On paths in grids with forbidden transitions

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Une transition dans un graphe est une paire d'arêtes incidentes à un même sommet. Etant donnés un graphe $G = (V,E)$, deux sommets $s,t \in V$ et un ensemble associé de transitions interdites $\mathcal{F} \subseteq E \times E$, le problème de chemin évitant des transitions interdites consiste à décider s'il existe un chemin élémentaire de s à t qui n'utilise aucune des transitions de $\mathcal{F}$. C'est-à-dire qu'il est interdit d'emprunter consécutivement deux arêtes qui soient une paire de $\mathcal{F}$. Ce problème est motivé par le routage dans les réseaux routiers (où une transition interdite représente une interdiction de tourner) ainsi que dans les réseaux optiques avec des noeuds asymétriques. Nous prouvons que le problème est NP-difficile dans les graphes planaires et plus particulièrement dans les grilles. Nous montrons également que le problème peut être résolu en temps polynomial dans la classe des graphes de largeur arborescente bornée.

Keywords: Forbidden transitions, planar graph, grid, asymmetric nodes

1 Introduction

Driving in New-York is not easy. Not only because of the rush hours and the taxi drivers, but because of the no-left, no-right and no U-turn signs. Even in a "grid-like" city like New-York, prohibited turns might force you to cross several times the same intersection before eventually reaching your destination. In this paper, we give hints explaining why it is difficult to deal with forbidden-turn signs when driving.

Let $G = (V,E)$ be a graph. A transition in G is a pair of two distinct edges incident to a same vertex. Let $\mathcal{F} \subseteq E \times E$ be a set of forbidden transitions in G. We say that a path $P = (v_0,\dots,v_q)$ is $\mathcal{F}$-valid if it contains none of the transitions of $\mathcal{F}$, i.e., $\{(v_{i-1},v_i),(v_i,v_{i+1})\} \notin \mathcal{F}$ for any $1 \leq i \leq q-1$. Given $s,t \in V$, the Path Avoiding Forbidden Transitions (PAFT) problem is to find an $\mathcal{F}$-valid s-t-path in G. This problem arises in many contexts. In optical networks, nodes can have asymmetric switching capabilities mostly due to cost-relevant reasons [CHW+13]. In this context, nodes have some restrictions on their internal connectivity : traffic on a certain ingress port can only reach a subset of the egress ports. Then, the optical nodes configured asymmetrically are vertices with forbidden transitions and routing is an application of PAFT. The study of PAFT is also motivated by its relevance to vehicle routing. In road networks, it is possible that some roads are closed due to traffic jams, construction, etc. It is also frequent to encounter no-left, no-right and no U-turn signs at intersections. These prohibited roads and turns can be modeled by forbidden transitions.

A distinction has to be made according to whether the path to find is elementary (cannot repeat vertices) or non-elementary. Indeed, PAFT can be solved in polynomial time [GM08] for the non-elementary case (using a simple BFS from t) while finding an elementary path avoiding forbidden transitions has been proved NP-complete in [Sze03]. This paper studies the elementary version of the PAFT problem in planar graphs and more particularly in grids. Planar graphs are not only closely related to road networks, they are also an interesting special case to study while trying to capture the difficulty of the problem. Furthermore, to the best of our knowledge, this case has not been addressed before in the literature.

Related work. PAFT is a special case of the problem of finding a path avoiding forbidden paths (PFP) introduced in [VD05]. Given a graph G, two vertices s and t, and a set $\mathcal{S}$ of forbidden paths, PFP aims at finding an s-t-path which contains no path of $\mathcal{S}$ as a subpath. When the forbidden paths are composed of

† Due to lack of space, proofs have been sketched or omitted. Full proofs are available here [KMMN15]

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exactly two edges, PFP is equivalent to PAFT. Many papers address the non-elementary version of PFP, proposing exact polynomial solutions [VD05, AL13]. The elementary counterpart has been recently studied in [PG13] where a mathematical formulation is given and two solution approaches are developed and tested. The computational complexity of the elementary PFP can be deduced from the complexity of PAFT which has been established in [Sze03]. Szeider proved in [Sze03] that finding an elementary path avoiding forbidden transitions is NP-complete and gave a complexity classification of the problem according to the types of the forbidden transitions. The NP-completeness proof in [Sze03] does not extend to planar graphs.

