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Dynamics of Anisotropic Universes

Jérôme Perez

Laboratory of Applied Mathematics, Ecole Nationale Supérieure de Techniques Avancées, 32 Bd Victor, 75739 Paris cedex 15 - jerome.perez@ensta.fr

Abstract. We present a general study of the dynamical properties of Anisotropic Bianchi Universes in the context of Einstein General Relativity. Integrability results using Kovalevskaya exponents are reported and connected to general knowledge about Bianchi dynamics. Finally, dynamics toward singularity in Bianchi type VIII and IX universes are showed to be equivalent in some precise sense.

Keywords: General relativity, Dynamical systems, Anisotropic universes, Analytical methods, Numerical methods

PACS: 98.80.Jk

HOMOGENEOUS UNIVERSE AND BIANCHI MODELS

Considering the usual synchronous frame of General Relativity\(^1\) \(ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{ij} dx^i dx^j - dt^2\). A Universe is said homogeneous when there exists an isometry group which preserves the infinitesimal spatial length \(dl^2 = \tilde{g}_{ij} dx^i dx^j\). A characterization of the isometry group is possible writing structure constants

\[
C_{ab}^c = \left( \partial_i e_j^c - \partial_j e_i^c \right) e_a^i e_b^j
\]

where \(dx^i = e^i_j dy^j\). Constants \(C_{ab}^c\) are tensorial, low components are antisymmetric, and follow the Jacobi rule:

\[
C_{ab}^c C_{cd}^e + C_{bc}^d C_{de}^c + C_{ca}^e C_{eb}^d = 0
\]

Decomposing \(C_{ab}^c = \varepsilon_{abc} N_{dc} + \delta_{ab} A_c\), where \(N_{ab}\) is a symmetric tensor, one can show that equivalence classes of homogeneous universes are equivalence classes of \(N_{ab}\) with \(N_{ab} A_b = 0\). Without less of generality, the symmetry of \(N_{ab}\) allows us to write

\[
N_{ab} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix}
\]

and \(A_b = [a, 0, 0]\)

Models split then into Class A with \(a = 0\) and Class B with \(a \neq 0\) and can be arranged in the well known Bianchi models

<table>
<thead>
<tr>
<th>(n_1)</th>
<th>(n_2)</th>
<th>(n_3)</th>
<th>(a)</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 is a triple eigenvalue of (N)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0 is a double eigenvalue of (N)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0 is a simple eigenvalue of (N)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0 is not an eigenvalue of (N)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0 is not an eigenvalue of (N)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^1\) without contrary indications greek indexes run from 0 to 3, latin indexes from 1 to 3, metric signature is \((-,-,+,-+)\), \(\varepsilon\) and \(\delta\) are respectively the completely antisymmetric Levi-Cevita tensor and the Kronecker symbol
EINSTEIN EQUATIONS

Following [1], one writes \(ds^2 = \gamma(\tau) \omega^i \omega^j - N^2(\tau) d\tau^2\) with \(\gamma(\tau) = \text{diag}\left[ e^{A_1(\tau)}, e^{A_2(\tau)}, e^{A_3(\tau)} \right]\) and \(dt = N(\tau) d\tau\).

The so called invariant differential forms basis \(\omega^i\) are linear combinations of \(dx_i\) with exponential or trigonometric coefficient in \(x_i\) (See [1]). Finally, \(\tau\) and \(N(\tau)\) are respectively the conformal time and the lapse function.

BKL Formalism

This formalism was introduced in the 70’s by [3]. Filling Universe by a barotropic fluid with pressure \(p\) such that \(P = (\Gamma - 1) \rho\), taking \(N^2(\tau) = V^2 = e^{A_1 + A_2 + A_3}\) for the lapse function, some algebra then gives from Einstein equations, the equations for the dynamics of Bianchi Universes

\[
\begin{align*}
0 &= E_c + E_p + E_m = H \\
\chi (2 - \Gamma) V^2 - \Gamma &= A''_1 + (n_1 e^{A_1})^2 - (n_2 e^{A_2} - n_3 e^{A_3})^2 \\
\chi (2 - \Gamma) V^2 - \Gamma &= A''_2 + (n_2 e^{A_2})^2 - (n_3 e^{A_2} - n_1 e^{A_1})^2 \\
\chi (2 - \Gamma) V^2 - \Gamma &= A''_3 + (n_3 e^{A_3})^2 - (n_1 e^{A_1} - n_2 e^{A_2})^2
\end{align*}
\]

(4)

where \(\chi = 8\pi G e^{-4}, E_c = \frac{1}{2} \sum_{i \neq j = 1}^3 A'_i A'_j, E_p = \sum_{i \neq j = 1}^3 n_i n_j e^{A_i + A_j} - \sum_{i = 1}^3 n_i^2 e^{A_i}, E_m = -4\chi \rho V^2\) and \(\Gamma\) stands for \(d/d\tau\).

