Ergodicity for multidimensional jump diffusions with position dependent jump rate

Eva Löcherbach, Victor Rabiet

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Abstract

We consider a jump type diffusion $X = (X_t)_t$ with infinitesimal generator given by

$$L\psi(x) = \frac{1}{2} \sum_{1 \le i,j \le d} a_{ij}(x) \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + g(x) \nabla \psi(x) + \int_{\mathbb{R}^d} (\psi(x + c(z,x)) - \psi(x)) \gamma(z,x) \mu(\mathrm{d}z)$$

where μ is of infinite total mass. We prove Harris recurrence of X using a regeneration scheme which is entirely based on the jumps of the process. Moreover we state explicit conditions in terms of the coefficients of the process allowing to control the speed of convergence to equilibrium in terms of deviation inequalities for integrable additive functionals.

Key words : Diffusions with jumps, Harris recurrence, Nummelin splitting, continuous time Markov processes, Additive functionals.

MSC 2000 : 60 J 55, 60 J 35, 60 F 10, 62 M 05

1 Introduction

Let $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$, defined on a probability space $(\Omega, \mathcal{A}, P)$ with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$, where μ is a $\sigma-$finite measure on $\mathbb{R}^d$ having infinite total mass. We consider a process $X = (X_t)_{t \ge 0}, X_t \in \mathbb{R}^d$, solution of

$$X_t = x + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_{[0,t]} \int_{\mathbb{R}^d \times \mathbb{R}_+} c(z, X_{s-}) 1\!\!1_{u \le \gamma(z, X_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \quad (1.1)$$

$x \in \mathbb{R}^d$, where W is an $m-$dimensional Brownian motion. The associated infinitesimal generator is given for smooth test functions by

$$L\psi(x) = \frac{1}{2} \sum_{1 \le i,j \le d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \psi(x) + g(x) \nabla \psi(x) + \int_{\mathbb{R}^d} (\psi(x + c(z,x)) - \psi(x)) \gamma(z,x) \mu(\mathrm{d}z) \quad (1.2)$$

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where $a = \sigma\sigma^*$. Notice that the jump rate at time t of process depends on the position of the process X_t itself, i.e. the intensity measure in the infinitesimal operator L is $\gamma(z, x)\mu(\mathrm{d}z)$. Moreover, since μ has infinite total mass, jumps occur with infinite activity, i.e. the process possesses infinitely many small jumps during any finite time interval $[0, T]$.

The principal aim of the present paper is to give easily verifiable conditions on the coefficients g, σ, c and γ under which the process is recurrent in the sense of Harris and satisfies the ergodic theorem, without imposing any non-degeneracy condition on the diffusive part. Recall that a process X is called recurrent in the sense of Harris if it possesses an invariant measure m such that any set A with $m(A) > 0$ is visited infinitely often almost surely (see Azéma, Duflo and Revuz [2] (1969)): For all $x \in \mathbb{R}^d$,

$$P_x\Big[\int_0^\infty 1_A(X_s)\,\mathrm{d}s = \infty\Big] = 1.$$

For classical jump diffusions there starts to be a huge literature on this subject, see e.g. Masuda [8] (2007) who works in a simpler situation where the "censure" (or "intensity") term $\mathbb{1}_{u \le \gamma(z, X_{s-})}$ is not present. In order to prove recurrence, Masuda follows the Meyn and Tweedie approach developed in [9] or [10]. Kulik [6] (2009) uses the stratification method in order to prove exponential ergodicity of jump diffusions, but the models he considers do not include the censured situation neither. Finally, let us mention Duan and Qiao [5] (2014) who are interested in solutions driven by non-Lipschitz coefficients.

On the contrary to the above mentioned papers, in our model, jumps occur at a given intensity depending both on the current position of the process and on an additional noise z. The presence of this intensity term $\gamma(z, X_{s-})$ in (1.1) is in fact quite natural, but it implies that the study of X is technically more involved than the non-censured situation when γ is lower-bounded and strictly positive.

The aim of the present paper is to show that the jumps themselves can be used in order to generate a splitting scheme which implies the recurrence of the process. The method we use is the so-called regeneration method which we apply to the big jumps. More precisely, for some suitable set E such that $\mu(E) < \infty$, we cut the trajectory of X into excursions between successive jumps appearing due to "noise" z belonging to E. In spirit of the splitting technique introduced by Nummelin [11] (1978) and Athreya and Ney [1] (1978), we state a non-degeneracy condition which guarantees that the jump operator associated to the big jumps possesses a Lebesgue absolutely continuous component. This amounts to imposing that the partial derivatives of the jump term c with respect to the noise z are sufficiently non-degenerate, see (2.7) and (2.8) below. We stress that we do not need any non-degeneracy condition for the diffusion coefficient σ.

In this way we are able to construct a sequence of regeneration times R_n such that the trajectories $(X_{(R_n+t), t < R_{n+1} - R_n})_{n \ge 1}$ are i.i.d. In particular, "regeneration generates independence immediately", i.e. at each regeneration time R_n, the "future" $X_{R_n+t}, t \ge 0$ is independent of the past $\sigma\{X_s, s < R_n\}$, without imposing any time lag as usual in the study of processes in continuous time.

Notice that we do not apply the splitting technique to an extracted sampled chain nor to the resolvent chain as in Meyn and Tweedie [10] (1993); the loss of memory needed for regeneration is produced only by big jumps. This approach is very natural in this context, but does not seem to be used so far in the literature, except for Xu [14] (2011), who works in a very specific frame where the jumps do not depend on the position of the process.

Our paper is organized as follows. In Section 2 we state our main assumptions, prove a lower bound which is of local Doeblin type and state our main results on Harris recurrence and speed of convergence to equilibrium of the process. Section 3 introduces the regeneration technique based on big jumps and proves the existence of certain (polynomial) moments of the associated regeneration times. Section 4 is devoted to an informal discussion on explicit and easily verifiable conditions stated in terms of the coefficients g, σ, c and γ which imply the Harris recurrence. Finally, we give in Section 5 a proof of the local Doeblin condition.

2 Notations

Consider a Poisson random measure $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$, defined on a probability space $(\Omega, \mathcal{A}, P)$, with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$, where μ is a $\sigma-$finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ of infinite total mass. Let $X = (X_t)_{t\geq 0}, X_t \in \mathbb{R}^d$, be a solution of

$$X_t = x + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_{[0,t]} \int_{\mathbb{R}^d \times \mathbb{R}_+} c(z, X_{s-}) \mathbb{1}_{u \leq \gamma(z, X_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \quad (2.3)$$

$x \in \mathbb{R}^d$, where W is an $m-$dimensional Brownian motion. Write $I\!\!F = (\mathcal{F}_t)_{t\geq 0}$ for the canonical filtration of the process given by

$$\mathcal{F}_t = \sigma\{W_s, N([0,s] \times A \times B), s \leq t, A \in \mathcal{B}(\mathbb{R}^d), B \in \mathcal{B}(\mathbb{R}_+)\}.$$

Throughout this paper, for any $x \in \mathbb{R}^d$, $|x|$ will denote the Euclidean norm on $\mathbb{R}^d$. Moreover, for $d \times -d$ matrices A, $\|A\|$ denotes the associated operator norm.

2.1 Assumptions

In order to grant existence and uniqueness of the above equation, throughout this article, we impose the following conditions on the coefficients g, σ, c and γ.

Assumption 2.1 *1. g and σ are globally Lipschitz continuous; σ is bounded.*

2. c and γ are Lipschitz continuous with respect to x, i.e.

$$|c(z,x) - c(z,x')| \leq L_c(z)|x - x'| \text{ and } |\gamma(z,x) - \gamma(z,x')| \leq L_\gamma(z)|x - x'|,$$

where $L_c,\ L_\gamma : \mathbb{R}^d \to \mathbb{R}_+$.

3. $\sup_x \int_{\mathbb{R}^d} (L_c(z)\gamma(z,x) + L_\gamma(z)|c(z,x)|)\mu(\mathrm{d}z) < \infty$.

4. $\sup_x \int_{\mathbb{R}^d} \gamma(z,x)|c(z,x)|\mu(\mathrm{d}z) < \infty$.

5. $\lim_{\Gamma\to\infty} \sup_x \int_{\mathbb{R}^d} \gamma(z,x)|c(z,x)| 1_{\{\Gamma \leq \gamma(z,x)\}} \mu(\mathrm{d}z) = 0$.

Under these assumptions, Theorem 3 of Bally and Rabiet [3] (2015) implies that (2.3) admits a unique non-explosive adapted solution which is Markov, having càdlàg trajectories.

Notice that our assumptions *do not imply* that there exists a finite total jump rate

$$\int_{\mathbb{R}^d} \gamma(z,x)\mu(dz)$$

for any $x \in \mathbb{R}^d$. In other words, the above integral might be equal to $+\infty$, and jumps occur with infinite activity. We also stress that due to the presence of the censure term $\mathbb{1}_{u \leq \gamma(z, X_{s-})}$

in equation (2.3) we are not in the classical frame of jump diffusions where the jump term depends in a smooth manner on z and x.

In the present article we are seeking for conditions ensuring that the process X is recurrent in the sense of Harris without using additional regularity of the coefficients, based on some minimal non-degeneracy of the jumps and without imposing any non-degeneracy condition on σ. Recall that a process X is called recurrent in the sense of Harris if it possesses an invariant measure m such that any set A of positive $m-$measure $m(A) > 0$ is visited infinitely often by the process almost surely (see Azéma, Duflo and Revuz [2] (1969)): For all $x \in \mathbb{R}^d$,

$$P_x\Big[\int_0^\infty 1_A(X_s)\,\mathrm{d}s = \infty\Big] = 1.$$

We will prove Harris recurrence by introducing a splitting scheme that is entirely based on the "big" jumps of X. In order to do so, we introduce the following additional assumption.

