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Random walks on graphs induced by aperiodic tilings

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Abstract In this paper, simple random walks on a class of graphs induced by quasi-periodic tilings of the Euclidean space $\mathbb{R}^d$ are investigated. Roughly speaking, these graphs are obtained by considering a d-dimensional slice of the Cayley graph of $\mathbb{Z}^N$. The quasi-periodicity of the underlying tilings implies that these graphs are not space homogeneous (roughly speaking, there is no transitive group action). In this context, we prove that the asymptotic entropy of the simple random walk is zero and characterize the type (recurrent or transient) of the simple random walk. These results are similar to the classical context of random walks on the integer lattice. In this sense, it suggests that a varying local curvature does not modify the global behavior of the simple random walk as long as the graph remains roughly globally flat.

Keywords Random walks . Aperiodic tilings . Aperiodic graphs . Non homogeneous spaces . Asymptotic entropy . Markov additive processes . Random walk with random transition probabilities . Recurrence . Transience . Isoperimetric inequalities

Mathematics Subject Classification (2010) 60F20 . 60G50 . 60J10 . 60J45 . 60J50 . 60K37 . 31C05

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1 Introduction and notations

In this paper, we study random walks on a class of graphs that are roughly a d-dimensional slice of the standard integer lattice $\mathbb{Z}^N$. These graphs are induced by aperiodic tilings of the Euclidean space and consequently are not homogeneous spaces — roughly speaking there is no transitive group action. Compared to the homogeneous context or more particularly the context of groups, new stochastic behaviors can be observed for the underlying random walks. For instance, in [5], the simple random walk on an example of directed graph with vertex set $\mathbb{Z}^2$ is shown to be transient.

Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1)$ be an *undirected graph* : $\mathbb{G}^0$ is a countable set named the set of *vertices* and $\mathbb{G}^1$ denotes the set of *edges*, that is a symmetric subset $\mathbb{G}^0 \times \mathbb{G}^0$, in symbols, a subset satisfying $(x, y) \in \mathbb{G}^1$ if and only if $(y, x) \in \mathbb{G}^1$ (such an undirected edge is also denoted by $\{x, y\}$). A *random walk* on $\mathbb{G}$ is nothing but a Markov chain on the countable state space $\mathbb{G}^0$ whose transitions $P(x, y)$ are strictly positive as soon as $(x, y) \in \mathbb{G}^1$. Let us denote the *degree* of a vertex $x \in \mathbb{G}^0$ by $\mathsf{deg}\ x$ and defined as the number of vertices $y \in \mathbb{G}^0$ such that $(x, y) \in \mathbb{G}^1$. A random walk is said to be *simple* if $P(x, y) = (\mathsf{deg}\ (x))^{-1}$ for any $y \in \mathbb{G}^0$ such that $(x, y) \in \mathbb{G}^1$, and zero otherwise. This definition implicitly assumes there is no isolated vertex so that $\mathsf{deg} \geq 1$. Finally, a random walk is said to be *reversible* if there exists a σ-finite measure m, satisfying $m(x) \in (0, +\infty)$ for all $x \in \mathbb{G}^0$, and such that $m(x)P(x, y) = m(y)P(y, x)$. Recall that the simple random walk on an undirected graph is reversible with $m = \mathsf{deg}$. For the terminology on reversible random walks, we mainly refer to [19].

The class of graphs (or groupoids) considered in this paper are obtained as 1-dimensional complexes by tiling the standard real vector space $\mathbb{R}^d$ with the help of the cut-and-project scheme. More precisely, let E be a d-dimensional vector subspace of $\mathbb{R}^N$, named the *real space*, and denote by $E^\perp$ the orthogonal complement of E, called the *internal space*. Let K be the unit cube in $\mathbb{Z}^N$. An edge in the Cayley graph of $\mathbb{Z}^N$ is accepted and projected on E (orthogonally) if it can be translated by a vector of E in the unit cube $K + t$, $t \in E^\perp$. Under suitable assumptions this method gives rise to a family of tilings $\mathcal{T}_t$ of the space E whose prototiles are the projections of the d-dimensional facets of the N-dimensional unit cube K. Moreover, depending on the orientations of the space E and $E^\perp$ those tilings will be periodic or aperiodic — the group of translations is given by $E \cap \mathbb{Z}^N$. Such a tiling naturally defines a connected graph embedded in the space $\mathbb{R}^d$, called *the cut-and-project graph* — the vertex and edge sets are respectively the sets of vertices and sides defining the tiles. In [17, 3], the authors considered random walks on the tiles (that are rhombuses) of the Penrose tiling so that, at each step, the walker can move to exactly four directions. In this sense, such a random walk is combinatorially similar to the classical random walk on $\mathbb{Z}^2$. On the contrary, as the main motivation of paper, random walks on the sides of the tiles are considered so that the local degree of the graph is no longer constant.

Among interesting properties about random walks, one of them is related to the *asymptotic entropy*. For a general Markov chain, the asymptotic entropy is intimately connected with the *tail* — or asymptotic — *σ-algebra*. Namely, the tail σ-algebra is trivial, *i.e.* it only contains measurable sets of probability 0 or 1, if and only if the asymptotic entropy is zero. This kind of result can be seen as an analogue of the Kolmogorov 0-1 law for independent and identically distributed random variables. In fact, the triviality of the tail σ-algebra can be interpreted as a certain asymptotic independence. Noting that the invariant σ-algebra is contained in the tail σ-algebra, the triviality of the former follows from the triviality of the latter.

For a general Markov chain, the tail σ-algebra is connected to the so-called *tail boundary* which is itself isometrically isomorphic to the space of sequences of bounded harmonic functions on the state space — for this general context, see [11]. Also, the invariant σ-algebra is intimately

related to the *Poisson boundary* which is isometrically isomorphic to the space of bounded harmonic functions. Thus, the tail (*resp.* the invariant) σ-algebra is trivial if and only if there is no non constant sequences of bounded harmonic functions (*resp.* bounded harmonic functions).

It turns out that the two σ-algebras coincide for a random walk on a locally compact topological group provided the starting distribution is a Dirac mass. As a consequence, the Poisson boundary is trivial — the random walk is also called *Liouville* — if and only if the asymptotic entropy is zero. However, in the general case of Markov chains, it can happen that the Poisson boundary is trivial whereas the asymptotic entropy remains strictly positive (see [11, Theorem 4.1] or the original reference [15] for a simple example and [4] for a more elaborated one). Consequently, even though the polynomial growth rate of balls in these graphs suggests the Poisson boundary is trivial, the computation of the asymptotic entropy still gives a valuable information on the tail boundary.