Hamiltonian formalism

This formalism was introduced by [4], at almost the same time than the precedent. It consists to diagonalize the quadratic form \(E_c\) by introducing new variables such that

\[
M := \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}
\end{bmatrix}
\]

and \(q := [q_1 q_2 q_3]^T = M [A_1 A_2 A_3]^T\)

\[
p := [p_1 p_2 p_3]^T = M [A'_1 A'_2 A'_3]^T = q' \quad (5)
\]

Dynamical equations for Bianchi Universes then becomes

\[
q'^{1,2} = - \frac{\partial H}{\partial p_{1,2}}, \quad p'^{1,2} = - \frac{\partial H}{\partial q_{1,2}} \quad \text{and} \quad q'^3 = \frac{\partial H}{\partial p_3}, \quad p'^3 = - \frac{\partial H}{\partial q_3}
\]

(6)

where \(H = \frac{1}{2} \langle p, p \rangle + \sum_{i=1}^7 k_i e^{(a_i, q)}\) with the following products

\[
\forall x, y \in \mathbb{R}^3 \quad \langle x, y \rangle := +x_1y_1 + x_2y_2 + x_3y_3 \quad \text{and} \quad \langle x, y \rangle := -x_1y_1 - x_2y_2 + x_3y_3
\]

(7)

and the constants \(k_1 := 2n_1n_2, \ k_2 := 2n_1n_3, \ k_3 := 2n_2n_3, \ k_4 := -n_1^2, \ k_5 := -n_2^2, \ k_6 := -n_3^2, \ k_7 := -4\rho \chi\). The set of vectors \(a_i = 1, \ldots, 7\) is highly symmetric (see Fig. 1): It allows to use algebraic techniques based on Lie algebra in some dynamical treatments of this problem.

INTEGRABILITY OF BIANCHI MODELS

Kovalevskaya stuff and autosimilar ODE

If an ODE

\[
\dot{x} = f(x) \quad \text{with} \ x \in \mathbb{R}^n \quad \text{and} \quad \cdot \equiv d/dt
\]

(8)
admits a self-similar solution
\[ \tilde{x} = [c_1 (t - t_0)^{-\gamma_1}, \ldots, c_n (t - t_0)^{-\gamma_n}]^T \] (9)
with the weight vector \( g \) and the constant vector \( c \) lying respectively \( \mathbb{Z}^n \) and \( \mathbb{R}^n \). One can then prove that the linearized system around \( \tilde{x} \), admits too an auto-similar solution
\[ z = [d_1 (t - t_0)^{k_1}, \ldots, d_n (t - t_0)^{k_n}]^T \] (10)
where \( d \) is a constant vector, and, this time the Kovalevskaya vector \( k \), whose components are the so-called Kovalevskaya exponents, lies \( \mathbb{C}^n \). In practice, one can compute this vector: it is easy to show that Kovalevskaya exponents are eigenvalues of
\[ K := D[f(x)](c) + \text{diag}(g) \].
A theorem by Poincaré shows that each component of the non linear general solution of equation (8) is on the form
\[ x_i(t) \propto (t - t_0)^{-\gamma_i} S(t - t_0)^{k_1}, \ldots, (t - t_0)^{k_n} \] (11)
where \( S[\cdot] \) stands for a multiple series. This result is at the basis of a sufficient condition (known as Yoshida’s Theorem see [5],[6]): if all the Kovalevskaya exponents are in \( \mathbb{Q} \), then the system is algebraically integrable.

**Kowalevskaya exponents and Bianchi Universes**

Such a work was Pioneering by [7], applied for the first time to some special cases by [8], and more recently, developed with some imprecisions and incompletude by [9]. We present here results obtained in a precise way in [10].

