed, (4.15) follows from (4.13), the commutativity of (4.14) and the construction of $\theta$.

Let $\mu_*\,:\,\mathrm{H}^1(X_0,\,\mathrm{At}_P(E_G\,,D_0))\,\longrightarrow\,\mathrm{H}^1(X_0,\,\mathcal{L})$ be the homomorphism induced by the homomorphism $\mu$ in (4.14). It is straight–forward to check that

$$\mu_*(\Phi)\,=\,S'(\phi(1_{X_0}))$$

(see (4.9), (2.4) and (4.11) for $S'$, $\phi(1_{X_0})$ and $\Phi$ respectively). Indeed, this follows from (4.13) and the definition of $S(\delta)$ in (4.6). Combining this with (4.15), we have

$$S'(\phi(1_{X_0})) = \mu_*(\Phi) = \mu_*(\xi_*(\theta))\,.$$

Since $\mu \circ \xi = 0$ (see (4.14)), it now follows that $S'(\phi(1_{X_0})) = 0$. □

## 5. Logarithmic connections and semistability of underlying principal bundle

Let $\mathcal{T}_{g,n}$ be the Teichmüller space of genus $g$ compact Riemann surfaces with $n$ ordered marked points. As before, we assume that $g > 0$, and if $g = 1$, then $n > 0$. Let

$$\tau : \mathcal{C} \longrightarrow \mathcal{T}_{g,n}$$

be the universal Teichmüller curve with $n$ ordered sections $\Sigma$. The fiber of $\mathcal{C}$ on any point $t \in \mathcal{T}_{g,n}$ will be denoted by $\mathcal{C}_t$. The ordered subset $\mathcal{C}_t \cap \Sigma \subset \mathcal{C}_t$ will be denoted by $\Sigma_t$.

Take a $n$–pointed Riemann surface $(C_0\,, \Sigma_0)$ of genus $g$, which is represented by a point of $\mathcal{T}_{g,n}$. Let

$$\nabla_0 \tag{5.1}$$

be a logarithmic connection on a principal $G$–bundle $F_G \longrightarrow C_0$ with polar divisor $\Sigma_0$. By Riemann–Hilbert correspondence, the connection $\nabla_0$ produces a flat (isomonodromic) logarithmic connection $\nabla$ on a holomorphic principal $G$–bundle $\mathcal{F}_G \longrightarrow \mathcal{C}$ with polar divisor $\Sigma$.

The following lemma is a special case of the main theorem in [GN] (see also [Sh]). Although the families of $G$–bundles considered in [GN] are algebraic, all arguments there go through in the analytic case of our interest with obvious modifications.

**Lemma 5.1** ([GN]). *Let $\mathcal{F}_G \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}_{g,n}$ be as above. For each Harder–Narasimhan type $\kappa$, the set*

$$\mathcal{Y}_\kappa := \{t \in \mathcal{T}_{g,n} \mid \mathcal{F}_G|_{\mathcal{C}_t} \text{ is of type } \kappa\}$$

*is a (possibly empty) locally closed complex analytic subspace of $\mathcal{T}_{g,n}$. More precisely, for each Harder–Narasimhan type $\kappa$, the union $\mathcal{Y}_{\leq\kappa} := \bigcup_{\kappa' \leq \kappa} \mathcal{Y}_{\kappa'}$ is a closed complex analytic subset of $\mathcal{T}_{g,n}$. Moreover, the principal $G$–bundle*

$$\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)} \longrightarrow \tau^{-1}(\mathcal{Y}_\kappa)$$

*possesses a canonical reduction of structure group inducing the Harder–Narasimhan reduction of $\mathcal{F}_G|_{\mathcal{C}_t}$ for each $t \in \mathcal{Y}_\kappa$.*

In the following two Sections 5.1 and 5.2, we will see that under certain assumptions, the only Harder-Narasimhan stratum $\mathcal{Y}_\kappa$ of maximal dimension $\dim(\mathcal{T}_{g,n}) = 3g - 3 + n$ is the trivial one, in the sense that the Harder–Narasimhan parabolic subgroup is $G$ itself. In other words, if the principal $G$–bundle $F_G$ is not semistable, and therefore has a non-trivial Harder–Narasimhan reduction $E_P \subset E_G$ to a certain parabolic subgroup $P \subsetneq G$, then there is always an isomonodromic deformation in which direction the reduction $E_P$ is obstructed.

5.1. **The case of $n = 0$.** In this subsection we assume that $n = 0$. So, we have $g > 1$.

**Theorem 5.2.** *There is a closed complex analytic subset $\mathcal{Y} \subset \mathcal{T}_{g,0}$ of codimension at least $g$ such that for any $t \in \mathcal{T}_{g,0} \setminus \mathcal{Y}$, the holomorphic principal $G$–bundle $\mathcal{F}_G|_{\mathcal{C}_t} \longrightarrow \mathcal{C}_t$ is semistable.*

*Proof.* Let $\mathcal{Y} \subset \mathcal{T}_{g,0}$ denote the (finite) union of all Harder-Narasimhan strata $\mathcal{Y}_\kappa$ as in Lemma 5.1 with nontrivial Harder-Narasimhan type $\kappa$. From Lemma 5.1 we know that $\mathcal{Y}$ is a closed complex analytic subset of $\mathcal{T}_{g,0}$.