Assumption 2.2 *1. Writing the Lebesgue decomposition $\mu = \mu_{ac} + \mu_s$, with $\mu_{ac}(\mathrm{d}z) = h(z)\,\mathrm{d}z$, for some measurable function $h \geq 0 \in \mathrm{L}^1{}_{loc}(\lambda)$, λ the Lebesgue measure on $\mathbb{R}^d$, we suppose that there exists $z_0 \in \mathbb{R}^d$ and $R > 0$ such that*

$$\inf_{z \in \mathbb{R}^d : |z - z_0| \leq R} h(z) > 0.$$

2. There exists a non-decreasing sequence $(E_n)_n$ of subsets of $\mathbb{R}^d$ and an increasing sequence of positive numbers $\bar{\gamma}_n$ with $\bar{\gamma}_n \uparrow +\infty$ as $n \to \infty$, such that $\bigcup E_n = \mathbb{R}^d$,

$$\int_{E_n} \gamma(z,x)\mu(\mathrm{d}z) =: \bar{\gamma}_n(x) \leq \bar{\gamma}_n < \infty \tag{2.4}$$

for all n.

2.2 A useful lower bound

We fix some n. Thanks to (2.4), we can couple the process X_t with a rate $\bar{\gamma}_n-$Poisson process $N^{[n]} = (N_t^{[n]})_{t\geq 0}$ such that jumps of X_t produced by noise $z \in E_n$,

$$\Delta X_t = \int_{E_n}\int_0^\infty c(z, X_{t-}) 1_{u \leq \gamma(z, X_{t-})} N(dt, dz, du),$$

can only occur at the jump times $T_k^{[n]}, k \geq 1$, of $N^{[n]}$.

Let $\Pi(x, dy) = \mathcal{L}(X_{T_k^{[n]}} | X_{T_k^{[n]}-} = x)(dy)$ be the associated transition kernel. Our aim is to obtain a local Doeblin condition of the type

$$\Pi(x, dy) \geq 1_C(x)\beta\nu(dy), \tag{2.5}$$

for a suitable measurable set C, some $\beta \in]0, 1[$ and a suitable probability measure ν.

First notice that $\bar{\gamma}_n$ in (2.4) is only an upper bound on the total jump rate produced by noise belonging to E_n. As a consequence, for any $k \geq 1$ and on the event that $X_{T_k^{[n]}-} = x$, jumps are only accepted with probability $\frac{\bar{\gamma}_n(x)}{\bar{\gamma}_n}$. Moreover, it is easy to see that the following lower bound holds. Write $\mathcal{K} = \overline{B(z_0, R)}$ with z_0 and R chosen according to Assumption 2.2 item 1. Then

$$\Pi(x, V) \geq \frac{1}{\bar{\gamma}_n} \int_{E_n \cap \mathcal{K}} \gamma(z, x) 1\!\!1_V(x + c(z, x)) \mu(\mathrm{d}z)$$
$$\geq \frac{1}{\bar{\gamma}_n} \int_{E_n \cap \mathcal{K}} \gamma(z, x) 1\!\!1_V(x + c(z, x)) h(z)\, \mathrm{d}z, \quad (2.6)$$

where h is the Lebesgue density of the absolute continuous part of μ. It is natural to use a change of variables in the r.h.s. of the above lower bound, i.e. to replace, for fixed initial position x, the argument $x + c(x, z))$ by $y = y(z)$, on suitable subsets of $\mathbb{R}^d$ where $z \mapsto x + c(x, z)$ is a diffeomorphism. The difficulty is to control the dependence on the starting point x, since we are seeking for uniform lower bounds (2.5), uniform in $x \in C$. This uniform control is achieved in the following proposition.

Proposition 2.3 *Grant Assumption 2.2. Suppose moreover that there exist $x_0 \in \mathbb{R}^d$ and $r > 0$ such that for all $x \in \overline{B(x_0, r)}$,*
i) there exists $A > 0$ with

$$|\nabla_z c(z_0, x) h| \geq A|h|, \qquad \forall h \in \mathbb{R}^d, \quad (2.7)$$

ii) there exists $K > 0$ such that for all $z \in \overline{B(z_0, R)}$,

$$\left\| \big(\nabla_z c(z_0, x)\big)^{-1} \right\| \sum_{i,j} \left| \frac{\partial^2 c}{\partial z_i \partial z_j}(z, x) \right| \leq \frac{K}{d}, \quad (2.8)$$

iii)

$$\inf_{z: |z - z_0| \leq R, x: |x - x_0| \leq r} \gamma(z, x) h(z) = \varepsilon > 0, \; S = \sup_{z: |z - z_0| \leq R} \sup_{x: |x - x_0| \leq r} \sup_i |\partial_{z_i} c(z, x)| < \infty, \quad (2.9)$$

where $h(z)$ is the Lebesgue density of the absolutely continuous part of μ.

Fix n_0 with $\overline{B(z_0, R)} \subset E_{n_0}$. Then there exist $\eta > 0$ and some ball $B \subset \mathbb{R}^d$ such that for all $n \geq n_0$,

$$\inf_{x \in B(x_0, \eta)} P[X_{T_k^{[n]}} \in V | X_{T_k^{[n]}-} = x] \geq \frac{1}{\bar{\gamma}_n S^d} \varepsilon \lambda(V \cap B). \quad (2.10)$$

As a consequence of Proposition 2.3, the local Doeblin condition (2.5) holds with $C = B$, $\beta = \frac{\lambda(B)\varepsilon}{\bar{\gamma}_n S^d} \wedge 1$ and $\nu(dy) = \frac{1}{\lambda(B)} 1_B(y) dy$. Notice that the set C is not a "petite" set in the sense of Meyn and Tweedie (1993) [10].

The main ingredient of the proof of Proposition 2.3 is the following result.

Lemma 2.4 *Let $\Psi_x(z) = x + c(z, x)$, $\mathcal{K} = \overline{B(z_0, R)}$, $\Psi_x(\mathcal{K}) = \{\Psi_x(z), z \in \mathcal{K}\}$ and $a_x = x + c(z_0, x) = \Psi_x(z_0)$. Put*

$$\rho = \frac{A}{2}\Big(R \wedge \frac{1}{2K}\Big). \quad (2.11)$$

Then there exists $\eta > 0$ such that

$$B(a_{x_0}, \frac{\rho}{2}) \subset \bigcap_{x \in B(x_0, \eta)} \Psi_x(\mathcal{K}). \quad (2.12)$$

Moreover, for all $x \in \overline{B(x_0, r)}$, $B(a_x, \rho) \subset \Psi_x(\mathcal{K})$, and there exists $\mathcal{K}_x \subset \mathcal{K}$ such that $z \mapsto \Psi_x(z)$ is a $\mathcal{C}^1$-diffeomorphism from $\mathcal{K}_x$ to $B(a_x, \rho)$.

The proof of this Lemma and of Proposition 2.3 is given in Section 5 below.

Remark 2.5 *The ball B appearing in* (2.10) *can be chosen as $B = B(a_{x_0}, \rho/2)$ with ρ as in* (2.11) *and $a_{x_0} = x_0 + c(z_0, x_0)$. If moreover*

$$L_c = \sup_{z:|z-z_0|\le R} L_c(z) < \infty,$$

then we can choose

$$\eta = \frac{\rho}{2(1+L_c)} \wedge r = \frac{A\Big(R \wedge \frac{1}{2K}\Big)}{4(1+L_c)} \wedge r.$$

We close this section with two examples where the ball B and the radius η are explicitly given.

Example 2.6 *We consider the one-dimensional case, with $\mu(\mathrm{d}z) = \mathrm{d}z$. Throughout this example, f will be a bounded $1-$Lipschitz function such that $|f(x)| \ge \underline{f} > 0$ for all $x \in \overline{B(x_0, r)}$.*
1. Suppose that $c(z,x) = e^{-|z|} f(x)$ for all $z \in \overline{B(z_0, R)}$ and that $|z_0| \ge R + a$, $a > 0$. Then for all $x \in \overline{B(x_0, r)}$, $|\nabla_z c(z_0, x) h| = \big|f(x) e^{-|z_0|} h\big| \ge A|h|$ with $A = \underline{f} e^{-|z_0|}$. Moreover

$$\big\|\big(\nabla_z c(z_0,x)\big)^{-1}\big\| \left|\frac{\partial^2 c}{\partial z^2}(z,x)\right| = \frac{e^{|z_0|}}{|f(x)|} |f(x)| e^{-|z|} \le e^{|z_0| - a} =: K, \qquad \forall z \in \overline{B(z_0, R)}.$$

Recall that $B = B(a_{x_0}, \rho/2)$ where $a_{x_0} = x_0 + c(z_0, x_0) = x_0 + e^{-|z_0|} f(x_0)$. We have

$$\frac{\rho}{2} = \frac{A}{4}\Big(R \wedge \frac{1}{2K}\Big) = \frac{\underline{f} e^{-|z_0|}}{4}\Big(R \wedge \frac{e^{a-|z_0|}}{2}\Big).$$

Finally, since $L_c = \sup_{z:|z-z_0|\le R} L_c(z) \le e^{-a}$,

$$\eta = \frac{\rho}{2(1+L_c)} \wedge r \ge \frac{\underline{f} e^{-|z_0|}}{4(1+e^{-a})}\Big(R \wedge \frac{e^{a-|z_0|}}{2}\Big) \wedge r.$$

2. Suppose now that $c(z,x) = \dfrac{f(x)}{1+z^2}$ and that $|z_0| \ge R + a$, $a > 0$. Then for all $x \in \overline{B(x_0, r)}$, $|\nabla_z c(z_0, x) h| = \Big|f(x) \frac{2z_0}{(z_0^2+1)^2} h\Big| \ge A|h|$ with $A = \frac{2\underline{f}|z_0|}{(z_0^2+1)^2}$. Moreover

$$\begin{aligned}\big\|\big(\nabla_z c(z_0,x)\big)^{-1}\big\| \left|\frac{\partial^2 c}{\partial z^2}(z,x)\right| &= \frac{(z_0^2+1)^2}{2|f(x)||z_0|} \times 2|f(x)| \frac{|3z^2-1|}{(z^2+1)^3} \\ &\le \frac{(z_0^2+1)^2 |3(|z_0|+R)^2+1|}{|z_0|(a^2+1)^3} = K, \qquad \forall z \in \overline{B(z_0, R)}.\end{aligned}$$

In this case,

$$\frac{\rho}{2} = \frac{\underline{f}|z_0|}{2(z_0^2+1)^2}\Big(R \wedge \frac{|z_0|(a^2+1)^3}{2(z_0^2+1)^2|3(|z_0|+R)^2+1|}\Big).$$

Since $L_c = \sup_{z:|z-z_0|\le R} L_c(z) \le \frac{1}{1+a^2}$, we have

$$\eta = \frac{\rho}{2(1+L_c)} \wedge r \ge \Big[\frac{\underline{f}|z_0|}{2(1+a^2)(z_0^2+1)^2}\Big(R \wedge \frac{|z_0|(a^2+1)^3}{2(z_0^2+1)^2|3(|z_0|+R)^2+1|}\Big)\Big] \wedge r.$$

2.3 Drift criteria

The set $C = B(x_0, \eta)$ appearing in the local Doeblin condition (2.5) and (2.10) will play the role of a small set in the sense of Nummelin [11] (1978) and Meyn-Tweedie [9] (1993). In order to be able to profit from the lower bound (2.10), we have to show that $(X_{T_k^{[n]}-})_k$ comes back to the set C i.o. For that sake, we introduce a drift condition in terms of the continuous time process, inspired by Douc, Fort and Guillin [4] (2009).