In the context of random walks on groups — see [10, 6] and references therein — or random walks on homogeneous spaces — see [12] — the asymptotic entropy satisfies the so-called fundamental inequality $\mathsf{h} \leq \ell \cdot v$ where h is the asymptotic entropy of the random walk, ℓ the linear rate of escape and v the exponential growth rate of the group. This inequality no longer holds in the general case of Markov chains for mainly two reasons. First, the asymptotic entropy of a Markov chain heavily depends on the initial distribution which is not the case in the context of random walks on groups — the Markov transition kernel related to such a random walk is actually invariant under the group action. In particular, this prevents from identifying the asymptotic entropy to the more tractable Avez entropy (see [2]). Secondly, the space of sequences of increments of random walks is structurally an ergodic Bernoulli shift, and from this observation a Shannon-McMillan-Breiman type theorem can be stated. In [9], this inequality is extended to the context of *Random Walk with Random Transition Probabilities*, for short RWRTP, that is a random process whose increments are still independent but no longer identically distributed. The distribution of an increment is chosen accordingly to the configuration of an ergodic dynamical system which permits to recover, somehow, the stationarity of increments as in the context of random walk on groups. More precisely, let (Ω, T, λ) be a probability measure preserving dynamical system T on Ω and $\{\mu^\omega\}_{\omega\in\Omega}$ a collection of probability measures on a group $\mathbf{G}$, a RWRTP is a Markov chain on the space $\Omega \times \mathbf{G}$ whose Markov operator R is given, for any real bounded measurable function f on $\Omega \times \mathbf{G}$ by

$$Rf(\omega, g) = \int_{\mathbf{G}} f(T\omega, gh) d\mu^\omega(h), \quad (\omega, g) \in \Omega \times \mathbf{G}.$$

In this sequel, it is shown that the simple random walk on a cut-and-project graph is a RWRTP whose underlying dynamics admits an invariant probability measure. At this level, we do not enter in the details of the assumptions which are generic (see Section 3.2).

Theorem 1.1. *Generically, the asymptotic entropy of the simple random walk on a cut-and-project is zero. Consequently, the tail and invariant σ-algebras are trivial.*

Note that the existence of an invariant probability measure for the underlying probability measure is essential. In fact, the polynomial growth rate of the balls in a cut-and-project graphs is not sufficient to ensure that the asymptotic entropy is zero. This is made precise in Remark 1.

Following a remark of [11], this result suggests that random walks on quasi-periodic graphs satisfy a Central Limit Theorem (CLT) or even an invariance principle. Such results have been obtained in [17, 3] in the substantially different context of random walks on the tiles of the Penrose tiling that are combinatorially analogous to standard random walks on $\mathbb{Z}^2$.

Another classical question in the field of random walks on graphs is the type problem : is the random walk recurrent or transient ? Note that this characterization can not be deduced, in general, from a CLT (it is worth noting that the estimates obtained in the following theorem are actually very close to those given in a Local Limit Theorem (LLT) so that the type problem is somehow an intermediate problem between CLT and LLT). Here again, avoiding the technical but generic assumptions (see Theorem 4.1 in Section 4 for further details), the following theorem gives a complete characterization of the type of the simple random walk on quasi-periodic graphs.

Theorem 1.2. *Denoting by $P^n(x, y)$ the n-step transition probability between two vertices x and y of a cut-and-project graph, and setting $d = \mathsf{dim}\ E$, generically, the following asymptotic estimates hold :*

- $P^{2n}(x, x) \geq C_0(n \log n)^{-d/2}$, *and*
- $P^n(x, y) \leq C_1 n^{-d/2}$,

for some constants $C_0, C_1 > 0$. Consequently, the simple random walk on the cut-and-project graph is recurrent if $d \leq 2$ and transient otherwise.

The proof of this theorem involves standard arguments in the context of random walks on undirected graphs. More precisely, it is based on an estimate of the growth rate of balls for the statement related to the lower bound of the return probability, whereas a d-dimensional isoperimetric inequality is stated in order to obtain the upper bound of the n-step transition probabilities. As far as we are considering the type problem, many technics such as "civilized" embedding — see [7] — or Nash-Williams inequalities — see [19] for a modern approach — or rough isometries are available. However, all of them require at some step an argument that is more or less equivalent to the statement on the growth rate of balls and spheres. In fact, all the strategies consist of proving that the cut-and-project graphs are of curvature zero globally — even though locally the curvature is varying.

Section 2.1 is devoted to the basic notation and definitions related to tilings and the description of the cut-and-project method. We also construct the induced cut-and-project graphs on which the simple random walks are investigated. In Section 3, it is shown that the simple random walk on a cut-and-project is a RWRTP and its asymptotic entropy is shown to be zero. Section 4 is related to the type problem.

2 Cut-and-project graphs

2.1 Tiling the standard d-dimensional real vector space

We start with the description of the cut-and-project scheme to tile the real line. We consider the standard integer lattice $\mathbb{Z}^2$ of $\mathbb{R}^2$. Let E be an irrational line in $\mathbb{R}^2$, *i.e.* satisfying $E \cap \mathbb{Z}^2 = \{0\}$, and denote by $E^\perp$ the orthogonal of E. We denote by K the unit square in $\mathbb{Z}^2$. Thus, the translation of K along E defines a strip (see Figure 1).

Consequently, we obtain a tiling of the space E with two types of segments (short and long). The short and long segments correspond to the projections of vertical and horizontal sides of the unit square which are entirely contained in the strip. Actually, it can be noticed that there is an ambiguity in the example of Figure 1 since two opposite sides of the same unit square are completely contained in the strip so that we have to choose which one we project. However, the strip can be translated by a vector $t \in E^\perp$ in such a way that there is no ambiguity. And since the projection of the lattice $\mathbb{Z}^2$ on the internal space is countable, there is no ambiguity for all but countably many $t \in E^\perp$.

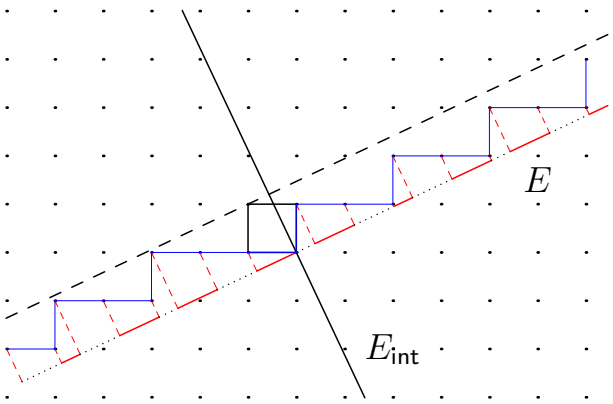


Figure 1: Quasi-periodic tiling of the real line within the cut-and-project scheme.

Finally, in a non-ambiguous case, we observe that there is a unique broken line which is completely contained in the strip. This is the theorem of [14], recalled at the end of this section, for the case of dimension 1. To keep the exposition self-contained, we recall some definitions introduced in [14].