Take any $t \in \mathcal{Y}_\kappa \subset \mathcal{Y}$. Let $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$ be the holomorphic principal $G$–bundle on $X_0 := \mathcal{C}_t$. The holomorphic connection on $E_G$ obtained by restricting $\nabla$ will be denoted by $\delta$. Since $E_G$ is not semistable, there is a proper parabolic subgroup $P \subsetneq G$ and a holomorphic reduction of structure group $E_P \subset E_G$ to $P$, such that $E_P$ is the Harder–Narasimhan reduction [Be], [AAB]. From Lemma 5.1 we know that $E_P$ extends to a reduction of structure group of the principal $G$–bundle $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to the subgroup $P$.

Let $\mathrm{ad}(E_P)$ and $\mathrm{ad}(E_G)$ be the adjoint vector bundles of $E_P$ and $E_G$ respectively. Consider the vector bundle

$$\mathrm{ad}(E_G)/\mathrm{ad}(E_P) \,=\, E_P(\mathfrak{g}/\mathfrak{p})$$

(see (3.2)). We know that

$$\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) \,<\, 0 \tag{5.2}$$

[AAB, p. 705] (see sixth line from bottom). In particular

$$\mathrm{degree}(E_P(\mathfrak{g}/\mathfrak{p})) \,<\, 0\,. \tag{5.3}$$

A holomorphic connection on $E_G$ induces a holomorphic connection on $\mathrm{ad}(E_G)$, hence $\mathrm{degree}(\mathrm{ad}(E_G)) = 0$. Combining this with (5.3) it follows that $\mathrm{degree}(\mathrm{ad}(E_P)) > 0$, because $\mathrm{ad}(E_G)/\mathrm{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$. Since $\mathrm{degree}(\mathrm{ad}(E_P)) \neq 0$, the principal $P$–bundle $E_P$ does not admit a holomorphic connection. Consequently, the second fundamental form $S(\delta)$ of $E_P$ for $\delta$ (see (4.6)) is nonzero.

Using the second fundamental form $S(\delta)$, construct the holomorphic line subbundle

$$\mathcal{L} \,\subset\, E_P(\mathfrak{g}/\mathfrak{p})$$

as done in (4.7). From (5.2) we have

$$\mathrm{degree}(\mathcal{L}) \,<\, 0\,. \tag{5.4}$$

Consider the short exact sequence of sheaves on $X_0$

$$0 \longrightarrow \mathrm{T}X_0 \xrightarrow{S(\delta)'} \mathcal{L} \longrightarrow T^\delta \,:=\, \mathcal{L}/(S(\delta)(\mathrm{T}X_0)) \longrightarrow 0\,, \tag{5.5}$$

where $S(\delta)'$ is constructed in (4.8); note that $T^\delta$ is a torsion sheaf because $S(\delta)' \neq 0$ (recall that $S(\delta) \neq 0$). From (5.4) it follows that

$$\mathrm{degree}(T^\delta) \,=\, \mathrm{degree}(\mathcal{L}) - \mathrm{degree}(\mathrm{T}X_0) \,<\, -\mathrm{degree}(\mathrm{T}X_0) \,=\, 2g-2\,.$$

So, we have

$$\dim \mathrm{H}^0(X_0,\, T^\delta) \,=\, \mathrm{degree}(T^\delta) \,<\, 2g-2 \,=\, \dim \mathrm{H}^1(X_0,\, \mathrm{T}X_0) + 1 - g\,.$$

This implies that

$$\dim \mathrm{H}^1(X_0,\, \mathrm{T}X_0) - \dim \mathrm{H}^0(X_0,\, T^\delta) \,\leq\, g\,. \tag{5.6}$$

Consider the long exact sequence of cohomologies

$$\mathrm{H}^0(X_0,\, T^\delta) \,\longrightarrow\, \mathrm{H}^1(X_0,\, \mathrm{T}X_0) \,\xrightarrow{\,\zeta\,}\, \mathrm{H}^1(X_0,\, \mathcal{L})$$

associated to the short exact sequence of sheaves in (5.5). From (5.6) it follows that

$$\dim \zeta(\mathrm{H}^1(X_0,\, \mathrm{T}X_0)) \,\geq\, g\,. \tag{5.7}$$

Since the reduction $E_P$ extends to a reduction of structure group of the principal $G$–bundle $\mathcal{F}_G\vert_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to the subgroup $P$, combining (5.7) and Proposition 4.3 we conclude that the codimension of the complex analytic subset $\mathcal{Y}_\kappa \,\subset\, \mathcal{T}_{g,0}$ is at least $g$. This completes the proof of the theorem. $\square$

5.2. **When $n$ is arbitrary.** Now we assume that $n \,>\, 0$.