Assumption 2.7 *There exists a continuous function $V : \mathbb{R}^d \to [1, \infty[$, an increasing concave positive function $\Phi : [1, \infty[\to (0, \infty)$ and a constant $b < \infty$ such that for any $s \geq 0$ and any $x \in \mathbb{R}^d$,*

$$E_x(V(X_s)) + E_x\left(\int_0^s \Phi \circ V(X_u)\,\mathrm{d}u\right) \leq V(x) + bE_x\left(\int_0^s 1\!\!1_{C'}(X_u)\,\mathrm{d}u\right), \tag{2.13}$$

where $C' = B(x_0, \frac{\eta}{2})$, η as in Proposition 2.3.

This drift condition ensures that the process comes back to the set $C' = B(x_0, \frac{\eta}{2})$ infinitely often. The choice of $\eta/2$ is on purpose and will be explained by Proposition 3.3 below.

If $V \in \mathcal{D}(L)$ belongs to the domain of the extended generator L of the process X, then Theorem 3.4 of Douc, Fort and Guillin [4] (2009) shows that the following condition

$$LV(x) \leq -\Phi \circ V(x) + b1\!\!1_B(x) \tag{2.14}$$

implies the above Assumption 2.13.

We discuss in Section 4 examples where (2.13) or (2.14) are verified.

Under Assumption 2.7, Douc, Fort and Guillin [4] (2009) give estimates on modulated moments of hitting times. Modulated moments are expressions of the type

$$E_x \int_0^\tau r(s) f(X_s)\,\mathrm{d}s,$$

where τ is a certain hitting time, r a rate function and f any positive measurable function. Knowledge of the modulated moments permits to interpolate between the maximal rate of convergence (taking $f \equiv 1$) and the maximal shape of functions f that can be taken in the ergodic theorem (taking $r \equiv 1$). In the present chapter we are interested in the maximal rate of convergence and hence we shall always take $f \equiv 1$.

For the function Φ of (2.13) put

$$H_\Phi(u) = \int_1^u \frac{\mathrm{d}s}{\Phi(s)},\ u \geq 1, \tag{2.15}$$

and

$$r_\Phi(s) = r(s) = \Phi \circ H_\Phi^{-1}(s). \tag{2.16}$$

If for instance $\Phi(v) = cv^\alpha$ with $0 \leq \alpha < 1$, this gives rise to polynomial rate functions

$$r(s) \sim Cs^{\frac{\alpha}{1-\alpha}};$$

$\alpha = 1$ yields $r(s) = ce^{cs}$. In most of the cases, we will deal with the case $\Phi(v) = cv^\alpha, 0 \leq \alpha < 1$ and thus work in the context of polynomial rates of convergence. In this situation, the most important technical feature about the rate function is the following sub-additivity property

$$r(t+s) \leq c(r(t) + r(s)), \tag{2.17}$$

for $t, s \geq 0$ and c a positive constant. We shall also use that

$$r(t+s) \leq r(t)r(s),$$

for all $t, s \geq 0$.

2.4 Main results

Theorem 2.8 *Grant the assumptions of Proposition 2.3, Assumptions 2.1, 2.2 and 2.7.*
1. The process X is recurrent in the sense of Harris having a unique invariant probability measure m such that $\Phi \circ V \in L^1(m)$. The invariant probability measure m is the unique solution of

$$\int_{\mathbb{R}^d} \left[\frac{1}{2} \sum_{1 \leq i,j \leq d} a_{ij}(x) \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + g(x) \nabla \psi(x) \right] m(\mathrm{d}x) =$$

$$\int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} (\psi(x) - \psi(x + c(z,x)))\gamma(z,x)\mu(\mathrm{d}z) \right] m(\mathrm{d}x), \quad (2.18)$$

for all $\psi \in C^2(\mathbb{R}^d)$ being of compact support.
2. Moreover, for any measurable function $f \in L^1(m)$, we have

$$\frac{1}{t} \int_0^t f(X_s)\, \mathrm{d}s \to m(f)$$

as $t \to \infty$, P_x-almost surely for any $x \in \mathbb{R}^d$.

The above ergodic theorem is an important tool e.g. for statistical inference based on observations of the process X in continuous time. In this direction, the following deviation inequality is of particular interest. Recall that ν is the measure given in the local Doeblin condition (2.5).

Theorem 2.9 *Grant the assumptions of Proposition 2.3, Assumptions 2.1, 2.2 and 2.7 with $\Phi(v) = cv^\alpha, 0 \leq \alpha < 1$. Put $p = 1/(1-\alpha)$. Let $f \in L^1(m)$ with $\|f\|_\infty < \infty$, x be any initial point and $0 < \varepsilon < \|f\|_\infty$. Then for all $t \geq 1$ the following inequality holds:*

$$P_x \left(\left| \frac{1}{t} \int_0^t f(X_s)\, \mathrm{d}s - m(f) \right| > \varepsilon \right) \leq K(p,\nu,X)\, V(x)\, t^{-(p-1)} \times$$

$$\times \left\{ \begin{array}{ll} \frac{1}{\varepsilon^{2(p-1)}} \|f\|_\infty^{2(p-1)} & \text{if } p \geq 2 \\ \frac{1}{\varepsilon^p} \|f\|_\infty^p & \text{if } 1 < p < 2 \end{array} \right\}. \quad (2.19)$$

Here $K(p,\nu,X)$ is a positive constant, different in the two cases, which depends on p, ν and on the process X, but which does not depend on f, t, ε.

Finally, we obtain the following quantitative control of the convergence of ergodic averages.

Proposition 2.10 *Grant the assumptions of Proposition 2.3, Assumptions 2.1, 2.2 and 2.7 with $\Phi(v) = cv^\alpha, 0 \leq \alpha < 1$. Then for any $x, y \in \mathbb{R}^d$,*

$$\|\frac{1}{t} \int_0^t P_s(x,\cdot)\, \mathrm{d}s - \frac{1}{t} \int_0^t P_s(y,\cdot) ds\|_{TV} \leq C \frac{1}{t} (V(x)^{(1-\alpha)} + V(y)^{(1-\alpha)}), \quad (2.20)$$

where $C > 0$ is a constant. In particular, if $\alpha \geq \frac{1}{2}$, then

$$\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - m\|_{TV} \leq C\frac{1}{t}V(x)^{(1-\alpha)}. \tag{2.21}$$

The proof of Theorems 2.8 and 2.9 and of Proposition 2.10 relies on the regeneration method that we are going to introduce now.

3 Regeneration for the chain of big jumps

3.1 Regeneration times

We show how the lower bound (2.10) on the jump kernel (2.5) allows us to introduce regeneration times for the process X.

We start "splitting" the jump transition kernel $\Pi(x, dy)$ of (2.5) in the following way. Since $\Pi(x, dy) \geq \beta 1_C(x)\nu(dy)$, we may introduce a split kernel $Q((x, u), dy)$, which is a transition kernel from $\mathbb{R}^d \times [0, 1]$ to $\mathbb{R}^d$, defined by

$$Q((x,u),dy) = \begin{cases} \nu(dy) & \text{if } (x,u) \in C \times [0,\beta] \\ \frac{1}{1-\beta}\left(\Pi(x,dy) - \beta\nu(dy)\right) & \text{if } (x,u) \in C \times]\beta, 1] \\ \Pi(x,dy) & \text{if } x \notin C. \end{cases} \tag{3.22}$$

Notice that

$$\int_0^1 Q((x,u),dy)du = \Pi(x,dy);$$

it is in this sense that $Q((x, u), dy)$ can be considered as "splitting" the original transition kernel Π by means of the additional "color" u.

We now show how to construct a version of the process X recursively over time intervals $[T_k^{[n]}, T_{k+1}^{[n]}[, k \geq 0$. We start at time $t = 0$ with $X_0 = x$ and introduce the process Z_t defined by

$$Z_t = x + \int_0^t g(Z_s)\,\mathrm{d}s + \int_0^t \sigma(Z_s)\,\mathrm{d}W_s + \int_0^t \int_{E_n^c} \int_0^\infty c(z, Z_{s-}) 1_{u \leq \gamma(z, Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

For $t < T_1^{[n]}$, we clearly have $Z_t = X_t$. Notice also that $T_1^{[n]}$ is independent of the rhs of the above equation and exponentially distributed with parameter $\bar{\gamma}_n$. We put $X_{T_1^{[n]}-} := Z_{T_1^{[n]}-}$ (notice that $Z_{T_1^{[n]}} = Z_{T_1^{[n]}-}$, since Z almost surely does not jump at time $T_1^{[n]}$). On $X_{T_1^{[n]}-} = x'$, we do the following.

1. We choose a uniform random variable $U_1 \sim U(0, 1)$, independently of anything else.
2. On $U_1 = u$, we choose a random variable $V_1 \sim Q((x', u), dy)$ and we put

$$X_{T_1^{[n]}} := V_1. \tag{3.23}$$

We then restart the above procedure with the new starting point V_1 instead of x.