Definition 2.1. A subset F of $\mathbb{R}^d$ is termed *regular* — for the usual topology of $\mathbb{R}^d$ — if it is bounded, has a non-empty interior $\mathsf{Int}(F)$, and is such that its closure $\mathsf{Clos}(F)$ coincides with $\mathsf{Clos}(\mathsf{Int}(F))$ and its interior $\mathsf{Int}(F)$ equals $\mathsf{Int}(\mathsf{Clos}(F))$.

Two regular subsets F_1 and F_2 are termed *congruent* if $F_1 = F_2$ up to a translation. The property of congruence induces an equivalence relation on the set of tiles; an equivalence class is termed a *prototile*.

Let $\mathcal{T}$ be a set of regular subsets of $\mathbb{R}^d$, we denote by $\mathcal{P}$ the corresponding set of prototiles, *i.e.* the factor set of $\mathcal{T}$ with respect to the equivalence relation of congruence.

Definition 2.2. A denumerable set $\mathcal{T} = \{F_i\}_{i\in I}$ of regular subsets is a *tiling* of $\mathbb{R}^d$ if

- the corresponding set $\mathcal{P}$ of prototiles is finite,
- $\mathbb{R}^d = \bigcup_{i\in I} F_i$, and
- $\mathsf{Int}(F_i) \cap \mathsf{Int}(F_j) \neq \emptyset$ for $i \neq j \in I$.

If $\mathcal{T}$ is a tiling, then a regular set $F \in \mathcal{T}$ is called a *tile*.

Let E be a d-dimensional subspace of $\mathbb{R}^N$, and $E^\perp$ be its orthogonal supplement in $\mathbb{R}^N$. The spaces E and $E^\perp$ are termed respectively *real space* and *internal space*. We will denote by $\mathfrak{p}$ and $\mathfrak{p}_\perp$ the canonical projections from $\mathbb{R}^N = E \oplus E^\perp$ to E and from $E \oplus E^\perp$ to $E^\perp$. Thus we have the following

$$E \xleftarrow{\ \mathfrak{p}\ } E \oplus E^\perp \xrightarrow{\ \mathfrak{p}_\perp\ } E^\perp \ .$$

We denote by K the unit cube in $\mathbb{Z}^N$, namely

$$K = \left\{ \sum_{i=1}^{N} \alpha_i \varepsilon_i : 0 \leq \alpha_i \leq 1 \right\},$$

where $(\varepsilon_1, \cdots, \varepsilon_N)$ is the canonical basis of $\mathbb{R}^N$.

Let p be an integer $0 \leq p \leq N$ and $M_p = \left\{I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, N\}\right\}$ be the set of index sets with p elements. The p-facets of the unit cube are indexed by M_p as follows :

$$K_I = \left\{ \sum_{i \in I} \alpha_i \varepsilon_i : \alpha_i \in [0,1] \right\} \text{ for all } I \in M_p \text{ and } p > 0,$$

and $K_\emptyset = \{0\}$. Obviously, the unit cube K admits the decomposition $K = K_I + K_{I^\complement}$.

Assumption 1. The decomposition $\mathbb{R}^N = E \oplus E^\perp$ is *non degenerated*, *i.e.* for any $I = \{i_1, \ldots, i_d\} \in M_d$, the system $\left\{\mathfrak{p}(\varepsilon_i), i \in I\right\}$ is of rank d and the system $\left\{\mathfrak{p}_\perp(\varepsilon_i), i \in I^\complement\right\}$ is of rank $N - d$.

Actually, the rank of the two systems of vectors are simultaneously maximal or not maximal by orthogonality. Moreover, maximality is a generic property and under this condition the d-facets of the unit cube are isomorphic to their projections on E by $\mathfrak{p}$, and also, the $(N-d)$-facets are isomorphic to their projections on $E^\perp$ by $\mathfrak{p}_\perp$. We will denote by D_I the projection $\mathfrak{p}(K_I)$ of the d-facet related to $I \in M_d$. According to [14], under Assumption 1, for (Lebesgue) almost every $t \in E^\perp$, the set

$$\mathcal{T}_t = \left\{ x + D_I : x = \mathfrak{p}(\xi), \text{ for some } \xi \in \mathbb{Z}^N, \mathfrak{p}_\perp(\xi) \in \mathfrak{p}_\perp(K_{I^\complement} + t), I \in M_d \right\} \tag{1}$$

is a tiling. It is, moreover, the projection of the unique d-dimensional faceted manifold entirely contained in the strip $\mathscr{K}_t = K + E + t$. A vector $t \in E^\perp$ for which this latter property does not hold is said to be ambiguous. Finally, the group of translation of $\mathcal{T}_t$ is given by $E \cap \mathbb{Z}^N$.

The Penrose's third tiling can be obtained by the cut-and-project method if we consider the real space E in $\mathbb{R}^5$ spanned by the two following vectors (see [14])

$$v_1 = (1, \cos(2\pi/5), -\cos(\pi/5), -\cos(\pi/5), \cos(2\pi/5)),$$

and,

$$v_2 = (0, \sin(2\pi/5), \sin(\pi/5), -\sin(\pi/5), -\sin(2\pi/5)).$$

For the icosahedral tiling of $\mathbb{R}^3$ — see [13] —, the vector subspaces E and $E^\perp$ of $\mathbb{R}^6$ are defined with the help of projectors

$$\mathfrak{p} = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & 1 & -1 & -1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & -1 & -1 & 1 \\ -1 & 1 & \sqrt{5} & 1 & -1 & 1 \\ -1 & -1 & 1 & \sqrt{5} & 1 & 1 \\ -1 & -1 & -1 & 1 & \sqrt{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{5} \end{pmatrix} \text{ and } \mathfrak{p}_\perp = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & -1 & 1 & 1 & -1 & -1 \\ -1 & \sqrt{5} & -1 & 1 & 1 & -1 \\ 1 & -1 & \sqrt{5} & -1 & 1 & -1 \\ 1 & 1 & -1 & \sqrt{5} & -1 & -1 \\ -1 & 1 & 1 & -1 & \sqrt{5} & -1 \\ -1 & -1 & -1 & -1 & -1 & \sqrt{5} \end{pmatrix}.$$

2.2 Constructive definition of a cut-and-project tiling

In the next section, it is proved that the simple random walk is in fact a RWRTP. The definition of this RWRTP relies on the constructive definition of cut-and-project tilings we present at this level.

Denote by $\mathbf{J} = \{\pm 1, \ldots, \pm N\}$ and let $t \in E^\perp$ be non ambiguous. Start from an arbitrary point $\xi \in \mathscr{K}_t \cap \mathbb{Z}^N$ and initialise the vertex, edge and d-dimensional facet sets by $\mathtt{V}_0^t = \{\xi\}$, $\mathtt{E}_0^t = \emptyset$ and $\mathtt{F}_0^t = \emptyset$. For $j \in \mathbf{J}$, denote by

$$V_j^t(\xi) = \begin{cases} \{\xi + \varepsilon_j\} & \text{if } \mathfrak{p}_\perp(\xi + \varepsilon_j) \in \mathfrak{p}_\perp(K) + t \\ \emptyset & \text{otherwise} \end{cases}$$

the set of nearest neighbours of ξ. In this definition, ε_i for a negative index $i \in \mathbf{J}$ has to be understood as $-\varepsilon_{|i|}$. Also, denote the set of indices corresponding to these neighbours by $J^t(\xi) = \{j \in \mathbf{J} : V_j^t(\xi) \neq \emptyset\}$.

