Erratum to “Nonparametric estimation of the stationary density and the transition density of a Markov chain”
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Erratum to


Claire Lacour

New proof of Proposition 1

The result of Proposition 1 is true but the proof must be modified in the following way. We replace Lemma 10 by

**Lemma 10** Under the assumptions of Proposition 1, and if \((X_n)\) has an atom \(A\),

\[
\sum_{\lambda \in \Lambda_m} \mathbb{E}|S_j(\varphi_\lambda)|^2 \leq r_0^2 \mathbb{E}_A(\tau^2) D_m.
\]

**Proof of Lemma 10:** Using a convex inequality, we can write

\[
\sum_{\lambda \in \Lambda_m} \mathbb{E}|S_j(\varphi_\lambda)|^2 \leq \sum_{\lambda \in \Lambda_m} \mathbb{E}_\mu \left| \sum_{i=\tau+1}^{\tau(2)} \varphi_\lambda(X_i) \right|^2 \leq \sum_{\lambda \in \Lambda_m} \mathbb{E}_\mu \left( (\tau(2) - \tau) \sum_{i=\tau+1}^{\tau(2)} \varphi_\lambda^2(X_i) \right)
\]

Assumption M2 entails \(\|\sum_{\lambda \in \Lambda_m} \varphi_\lambda\|_\infty \leq r_0^2 D_m\). Then

\[
\sum_{\lambda \in \Lambda_m} \mathbb{E}|S_j(\varphi_\lambda)|^2 \leq \mathbb{E}_\mu \left( (\tau(2) - \tau) \sum_{i=\tau+1}^{\tau(2)} r_0^2 D_m \right) \leq r_0^2 \mathbb{E}_\mu \left( (\tau(2) - \tau)^2 \right) D_m
\]

To conclude, recall that by the Markov property,

\[
\mathbb{E}_\mu \left( (\tau(2) - \tau)^2 \right) = \sum_k \sum_{l>k} (l-k)^2 \mathbb{P}_\mu(\tau = k, \tau(2) = l)
\]

\[
= \sum_k \sum_{l>k} (l-k)^2 \mathbb{P}(X_{k+1} \notin A, \ldots, X_{l-1} \notin A, X_l \in A|X_k \in A) \mathbb{P}(X_1 \notin A, \ldots, X_k \in A) \mathbb{P}_\mu(\tau = k)
\]

\[
= \sum_k \sum_{l>k} (l-k)^2 \mathbb{P}_A(\tau = k)
\]

\[
= \sum_k \sum_{j>0} j^2 \mathbb{P}_A(\tau = j) \mathbb{P}_\mu(\tau = k) = \mathbb{E}_A(\tau^2).
\]
We can then give the bound
\[
\sum_{\lambda \in \Lambda_m} \mathbb{E}(\nu_n^{(3)}(\varphi_\lambda)^2) \leq \frac{r_0^2 \mathbb{E}_A(\tau^2) D_m}{n}.
\]
Finally \(E\|f_m - \hat{f}_m\|^2 \leq CD_m/n\) with \(C = 4[8r_0^2(\mathbb{E}_\mu(\tau^2) + \mu(A)\mathbb{E}_A(\tau^4)) + r_0^2\mathbb{E}_A(\tau^2)].\)

New proof of Theorem 3

The result of Theorem 3 is true but the proof must be modified in the following way. Proposition 12 must be replaced by:

Proposition 12 Let \((X_n)\) be a Markov chain which satisfies A1–A5 and \((S_m)_{m \in \mathcal{M}_n}\) be a collection of models satisfying M1–M3. We suppose that \((X_n)\) has an atom \(A\). Let \(B(m, m') = \{t \in S_m + S_{m'}, \|t\| = 1\}\) and

\[
p(m, m') = K\mu(A)\mathbb{E}_A(\tau^2)r_0^2\frac{\dim(S_m + S_{m'})}{n}
\]
(where \(K\) is a numerical constant). Then

\[
\sum_{m' \in \mathcal{M}_n} \mathbb{E} \left[ \sup_{t \in B(m, m')} \nu_n^2(t) - p(m, m') \right]_+ = O(n^{-1}).
\]