We will write $\mathbf{X_t}$ for the process with additional color U_k, defined by

$$\mathbf{X_t} = \sum_{k \geq 0} 1_{[T_k^{[n]}, T_{k+1}^{[n]}[}(t)(X_t, U_k).$$

Remark 3.1 *Notice that the above splitting procedure does not even use the strong Markov property of the underlying process. It only uses the independence properties of the driving Poisson random measure.*

This new process is clearly Markov with respect to its filtration, and by abuse of notation we will not distinguish between the original filtration $I\!\!F$ introduced in Section 2 and the canonical filtration of $\mathbf{X_t}$. In this richer structure, where we have added the component U_k to the process, we obtain regeneration times for the process $\mathbf{X}$. More precisely, write

$$A := C \times [0, \beta]$$

and put

$$R_0 := 0,\ R_{k+1} := \inf\{T_m^{[n]} > R_k : \mathbf{X}_{T_m^{[n]}-} \in A\}. \tag{3.24}$$

Then we clearly have

Proposition 3.2 *a)* $\mathbf{X}_{R_k} \sim \nu(dx)U(\mathrm{d}u)$ *on* $R_k < \infty$, *for all* $k \geq 1$.
b) $\mathbf{X}_{R_k+\cdot}$ *is independent of* $\mathcal{F}_{R_k-}$ *on* $R_k < \infty$, *for all* $k \geq 1$.
c) If $R_k < \infty$ *for all* k, *then the sequence* $(\mathbf{X}_{R_k})_{k\geq 1}$ *is i.i.d.*

It is clear that in this way the speed of convergence to equilibrium of the process is determined by the moments of the extended stopping times R_k. In the next section we show that the drift condition of Assumption 2.7 ensures in particular that $R_k < \infty$ P_x-almost surely for any x.

3.2 Existence of moments of the regeneration times

Recall the local Doeblin condition (2.5) , the definition of the set C and of $C' = B(x_0, \eta/2)$. Let $\tau_{C'} = \inf\{t \geq 0 : X_t \in C'\}$ be the first hitting time of C'. It is known (Douc, Fort and Guillin [4] (2009)) that the condition (2.13) implies that

$$E_x \int_0^{\tau_{C'}} r(s)\,\mathrm{d}s \leq V(x), \tag{3.25}$$

where r is given as in (2.16).

Return times to C

In particular, equation (3.25) implies that $\tau_{C'} < \infty$ P_x-surely for all x. We show that this implies that the regeneration times R_k introduced in (3.24) above are finite almost surely. Recall that $T_k^{[n]}$ are the successive jump times of the dominating Poisson point process $N^{[n]}$ having rate $\bar{\gamma}_n$. The regeneration times R_k are expressed in terms of the jump chain $X_{T_k^{[n]}-}, k \geq 0$. We have to ensure that the control of return times to C' for the continuous time process imply analogous moments for the jump chain.

Before stating the first result going into this direction, we have to introduce the following objects. Let $\|\sigma\|_\infty$ be the sup-norm of the diffusion coefficient σ and let B be such that $|g(x)| \leq B(1 + |x|)$, $\forall x \in \mathbb{R}^d$. Since g is supposed to be globally Lipschitz continuous, such a constant B clearly exists. Finally, we choose n sufficiently large such that

$$\bar{\gamma}_n > B \tag{3.26}$$

(recall that $\bar\gamma_n \to \infty$ as $n \to \infty$) and such that

$$\|\sigma\|_\infty \frac{\sqrt{\pi}}{2} \frac{\bar\gamma_n}{(\bar\gamma_n - B)^{\frac{3}{2}}} + B_\eta \frac{\bar\gamma_n}{(\bar\gamma_n - B)^2} < \frac{\eta}{4}, \tag{3.27}$$

where $B_\eta = \sup_x \int_{\mathbb{R}^d} c(z,x)\gamma(z,x)\,\mathrm{d}\mu(z) + B\big(1 + |x_0| + \frac{\eta}{2}\big)$.

Proposition 3.3 *For any n verifying* (3.26) *and* (3.27),

$$\inf_{x\in C'} P_x(X_{T_1^{[n]}-} \in C) \geq \frac{1}{2}. \tag{3.28}$$

Remark 3.4 *The choice $\frac{1}{2}$ in the above lower bound is arbitrary, by choosing larger values of n, we could achieve any bound $1-\epsilon$ on the right hand side of* (3.28).

Proof Recall the process Z_t defined by

$$Z_t = x + \int_0^t g(Z_s)\,\mathrm{d}s + \int_0^t \sigma(Z_s)\,\mathrm{d}W_s + \int_0^t \int_{E_n^c} \int_0^\infty c(z, Z_{s-}) 1_{u\leq\gamma(z,Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and recall that for any $t < T_1^{[n]}$, $Z_t = X_t$. Recall also that $T_1^{[n]}$ is independent of the rhs of the above equation, exponentially distributed with parameter $\bar\gamma_n$. Now let $x \in C'$ and upper-bound

$$\mathrm{P}_x[X_{T_1^{[n]}-} \notin C] = \mathrm{P}_x[Z_{T_1^{[n]}-} \notin C].$$

Clearly, $\mathrm{P}_x\left[|Z_t - x| \geq \frac{\eta}{2}\right] \leq \frac{2}{\eta} E_x[|Z_t - x|]$.

Let $T > 0$. Then, with $Z_T^* = \sup_{t\in[0,T]} |Z_t|$ and $m \in \mathbb{N}$,

$$\begin{aligned}
E_x[|Z_t - x| \mathbb{1}_{Z_T^*<m}] &\leq E_x\Big[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s| \mathbb{1}_{Z_T^*<m}\Big] + E_x\Big[\int_0^t |g(Z_s)|\,\mathrm{d}s \mathbb{1}_{Z_T^*<m}\Big] \\
&\quad + E_x\Big[\int_0^t \int_{E_n^c} \int_0^\infty |c(z, Z_{s-})| 1_{u\leq\gamma(z,Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \mathbb{1}_{Z_T^*<m}\Big] \\
&\leq E_x\Big[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s|\Big] + E_x\Big[\int_0^t |g(Z_s)|\,\mathrm{d}s \mathbb{1}_{Z_T^*<m}\Big] \\
&\quad + E_x\Big[\int_0^t \int_{E_n^c} \int_0^\infty |c(z, Z_{s-})| 1_{u\leq\gamma(z,Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\Big]
\end{aligned}$$

with (using the Itô isometry and the fact that σ is bounded)

$$E_x\Big[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s|\Big] \leq \sqrt{E_x\Big[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s|^2\Big]} \leq \|\sigma\|_\infty \sqrt{t}.$$

Moreover, for $x \in B(x_0, \frac{\eta}{2}) = C'$,

$$\begin{aligned}
E_x\Big[\int_0^t |g(Z_s)|\,\mathrm{d}s \mathbb{1}_{Z_T^*<m}\Big] &\leq E_x\Big[\int_0^t B(1 + |Z_s|)\,\mathrm{d}s \mathbb{1}_{Z_T^*<m}\Big] \\
&= Bt + B\int_0^t E_x\big[|Z_s| \mathbb{1}_{Z_T^*<m}\big]\,\mathrm{d}s \\
&\leq Bt(1 + |x|) + B\int_0^t E_x\big[|Z_s - x| \mathbb{1}_{Z_T^*<m}\big]\,\mathrm{d}s \\
&\leq Bt\big(1 + |x_0| + \frac{\eta}{2}\big) + B\int_0^t E_x\big[|Z_s - x| \mathbb{1}_{Z_T^*<m}\big]\,\mathrm{d}s.
\end{aligned}$$

We upper bound

$$
\begin{aligned}
&E_x\Big[\int_0^t\int_{E_n^c}\int_0^\infty |c(z,Z_{s-})|1_{u\le\gamma(z,Z_{s-})}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\Big]\\
&\qquad\qquad = E_x\Big[\int_0^t\int_{E_n^c}\int_0^\infty |c(z,Z_{s-})|1_{u\le\gamma(z,Z_{s-})}\,\mathrm{d}z\,\mathrm{d}\mu(z)\,\mathrm{d}u\Big]\\
&\qquad\qquad\quad \le t\sup_x\int_{E_n^c} c(z,x)\gamma(z,x)\,\mathrm{d}\mu(z)\le t\sup_x\int_{\mathbb{R}^d} c(z,x)\gamma(z,x)\,\mathrm{d}\mu(z)
\end{aligned}
$$

and put $B_\eta=\sup_x\int_{\mathbb{R}^d} c(z,x)\gamma(z,x)\,\mathrm{d}\mu(z)+B\big(1+|x_0|+\frac{\eta}{2}\big)$. Then

$$E_x[|Z_t-x|\mathbb{1}_{Z_T^*<m}]\le\|\sigma\|_\infty\sqrt{t}+B_\eta t+B\int_0^t E_x\big[|Z_s-x|\mathbb{1}_{Z_T^*<m}\big]\,\mathrm{d}s.$$

Then Gronwall's lemma (see Proposition 5.5 in the appendix) implies that

$$E_x[|Z_t-x|\mathbb{1}_{Z_T^*<m}]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt},$$

for all $t\le T$.

Since Z_t is a càdlàg process, Z_T^* is finite almost surely. Therefore $|Z_t-x|\mathbb{1}_{Z_T^*<m}$ tends to $|Z_t-x|$ almost surely as $m\to\infty$, and monotone convergence implies that

$$E_x[|Z_t-x|]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt}, \tag{3.29}$$

for all $t\le T$. In the above rhs, the constants do not depend on T, hence (3.29) is actually true for any $t\ge 0$.

Furthermore

$$E_x[|Z_{t-}-x|]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt}, \tag{3.30}$$

which can be seen as follows. Using (3.29), we have, for $s<t$,

$$E_x[|Z_s-x|\mathbb{1}_{Z_t^*<m}]\le E_x[|Z_s-x|]\le(\|\sigma\|_\infty\sqrt{s}+B_\eta s)e^{Bs}\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt}$$

so, using dominated convergence when s tends to t from inferior values,

$$E_x[|Z_{t-}-x|\mathbb{1}_{Z_t^*<m}]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt},$$

and letting $m\to+\infty$, monotone convergence gives (3.30).