Introducing new variables
\[ \{q,p\} \mapsto \{u,v\} \text{ where } \begin{cases} u \in \mathbb{R}^7, u_{i=1,\ldots,7} := (a_i, p) \\ v \in \mathbb{R}^7, v_{i=1,\ldots,7} := \exp(a_i, q) \end{cases} \] (12)
Einstein’s Hamiltonian equations become
\[ \forall i = 1, \ldots, 7 \begin{cases} v_i' = u_i v_i \\ u_i' = \sum_{j=1}^7 W_{ij} v_j \end{cases} \text{ with } W_{ij} := -k_j \langle a_i, a_j \rangle \] (13)
This dynamical system admits an autosimilar solution \( \tilde{x} = [\lambda t^{-1}, \mu t^{-2}]^T \) where the constants \( [\lambda, \mu] \in \mathbb{R}^7 \times \mathbb{R}^7 \) are solutions of the algebraic system
\[ \forall i = 1, \ldots, 7 \begin{cases} \sum_{j=1}^7 W_{ij} \mu_j = -\lambda_i \\ \lambda_i = -2 \mu_i \end{cases} \] (14)
Taking into account that \( \text{Rank}(W) = 3 \), it exists 45 distincts non trivial solution for system (14), and then 45 sets of 14 Kovalevskaya exponents for all Bianchi Universes. Analysis of such sets let us claim that:

- In vacuum or with stiff matter \((\Gamma = 2)\), excepted \(B_{IX}\) and \(B_{VIII}\), all other class A Bianchi models have fractional Kovalevskaya exponents.
- With non stiff matter \((0 \leq \Gamma < 2)\) class A Bianchi models have at least one real or complex Kovalevskaya exponent, except for \(B_I\) with fractional values of \(\Gamma\) and \(B_{II}\) with \(\Gamma\) fractional in \(h_0\) and \(p_7\):

These integrability indications are conformed by [12] for \(B_{VIII}\) using Morales-Ruis Theory and by [11] using Painlevé’s analysis for \(B_{IX}\).

**EXACT SOLUTIONS**

**B_I dynamics : The fundamental state**

In vacuum, the general solution of \(B_I\) model could be explicited. The line element is

\[
\text{ds}^2 = t^{2p_1} \text{dx}_1^2 + t^{2p_2} \text{dx}_2^2 + t^{2p_3} \text{dx}_3^2 - dt^2
\]

where

\[
\exists! u \in [1, +\infty[ \text{ such that } \begin{cases} p_1 = -\Omega u/(1 + u + u^2) \\
p_2 = \Omega(1 + u)/(1 + u + u^2) \\
p_3 = \Omega u(1 + u)/(1 + u + u^2) \end{cases}
\]

The real \(\Omega\) is directly proportional to Universe Volume variation which is constant in \(B_I\). Toward singularity \((t \to 0)\), \(x_1\) expands when \(x_2\) and \(x_3\) contracts, all with a constant exponential rate. Any couple \([u, \Omega] \in [1, +\infty[ \times \mathbb{R}\) associated to an order of axis defines a Kasner state.

**B_{II} dynamics : One Kasner transition**

As noted by [3], \(B_{II}\) dynamics corresponds to a transition between 2 asymptotic Kasner states:

\[
\begin{array}{c}
P_1 \ \square \ < \ P_2 \ \triangle \ < \ P_3 \ \heartsuit \\
\text{Initial Kasner State : } t \to +\infty
\end{array} \quad \leadsto \quad \begin{array}{c}
[u-1, \Omega(1-2p_1)] \ \bigtriangleup \square \ \text{if } u \geq 2 \\
(u-1)^{-1}, \Omega(1-2p_1) \ \bigtriangleup \bigtriangleup \ \text{if } u < 2
\end{array} \quad \text{Final Kasner State : } t \to 0
\]

**Conjectures (partially proven)**

As indicated by the rigorous proof by [16], all classes of Bianchi models converge generically toward a simple Kasner state or a closure of Kasner states when \(t \to 0\). Numerical analysis, more precisely Billiard analogy (see next section), let us think that:

- \(B_{VI}\) and \(B_{VII}\) dynamics correspond to a finite number of Kasner transitions
- \(B_{VIII}\) and \(B_{IX}\) dynamics correspond to an infinite number of Kasner transitions
BILLIARD ANALOGY