Remark 1 This gives a penalty in Theorem 3 of the form

\[
\text{pen}(m) = K\mu(A)\mathbb{E}_A(\tau^2)r_0^2\frac{D_m}{n}, \quad \text{for some } K > K_0
\]
with \(K_0\) a numerical constant. Note that this penalty is simpler than in the previous version of this theorem. In particular, it does not depend on \(\|f\|_\infty\).

Remark 2 As it can be seen in the proof, Assumption M1 can be relaxed, it is now sufficient to assume that each \(S_m\) is a linear subspace of \((L^\infty \cap L^2)([0, 1])\) with dimension \(D_m \leq n\). This entails an improvement on the smoothness assumption for Corollary 5: \(\alpha > 0\) is sufficient. In the same way, M1’ can be relaxed and the condition for Corollary 8 is only \(\alpha > 0\).

Proof of Proposition ??: The heart of the proof is to use Theorem 7 in \(\text{??}\) which is a concentration inequality for Markov chains. In our case \(T_1 = \tau(1) = \tau\) and \(T_2 = \tau(2) - \tau(1)\). Let us check that our assumptions allow us to use this theorem.
• We can easily prove that our assumption A4 implies the Minorization Condition with \( m = 1 \) in ?. Indeed, since \( \int h d\mu > 0 \), there exists \( C \) with measure \( \mu(C) > 0 \) and \( \delta > 0 \) such that \( h \) is larger than \( \delta \) on \( C \). Then for all \( x \in C \) and all event \( B \), \( P(x, B) \geq h(x)\nu(B) \geq \delta \nu(B) \). Moreover, fixing \( x \in \mathbb{R} \), for \( n \) large enough, the ergodicity of the chain gives

\[ |P^n(x, C) - \mu(C)| \leq \frac{\mu(C)}{2}, \]

which implies \( P^n(x, C) \geq \mu(C)/2 > 0 \).

• As noticed at the very beginning of his Section 3.5, the assumption of finiteness of Orlicz norm of \( T_1 \) and \( T_2 \), which is required to apply the theorem, is equivalent to existence of a number \( s > 1 \) such that

\[ \mathbb{E}_\mu(s^\tau) < \infty, \quad \mathbb{E}_\nu(s^\tau) < \infty. \tag{1} \]

Now, we use condition A5 of geometric ergodicity. Theorem 15.4.2 in ? shows that there exists a full absorbing set \( S \) such that \( S \) is geometrically regular, i.e. \( \sup_{x \in S} \mathbb{E}_\nu(s^\tau) < \infty \) for some \( s > 1 \) (depending on \( A \)). Since \( S \) is full absorbing, and \( \mu \) is the limit distribution of the chain \( \mu(S) = 1 \). Moreover \( \mu(C \cap S) > 0 \), where \( C \) is the set introduced in the Minorization Condition. So we can find an \( x \in C \cap S \) and \( \delta \nu(S^c) \leq P(x, S^c) = 0 \). Thus \( \nu(S) = 1 \) too. This implies condition (??).

Now we write an integrated version of the concentration inequality. We denote \( \nu_n(t) = n^{-1} \sum_{i=1}^n [t(X_i) - \langle t, f \rangle] \) where \( f \) is the stationary density of the chain and we consider a countable class \( B \) of measurable functions \( t \). Let \( a \) and \( H \) such that

\[ \sup_{t \in B} ||t - \langle t, f \rangle||_\infty \leq a, \quad \mathbb{E} \left( \sup_{t \in B} |\nu_n(t)| \right) \leq H. \]

Let the variance term

\[ \sigma^2 = \mathbb{E}_A(\tau)^{-1} \sup_{t \in B} \mathbb{E}_A \left( \left( \sum_{i=1}^\tau t(X_i) - \langle t, f \rangle \right)^2 \right). \]