Allowed undirected egdes are given locally by the set $E^t(\xi) = \big\{ \{\xi, \xi'\} : \xi' \in \cup_{j\in\mathbf{J}} V_j^t(\xi)\big\}$ whereas allowed d-dimensional facets are given locally by

$$F^t(\xi) = \Big\{ K_I : I \subset J^t(\xi), \mathsf{card}\ I = d, \mathsf{dim\ span}\ (\varepsilon_i, i \in I) = d, \mathfrak{p}_\perp(\xi + K_I) \in \mathfrak{p}_\perp(K) + t \Big\}.$$

The recursion is then given for $n \geq 0$ by

$$\begin{cases} \mathtt{V}_{n+1}^t = \mathtt{V}_n^t \cup \Big[\bigcup_{\xi\in\mathtt{V}_n^t} \bigcup_{j\in J^t(\xi)} V_j^t(\xi)\Big] \\ \mathtt{E}_{n+1}^t = \bigcup_{\xi\in\mathtt{V}_n^t} E^t(\xi) \\ \mathtt{F}_{n+1}^t = \bigcup_{\xi\in\mathtt{V}_n^t} \bigcup_{K\in F^t(\xi)} (\xi + K). \end{cases}$$

Furthermore, one may define the vertex and edge sets of the resulting cut-and-project graph $\mathbb{G}_t$ as follows

$$\mathbb{G}_t^0 = \mathfrak{p}\left(\cup_{n\geq 0}\mathtt{V}_n^t\right) \quad \text{and} \quad \mathbb{G}_t^1 = \mathfrak{p}\left(\cup_{n\geq 0}\mathtt{E}_n^t\right),$$

whereas the tiling $\mathcal{T}_t$ is given by $\mathfrak{p}(\cup_{n\geq 0}\mathtt{F}_n^t)$. Note that the characterization of projected facets are slightly different between this definition of $\mathcal{T}_t$ and the one of (1). Still the two characterizations are equivalent as shown in [14, Theorem VI.1].

Remark that $\mathsf{card}\ J^t(\xi)$ is nothing but the degree $\mathsf{deg}\ \xi$ of ξ in the subgraph of the Cayley graph of $\mathbb{Z}^N$ whose vertex and edge sets are given respectively by $\mathtt{V}^t = \cup_{n\geq 0}\mathtt{V}_n^t$ and $\mathtt{E}^t = \cup_{n\geq 0}\mathtt{E}_n^t$. Moreover, the simple random walk on the graph $\mathbb{G}_t$, denoted by $(M_n)_{n\geq 0}$ in the sequel, and the random walk on $(\mathtt{V}^t, \mathtt{E}^t)$ are obviously combinatorially identical since the restriction to the set $\mathscr{K}_t \cap \mathbb{Z}^N$ of the map $\mathfrak{p} : E \oplus E^\perp \to E$ is bijective.

3 The asymptotic entropy estimate

In this section, we shall estimate the asymptotic entropy of the simple random walk defined in the previous section. First, we shall prove that this random walk is the projection on E by $\mathfrak{p}$ of a RWRTP. The underlying dynamics is given by a Markov chain whose Markov operator Q is defined by (3). It turns out that this Markov chain admits an invariant probability measure (see Proposition 3.2). Proposition 3.3 explicits how the original random walk $(M_n)_{n\geq 0}$ is related to the RWRTP. It follows that the asymptotic entropy of the initial random walk can be expressed in terms of the asymptotic entropy of the RWRTP (see Proposition 3.4). Finally, Theorem 3.5 gives the statement announced in the introduction.

3.1 Random walk with random transition probabilities

Let $t \in E^\perp$ be non ambiguous and set $\mathbb{X}_t = \mathfrak{p}_\perp(K) + t$. The definition of $V_j^t(\xi)$, $\xi \in \mathscr{K}_t \cap \mathbb{Z}^N$, $j \in \mathbf{J}$, can be rewritten as follows

$$V_j^t(x) = \begin{cases} \{x + \mathfrak{p}_\perp(\varepsilon_j)\} & \text{if } x + \mathfrak{p}_\perp(\varepsilon_j) \in \mathbb{X}_t \\ \emptyset & \text{otherwise,} \end{cases} \qquad \text{where} \quad x = \mathfrak{p}_\perp(\xi). \tag{2}$$

Thus, this set is *a priori* defined for $x \in \mathbb{X}_t \cap \mathfrak{p}_\perp(\mathbb{Z}^N)$ but still makes sense on the whole set $\mathbb{X}_t$. The set of allowed indices is denoted by $J^t(x)$.

A point $x \in \mathbb{X}_t$ is termed ambiguous if $\mathfrak{p}_\perp(\mathbb{Z}^N) \cap \{x+u\} \neq \emptyset$ for some $u \in E^\perp$ such that $u - t$ is ambiguous for the cut-and-project scheme. The set of ambigous points is denoted by $\mathcal{N}_t$. The

set $\mathcal{N}_t$ inherits the properties of the set of ambiguous $t \in E^\perp$ for the cut-and-project scheme. It is shown in [14] to be a closed with empty interior negligible (with respect to the Lebesgue measure on $E^\perp$) set. Let us also point out that for any $x \in \mathbb{X}_t \setminus \mathcal{N}_t$, $j \in \mathbf{J}$, $V_j^t(x) \cap \mathcal{N}_t = \emptyset$ and for any $t, t' \in E^\perp$ it holds $\mathcal{N}_{t'} = \mathcal{N}_t - (t + t')$. The ambigous points are pathological since they correspond in the cut-and-project scheme to the case for which the map $\mathfrak{p}$, when restricted to the strip $\mathscr{K}_t \cap \mathbb{Z}^N$, is no longer injective. Consequently, these points shall not be considered and the definitions below have to be understood up to this set of pathological points.

With these notations, one can define, respectively, the conductance and the Markov kernels for $x \in \mathbb{X}_t \setminus \mathcal{N}_t$

$$C_t(x, dy) = \sum_{j \in \mathbf{J}} \delta_{V_j(x)}(dy), \quad Q_t(x, dy) = \frac{C_t(x, dy)}{m_t(x)} \quad \text{with} \quad m_t(x) = \int_{\mathbb{X}_t} C_t(x, dy). \tag{3}$$

Remark that $m_t(x) = \mathsf{card}\ J^t(x) \leq 2N$. These quantities are invariant by translation as states in the following proposition.