Those works was pioneered by [4], and more recently by [13] or [15]. Introducing the super-time \( \tilde{t} \) such that \( dt = V^{1/3} dt \), a “mass” \( m = V^{4/3} \) and an “energy” \( E = (dV/dt)^2/2V^{2/3} \), dynamical equation (6) in vacuum becomes

\[
\begin{align*}
\frac{dq_{1,2}}{dt} &= \frac{p_{1,2}}{m} = \frac{\partial E}{\partial q_{1,2}}, \\
\frac{dp_{1,2}}{dt} &= -\frac{\partial \xi}{\partial q_{1,2}} = \frac{\partial E}{\partial p_{1,2}}
\end{align*}
\]

with \( E = \frac{p_1^2 + p_2^2}{2m} - \xi(q) \), \( \xi(q) = \sum_{i=1}^{6} k_i e^{\pi(a_i)q} \) \( q \in \mathbb{R}^2, \pi : \) normal projector onto \((e_1, e_2)\) plane

\[ \text{(18)} \]

Toward singularity, \( E \to +\infty \) and \( m \to 0 \). When we go through time backward, equations (18) are ones of a ball with a decreasing mass \( m \) and with an increasing energy. The exponential nature of the potential \( \xi \) allows a precise description of he dynamics, namely, the billiard analogy. When \( \xi \) is exponentially negligible, solutions of (18) are almost straight lines in phase plane \((q_1, q_2)\); these are almost Kasner states in BKL representation. When \( \xi \) cannot be neglected anymore, it is mainly due to one of it exponential term, equation of motion − for a linear combination \( y \) of \((q_1, q_2)\) − takes a form equivalent to

\[
\frac{d^2y}{dx^2} = -k^2 e^y \quad \text{with} \quad \frac{dy}{dx} \bigg|_{x=0} = 0
\]

This equation could be solved analytically in

\[ y(x) = \ln \left[ 1 - \frac{1}{2} k^2 \left( \frac{kx}{\sqrt{2}} \right)^2 \right] = -2 \ln \left[ \cosh \left( \frac{kx}{\sqrt{2}} \right) \right] \]

\[ \text{(20)} \]

It shows clearly a transition between the two Kasner states associated to the straight lines which are the asymptotes of this solution (see Fig. 2). This generalized \( B_{1I} \) transition corresponds to a bounce of a ball in an amazing billiard. As a matter of fact, when \( p_1^2 + p_2^2 = 0 \), all ball’s energy is concentrated in the potential term \( \xi \), this situation corresponds to the maximum of the \( y \) function plotted in Fig. 2: this is a bounce. It happens effectively against the isocountour line \( \xi = \xi(q^*) \) where \( dq^*/dt = 0 \). As we go toward singularity (\( \tilde{t} \to 0 \)) these lines move outward as one can see in Fig. 3. The Universe ball then moves in a expanding billiard. It is easy to prove that ball’s velocity norm \( |dq/d\tilde{t}| \) is always greater than the one of cushions which is the energy variation (see [18] for instance). This property remains true when barotropic matter fills Universe provided that \( \Gamma < 2 \). The amplitude of the bounce depends on \( k \); the largest \( k \) is, the smallest is the bounce angle between the two asymptotic linear regimes. Between the two ideal cases (quasi Kasner state and bounce) the parameter \( k \) is dynamical, its variations remains negligible during each phase.

DYNAMICAL PROPERTIES OF BIANCHI BILLIARDS

In the billiard formulation, dynamics of Bianchi Universe is fully understood considering isocountours of the potential \( \xi \) which are presented on Fig. 3. Excluding \( B_{1X} \) and \( B_{VIII} \) all isocountours are open curves. Hence after a finite number of bounces, the ball representing Universe find a hole and go away to reach the asymptotic regime solution of equation (19) which is a Kasner state. By opposite, for the two closed curves potential, there is no hole in the billiard, cushion allows expands less quickly than the ball, and there is an infinite sequence of \( B_{1I} \) transitions toward singularity. Although \( B_{VIII} \) cushions seems open along a vertical channel, the bounce process makes the ball which enter the channel goes back. As sketched on Fig. 2, in the channel \( k \sim q_2 \), then as the ball sinks into this region, \( q_2 \) grows and bounces are more and more pinched. By consequence, as the cushion is not strictly vertical, the ball finally goes back and leaves the channel. These results are surely well known and one remaining question could be : Is \( B_{VIII} \) dynamics equivalent in some precise sence to \( B_{1X} \) one ? In terms of complexity the two dynamics seems equivalent as they have exactly the same set of Kovalevskaya exponents. Let us show that they are also equivalent by the fact that they present the same kind of transition to chaos, and they have the same kind of attractor.