Now, $T_1^{[n]}$ is independent from $(Z_t)_t$, exponentially distributed with parameter $\bar\gamma_n$. By choice of n, $\bar\gamma_n>B$. Then

$$
\begin{aligned}
E_x[|Z_{T_1^{[n]}-}-x|]&\le\int_0^{+\infty}(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt}\bar\gamma_n e^{-\bar\gamma_n t}\,\mathrm{d}t\\
&=\bar\gamma_n\Big(\|\sigma\|_\infty\int_0^{+\infty}\sqrt{t}e^{-(\bar\gamma_n-B)t}\,\mathrm{d}t+B_\eta\int_0^{+\infty}te^{-(\bar\gamma_n-B)t}\,\mathrm{d}t\Big)\\
&=\bar\gamma_n\Big(\|\sigma\|_\infty\frac{\Gamma(\frac32)}{(\bar\gamma_n-B)^{\frac32}}+\frac{B_\eta\Gamma(2)}{(\bar\gamma_n-B)^2}\Big)=\bar\gamma_n\Big(\|\sigma\|_\infty\frac{\sqrt\pi}{2}\frac{1}{(\bar\gamma_n-B)^{\frac32}}+\frac{B_\eta}{(\bar\gamma_n-B)^2}\Big)
\end{aligned}
$$

for every $x\in B(x_0,\frac{\eta}{2})$. Since n was chosen to have

$$\|\sigma\|_\infty\frac{\sqrt\pi}{2}\frac{\bar\gamma_n}{(\bar\gamma_n-B)^{\frac32}}+B_\eta\frac{\bar\gamma_n}{(\bar\gamma_n-B)^2}<\frac{\eta}{4},$$

we obtain

$$\sup_{x\in B(x_0,\frac{\eta}{2})} \mathrm{P}_x[X_{T_1^{[n]}-} \notin C] \le \sup_{x\in B(x_0,\frac{\eta}{2})} \mathrm{P}_x[|Z_{T_1^{[n]}-} - x| \ge \frac{\eta}{2}]$$
$$\le \sup_{x\in B(x_0,\frac{\eta}{2})} \frac{2}{\eta} E_x[|Z_{T_1^{[n]}-} - x|] < \frac{1}{2}.$$

•

The above arguments imply the following statement.

Corollary 3.5 *Let* $S_1 = \inf\{T_k^{[n]}, k \ge 1 : X_{T_k^{[n]}-} \in C\}$. *Then* $P_x(S_1 < \infty) = 1$ *for all* x.

Proof We introduce the following sequence of stopping times.

$$t_1 = \tau_{C'}, s_1 = \inf\{T_k^{[n]} > t_1\}, \ldots, t_l = \inf\{s \ge s_{l-1} : X_s \in C'\}, s_l = \inf\{T_k^{[n]} > t_l\}.$$

The above stopping times are all finite almost surely. We put

$$\tau_* = \inf\{l : X_{s_l-} \in C\}.$$

Then, using (3.28), for any $x \in \mathbb{R}^d$,

$$P_x(\tau_* \ge n_0) \le \left(\frac{1}{2}\right)^{n_0},$$

which shows that $\tau_* < \infty$ P_x-almost surely for all x. In particular,

$$S_1 \le s_{\tau_*} < \infty$$

P_x-almost surely for all x. •

The above proof shows in particular that the polynomial control obtained for the first entrance time in C', obtained in (3.25) remains true for S_1. Moreover we have the following control on polynomial moments of the regeneration times.

Proposition 3.6 *Grant Assumption 2.7 with* $\Phi(v) = cv^\alpha, 0 \le \alpha < 1$. *Let* $p = \frac{1}{1-\alpha}$. *Then there exists a constant c such that*

$$E_x(S_1^p) \le cV(x). \tag{3.31}$$

Proof We adopt the notation of the proof of Corollary 3.5.

1. In what follows, c will denote a constant that might change from line to line. We start by studying $E_x \int_0^{s_1} r(s)\,\mathrm{d}s$, where r is as in (2.16) and $s_1 = \inf\{T_k^n > \tau_{C'}\}$. Let

$$\lambda = \bar{\gamma}_n$$

be the rate of the Poisson process associated to $T_k^{[n]}, k \ge 1$. Then by definition of s_1,

$$E_x \int_0^{s_1} r(s)\,\mathrm{d}s = E_x \int_0^{\tau_{C'}} r(s)\,\mathrm{d}s + E_x \int_{\tau_{C'}}^{s_1} r(s)\,\mathrm{d}s \le V(x) + E_x \int_{\tau_{C'}}^{s_1} r(s)\,\mathrm{d}s,$$

where we have used (3.25).

Now, using that $s_1 - \tau_{C'}$ is independent of $\mathcal{F}_{\tau_{C'}}$, exponentially distributed with parameter λ, we upper-bound

$$E_x \int_{\tau_{C'}}^{s_1} r(s)\,\mathrm{d}s = E_x E_{X_{\tau_{C'}}} \int_0^{s_1-\tau_{C'}} r(\tau_{C'}+s)\,\mathrm{d}s$$
$$\leq E_x(r(\tau_{C'}))E(r(s_1-\tau_{C'})) = cE_x(r(\tau_{C'})),$$

since $E_{X_{\tau_{C'}}}(r(s_1-\tau_{C'})) = \int_0^\infty \lambda e^{-\lambda t} r(t)\,\mathrm{d}t < \infty$ does not depend on $X_{\tau_{C'}}$.
Using that

$$r(t) \leq c + \int_0^t r(s)\,\mathrm{d}s, \tag{3.32}$$

we obtain

$$E_x(r(\tau_{C'})) \leq c + E_x \int_0^{\tau_{C'}} r(s)\,\mathrm{d}s \leq c + V(x).$$

Therefore,

$$E_x \int_0^{s_1} r(s)\,\mathrm{d}s \leq c + cV(x) \leq cV(x), \tag{3.33}$$

where we have used that $V(x) \geq 1$.

2. We now use $r(t+s) \leq r(t)r(s)$ in order to obtain a control of $E_x \int_0^{t_{\tau_*}} r(s)\,\mathrm{d}s$. We certainly have

$$\begin{aligned}
E_x \int_0^{t_{\tau_*}} r(s)\,\mathrm{d}s &= E_x \int_0^{t_1} r(s)\,\mathrm{d}s + \sum_{n\geq 1} E_x \int_{t_n}^{t_{n+1}} r(s)\,\mathrm{d}s 1_{\{n<\tau_*\}} \\
&\leq V(x) + \sum_{n\geq 1} E_x \left(1_{\{n-1<\tau_*\}} r(t_n) \int_0^{t_{n+1}-t_n} r(s)\,\mathrm{d}s \right) \\
&= V(x) + \sum_{n\geq 1} E_x \left(1_{\{n-1<\tau_*\}} r(t_n) E_{X_{t_n}} \int_0^{t_1} r(s)\,\mathrm{d}s \right) \\
&\leq V(x) + \sum_{n\geq 1} E_x \left(1_{\{n-1<\tau_*\}} r(t_n) V(X_{t_n}) \right),
\end{aligned} \tag{3.34}$$

where we have used (3.25) and the fact that $1_{\{n-1<\tau_*\}}$ is $\mathcal{F}_{s_{n-1}}-$measurable. Now, X_{t_n} belonging to C', we can upper-bound $V(X_{t_n}) \leq \|V\|_{C'} = c$, and obtain

$$E_x \int_0^{t_{\tau_*}} r(s)\,\mathrm{d}s \leq V(x) + c \sum_{n\geq 1} E_x \left(1_{\{n-1<\tau_*\}} r(t_n) \right). \tag{3.35}$$

We use $r(t+s) \leq r(t)r(s)$ and the Markov property with respect to t_1 to obtain

$$E_x \left(1_{\{n-1<\tau_*\}} r(t_n) \right) \leq E_x r(t_1) \sup_{y\in C'} E_y (r(t_{n-1}) 1_{\{n-2<\tau_*\}}).$$

Using (3.32), the first factor can be treated as follows

$$E_x r(t_1) \leq c + E_x \int_0^{t_1} r(s)\,\mathrm{d}s \leq c + V(x) \leq cV(x),$$

since $V(x) \geq 1$.

Further, let $p \in]\frac{1-\alpha}{\alpha} \vee 1, \frac{1}{\alpha}[$, and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, using that $P(n-2 < \tau_*) \leq (\frac{1}{2})^{n-2}$,

$$E_y \left(1_{\{n-2<\tau_*\}} r(t_{n-1}) \right) \leq E_y(r^p(t_{n-1}))^{1/p} \left(\frac{1}{2} \right)^{(n-2)/q}.$$

We have, by definition of r that $r^p(t) \leq c t^{\frac{\alpha}{1-\alpha} p}$, where $\frac{\alpha}{1-\alpha} p > 1$ by choice of p. Using Jensen's inequality we obtain

$$r^p(t_{n-2}) \leq c(n-2)^{\frac{\alpha}{1-\alpha}p-1} (t_1^{\frac{\alpha}{1-\alpha}p} + \ldots + (t_{n_2} - t_{n-3})^{\frac{\alpha}{1-\alpha}p}).$$

We now use, by choice of p, that $\frac{\alpha}{1-\alpha} p - 1 \leq \frac{\alpha}{1-\alpha} \frac{1}{\alpha} - 1 = \frac{\alpha}{1-\alpha}$, and therefore

$$t^{\frac{\alpha}{1-\alpha}p} = \frac{1}{\frac{\alpha}{1-\alpha}p - 1} \int_0^t s^{\frac{\alpha}{1-\alpha}p-1} \, \mathrm{d}s \leq c \int_0^t r(s) \, \mathrm{d}s.$$

This allows to rewrite

$$r^p(t_{n-2}) \leq c(n-2)^{\frac{\alpha}{1-\alpha}p-1} \left(\int_0^{t_1} r(s) \, \mathrm{d}s + \ldots + \int_0^{t_{n_2} - t_{n-3}} r(s) \, \mathrm{d}s \right).$$

Using successively the Markov property at times $t_1, t_2, \ldots, t_{n-3}$, we obtain

$$E_y r^p(t_{n-2}) \leq c(n-2)^{\frac{\alpha}{1-\alpha}p-1} (n-2) \sup_{z \in C'} E_z \int_0^{t_1} r(s) \, \mathrm{d}s.$$

Finally, by (3.25), $\sup_{z \in C'} E_z \int_0^{t_1} r(s) \, \mathrm{d}s \leq \sup_{z \in C'} V(z) = c$, and therefore

$$(E_y r^p(t_{n-2}))^{1/p} \leq c(n-2)^{\frac{\alpha}{1-\alpha}}.$$

Coming back to (3.35) we conclude that

$$E_x \int_0^{t_{\tau^*}} r(s) \, \mathrm{d}s \leq V(x) + cV(x) \sum_{n \geq 1} \left(\frac{1}{2} \right)^{\frac{n-2}{q}} (n-2)^{\frac{\alpha}{1-\alpha}} \leq cV(x).$$

3. We now argue as follows.

$$\begin{aligned} E_x \int_0^{s_{\tau^*}} r(s) \, \mathrm{d}s &= E_x \int_0^{t_{\tau^*}} r(s) \, \mathrm{d}s + E_x \int_{t_{\tau^*}}^{s_{\tau^*}} r(s) \, \mathrm{d}s \\ &\leq cV(x) + \sum_{n \geq 1} E_x 1_{\{\tau^* = n\}} \int_{t_n}^{s_n} r(s) \, \mathrm{d}s \\ &\leq cV(x) + \sum_{n \geq 1} E_x 1_{\{\tau^* > n-1\}} r(t_n) \int_0^{s_n - t_n} r(s) \, \mathrm{d}s \\ &\leq cV(x) + \sum_{n \geq 1} E_x [1_{\{\tau^* > n-1\}} r(t_n) \sup_{y \in C'} \int_0^{s_1} r(s) \, \mathrm{d}s] \\ &\leq cV(x) + c \sum_{n \geq 1} E_x 1_{\{\tau^* > n-1\}} r(t_n), \end{aligned}$$

where we have used the Markov property with respect to t_n and (3.33). The last sum is treated as (3.35), which concludes our proof, since $r(s) \geq c s^{\frac{\alpha}{1-\alpha}}$.