Proposition 3.1. *Let $t, t' \in E^\perp$ be non ambiguous and $x \in \mathbb{X}_t \setminus \mathcal{N}_t$, then $x - (t - t') \in \mathbb{X}_{t'} \setminus \mathcal{N}_{t'}$ and for any measurable subset $A \subset \mathbb{X}_t \setminus \mathcal{N}_t$*

$$Q_t(x, A) = Q_{t'}(x - (t - t'), A - (t - t')).$$

Proof. Let $j \in \mathbf{J}$. Obviously, $x - (t - t') \in \mathbb{X}_{t'} \setminus \mathcal{N}_{t'}$. Furthermore, since $x + \mathfrak{p}_\perp(\varepsilon_j) \in \mathbb{X}_t$ if and only if $x - (t - t') + \mathfrak{p}_\perp(\varepsilon) \in \mathbb{X}_{t'}$, it follows that $V_j^t(x) = V_j^{t'}(x - (t - t'))$ and $J^t(x) = J^{t'}(x - (t - t'))$. Then, it is straightforward that the following equalities hold for any measurable set $A \subset \mathbb{X}_t$

$$C_t(x, A) = C_{t'}(x - (t - t'), A - (t - t')), \quad m_t(x) = m_{t'}(x - (t - t')).$$

□

This proposition implies that the Markov kernels Q_t and $Q_{t'}$ define the same Markov chain. In the sequel, the indices t and t' in the notations shall be dropped and implicitly understood as $t = 0$.

Let us define the following finite measure on $\mathbb{X}$

$$\tilde{\pi}(dx) = \mathbf{1}_{\mathbb{X}}(x) m(x) \mathsf{Leb}(dx)$$

where Leb is the Lebesgue measure on $E^\perp$, and denote by π the probability measure obtained after renormalization.

Proposition 3.2. *The probability measure π is stationary with respect to Q.*

Proof. Let $f : \mathbb{X} \to \mathbb{R}$ be a non negative real measurable function and compute

$$\begin{aligned}
\tilde{\pi} Q f &= \int_{\mathbb{X} \times \mathbb{X}} \tilde{\pi}(dx) Q(x, dy) f(y) \\
&= \sum_{j \in \mathbf{J}} \int_{\mathbb{X}} \mathbf{1}_{\mathbb{X}}(x) \mathbf{1}_{V_j(x)}(x + \mathfrak{p}_\perp(\varepsilon_j)) f(x + \varepsilon_j) \mathsf{Leb}(dx) \\
&= \sum_{j \in \mathbf{J}} \int_{\mathbb{X}} \mathbf{1}_{\mathbb{X}}(z) \mathbf{1}_{V_j(z - \mathfrak{p}_\perp(\varepsilon_j))}(z) f(z) \mathsf{Leb}(dz)
\end{aligned}$$

after the change of variable $z = x + \mathfrak{p}_\perp(\varepsilon_j)$ in each integral terms. Remarking that

$$\mathbf{1}_{\mathbb{X}}(z - \mathfrak{p}_\perp(\varepsilon_j)) \mathbf{1}_{V_j(z - \mathfrak{p}_\perp(\varepsilon_j))}(z) = \mathbf{1}_{\mathbb{X}}(z) \mathbf{1}_{V_{-j}(z)}(z + \mathfrak{p}_\perp(\varepsilon_{-j})),$$

one deduce that

$$\tilde{\pi}Qf = \sum_{j\in\mathbf{J}} \int_{\mathbb{X}} \mathbf{1}_{\mathbb{X}}(z)\mathbf{1}_{V_{-j}(z)}(z+\mathfrak{p}_{\perp}(\varepsilon_{-j}))f(z)\mathsf{Leb}(dz) = \tilde{\pi}f.$$

□

On the state space $\mathbb{X}\times\mathbb{Z}^N$, we define the Markov chain $(X_n,\tilde{M}_n)_{n\geq 0}$ whose initial distribution is $\delta_x\otimes\delta_0$ for some $x\in\mathbb{X}\setminus\mathcal{N}$ and transition kernel R is given by

$$R((x,z),(y,z')) = Q(x,y)\mu^{x,y}(z'-z),$$

where $\{\mu^{x,y}\}_{x,y\in\mathbb{X}}$ is the family of distributions defined by

$$\mu^{x,y} = \frac{\sum_{j\in\mathbf{J}:\mathfrak{p}_{\perp}(\varepsilon_j)=y-x}\delta_{\varepsilon_j}}{\mathsf{card}\{j\in\mathbf{J}:\mathfrak{p}_{\perp}(\varepsilon_j)=y-x\}}. \tag{4}$$

As a matter of fact, the Markov chain $(X_n,\tilde{M}_n)_{n\geq 0}$ is a RWRTP : the underlying dynamical system is given by the time-shift T on $\Omega=\mathbb{X}^{\mathbb{N}}$ preserving the Markovian probability measure $\mathbf{Q}^{\pi}$, that is the canonical probability measure on the path space Ω induced by the initial distribution π and the Markov operator Q. In addition, the probability measures μ^{ω}, $\omega\in\Omega$, defined in (4), actually depend only on the two first coordinates $(\omega_0,\omega_1)\in\mathbb{X}^2$. In fact, $(X_n,\tilde{M}_n)_{n\geq 0}$ is more specifically a Markov additive process.

If $t\in E^{\perp}$ is chosen non ambiguous, the map $\mathfrak{p}$ realizes a bijection between the subset $\mathbb{Z}^N\cap\mathscr{H}_t$ and $\mathbb{G}_t^0$. Therefore, for $x\in\mathbb{G}_t^0$, we shall denote by $\tilde{x}$ the unique point of $\mathbb{Z}^N\cap\mathscr{H}_t$ such that $\mathfrak{p}(\tilde{x})=x$ and $\bar{x}=\mathfrak{p}_{\perp}(\tilde{x}-t)\in\mathbb{X}$. In a similar way, suppose that $f:\mathbb{G}_t^0\to\mathbb{R}$ satisfies $f(x)=f(y)$, $x,y\in\mathbb{G}_t^0$, as soon as $\bar{x}=\bar{y}$, then $\bar{f}:\mathbb{X}\to\mathbb{R}$ is the function satisfying $f(x)=\bar{f}(\bar{x})$, for all $x\in\mathbb{G}_t^0$. Finally, we shall make use of the following compact notation for the convolution of $\mu^{\omega},\ldots,\mu^{T^n\omega}$:

$$\mu^{\omega}*\cdots*\mu^{T^n\omega} = \mu^{\omega}_{0,n}.$$

Obviously, the choice of the set $\mathbb{X}_0$ as a representative of $\mathbb{X}$ is irrelevant, other choices would only lead to a different formula for $\bar{x}$ and $\bar{f}$.

With these notations we can now state the following proposition which makes precise how the RWRTP and the original simple random walk on a cut-and-project graph are related.