Then we prove the existence of a numerical constant \( c > 0 \) such that

\[ \mathbb{E}[\sup_{t \in B} |\nu_n(t)|^2 - cH^2]_+ \leq K_1 \left( \frac{1}{n^2} + \frac{\sigma^2}{n} e^{-K_2 \frac{a^2}{\sigma^2}} + \frac{a^2 (\log n)^2}{n^2} e^{-K_3 \frac{a}{n \log n}} \right) \tag{2} \]

where \( K_1, K_2, K_3 \) are depending on the chain. Indeed we compute, for \( c = 8K^2 \),

\[ \mathbb{E} \left[ \sup_{t \in B} |\nu_n(t)|^2 - cH^2 \right]_+ = \int_0^\infty P \left( \sup_{t \in B} |\nu_n(t)|^2 \geq cH^2 + x \right) dx \]

\[ \leq \int_0^\infty P \left( \sup_{t \in B} |\nu_n(t)| \geq \sqrt{c/2}H + \sqrt{x/2} \right) dx \leq \int_0^\infty P \left( Z \geq \sqrt{c/2}EZ + n\sqrt{\frac{x}{2}} \right) dx \]

\[ \leq \int_0^\infty P \left( Z \geq KEZ + KEZ + n\sqrt{\frac{x}{2}} \right) dx \]

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where $Z = n \sup_{t \in B} |\nu_n(t)|$. If $x \geq 2n^{-2}$, $t = K\mathbb{E}Z + n\sqrt{x/2} \geq 1$ so that we can apply Theorem 7. Moreover

$$
\int_0^{2n^{-2}} P\left(Z \geq K\mathbb{E}Z + K\mathbb{E}Z + n\sqrt{x/2}\right) dx \leq 2n^{-2}.
$$

Thus

$$
\mathbb{E} \left[ \sup_{t \in B} |\nu_n(t)|^2 \right] \geq \frac{2}{n^2}
$$

$$
+ \int_0^{\infty} K \exp \left(-\frac{1}{K} \min \left(\frac{[K\mathbb{E}Z + n\sqrt{x/2}]^2}{n\sigma^2}, \frac{K\mathbb{E}Z + n\sqrt{x/2}}{a \log n}\right)\right) dx
$$

$$
\leq \frac{2}{n^2} + \frac{1}{K^2} e^{-\frac{K^2\mathbb{E}Z^2}{n\sigma^2}} \int_0^{\infty} e^{-\frac{K^2\sigma^2}{2}} dx + \frac{1}{K^3} e^{-\frac{K\mathbb{E}Z}{a \log n}} \int_0^{\infty} e^{-\frac{K\mathbb{E}Z}{a \log n}} dx
$$

$$
\leq \frac{2}{n^2} + K_4 \frac{\sigma^2}{n} e^{-\frac{K_4H^2}{\sigma^2}} + K_5 \frac{(a \log n)^2}{n^2} e^{-\frac{K_5H}{a \log n}}
$$

This gives inequality (??). This result can be extended to a non-countable class $B$ with classical density arguments. So we apply it with $B = B(m, m')$. Moreover, the result of ?? is also true when replacing $\mathbb{E}Z = n\mathbb{E}(\sup_{t \in B} |\nu_n(t)|)$ by $n\mathbb{E}(\sup_{t \in B} |\nu'_n(t)|)$ with

$$
\nu'_n(t) = \frac{1}{n} \sum_{j=1}^{[3n/\mathbb{E}_A(\tau)]} S_j(t)
$$

(see in the proof of Theorem 7, p 1020). Thus (??) is also valid with $H \geq \mathbb{E}(\sup_{t \in B} |\nu'_n(t)|)$. It remains to compute $a, H$ and $\sigma^2$. We denote $D(m, m') = \max(D_m, D_{m'})$ the dimension of the space $S_m + S_{m'}$ (recall that the models are nested) and $(\varphi_\lambda)_{\lambda \in \Lambda(m, m')}$ an orthonormal basis of $S_m + S_{m'}$.