PAFT is also a generalization of the problem of finding a properly colored path in an edge-colored graph (PEC). Given an edge-colored graph G^c and two vertices s and t, the PEC problem aims at finding an s-t-path such that any two consecutive edges have different colors. It is easy to see that PEC is equivalent to PAFT when the set of forbidden transitions consists of all pairs of adjacent edges that have the same color. The PEC problem is proved to be NP-complete in directed graphs [GLMM13] which directly implies that the PAFT problem is NP-complete in directed graphs [§].

Contributions. Our main contribution is proving that the PAFT problem is NP-complete in grids. We also prove that the problem can be solved in time $O((3\Delta(k+1))^{2k+O(1)}n))$ in n-node graphs with treewidth at most k and maximum degree Δ. In other words, we prove that the PAFT problem is FPT in $k+\Delta$.

2 Complexity of the PAFT problem

We start by proving that the PAFT problem is NP-complete in grids. For this purpose, we first prove that it is NP-complete in planar graphs with maximum degree at most 8 by a reduction from 3-SAT. Then, we propose simple transformations to reduce the degree of the vertices and prove that the PAFT problem is NP-complete in planar graphs with degree at most 4. Finally, we prove it is NP-complete in grids.

Lemma 1 *The PAFT problem is NP-complete in planar graphs with maximum degree* 8.

Sketch of proof. The problem is clearly in NP. We prove the hardness using a reduction from the 3-SAT problem. We do the proof for multi-graphs for ease of presentation but since a multi-graph can be easily transformed to a graph by subdividing the edges, the lemma follows. Let Φ be an instance of 3-SAT, i.e., Φ is a boolean formula with variables $\{v_1,\cdots,v_n\}$ and clauses $\{C_1,\cdots,C_m\}$. We build a grid-like planar graph G where rows correspond to clauses and columns correspond to variables. In what follows, the colors are only used to make the presentation easier. Moreover, we consider undirected graphs but, since the forbidden transitions can simulate orientations, the figures are depicted with directed arcs for ease of presentation.

Gadget G_{ij}. For any $i \le n$ and $j \le m$, we define the gadget G_{ij} depicted in Figure 1(left) and that consists of 4 edge-disjoint paths from s_{ij} to t_{ij} : 2 "blue" paths $\mathcal{B}T_{ij}$ and $\mathcal{B}F_{ij}$, and 2 "red" paths $\mathcal{R}T_{ij}$ and $\mathcal{R}F_{ij}$. The forbidden transitions are defined in such a way that the only way to go from s_{ij} to t_{ij} is by following one of the paths in $\{\mathcal{B}T_{ij},\mathcal{B}F_{ij},\mathcal{R}T_{ij},\mathcal{R}F_{ij}\}$. Intuitively, assigning the variable v_i to $True$ will be equivalent to choosing one of the paths $\mathcal{B}T_{ij}$ or $\mathcal{R}T_{ij}$ (called *positive* paths) depicted with full lines in Fig. 1(left). Respectively, assigning v_i to $False$ will correspond to choosing one of the paths $\mathcal{B}F_{ij}$ or $\mathcal{R}F_{ij}$ (called *negative* paths) and depicted by dotted line in Fig. 1(left).

So far, it is *a priori* not possible to start from s_{ij} by one path and arrive in t_{ij} by another path. In particular, the color by which s_{ij} is left must be the same by which t_{ij} is reached. If Variable v_i appears in Clause C_j, we add one edge to G_{ij} as follows. If v_i appears positively in C_j, we add the *brown* edge $\{\alpha_{ij},\beta_{ij}\}$ that creates a "bridge" between $\mathcal{B}T_{ij}$ and $\mathcal{R}T_{ij}$. When Brown edge is present, the forbidden transitions are defined such that it is possible to switch between the positive paths $\mathcal{B}T_{ij}$ and $\mathcal{R}T_{ij}$ when going from s_{ij} to t_{ij}. Similarly, v_i appears negatively in C_j, we add the *green* edge $\{\gamma_{ij},\delta_{ij}\}$ that creates a "bridge" between $\mathcal{B}F_{ij}$ and $\mathcal{R}F_{ij}$. Hence, if v_i appears in C_j, it will be possible to start in s_{ij} by some color and finish in t_{ij} with a different one. Note that, the type of path (positive or negative) cannot be modified between s_{ij} and t_{ij}.