•

The above result implies an analogous control for moments of the regeneration times R_k of (3.24). More precisely, we can now define

$$S_l = \inf\{T_k^{[n]} > S_{l-1} : X_{T_k^{[n]}-} \in C\}, l \geq 2,$$

and let

$$R_1 = \inf\{S_l : U_l \leq \beta\}, R_{k+1} = \inf\{S_l > R_k : U_l \leq \beta\}. \tag{3.36}$$

An analogous argument as the one used in the proof of Proposition 3.6 then implies

Theorem 3.7 *Grant Assumption 2.7 with $\Phi(v) = cv^\alpha, 0 \leq \alpha < 1$ and let $p = 1/(1-\alpha)$. Then*

$$E_x R_1^p \leq cV(x). \tag{3.37}$$

We are now ready to prove Theorems 2.8 and 2.9.

3.3 Proof of Theorems 2.8 and 2.9

Proof of Theorem 2.8.
Let

$$\mathbf{m}(O) := E \int_{R_1}^{R_2} 1_O(\mathbf{X_s})\, \mathrm{d}s,$$

for any measurable set O. By the strong law of large numbers, any set O with $\mathbf{m}(O) > 0$ is visited i.o. P_x-almost surely by the process $\mathbf{X}$, for any starting point $(x, u) \in \mathbb{R}^d \times [0, 1]$. Hence, the process is recurrent in the sense of Harris, and by the Kac occupation time formula, $\mathbf{m}$ is the unique invariant measure of the process (unique up to multiplication with a constant).

Now, recall that ν is of compact support, hence $V \in L^1(\nu)$. Using (3.37) in the case $\alpha = 0$, we obtain $\mathbf{m}(\mathbb{R}^d \times [0, 1]) = E(R_2 - R_1) = E_\nu R_1 \leq c\nu(V) < \infty$. This implies that $\mathbf{X}$ is positive recurrent.

The invariant measure m of the original process X is the projection onto the first coordinate of $\mathbf{m}$. In particular, X is also positive Harris recurrent, and m can be represented as

$$m(f) = E \int_{R_1}^{R_2} f(X_s)\, \mathrm{d}s.$$

The ergodic theorem is then simply a consequence of the positive Harris property of X. Finally, the fact that $\Phi \circ V \in L^1(m)$ is an almost immediate consequence of (2.13), based on Dynkin's formula. •

Proof of Theorem 2.9.
Theorem 2.9 follows from Theorem 5.2 of Löcherbach and Loukianova (2013) in [7] together with Proposition 3.6. •

We finally proceed to the proof of Proposition 2.10.

Proof of Proposition 2.10.
Let X and Y be a copies of the process, issued from x (from y respectively) at time 0. Let R_1 and R_1' be the respective regeneration times. Using the same realization V_k for X and for

Y (recall (3.23)), it is clear that R_1 and R'_1 are shift-coupling epochs for X and for Y, i.e. $X_{R_1+\cdot} = Y_{R'_1+\cdot}$. If follows then from Thorisson [13] (1994), see also Roberts and Rosenthal [12] (1996), Proposition 5, that

$$\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - \frac{1}{t}\int_0^t P_s(y,\cdot)ds\|_{TV} \le C\frac{1}{t}(E_x(R_1) + E_y(R'_1)). \tag{3.38}$$

Recall that $p = 1/(1-\alpha)$. Then

$$E_x(R_1) \le (E_x R_1^p)^{1/p} \le c(V(x))^{(1-\alpha)}.$$

Now, if $\alpha \ge \frac{1}{2}$, then $1-\alpha \le \alpha$ and therefore,

$$E_x(R_1) \le c\,\Phi \circ V(x) \in L^1(m).$$

In this case, we can integrate (3.38) against $m(dy)$ and obtain the second part of the assertion. •

4 Discussing the drift condition

In this section, we discuss in an informal way several easily verifiable sufficient conditions implying Assumption 2.7 with $\Phi(v) = cv^\alpha, 0 < \alpha \le 1$. These conditions will involve different coefficients of the process. Recall that the infinitesimal generator L of the process X is given for every $\mathcal{C}^2$-function ψ with compact support on $\mathbb{R}^d$ by

$$L\psi(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\psi(x) + b(x)\nabla\psi(x) + \int_{\mathbb{R}^d}\left[\psi(x + c(z,x)) - \psi(x)\right] K(x,\mathrm{d}z),$$

where $a = \sigma\sigma^*$ and $K(x,\mathrm{d}z) = \gamma(z,x)\mu(\mathrm{d}z)$. In order to grant Assumption 2.7, we are seeking for conditions implying that

$$LV \le -cV^\alpha(x) + b\mathbb{1}_{C'}(x), \tag{4.39}$$

for some $0 \le \alpha \le 1$, with $C' = B(x_0, \frac{\eta}{2})$ and $b, c > 0$.

Example 4.1 *If we choose for instance $V(x) = |x - x_0|^2$ and $\alpha = \frac{1}{2}$ it suffices to impose that for all $x \in \mathbb{R}^d \setminus C'$,*

$$\mathrm{Tr}(\sigma\sigma^*) + 2\langle g(x), x - x_0\rangle + \int_{\mathbb{R}^d}\langle 2(x - x_0) + c(z,x), c(z,x)\rangle\gamma(z,x)\mu(\mathrm{d}z) \le -c|x - x_0|. \tag{4.40}$$

We now discuss several concrete sufficient conditions implying (4.39). In this context, it is interesting to notice that the influence of the different coefficients can be quite different. Some coefficients can work in a favorable way in order to ensure (4.39). In that case we will say that they are "pushing" the diffusion into the set C'. Other coefficients might play a neutral role or even work against (4.39). Since we have three natural parts of coefficients (diffusion part, drift and the jump part), we will discuss here the following cases: "pushing" with the jumps only, "pushing" with jumps and drift together[1] and "pushing" with the drift only.

[1]this will be the most interesting case

Pushing with the jumps

Consider first a pure jump process, i.e. the case when $a = g = 0$. We choose $V(x) = |x - x_0|^2$ and propose the following conditions.
1. Global condition with respect to z. $\forall z \in \mathbb{R}^d$, $\forall x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\langle c(z,x) + 2(x - x_0), c(z,x)\rangle \leq 0. \tag{4.41}$$

2. Local conditions with respect to z on some set $\mathcal{K}$. There exists a set $\mathcal{K}$ such that the following holds.

1. there exists $\xi > 0$ such that for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\int_{\mathcal{K}} |c(z,x)|\gamma(z,x)\mu(\mathrm{d}z) > \xi. \quad {}^{2} \tag{4.42}$$

2. There exists $\zeta \in (0,1]$ such that for all $z \in \mathcal{K}$ and for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$.

$$\langle c(z,x) + 2(x - x_0), c(z,x)\rangle \leq -\zeta |c(z,x) + 2(x - x_0)||c(z,x)|, \tag{4.43}$$

3. For all $z \in \mathcal{K}$ and for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$|c(z,x)| \leq |x - x_0|. \tag{4.44}$$

Notice that this last condition implies in particular that $|c(z,x) + 2(x - x_0)| \geq |x - x_0|$. Then under the above conditions, for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\begin{aligned}
LV(x) &= \int_E (V(x + c(z,x)) - V(x))\gamma(z,x)\mu(\mathrm{d}z) \\
&= \int_E \langle c(z,x) + 2(x - x_0), c(z,x)\rangle \gamma(z,x)\mu(\mathrm{d}z) \\
&\leq -\zeta \int_{\mathcal{K}} |c(z,x) + 2(x - x_0)||c(z,x)|\gamma(z,x)\mu(\mathrm{d}z) \\
&\leq -\zeta |x - x_0| \int_{\mathcal{K}} |c(z,x)|\gamma(z,x)\mu(\mathrm{d}z) \\
&\leq -\zeta |x - x_0| \xi = -c(V(x))^{\frac{1}{2}}
\end{aligned}$$

with $c = \zeta\xi > 0$.