Proposition 3.3. *Let $t\in E^{\perp}$ be non ambiguous, $\mathbb{G}_t$ the corresponding cut-and-project graph and $x\in\mathbb{G}_t^0$. Then for any $y\in\mathbb{G}_t^0$, the following holds:*

(i) $P(x,y)=\mathbf{E}^{\bar{x}}(\mu^{\omega}(\tilde{y}-\tilde{x}))$,

(ii) *for $n\geq 1$,* $\delta_x P^n(y)=\mathbf{E}^{\bar{x}}(\mu^{\omega}_{0,n-1}(\tilde{y}-\tilde{x}))$,

(iii) $Pf(x)=Q\bar{f}(\bar{x})$.

where $\mathbf{E}^{\bar{x}}$ stands for the expectation with respect to $\mathbf{Q}^{\bar{x}}$.

Proof. First recall that the map $\mathfrak{p}:\mathscr{H}_t\cap\mathbb{Z}^N\to\mathbb{G}_t^0$ is bijective. Let $x,y\in\mathbb{G}_t^0$ for some non ambiguous $t\in E^{\perp}$ and compute

$$\begin{aligned}
\mathbf{E}^{\bar{x}}(\mu^{\omega}(\tilde{y}-\tilde{x})) &= \int_{\mathbb{X}} Q(\bar{x},dz)\mu^{\bar{x},z}(\tilde{y}-\tilde{x}) \\
&= \frac{1}{\mathsf{card}\ J(\bar{x})}\sum_{j\in\mathbf{J}}\mathbf{1}_{V_j(\bar{x})}(\bar{x}+\mathfrak{p}_{\perp}(\varepsilon_j))\frac{\sum_{j'\in\mathbf{J}}\mathbf{1}_{\{\mathfrak{p}_{\perp}(\epsilon_j)\}}(\mathfrak{p}_{\perp}(\varepsilon_{j'}))\mathbf{1}_{\{\tilde{y}-\tilde{x}\}}(\varepsilon_{j'})}{\mathsf{card}\ \{j'\in\mathbf{J}:\mathfrak{p}_{\perp}(\varepsilon_{j'})=\mathfrak{p}_{\perp}(\varepsilon_j)\}} \\
&= \frac{1}{\mathsf{card}\ J(\bar{x})}\sum_{j,j'\in\mathbf{J}}\frac{\mathbf{1}_{V_j(\bar{x})}(\bar{x}+\mathfrak{p}_{\perp}(\varepsilon_j))\mathbf{1}_{\{\mathfrak{p}_{\perp}(\varepsilon_j)\}}(\mathfrak{p}_{\perp}(\varepsilon_{j'}))\mathbf{1}_{\{\tilde{y}-\tilde{x}\}}(\mathfrak{p}_{\perp}(\varepsilon_{j'}))}{\mathsf{card}\ \{j\in\mathbf{J}:\mathfrak{p}_{\perp}(\varepsilon_{j'})=\varepsilon_j\}}.
\end{aligned}$$

Then, one of the three following distinct cases must prevail.

a) For all $j \in \mathbf{J}$, $\tilde{y} \neq \tilde{x} + \varepsilon_j$ and obviously $\mathbf{E}^{\bar{x}}(\mu^\omega(\tilde{y} - \tilde{x})) = 0$.

b) There exists a $j_0 \in \mathbf{J}$ (necessarily unique) such that $\tilde{y} = \tilde{x} + \varepsilon_{j_0}$ and $\bar{x} + \mathfrak{p}_\perp(\varepsilon_{j_0}) \notin \mathfrak{p}_\perp(K)$. Obviously, for all $j \in \mathbf{J}$ such that $\mathfrak{p}_\perp(\varepsilon_j) = \mathfrak{p}_\perp(\varepsilon_{j_0})$, $\bar{x} + \mathfrak{p}_\perp(\varepsilon_j) \notin \mathfrak{p}_\perp(K)$ so that $\mathbf{E}^{\bar{x}}(\mu^\omega(\tilde{y} - \tilde{x})) = 0$.

c) There exists a $j_0 \in \mathbf{J}$ such that $\tilde{y} = \tilde{x} + \varepsilon_{j_0}$ but $\bar{x} + \mathfrak{p}_\perp(\varepsilon_{j_0}) \in \mathfrak{p}_\perp(K)$. With the same argument as in the case b), it comes that $\mathbf{E}^{\bar{x}}(\mu^\omega(\tilde{y} - \tilde{x})) = 1/\mathsf{card}\ J(\bar{x})$.

In case a), $P(x, y) = 0$. Consider the case b) and suppose $\tilde{y} = \tilde{x} + \varepsilon_{j_0}$ is an admissible neighbour of $\tilde{x}$, then $\mathfrak{p}_\perp(\tilde{x} + \varepsilon_{j_0}) \in \mathfrak{p}_\perp(K) + t$ and also $\bar{x} + \mathfrak{p}_\perp(\varepsilon_{j_0}) \in \mathfrak{p}_\perp(K)$ which is a contradiction. Consequently, $P(x, y) = 0$. Finally, in case c), $\tilde{y} = \tilde{x} + \varepsilon_{j_0}$ is obviously an admissible neighbour of $\tilde{x}$. The equalities

$$\mathsf{deg}\ x = \mathsf{deg}\ \tilde{x} \quad \text{and} \quad \big\{j \in \mathbf{J} : \mathfrak{p}_\perp(\tilde{x} + \varepsilon_j) \in \mathfrak{p}_\perp(K) + t\big\} = \big\{j \in \mathbf{J} : \bar{x} + \mathfrak{p}_\perp(\varepsilon_j) \in \mathfrak{p}_\perp(K)\big\}$$

imply that $\mathsf{deg}\ x = \mathsf{card}\ J(\bar{x})$ and ends the proof of point (i).

By equality (i), the equality (ii) holds for $n = 1$. We start with

$$P^{n+1}(x, y) = \sum_{\xi \in \mathbb{Z}^N \cap \mathscr{K}_t} P(x, \mathfrak{p}(\xi)) P^n(\mathfrak{p}(\xi), y).$$

By the point (i) and the induction assumption, it follows

$$P^{n+1}(x, y) = \sum_{\xi \in \mathbb{Z}^N \cap \mathscr{K}_t} Q(\bar{x}, \bar{\xi}) \mu^{\bar{x}, \bar{\xi}}(\xi - \tilde{x}) \mathbf{E}^{\bar{\xi}}(\mu^\omega_{0,n-1}(\tilde{y} - \xi)), \quad \bar{\xi} = \mathfrak{p}_\perp(\xi - t).$$

Remark that for $z \in \mathbb{Z}^N$, we have the following equivalence

$$\mu^{\bar{x}, \bar{z}}(\xi - \tilde{x}) \neq 0 \Longleftrightarrow \bar{\xi} = \bar{z}. \tag{5}$$