- Computation of $a$. If $t \in S_m + S_{m'}, \|t\|_\infty \leq r_0 \sqrt{D(m, m')} \|t\|$. Then $a = 2r_0 \sqrt{D(m, m')}$.  

- Computation of $H^2$. Since any $t \in B(m, m')$ can be written $t = \sum_{\lambda \in \Lambda(m, m')} a_\lambda \varphi_\lambda$, 

$$
\mathbb{E} \left( \sup_{t \in B(m, m')} \nu'_n(t)^2 \right) \leq \sum_{\lambda \in \Lambda(m, m')} \mathbb{E}(\nu'_n(\varphi_\lambda)^2) \leq \sum_{\lambda \in \Lambda(m, m')} \mathbb{E} \left( \left( \frac{1}{n} \sum_{j=1}^{[3n/\mathbb{E}_A(\tau)]} S_j(\varphi_\lambda) \right)^2 \right).
$$

Recall that the $S_j(t)$ are independent identically distributed and centered. Then, using (new) Lemma 10,

$$
\mathbb{E} \left( \sup_{t \in B(m, m')} \nu'_n(t)^2 \right) \leq \frac{[3n/\mathbb{E}_A(\tau)]}{n^2} r_0^2 \mathbb{E}_A(\tau^2) D(m, m').
$$

Finally, since $\mu(A) = \mathbb{E}_A(\tau)^{-1}$, we set $H^2 = CD(m, m')/n$ with $C = 3\mu(A)\mathbb{E}_A(\tau^2)r_0^2$.  


• Computation of $\sigma^2$. We use the following inequality, given in ?, subsection 17.4.3:

$$\mu(A)E_A \left[ \left( \sum_{i=1}^{\tau} t(X_i) - \langle t, f \rangle \right)^2 \right] = 2 \int (t - \langle t, f \rangle) \hat{t}d\mu - \int (t - \langle t, f \rangle)^2 d\mu$$

where

$$\hat{t}(x) := E_x \left( \sum_{i=0}^{\sigma_A} t(X_i) - \langle t, f \rangle \right)$$

and $\sigma_A = \inf\{n \geq 0, X_n \in A\}$. Then, since $\mu(A) = E_A (\tau)^{-1}$,

$$\sigma^2 \leq \sup_{t \in B(m,m')} 2 \int (t - \langle t, f \rangle) \hat{t}d\mu \leq \sup_{t \in B(m,m')} 2 \left( \int (t - \langle t, f \rangle)^2 d\mu \int \hat{t}^2 d\mu \right)^{1/2}.$$ 

But $\int (t - \langle t, f \rangle)^2 d\mu \leq \int t^2 f \leq \|f\|_\infty \|t\|^2$ and

$$\hat{t}^2(x) \leq E_x \left( \left( \sum_{i=0}^{\sigma_A} t(X_i) - \langle t, f \rangle \right) \right)^2 \leq 4\|t\|_\infty^2 E_x((\sigma_A + 1)^2).$$

with $E_x((\sigma_A + 1)^2) \leq E_x((\tau + 1)^2)$. Then

$$\sigma^2 \leq 4 \sqrt{E_x((\tau + 1)^2)} \sqrt{\|f\|_\infty} \sup_{t \in B(m,m')} \|t\|_\infty \|t\|$$

so that

$$\sigma^2 \leq 4 \sqrt{E_x((\tau + 1)^2)} \sqrt{\|f\|_\infty r_0} \sqrt{D(m,m')}.$$ 