Remark 4.2 *1. Using the Cauchy-Schwarz inequality,* (4.41) *implies that for all* $z \in \mathbb{R}^d$ *and for all* $x \in B\big(x_0, \frac{\eta}{2}\big)^c$, $|c(z,x)| \leq 2|x - x_0|$. *In particular for all* $x \in B\big(x_0, \frac{\eta}{2}\big)^c$, $\sup_{z \in \mathbb{R}^d} |c(z,x)| < +\infty$.
2. The condition (4.44) *is a natural condition to force the process to enter into the set* $B\big(x_0, \frac{\eta}{2}\big)$.
3. There is a simple geometric interpretation of the conditions (4.43) *and* (4.44). *Indeed, they lead to the (effective) condition*

$$\langle c(z,x) + 2(x - x_0), c(z,x)\rangle \leq -\zeta |x - x_0||c(z,x)|$$

[2] This condition has to be seen in relation with Condition (2.9).

or

$$2\langle (x-x_0), c(z,x)\rangle + |c(z,x)|^2 \le -\zeta |x-x_0||c(z,x)|.$$

On the one hand, this implies that $\langle (x-x_0), c(z,x)\rangle \le -\frac{\zeta}{2}|x-x_0||c(z,x)|$, *which means that* $c(z,x)$ *belongs to the convex cone of direction* $(x-x_0)$ *and angle* $\arccos\big(-\frac{\zeta}{2}\big)$. *On the other hand, using* (4.44), *the following condition*

$$2\langle (x-x_0), c(z,x)\rangle + |c(z,x)||x-x_0| \le -\zeta |x-x_0||c(z,x)|$$

is a sufficient (but not necessary!) condition which leads to $\langle (x-x_0), c(z,x)\rangle \le -\frac{(1+\zeta)}{2}|x-x_0||c(z,x)|$. *In other words, it suffices that* $c(z,x)$ *belongs to the convex cone of direction* $(x-x_0)$ *and angle* $\arccos\big(-\frac{(1+\zeta)}{2}\big)$.

The above conditions on the jump mechanism are naturally quite restrictive since they ensure that from everywhere in $\mathbb{R}^d \setminus C'$, the jumps force the process into the set C'. Nevertheless, this example is useful, and we will come back to these arguments later when discussing the influence of the drift coefficient.

In a next step, let us suppose that $\sigma \neq 0$. Then under the above conditions, for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\begin{aligned} LV(x) &= \mathrm{Tr}(\sigma(x)\sigma^*(x)) + \int_E \langle c(z,x) + 2(x-x_0), c(z,x)\rangle \gamma(z,x)\mu(\mathrm{d}z) \\ &\le \mathrm{Tr}(\sigma(x)\sigma^*(x)) - \zeta|x-x_0|\xi. \end{aligned}$$

Let $\Sigma = \sup_{x\in B\left(x_0,\frac{\eta}{2}\right)^c} \frac{|\mathrm{Tr}(\sigma(x)\sigma^*(x))|}{|x-x_0|}$ and suppose that $\mathcal{K}$ is such[3] that $\zeta\xi > \Sigma$. Then

$$LV(x) \le -c(V(x))^{\frac{1}{2}}$$

with $c = \zeta\xi - \Sigma$.

Finally, if $g \neq 0$, the minimal additional condition $\langle x-x_0, g(x)\rangle < 0$, for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$, ensures that the above result will remain true.

Pushing with both jumps and drift part

The conditions we made on the jump mechanism in the above paragraph are of course very strong. In this paragraph, we will therefore consider that these conditions hold only for x belonging to some set E_1. Moreover, we will suppose that the drift coefficient contributes to force the diffusion into C' when x belongs to another set E_2.

More precisely, we suppose that $E_1 \subset B\big(x_0, \frac{\eta}{2}\big)^c$ and put $E_2 = B\big(x_0, \frac{\eta}{2}\big)^c \setminus E_1$. We will impose the global condition (4.41) but aim to weaken the conditions (4.42), (4.43) and (4.44) by replacing $x \in B\big(x_0, \frac{\eta}{2}\big)^c$ by $x \in E_1$. For $x \in E_2$, we assume additionally that

$$\mathrm{Tr}(\sigma\sigma^*) + 2\langle g(x), x-x_0\rangle \le -c|x-x_0|.$$

Such a condition is true for example if

$$g(x) = -\frac{1}{2}(c+\Sigma)\frac{x-x_0}{|x-x_0|},$$

where we recall that $\Sigma = \sup_{x\in B\left(x_0,\frac{\eta}{2}\right)^c} \frac{|\mathrm{Tr}(\sigma(x)\sigma^*(x))|}{|x-x_0|}$.

[3] Actually it is always possible to multiply $\gamma(z,x)$ with a sufficiently large constant ensuring that $\xi > \Sigma/\zeta$.

Example 4.3 *We continue Example 2.6 item 1. and consider the one-dimensional case with* $\mu(\mathrm{d}z) = \mathrm{d}z$ *and* $c(z,x) = e^{-|z|} f(x)$*. We suppose that* $x_0 = 0$ *and let* $E_1 = [-M - \frac{\eta}{2}] \cup [\frac{\eta}{2} + M]$*. Moreover we choose* $\mathcal{K} = [a, a+2R]$ *in such a way that* $\int_{\mathcal{K}} e^{-|z|}\,\mathrm{d}z = \frac{1}{2}$*. Finally we will suppose that for all* $(z,x) \in \mathcal{K} \times E_1$

$$\gamma(z,x) \geq \underline{\gamma} > 0 \qquad \text{and} \qquad f(x) \geq \underline{f} > 0 \tag{4.45}$$

with

$$\underline{f} \cdot \underline{\gamma} > 2\frac{\Sigma}{\zeta}. \tag{4.46}$$

It is clear that (4.42) *is verified for all* $(z,x) \in \mathcal{K} \times E_1$*, and, moreover, that the jumps are strong enough to ensure the drift condition even in presence of the Brownian part.*

If we impose moreover that for all $x \in B\big(0, \frac{\eta}{2}\big)^c$, $|f(x)| \leq |x|$, *then* (4.44) *is satisfied. Adding finally the condition that for all* $x \in B\big(0, \frac{\eta}{2}\big)^c$, $\operatorname{sgn}(f(x)) = -\operatorname{sgn}(x)$, (4.41) *is true as well and* (4.43) *follows with* $\zeta = 1$.

Pushing partially with both

In the last paragraph we supposed that on the subset E_2 where the drift is driving the process towards C', the jumps do not act in a contradictory way – this is actually ensured by the condition (4.41). Notice that it is a priori not possible to weaken this assumption on E_2. Indeed, without condition (4.41) we have the following structural problem: we cannot even be sure that

$$\int_E |c(z,x)|^2 \gamma(z,x) \mu(\mathrm{d}z) < +\infty. \tag{4.47}$$

Moreover if we do not suppose a global condition as (4.41), it will be necessary to compensate the possible non-negative part $\int_{E \setminus \mathcal{K}} \langle c(z,x) + 2(x - x_0), c(z,x) \rangle \gamma(z,x) \mu(\mathrm{d}z)$ due to jumps in order to obtain a suitable control for $LV(x)$.

Pushing only with the drift

If we decide to ensure the Lyapunov condition by means of the drift coefficient g only, in the same spirit as above, we could take $E_1 = \emptyset$, but would have to keep global conditions, like the condition (4.41), if we use the same Lyapunov function.

However, if we choose another Lyapunov function, the situation might be more favorable as we are going to explain now. Let for example $V(x) = |x|$ $(= \sqrt{x_1^2 + \cdots + x_d^2})$ for $x \in B\big(x_0, \frac{\eta}{2}\big)^c$. Then

$$\nabla V(x) = \frac{x}{|x|}, \qquad \frac{\partial^2}{\partial_i \partial_j} V(x) = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}.$$

Let D be such that

$$\int_E |c(z,x)| \gamma(z,x) \mu(\mathrm{d}z) \leq D$$

and $|a_{ij}| < D$, where $a = \sigma\sigma^*$. With $\gamma > 0$ such that $\gamma |x|_1 \leq |x|$ (where $|x|_1 = |x_1| + \cdots + |x_d|$) and $\tilde{D} \overset{\text{def}}{=} \frac{D}{2}\big(d + \frac{1}{\gamma^2}\big)$ we assume that g verifies, for every $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\langle x, g(x) \rangle \leq -c|x|^{1+\alpha} - D|x| - \tilde{D}, \tag{4.48}$$

for some $0 < \alpha \leq 1$. Then it is immediate to see that

$$
\begin{aligned}
LV(x) &\leq \frac{D}{2}\Big(\frac{d}{|x|} + \sum_{1\leq i,j\leq d} \frac{|x_i||x_j|}{|x|^3}\Big) + \frac{\langle x, g(x)\rangle}{|x|} + \underbrace{\int_E |c(z,x)|\gamma(z,x)\mu(\mathrm{d}z)}_{\leq D} \\
&= \frac{D}{2}\Big(\frac{d}{|x|} + \frac{|x|_1^2}{|x|^3}\Big) + \frac{\langle x, g(x)\rangle}{|x|} + D \\
&\leq \frac{\tilde{D}}{|x|} + \frac{\langle x, g(x)\rangle}{|x|} + D \\
&\leq \frac{\tilde{D}}{|x|} - \frac{1}{|x|}(c|x|^{1+\alpha} + D|x| + \tilde{D}) + D \leq -c|x|^{\alpha}.
\end{aligned}
$$

5 Proofs

5.1 Proof of Proposition 2.3

Proof We first admit Lemma 2.4 and we put $\mathcal{K} = \overline{B(z_0, R)}$. As a consequence, there exists a ball $B(x_0, \eta)$ such that for all $x \in B(x_0, \eta)$, $B(a_{x_0}, \frac{\rho}{2}) \subset \Psi_x(\mathcal{K})$. Choose $\mathcal{K}'' \subset \mathcal{K}$ such that $\Psi_x : \mathcal{K}'' \to B(a_{x_0}, \frac{\rho}{2})$ is a $\mathcal{C}^1$-diffeomorphism for all $x \in B(x_0, \eta)$.[4] Since for all $(z,x) \in \mathcal{K} \times B(x_0, \eta)$, $\gamma(z,x)h(z) \geq \varepsilon$, we now have

$$
\begin{aligned}
\int_{E_n} \mathbb{1}_V(\psi_x(z))\gamma(z,x)\,\mathrm{d}\mu(z) &\geq \varepsilon \int_{K''} \mathbb{1}_V(\psi_x(z))\,\mathrm{d}z \\
&= \varepsilon \int_{B(a_{x_0}, \frac{\rho}{2})} \mathbb{1}_V(y)|J_{\psi_x^{-1}}(y)|\,\mathrm{d}y.
\end{aligned}
$$

Put $z = \psi_x^{-1}(y)$, then

$$
|J_{\psi_x^{-1}}(y)| = \frac{1}{|J_{\psi_x}(z)|} = \frac{1}{|\det(\nabla_z c(z,x))|}
$$

and, using Hadamard's Inequality,

$$
|\det(\nabla_z c(z,x))| \leq \prod_{i=1}^{d} |\partial_{z_i} c(z,x)|.
$$

As a consequence, we obtain

$$
\int_{E_n} \mathbb{1}_V(\psi_x(z))\gamma(z,x)\,\mathrm{d}\mu(z) \geq \frac{\varepsilon}{S^d}\lambda\Big(V \cap B\big(a_{x_0}, \frac{\rho}{2}\big)\Big) \tag{5.49}
$$

which, together with (2.6), ends the proof. •

It remains to give a proof of Lemma 2.4. This proof goes through several intermediate steps which are given now.