Clause-graph G_j. For any $j \le m$, the Clause-gadget G_j is built by combining the graphs G_{ij}, $i \le n$, in a "line" (see Fig. 2). The subgraphs G_{ij} are combined from "left to right" (for $i=1$ to n) if j is odd and from

§. Note that, in [GLMM13], the authors state that their result can be extended to planar graph. However, there is a mistake in the proof of the corresponding Corollary 7 : to make their graph planar, vertices are added when edges intersect. Unfortunately, this transformation does not preserve the fact that the path is elementary.

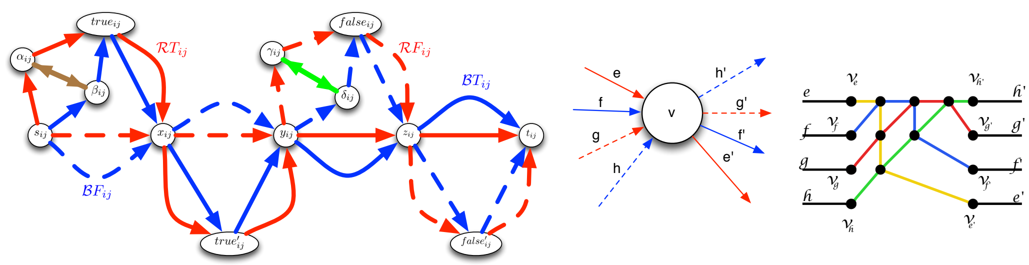

FIGURE 1: (left) Example of the Gadget-graph G_{ij} for Variable v_i, and $j \leq m$. Brown (resp. green) edge is added if v_i appears positively (resp., negatively) in C_j. If $v_i \notin C_j$, none of the green nor brown edge appear. (middle) Example of degree-8 node and allowed transtions $\{\{e,e'\}, \{f,f'\},\{g,g'\},\{h,h'\}\}$, and (*right*) corresponding gadget g_v.

"right to left" (for $i = n$ to 1) otherwise. For any $j \leq m$ odd, the subgraph G_j starts with a red edge $\{s_j, s_{1j}\}$ and then, for $1 < i \leq n$, the nodes s_{ij} and $t_{i-1,j}$ are identified. Finally, there is a blue edge from t_{nj} to a new node t_j. For any $j \leq m$ even, the subgraph G_j starts with a blue edge $\{s_j, s_{nj}\}$ and then, for $1 < i \leq n$, the nodes t_{ij} and $s_{i-1,j}$ are identified. Finally, there is a red edge from t_{1j} to a new node t_j. Forbidden transitions are defined such that, when passing from a gadget G_{ij} to the next one, the same color must be used (entering in $t_{ij} = s_{i,j+1}$ by an edge with some color, the same color must be used to leave this node). However, in such nodes, we can change the type (positive or negative) of path.

Note that if we enter a Clause-graph with a red (resp. blue) edge, we can only leave it with a blue (resp. red) edge. This means that a path must change its color inside the Clause-graph, and must hence use a brown or green edge. The use of a brown (resp. green) forces a variable that appears positively (resp. negatively) in the clause to be set to true (resp. false) and validates the Clause.

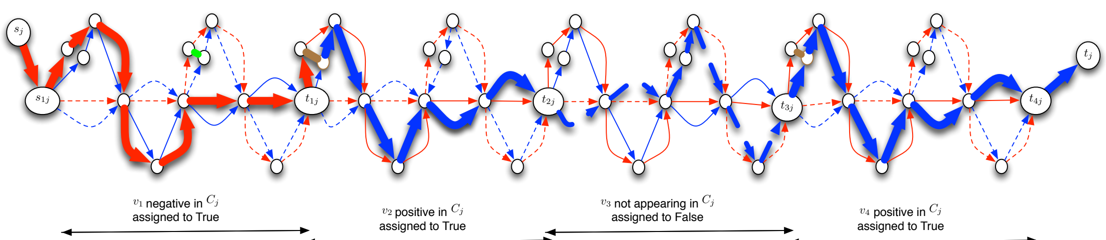

FIGURE 2: Case j odd. Clause-graph G_j for a Clause $C_j = \bar{v}_1 \vee v_2 \vee v_4$ in a formula with 4 Variables. The bold path corresponds to an assignment of v_1, v_2 and v_4 to *True*, and of v_3 to *False*.