Now, we can use inequality (??): it implies existence of positive constants $K_1, K_2, K_3$ such that

$$E[\sup_{t \in B} |\nu_n(t)|^2 - cCD(m,m')/n]_+ \leq$$

$$K_1 \left( \frac{1}{n^2} + \sqrt{\frac{D(m,m')}{n} e^{-K_3^3 \sqrt{D(m,m')}}} + \frac{D(m,m')(\log n)^2}{n^2} e^{-K_3^3 \sqrt{\log n}} \right).$$

Using that $D(m,m') = \max(D_m, D'_m) \leq n$, we obtain that $\sum_{m' \in M_n} \sqrt{D(m,m')} e^{-K_3^3 \sqrt{D(m,m')}}$ and $\sum_{m' \in M_n} D(m,m')(\log n)^2 n^{-1} e^{-K_3^3 \sqrt{\log n}}$ are bounded. Moreover $|M_n| n^{-2} = O(n^{-1})$. Thus

$$\sum_{m' \in M_n} E[\sup_{t \in B} |\nu_n(t)|^2 - cCD(m,m')/n]_+ = O(n^{-1})$$

□
New proof of Theorem 9

The result of Theorem 9 is true but the proof must be modified in the following way. Recall that we denote $E_n = \{\|f - \hat{f}\|_\infty \leq \chi/2\}$ and $E_n^c$ its complementary. We have

$$\mathbb{E}\|\hat{\pi} - \bar{\pi}\|^2 \leq \frac{8}{\chi^2} \left( \mathbb{E}\|g - \bar{g}\|^2 + \mathbb{E}\|f - \tilde{f}\|^2 \right) + (a_n + \|\pi\|_\infty)^2 P(E_n^c)$$

so that it is sufficient to bound $(a_n + \|\pi\|_\infty)^2 P(E_n^c)$. We have proven that, for $n$ large enough,

$$P(E_n^c) \leq P \left( \|f_m - \hat{f}_m\|_\infty > \frac{\chi}{4} \right) \leq P \left( \|f_m - \hat{f}_m\| > \frac{\chi}{4r_0\sqrt{D_m}} \right).$$

But

$$\|f_m - \hat{f}_m\| = \sup_{t \in S_m, \|t\| \leq 1} \int t(\hat{f}_m - f_m) = \sup_{t \in S_m, \|t\| \leq 1} \nu_n(t).$$

Let $S_{m_0}$ the largest model with dimension $D_{m_0} \leq n^{1/4}$.

$$P(E_n^c) \leq P \left( \sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 > \frac{\chi^2}{16r_0^2 D_{m_0}} \right) \leq P \left( \sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 > \frac{\chi^2}{16r_0^2 D_{m_0}} \right).$$

As shown in the (new) proof of Proposition 12, our assumptions allow us to use Theorem 7 in ?. Then, reasoning as in the proof of Proposition 12, we can show the existence of a numerical constant $c > 0$ and constants depending on the chain $K_1, K_2, K_3 > 0$ such that

$$P \left( \sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 \geq \frac{c}{2} H^2 \right) \leq K_1 \left( e^{-K_2\sqrt{D_{m_0}}} + e^{-K_3\sqrt{n}/\log(n)} \right)$$

where $H^2 = 3\mu(A)\mathbb{E}_A(\tau^2)r_0^2 D_{m_0}/n$. Now, for $n$ large enough, since $D_{m_0}^2 = o(n)$,

$$\frac{\chi^2}{16r_0^2 D_{m_0}} \geq \frac{3c\mu(A)\mathbb{E}_A(\tau^2)r_0^2 D_{m_0}}{2 \frac{2}{n}}.$$

Then

$$P(E_n^c) \leq P \left( \sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 \geq \frac{c}{2} H^2 \right) \leq K_1 \left( e^{-K_2\sqrt{D_{m_0}}} + e^{-K_3\sqrt{n}/\log(n)} \right)$$

so that $(a_n + \|\pi\|_\infty)^2 P(E_n^c) = o(n^{-1})$. Note that it is sufficient to have $D_{m_0} = |n^{1/2-\epsilon}|$ to obtain the result.
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References