A logarithmic connection $\eta$ on a holomorphic principal $G$–bundle $F_G \,\longrightarrow\, X_0$ is called *reducible* if there is pair $(P\,, F_P)$, where $P \,\subsetneq\, G$ is a parabolic subgroup and $F_P \,\subset\, F_G$ is a holomorphic reduction of structure group of $F_G$ to $P$, such that $\eta$ is induced by a connection on $F_P$. Note that $\eta$ is induced by a connection on $F_P$ if and only if the second fundamental form of $F_P$ for $\eta$ vanishes identically. A connection is called *irreducible* if it is not reducible or, equivalently, if the monodromy representation of the corresponding flat principal $G$–bundle does not factor through a proper parabolic subgroup of $G$.

**Proposition 5.3.** *Assume that the logarithmic connection $\nabla_0$ in* (5.1) *is irreducible. Then there is a closed complex analytic subset $\mathcal{Y} \,\subset\, \mathcal{T}_{g,n}$ of codimension at least $g$ such that for any $t \,\in\, \mathcal{T}_{g,n} \setminus \mathcal{Y}$, the holomorphic principal $G$–bundle $\mathcal{F}_G\vert_{\mathcal{C}_t}$ is semistable.*

*Proof.* The proof of Theorem 5.2 goes through after some obvious modifications. As before, let $\mathcal{Y} \,\subset\, \mathcal{T}_{g,n}$ be the locus of all points $t$ such that the principal $G$–bundle $\mathcal{F}_G\vert_{\mathcal{C}_t}$ is not semistable. Take any $t \,\in\, \mathcal{Y}_\kappa \,\subset\, \mathcal{Y}$. Let $E_G \,=\, \mathcal{F}_G\vert_{\mathcal{C}_t}$ be the holomorphic principal $G$–bundle on $X_0 \,:=\, \mathcal{C}_t$. The logarithmic connection on $E_G$ with polar divisor $D_0 \,:=\, \Sigma_t$ obtained by restricting $\nabla$ will be denoted by $\delta$.

Let $E_P \,\subset\, E_G$ be the Harder–Narasimhan reduction. Since $\nabla_0$ is irreducible, the second fundamental form $S(\delta)$ of $E_P$ for $\delta$ (see (4.6)) is nonzero. Consider the short exact sequence of sheaves on $X_0$

$$0 \,\longrightarrow\, \mathrm{T}X_0(-D_0) \,\xrightarrow{S(\delta)'}\, \mathcal{L} \,\longrightarrow\, T^\delta \,:=\, \mathcal{L}/(S(\delta)(\mathrm{T}X_0(-D_0))) \,\longrightarrow\, 0\,, \tag{5.8}$$

where $S(\delta)'$ is constructed in (4.8). As before, degree$(\mathcal{L}) \,<\, 0$, because $\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) \,<\, 0$. So, degree$(T^\delta) \,<\, -$degree$(\mathrm{T}X_0(-D_0)) \,=\, 2g-2+n$. We now have

$$\dim \mathrm{H}^0(X_0,\, T^\delta) \,=\, \mathrm{degree}(T^\delta) \,<\, 2g-2+n \,=\, \dim \mathrm{H}^1(X_0,\, \mathrm{T}X_0(-D_0)) + 1 - g\,.$$

Hence the dimension of the image of the homomorphism

$$\mathrm{H}^1(X_0,\, \mathrm{T}X_0(-D_0)) \,\longrightarrow\, \mathrm{H}^1(X_0,\, \mathcal{L}) \tag{5.9}$$

in the long exact sequence of cohomologies associated to (5.8) is at least $g$. Since the reduction $E_P$ extends to a reduction of $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to $P$, and the dimension of the image of the homomorphism in (5.9) is at least $g$, from Proposition 4.3 we conclude that the codimension of $\mathcal{Y}_\kappa \subset \mathcal{T}_{g,0}$ is at least $g$. $\square$

5.3. **Stability of underlying principal bundle.** We now assume that $g \geq 2$.

**Proposition 5.4.** *Assume that the logarithmic connection $\nabla_0$ in (5.1) is irreducible. There is a closed analytic subset $\mathcal{Y}' \subset \mathcal{T}$ of codimension at least $g-1$ such that for any $t \in \mathcal{T}_g \setminus \mathcal{Y}$, the holomorphic principal $G$–bundle $\mathcal{F}_G|_{\mathcal{C}_t}$ is stable.*

*Proof.* The proof is identical to the proof of Proposition 5.3. If $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$ is not stable, there is a maximal parabolic subgroup $P \subsetneq G$ and a holomorphic reduction of structure group $E_P \subset E_G$ to $P$, such that the quotient bundle

$$\mathrm{ad}(E_G)/\mathrm{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$$

is semistable of degree zero. Therefore, we have degree$(\mathcal{L}) \leq 0$. This implies that

$$\mathrm{degree}(T^\delta) \leq 2g-2-n\,.$$

Hence the dimension of the image of the homomorphism $\mathrm{H}^1(X_0,\, \mathrm{T}X_0(-D_0)) \longrightarrow \mathrm{H}^1(X_0,\, \mathcal{L})$ in the long exact sequence of cohomologies associated to (5.8) is at least $g-1$. $\square$

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