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On paths in grids with forbidden transitions

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Une transition dans un graphe est une paire d’arêtes incidentes à un même sommet. Étant donnés un graphe $G = (V,E)$, deux sommets $s,t \in V$ et un ensemble associé de transitions interdites $\mathcal{F} \subseteq E \times E$, le problème de chemin évitant des transitions interdites consiste à décider s’il existe un chemin élémentaire de $s$ à $t$ qui n’utilise aucune des transitions de $\mathcal{F}$. C’est-à-dire qu’il est interdit d’emprunter consécutivement deux arêtes qui soient une paire de $\mathcal{F}$. Ce problème est motivé par le routage dans les réseaux routiers (où une transition interdite représente une interdiction de tourner) ainsi que dans les réseaux optiques avec des noeuds asymétriques. Nous prouvons que le problème est NP-difficile dans les graphes planaires et plus particulièrement dans les grilles. Nous montrons également que le problème peut être résolu en temps polynomial dans la classe des graphes de largeur arborescente bornée.

Keywords: Forbidden transitions, planar graph, grid, asymmetric nodes

1 Introduction

Driving in New-York is not easy. Not only because of the rush hours and the taxi drivers, but because of the no-left, no-right and no U-turn signs. Even in a “grid-like” city like New-York, prohibited turns might force you to cross several times the same intersection before eventually reaching your destination. In this paper, we give hints explaining why it is difficult to deal with forbidden-turn signs when driving.

Let $G = (V,E)$ be a graph. A transition in $G$ is a pair of two distinct edges incident to a same vertex. Let $\mathcal{F} \subseteq E \times E$ be a set of forbidden transitions in $G$. We say that a path $P = (v_0, \ldots, v_q)$ is $\mathcal{F}$-valid if it contains none of the transitions of $\mathcal{F}$, i.e., $\{(v_{i-1}, v_i), (v_i, v_{i+1})\} \notin \mathcal{F}$ for any $1 \leq i \leq q-1$. Given $s,t \in V$, the Path Avoiding Forbidden Transitions (PAFT) problem is to find an $\mathcal{F}$-valid $s$-$t$-path in $G$. This problem arises in many contexts. In optical networks, nodes can have asymmetric switching capabilities mostly due to cost-relevant reasons \cite{CHW+13}. In this context, nodes have some restrictions on their internal connectivity: traffic on a certain ingress port can only reach a subset of the egress ports. Then, the optical nodes configured asymmetrically are vertices with forbidden transitions and routing is an application of PAFT. The study of PAFT is also motivated by its relevance to vehicle routing. In road networks, it is possible that some roads are closed due to traffic jams, construction, etc. It is also frequent to encounter no-left, no-right and no U-turn signs at intersections. These prohibited roads and turns can be modeled by forbidden transitions.

A distinction has to be made according to whether the path to find is elementary (cannot repeat vertices) or non-elementary. Indeed, PAFT can be solved in polynomial time \cite{GM08} for the non-elementary case (using a simple BFS from $t$) while finding an elementary path avoiding forbidden transitions has been proved NP-complete in \cite{Sze03}. This paper studies the elementary version of the PAFT problem in planar graphs and more particularly in grids. Planar graphs are not only closely related to road networks, they are also an interesting special case to study while trying to capture the difficulty of the problem. Furthermore, to the best of our knowledge, this case has not been addressed before in the literature.

Related work. PAFT is a special case of the problem of finding a path avoiding forbidden paths (PFP) introduced in \cite{VD05}. Given a graph $G$, two vertices $s$ and $t$, and a set $\mathcal{S}$ of forbidden paths, PFP aims at finding an $s$-$t$-path which contains no path of $\mathcal{S}$ as a subpath. When the forbidden paths are composed of...
Hence, if \( v \) appears positively in Clause \( i \), we add the "blue" edge \( \{a_{ij}, b_{ij}\} \) that creates a "bridge" between \( BT_{ij} \) and \( RT_{ij} \). When Brown edge is present, the forbidden transitions are defined such that it is possible to switch between the positive paths \( BT_{ij} \) and \( RT_{ij} \) when going from \( s_{ij} \) to \( t_{ij} \). Similarly, \( v \) appears negatively in \( C_j \), we add the "green" edge \( \{\gamma_{ij}, \delta_{ij}\} \) that creates a "bridge" between \( BF_{ij} \) and \( RF_{ij} \). Hence, if \( v \) appears in \( C_j \), it will be possible to start in \( s_{ij} \) by some color and finish in \( t_{ij} \) with a different one. Note that, the type of path (positive or negative) cannot be modified between \( s_{ij} \) and \( t_{ij} \).

### Clause-graph \( G_j \)

For any \( j \leq m \), the Clause-gadget \( G_j \) is built by combining the graphs \( G_{ij} \), \( i \leq n \), in a "line" (see Fig. 2). The subgraphs \( G_{ij} \) are combined from "left to right" (for \( i = 1 \) to \( n \)) if \( j \) is odd and from

\[ \Phi = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} C_i \]

Contributions. Our main contribution is proving that the PAFT problem is NP-complete in grids. We also prove that the problem can be solved in time \( O((3\Delta(k + 1))^{2k+O(1)} n) \) in \( n \)-node graphs with treewidth at most \( k \) and maximum degree \( \Delta \). In other words, we prove that the PAFT problem is FPT in \( k + \Delta \).