Consequently, the simple sum above can be rewritten as the following double sum

$$\sum_{z \in \mathbb{Z}^N \cap \mathscr{K}_t} \sum_{\xi \in \mathbb{Z}^N \cap \mathscr{K}_t} Q(\bar{x}, \bar{z}) \mu^{\bar{x}, \bar{z}}(\xi - \tilde{x}) \mathbf{E}^{\bar{z}}(\mu^\omega_{0,n-1}(\tilde{y} - \xi)),$$

which, by Markov property, is nothing but

$$\sum_{z, \xi \in \mathbb{Z}^N \cap \mathscr{K}_t} Q(\bar{x}, \bar{z}) \mathbf{E}^{\bar{x}}(\mu^\omega(\xi - \tilde{x}) | X_1 = \bar{z}) \mathbf{E}^{\bar{x}}(\mu^{T\omega}_{0,n-1}(\tilde{y} - \xi) | X_1 = \bar{z}).$$

Since past and future of a Markov chain, conditionally to the present, are independent, the sum becomes

$$\sum_{z \in \mathbb{Z}^N \cap \mathscr{K}_t} Q(\bar{x}, \bar{z}) \mathbf{E}^{\bar{x}}(\mu^\omega_{0,n}(\tilde{y} - \tilde{x}) | X_1 = \bar{z}) = \mathbf{E}^{\bar{x}}(\mu^\omega_{0,n}(\tilde{y} - \tilde{x})),$$

which ends the proof of (ii).

The last point can be proved almost similarly, in fact we write

$$\begin{aligned} Pf(x) &= \sum_{\xi \in \mathbb{Z}^N \cap \mathscr{K}_t} P(x, \mathfrak{p}(\xi)) f(\mathfrak{p}(\xi)) \\ &= \sum_{\xi \in \mathbb{Z}^N \cap \mathscr{K}_t} Q(\bar{x}, \bar{\xi}) \mu^{\bar{x}, \bar{\xi}}(\xi - \tilde{x}) \bar{f}(\bar{\xi}). \end{aligned}$$

But with the equivalence (5), this sum is equal to the double sum

$$\sum_{z, \xi \in \mathbb{Z}^N \cap \mathscr{K}_t} Q(\bar{x}, \bar{z}) \mu^{\bar{x}, \bar{z}}(\xi - \tilde{x}) \bar{f}(\bar{z}).$$

Since $\mu^{\bar{x}, \bar{z}}$ is a probability, the result follows. □

3.2 Entropy estimate of RWRTP

We now have the ingredients to state a generic estimate of the asymptotic entropy of the simple random walk on a cut-and-project graph. The notion of genericity will be made precise at the end of this section.

For a discrete measure μ, we define its *entropy* $H(\mu)$ by

$$H(\mu) = -\sum_{z\in\mathbb{Z}^N} \mu(z)\log\mu(z).$$

The asymptotic entropy of the simple random walk on a cut-and-project graphe $\mathbb{G}_t$, $t \in E^\perp$ non ambiguous, is given (see [11] for instance for a definition in the context of Markov chains) for any $x \in \mathbb{G}_t^0$ by

$$h(x) = \lim_{n\to\infty} H(\delta_x P^n) - \sum_{y\in\mathbb{G}_0} P(x,y)H(\delta_y P^{n-1}). \tag{6}$$

Proposition 3.4. *Let $t \in E^\perp$ and $x \in \mathbb{G}_t^0$. Then, the asymptotic entropy h depends on $x \in \mathbb{G}_t^0$ only through $\bar{x} = \mathfrak{p}_\perp(\tilde{x} - t)$ where $\tilde{x}$ is the unique point in $\mathbb{Z}^N \cap \mathscr{H}_t$ such that $\mathfrak{p}(\tilde{x}) = x$.*

Proof. By Proposition 3.3 we get that

$$H(\delta_y P^{n-1}) = H\left(\int_\Omega \mathbf{Q}^{\bar{y}}(d\omega)\mu^\omega_{0,n-2}\right),$$

and also that

$$h(x) = \lim_{n\to\infty} H\left(\int_\Omega \mathbf{Q}^{\bar{x}}(d\omega)\mu^\omega_{0,n-1}\right) - \sum_{\tilde{y}\in\mathbb{Z}^N} Q(\bar{x},\bar{y})H\left(\int_\Omega \mathbf{Q}^{\bar{y}}(d\omega)\mu^\omega_{0,n-2}\right). \tag{7}$$

This form only depends on $\bar{x} \in \mathbb{X}$. □

Theorem 3.5. *There exists a subset $\mathcal{W}_\pi$ of $\mathbb{X}$ of full measure such that $h(x) = 0$ for all $x \in \mathbb{G}_t^0$, $t \in E^\perp$ non ambiguous, satisfying $\bar{x} \in \mathcal{W}_\pi$.*

This theorem states that asymptotic entropy of the simple random walk on a cut-and-project graph is generically zero. In fact, if a point $x \in \mathbb{X}$ is non ambiguous thus any $u \in E^\perp$ such that $\mathfrak{p}_\perp(\mathbb{Z}^N) \cap \{x + u\} \neq \emptyset$ is non ambiguous for the cut-and-project scheme. Consequently, one can define the graph $\mathbb{G}_u$ and it follows that any non ambiguous point $x \in \mathbb{X}$ corresponds to at least one vertex of some cut-and-project graph. Thus, genericity in the set $\mathbb{X}$ give rises to genericity for random walks on cut-and-project graphs.

Proof. Integrating in (7) with respect to the stationary measure π, and exchanging the limit and the integral (the quantity inside the limit is non negative and monotonically decreasing, see [11]), we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{X}} \pi(d\bar{x}) H\left(\int_\Omega \mathbf{Q}^{\bar{x}}(d\omega)\mu^\omega_{0,n-1}\right) - \int_{\mathbb{X}} Q(\bar{x},d\bar{y}) H\left(\int_\Omega \mathbf{Q}^{\bar{y}}(d\omega)\mu^\omega_{0,n-2}\right) \\
&\quad = \lim_{n\to\infty} \int_{\mathbb{X}} \pi(d\bar{x}) H\left(\int_\Omega \mathbf{Q}^{\bar{x}}(d\omega)\mu^\omega_{0,n-1}\right) - H\left(\int_\Omega \mathbf{Q}^{\bar{x}}(d\omega)\mu^\omega_{0,n-2}\right) \\
&\quad = \lim_{n\to\infty} \frac{\int_{\mathbb{X}} d\pi(d\bar{x}) H\left(\int_\Omega \mathbf{Q}^{\bar{x}}(d\omega)\mu^\omega_{0,n-1}\right)}{n},
\end{aligned} \tag{8}$$

by stationarity of π. The polynomial growth rate of $\mathbb{Z}^N$ gives the majorization

$$H\left(\int_\Omega \mathbf{Q}^{\bar{x}}(d\omega)\mu^\omega_{0,n-1}\right) \leq N\log(n) + \log(\kappa),$$

for some constant $\kappa > 0$. Consequently, the limit in (8) is zero π-a.s. and the theorem follows. □

Remark 1. It is worth noting that arguments involving a polynomial growth rate of balls is useless in general because of the difference in (6) and the fact there is no non trivial lower bound for the Shannon entropy. Also, the existence of a stationary probability measure is essential to deal with the more tractable expression of the asymptotic entropy appearing in (8) that can be majorized using the polynomial growth rate of balls. For an example of graphs with linear growth rate with positive asymptotic entropy, we refer to [11, Theorem 4.1].