---

\[ ^2 \text{The \( x \) variable represents the supertime \( \tilde{t} \) such that when \( \tilde{t} \sim x \) one has} \quad \xi \sim -k^2 e^x \]
**B_{IX} and B_{VIII} Poincaré map analysis**

Detailed analysis show that Bianchi Universe volume is a regular variable. It is the only variable attached to cushions expansion in the billiard formalism. Therefore, the complexity of the dynamics is contained in the cushions’s shape. In order to study this complexity in the two relevant cases which are $B_{IX}$ and $B_{VIII}$ Universes, we have studied dynamical propertied of temporal sections of the hamiltonian $E$ involving in equation (18). As $V$, $t$, $i$ and $\tau$ are all related in a bijective way, by temporal section we mean sections of the dynamics described by equation (18) with $m = \text{cst}$ and $E = \text{cst}$. For convenience we use always a ball with unit mass, we focuss attention under Poincaré maps associated to growing values of $E$. Such maps are obtained by considering the intersection between an orbit (a solution of the 4D differential system) and a fixed plane, namely $\Pi = \{q_2 = 0, p_2 = 0\}$ for maps plotted in Fig. 4. The greater are values of the energy $E$, the closest from singularity sections are. Analysis of such maps is clear using KAM Theory : When periodic or quasiperiodic orbits correspond to Poincaré sections disposed along curves, chaotic orbits sections
FIGURE 4. Poincaré maps toward singularity of temporal sections of $m = \text{cst}$ and $E = \text{cst}$ of $B_{IX}$ (top: $m = 1$ and $E = -2.8, -2.1.0$ from left to right) and $B_{VIII}$ (bottom: $m = 1$ and $E = 5.0, 10.0$ from left to right) dynamics. For each map a set of 9 homogeneously distributed initial conditions have been chosen.

fill dense regions. We can then conclude that far from singularity orbits are regular in both $B_{IX}$ and $B_{VIII}$. Approaching singularity, a transition to chaos seems to appear in the dynamics of both these closed cushions expanding billiards. In order to produce a comparative measure of this chaos we propose to produce an analysis base on $B_{IX}$ and $B_{VIII}$ Universes truncated dynamics.

$B_{IX}$ and $B_{VIII}$ fractal attractors

As suggered by all previous indicators (Kovalevskaya exponents, Poincaré maps), we suspect $B_{VIII}$ dynamics to possess the same kind of attractor exhibited initially by [17] for $B_{IX}$. In order to confirm this suspition we have applied the truncated dynamical technique used by [17] in the context of Hamiltonian formalism described above. Initial conditions we used are $q_o = [0, -\sqrt{6}\ln 2, \Omega]$ and $p_o = [\cos \theta, \sin \theta, \omega]$ where the constraint $H = 0$ is fullfilled provided that

$$\Omega = \frac{3}{\sqrt{6}} \ln \left[ \pm \frac{(1 - \omega^2)}{2^\alpha (1 + 2)} \right]$$

(21)

The upper and lower signs are associated respectively to $B_{IX}$ and $B_{VIII}$ Universes. The two free parameters $\omega$ and $\theta$ fixe respectively the initial variation rate of Univers volume (or the initial escape speed of cushions in the billiard formalism) and the orientation of the initial velocity vector. These initial conditions are represented on Fig. 5. The numerical integration of the dynamics from these initial conditions shows an unlimited sequence of $B_{II}$ transitions between Kasner states (see formula (17)). We stop numerical integration when the Kasnerian $u$ parameter is greater than a critical value (for maps presented on Fig. 6 we have used $u_{\text{crit}} = 8$). When the dynamic is stopped, we affect to the corresponding point in the $\theta - \omega$ plane a color which correspond to the expanding axe of stopped Kasner state (Red $\leftrightarrow A_1$, Blue $\leftrightarrow A_2$ and Green $\leftrightarrow A_3$). We have studied the $[0, \pi/2] \times [-2, -3]$ portion of the $\theta - \omega$ plane, projected on a regular $500 \times 500$ grid. The corresponding maps are presented in Fig. 6. From this map we can compute the 3 Hausdorff dimensions associated to each color set. We can resume these three numbers into one by averaging each dimension weighted by the surface proportion occupied by the corresponding color. We then obtain a mean Hausdorff dimension of each map which is $d = 1.6976 \pm 8.9 \times 10^{-3}$ for $B_{VIII}$’s map and $d = 1.7141 \pm 8.5 \times 10^{-3}$ for $B_{IX}$’s
map. These two numbers are then numerically indistinguishable. In this sense we claim that $B_{\text{VIII}}$ and $B_{\text{IX}}$ universes are dynamically equivalent.

REFERENCES