## 2 Complexity of the PAFT problem

We start by proving that the PAFT problem is NP-complete in grids. For this purpose, we first prove that it is NP-complete in planar graphs with maximum degree at most 8 by a reduction from 3-SAT. Then, we propose simple transformations to reduce the degree of the vertices and prove that the PAFT problem is NP-complete in planar graphs with degree at most 4. Finally, we prove it is NP-complete in grids.

#### Lemma 1

The PAFT problem is NP-complete in planar graphs with maximum degree 8.

**Sketch of proof.** The problem is clearly in NP. We prove the hardness using a reduction from the 3-SAT problem. We do the reduction for multi-graphs for ease of presentation but since a multi-graph can be easily transformed to a graph by subdividing the edges, the lemma follows. Let \( \Phi \) be an instance of 3-SAT, i.e., \( \Phi \) is a boolean formula with variables \( \{v_1, \ldots, v_n\} \) and clauses \( \{C_1, \ldots, C_m\} \). We build a grid-like planar graph \( G \) where rows correspond to clauses and columns correspond to variables. In what follows, the colors are only used to make the presentation easier. Moreover, we consider undirected graphs but, since the forbidden transitions can simulate orientations, the figures are depicted with directed arcs for ease of presentation.

**Gadget \( G_{ij} \).** For any \( i \leq n \) and \( j \leq m \), we define the gadget \( G_{ij} \) depicted in Figure 1 (left) and that consists of 4 edge-disjoint paths: \( s_{ij} \) to \( t_{ij} \): 2 "blue" paths \( BT_{ij} \) and \( BF_{ij} \), and 2 "red" paths \( RT_{ij} \) and \( RF_{ij} \). The forbidden transitions are defined in such a way that the only way to go from \( s_{ij} \) to \( t_{ij} \) is by following one of the paths in \( \{ BT_{ij}, BF_{ij}, RT_{ij}, RF_{ij} \} \). Intuitively, assigning the variable \( v_i \) to \( True \) will be equivalent to choosing one of the paths \( BT_{ij} \) or \( RT_{ij} \) (called positive paths) depicted with full lines in Fig. 1 (left). Respectively, assigning \( v_i \) to \( False \) will correspond to choosing one of the paths \( BF_{ij} \) or \( RF_{ij} \) (called negative paths) and depicted by dotted line in Fig. 1 (left).

So far, it is a priori not possible to start from \( s_{ij} \) by one path and arrive in \( t_{ij} \) by another path. In particular, the color by which \( s_{ij} \) is left must be the same by which \( t_{ij} \) is reached. If Variable \( v_i \) appears in Clause \( C_j \), we add one edge to \( G_{ij} \) as follows. If \( v_i \) appears positively in \( C_j \), we add the brown edge \( \{a_{ij}, b_{ij}\} \) that creates a "bridge" between \( BT_{ij} \) and \( RT_{ij} \). When Brown edge is present, the forbidden transitions are defined such that it is possible to switch between the positive paths \( BT_{ij} \) and \( RT_{ij} \) when going from \( s_{ij} \) to \( t_{ij} \). Similarly, \( v_i \) appears negatively in \( C_j \), we add the green edge \( \{\gamma_{ij}, \delta_{ij}\} \) that creates a "bridge" between \( BF_{ij} \) and \( RF_{ij} \).
Lemma 2 The PAFT problem is NP-complete in planar graphs with maximum degree 4.

Sketch of proof. The graph $G$ built in the proof of Lemma 1 is planar and each vertex $v$ of $G$ has either degree at most 4, degree 5 or 8. Vertices of degree 5 can be modified to have only degree 3. Then, using the specific structure of forbidden transitions around $v$, we can replace each degree-8 vertex $v$ of $G$ by a gadget $g_v$ made of vertices of degree at most 4. Gadget $g_v$ is designed such that it can be crossed at most
The problem of finding a path avoiding forbidden transitions is NP-complete in grids.

Sketch of proof. A planar grid embedding of a graph $G$ maps $G$ into a grid such that each vertex of $G$ is mapped into a distinct vertex of the grid and each edge $e$ of $G$ into a path of the grid whose endpoints are mappings of vertices linked by $e$. Two paths of the grid corresponding to two edges of $G$ are vertex-disjoint, except, possibly, at the endpoints. Starting from the graph defined in the reduction presented above, we use the fact that any $n$-node graph $G$ with maximum degree at most 4 can be mapped into a grid of size at most $O(n^2)$ in polynomial-time [Val81]. The key point is that the initial graph has maximum degree at most 4 (see Lemma 2) which allows us to transfer the forbidden transitions into the grid.

On the positive side, by using dynamic programming on a tree-decomposition of the input graph, we prove:

Theorem 2 The problem of finding a path avoiding forbidden transitions is FPT when parameterized by $k + \Delta$ where $k$ is the treewidth and $\Delta$ is the maximum degree. In particular, there exists an algorithm that finds the shortest path avoiding forbidden transitions between two vertices in time $O((3\Delta(k+1))^{2k+O(1)}n)$.

Références


[KMMN15] M. M. Kanté, F. Z. Moataz, B. Momège, and N. Nisse. Finding paths in grids with forbidden transitions, 2015. [https://hal.inria.fr/hal-01115395/document](https://hal.inria.fr/hal-01115395/document).