Main graph. To conclude, we have to be sure that the assignment of the variables is coherent between the clauses. For this purpose, let us combine the subgraphs G_j, $j \leq m$, as follows (see Fig 3). First, for any $1 \leq j < m$, let us identify t_j and s_{j+1}. Then, some nodes (depicted in grey in Fig 3) of G_{ij} are identified with nodes of $G_{i,j+1}$ in such a way that using a positive (resp., negative) path in G_{ij} forces to use the same type of path in $G_{i,j+1}$. That is, the choice of the path used in G_{ij} is transferred to $G_{i,j+1}$ and therefore it corresponds to a truth assignment for Variable v_i. Finally, forbidden transitions are defined in order to forbid "crossing" a grey node, i.e., it is not possible to go from $G_{i,j}$ to $G_{i,j+1}$ via a grey node.

Finally, we prove that there is an elementary s_1-t_m path in G if and only if Φ is satisfiable. □

Lemma 2 *The PAFT problem is NP-complete in planar graphs with maximum degree* 4.

Sketch of proof. The graph G built in the proof of Lemma 1 is planar and each vertex v of G has either degree at most 4, degree 5 or 8. Vertices of degree 5 can be modified to have only degree 3. Then, using the specific structure of forbidden transitions around v, we can replace each degree-8 vertex v of G by a gadget g_v made of vertices of degree at most 4. Gadget g_v is designed such that it can be crossed at most

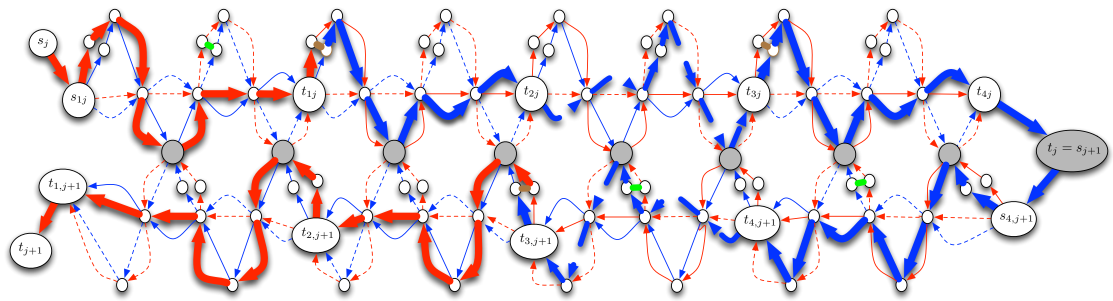

FIGURE 3: Combining $C_j = \bar{v}_1 \vee v_2 \vee v_4$ and $C_{j+1} = v_2 \vee \bar{v}_3 \vee \bar{v}_4$ (Case j odd).

once by a path and only if the edges used to enter and leave g_v correspond to an allowed transition around v. Fig. 1(middle) and 1(right) give an example of a vertex v in G and the corresponding gadget g_v in G'. □

Theorem 1 *The problem of finding a path avoiding forbidden transitions is NP-complete in grids.*

Sketch of proof. A planar grid embedding of a graph G maps G into a grid such that each vertex of G is mapped into a distinct vertex of the grid and each edge e of G into a path of the grid whose endpoints are mappings of vertices linked by e. Two paths of the grid corresponding to two edges of G are vertex-disjoint, except, possibly, at the endpoints. Starting from the graph defined in the reduction presented above, we use the fact that any n-node graph G with maximum degree at most 4 can be mapped into a grid of size at most $O(n^2)$ in polynomial-time [Val81]. The key point is that the initial graph has maximum degree at most 4 (see Lemma 2) which allows us to transfer the forbidden transitions into the grid. □

On the positive side, by using dynamic programming on a tree-decomposition of the input graph, we prove :

Theorem 2 *The problem of finding a path avoiding forbidden transitions is FPT when parameterized by $k+\Delta$ where k is the treewidth and Δ is the maximum degree. In particular, there exists an algorithm that finds the shortest path avoiding forbidden transitions between two vertices in time $O((3\Delta(k+1))^{2k+O(1)}n))$*

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