Lemma 5.1 *Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{C}^2$-function such that*

[4] Indeed, from Lemma 2.4, there exists $\mathcal{K}' \subset \mathcal{K}$ such that $\Psi_x : \mathcal{K}' \to B(a_x, \rho)$ is a $\mathcal{C}^1$-diffeomorphism, and since $B(a_{x_0}, \frac{\rho}{2}) \subset B(a_x, \rho)$, there exists $\mathcal{K}'' \subset \mathcal{K}' \subset \mathcal{K}$ such that $\Psi_x : \mathcal{K}'' \to B(a_{x_0}, \frac{\rho}{2})$ is a $\mathcal{C}^1$-diffeomorphism.

1. $g(0) = 0$,

2. $dg_0 = \mathrm{Id}$,

3. *there exist* $R, K > 0$ *such that for all* $z \in B(0, R)$,

$$\sum_{i,j,k} \left| \frac{\partial^2 g_k}{\partial z_i \partial z_j}(z) \right| \leq K$$

.

Put $\tilde{R} = R \wedge \frac{1}{2K}$. *Then* $B\Big(0, \frac{\tilde{R}}{2}\Big) \subset g\big(B(0, \tilde{R})\big)$.

Proof The third condition allows to apply the Mean Value Inequality to $z \mapsto dg_z$ since

$$|||d(dg)_z||| \leq K, \qquad \forall z \in B(0, R).$$

Therefore, with $\tilde{R} = R \wedge \frac{1}{2K}$,

$$\|dg_z - \mathrm{Id}\| = \|dg_z - dg_0\| \leq K|z| \leq \frac{1}{2}, \qquad \forall z \in B(0, \tilde{R}).$$

Let now $y \in B\Big(0, \frac{\tilde{R}}{2}\Big)$ and set $h : \overline{B(0, \tilde{R})} \to \mathbb{R}^d$, $z \mapsto h(z) := y + z - g(z)$. We have

$$\|dh_z\| = \|\mathrm{Id} - dg_z\| \leq \frac{1}{2}, \qquad \forall z \in B(0, \tilde{R}).$$

Using again the Mean Value Inequality, we obtain for all $z, z' \in \overline{B(0, \tilde{R})}$,

$$|h(z) - h(z')| \leq \frac{1}{2}|z - z'|.$$

In particular $|h(z)| \leq \frac{1}{2}|z - z'| + |h(z')|$, so $|h(z)| \leq \frac{1}{2}|z| + |h(0)| = \frac{1}{2}|z| + |y| < \tilde{R}$, for all $z \in \overline{B(0, \tilde{R})}$.

This last result highlights two facts. First, h is an $\frac{1}{2}$-contraction from the complete space $\overline{B(0, \tilde{R})}$ into itself, so the fixed-point theorem gives us the existence of $z \in \overline{B(0, \tilde{R})}$ such that $h(z) = z$, and, secondly, the range of h defined on $\overline{B(0, \tilde{R})}$ is $B(0, \tilde{R})$, so we have in fact the existence of $z \in B(0, \tilde{R})$ such that $h(z) = z$, or equivalently, $g(z) = y$, which ends the proof. •

Remark 5.2 *1. g is in fact a* $\mathcal{C}^1$*-diffeomorphism from* $V = B(0, \tilde{R}) \cap g^{-1}\Big(B\Big(0, \frac{\tilde{R}}{2}\Big)\Big)$ *to* $B\Big(0, \frac{\tilde{R}}{2}\Big)$.

2. We could have taken, of course, $\tilde{R} = R \wedge \frac{1-\varepsilon'}{K}$ *for any* $\varepsilon' \in]0, 1[$.

Lemma 5.3 *Let* A *be a* $d \times d$ *matrix such that*

$$\forall h \in \mathbb{R}^d, \qquad |Ah| \geq K|h|.$$

Then

$$B(Au, K\tilde{R}) \subset A(B(u, \tilde{R})).$$

Proof Notice first that A is clearly invertible. Let now $y \in B(Au, K\tilde{R})$. Then for $v \in \mathbb{R}^d$,

$$|v| = |A(A^{-1}v)| \geq K|A^{-1}v|,$$

so, with $v = y - Au$,

$$K\tilde{R} \geq |y - Au| \geq K|A^{-1}(y - Au)| = K|A^{-1}y - u|,$$

or, equivalently, $\tilde{R} \geq |A^{-1}y - u|$ implying that $A^{-1}y \in B(u, \tilde{R})$ and $y \in A(B(u, \tilde{R}))$. •

We now have the following extension of Lemma 5.1.

Proposition 5.4 *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ a $\mathcal{C}^2$-function and $a \in \mathbb{R}^d$ such that*

1. *$|df_a h| \geq A|h|$ for all $h \in \mathbb{R}^d$,*
2. *there exist $R, K > 0$ such that for all $y \in B(a, R)$,*

$$\|df_a^{-1}\| \sum_{i,j} \left| \frac{\partial^2 f}{\partial z_i \partial z_j}(y) \right| \leq \frac{K}{d}.$$

Then, with $\tilde{R} = R \wedge \frac{1}{2K}$,

$$B\Big(f(a), A\frac{\tilde{R}}{2}\Big) \subset f\big(B(a, \tilde{R})\big).$$

Proof 1) We use Lemma 5.1 with

$$g(z) = df_a^{-1}(f(a + z) - f(a)).$$

All hypotheses needed in Lemma 5.1 are satisfied since

$$\frac{\partial^2 g}{\partial z_i \partial z_j}(z) = df_a^{-1} \frac{\partial^2 f}{\partial z_i \partial z_j}(a + z).$$

Thus

$$B\Big(0, \frac{\tilde{R}}{2}\Big) \subset g\big(B(0, \tilde{R})\big).$$

2) Since $f(y) = df_a g(y - a) + f(a)$, using Lemma 5.3,

$$B\Big(0, A\frac{\tilde{R}}{2}\Big) \subset df_a\Big(B\Big(0, \frac{\tilde{R}}{2}\Big)\Big) \subset df_a g\big(B(0, \tilde{R})\big),$$

where we have used the preceding step in order to obtain the last inclusion. Therefore,

$$B\Big(f(a), A\frac{\tilde{R}}{2}\Big) \subset f\big(B(a, \tilde{R})\big).$$

•

We are now able to prove Lemma 2.4.

Proof [of Lemma 2.4] **1)** Let $x \in \overline{B(x_0, r)}$. We can apply Proposition 5.4 with $a = z_0$, $f = \Psi_x$ which gives $\rho = \frac{A}{2}\Big(R \wedge \frac{1}{2K}\Big)$ such that

$$B(a_x, \rho) \subset \Psi_x\Big(B(z_0, \frac{2\rho}{A})\Big) \subset \Psi_x(\mathcal{K}),$$

where we recall that $\mathcal{K} = \overline{B(z_0, R)}$. Since our conditions are uniform in x, the radius ρ will be the same for all $x \in \overline{B(x_0, r)}$.
2) The previous point implies in particular that

$$B(a_{x_0}, \rho) \subset \Psi_{x_0}(\mathcal{K}).$$

Since $x \mapsto \Psi_x(z_0)$ is continuous, there exists η with $r > \eta > 0$ such that

$$|x - x_0| < \eta \quad \Longrightarrow \quad |\Psi_x(z_0) - \Psi_{x_0}(z_0)| < \frac{\rho}{2}. \tag{5.50}$$

Therefore,

$$\bigcap_{y \in B(x_0,\eta)} B(a_y, \rho) \subset \Psi_x(\mathcal{K}),$$

so it is sufficient to prove that

$$B(a_{x_0}, \frac{\rho}{2}) \subset \bigcap_{y \in B(x_0,\eta)} B(a_y, \rho)$$

which can be seen as follows. Let $y \in B(a_{x_0}, \frac{\rho}{2})$ and $x \in B(x_0, \eta)$, then

$$\begin{aligned} |a_x - y| &\le |a_{x_0} - y| + |a_x - a_{x_0}| \\ &= |a_{x_0} - y| + |\Psi_x(z_0) - \Psi_{x_0}(z_0)| \\ &< \frac{\rho}{2} + \frac{\rho}{2} = \rho, \end{aligned}$$

so $y \in B(a_x, \rho)$, for every $x \in B(x_0, \eta)$ and the statement is proved. •

Proof [of Remark 2.5] Recall that we have imposed the additional hypothesis $L_c = \sup_{z \in \mathcal{K}} L_c(z) < \infty$. Since

$$|c(z, x) - c(z, y)| \le L_c(z)|x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad \forall z \in E,$$

it is sufficient to set

$$\eta = \frac{\rho}{2(1 + L_c)} \wedge r,$$

in order to grant (5.50). •

Appendix

In this paper, we have used the following version of Gronwall's lemma.

Proposition 5.5 *If a measurable function $g : [0, T] \to \mathbb{R}^+$ is such that*
1. $G = \sup_{t \in [0,T]} g(t) < +\infty$;
2. *for all $t \in [0, T]$,*

$$g(t) \le A + B \int_0^t g(s)\, \mathrm{d}s$$

then, for all $t \in [0, T]$,

$$g(t) \le A \exp(Bt).$$

Proof It is easy to obtain by induction that, for every $n \in \mathbb{N}^*$,

$$g(t) \le A\Big(1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + B^n \int_0^t \int_0^{t_1} ... \int_0^{t_{n-1}} g(t_n)\, \mathrm{d}t_n \cdots \mathrm{d}t_1\, \mathrm{d}t\Big),$$

which implies

$$g(t) \le A\Big(1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + G\frac{(Bt)^n}{n!}\Big).$$

Since $\lim_{n\to+\infty} G\frac{(Bt)^n}{n!} = 0$, the assertion follows. •

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