4 The type problem

In this section, the following theorem is proved. It gives a characterization of the recurrent or transient behavior of the simple random walk induced by a cut-and-project tiling.

Theorem 4.1. *Set $d = \dim E \leq N$ and $\mu_\perp$ the Lebesgue measure on $E^\perp$. Under Assumption 1, consider the simple random walk $(M_n)_{n\geq 0}$ on the cut-and-project graph $\mathbb{G}_t$ induced by the tiling $\mathcal{T}_t$ for some $t \in E^\perp$ non ambiguous. Then, for $\mu_\perp$-a.e. $t \in E^\perp$, the following estimates hold for $x, y \in \mathbb{G}^0$,*

- $P^{2n}(x,x) \geq C_0(n\log n)^{-d/2}$, *and*
- $P^n(x,y) \leq C_1 n^{-d/2}$,

for some constants $C_0, C_1 > 0$. Obviously, we obtain the following dichotomy:

- *if $\dim E \leq 2$ then $(M_n)_{n\geq 0}$ is recurrent,*
- *if $\dim E \geq 3$ then $(M_n)_{n\geq 0}$ is transient.*

Even though the theorem is stated for simple random walk, it can be trivially extended to strongly reversible, uniformly irreducible with bounded range random walks (see [1] for instance).

The proof of Theorem 4.1 relies on a suitable estimate of the growth rate of balls for the statement related to the lower bound of the return probability and d-dimensional isoperimetric inequalities for the upper bound of the decreasing rate of the heat kernel. In particular, we shall apply in Section 4.2 the standard machinery related to reversible random walk. Intuitively, the vertices of cut-and-project graphs satisfy an equirepartition theorem. This result is shown in [8] (a slightly more general statement is also given in [16]) and Theorem 4.2 is a version adapted to our context and notations.

At this level, we need to stress that other strategies, involving argument such as "civilized embedding" (see [7], Nash-Williams inequalities, rough isometries or Dirichlet forms (see [19] for a modern approach), eventually rely on estimates of the growth rate of balls and spheres. This growth rate reveals the global flatness (in a combinatorial sense) of the cut-and-project graph that can not be seen directly since locally the curvature is varying.

4.1 Equirepartition theorem

Fixing a basis $\{a^{(1)}, \ldots, a^{(d)}\}$ of E, we denote by $\|\cdot\|_p$ the p-norm on E, namely,

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, \quad \text{for} \quad x = \sum_{j=1}^{d} x_j a^{(j)}$$

Also, the p-metric induced by the p-norm is denoted by d_p. We denote by $B_p(x,r)$ the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$, and by $\partial B_p(x,r)$ the corresponding sphere. We simply write

B and ∂B, without subscripts, if the choice of a specific metric is irrelevant to the statement of the result. Finally, the space E (*resp.* $E^\perp$) comes with the d-dimensional (*resp.* $(N-d)$-dimensional) Lebesgue measure denoted in the sequel μ (*resp.* $\mu_\perp$). Then, the following theorem is a slight generalization of the equirepartition theorem given in [8].

Theorem 4.2. *Let $t = t_E + t_{E^\perp} \in \mathbb{R}^N = E \oplus E^\perp$ with $\dim E = d$. Then, under Assumption 1, there exists a non negative function ℓ on $E^\perp$ satisfying $\mu_\perp(\ell = 0) = 0$ such that*

$$\lim_{r\to\infty} \frac{\mathsf{card}(\mathbb{Z}^N \cap (t + B(0,r) + \mathbb{X}_0))}{\mu(B(0,r))} = \ell(t_{E^\perp}), \tag{9}$$

for all $t_E \in E$ uniformly.

Remark 2. The theorem in [8] is stated under the additional assumption $\mathbb{Z}^N \cap E^\perp = \{0\}$ so that the limit holds uniformly. Also, in this case, the estimates given in Theorem 4.1 hold for all non ambiguous $t \in E^\perp$. This condition is satisfied, for instance, for the icosahedral tilings of $\mathbb{R}^3$ and not satisfied for the Penrose's third tiling (the vector $(1,1,1,1,1)$ belongs to $E^\perp$).

The proof of Theorem 4.2 is adapted from the one in [8]. In fact, our contribution consists of identifying the limit in (10) which is no longer constant as soon as $\mathbb{Z}^N \cap E^\perp \neq \{0\}$.

Proof. Let $\Gamma = \mathbb{R}^N / \mathbb{Z}^N$ be endowed with the Haar probability measure, denoted by λ. Theorem 4.2 is in fact a consequence of the ergodic theorem applied to the flow $\{T_u\}_{u\in\mathbb{R}^d}$, defined for $x \in \Gamma$ by

$$T_u x = \pi_\Gamma\left(x + \sum_{j=1}^d u_j a^{(j)}\right), \quad u = (u_1, \cdots, u_d) \in \mathbb{R}^d,$$

where π_Γ is the canonical projection of $\mathbb{R}^N$ on Γ.

In [8], it is shown, under the assumption $\mathbb{Z}^N \cap E^\perp = \{0\}$ (which is equivalent to the ergodicity of the flow $\{T_u\}_{u\in\mathbb{R}^d}$ on Γ),that

$$\lim_{r\to\infty} \frac{\mathsf{card}(\mathbb{Z}^N \cap (t + B_1(0,r) + \mathbb{X}_0))}{\mu(B_1(0,r))} = \delta^{-d}\lambda(\pi_\Gamma(A \times B)), \tag{10}$$

where $A \subset E^\perp$ is contained in a translate of the fundamental domain of Γ and $B = [0,\delta)^d$ with $\delta > 0$ chosen so small that $\mathsf{Leb}_N(A \times B) = \lambda(\pi_\Gamma(A \times B))$. If the flow is no longer ergodic, the limit is given by $\lambda_{\mathcal{I}}(\pi_\Gamma(A \times B))$ with $\lambda_{\mathcal{I}}$ the conditional probability measure with respect to the σ-algebra $\mathcal{I}$ of measurable sets invariant under the flow.

Thus, we need to compute this limit (which is no longer constant). Defining $P_{\mathcal{I}}$ as the projector in $\mathbf{L}^2(\Gamma)$ on the closed subspace $\mathsf{ker}\ (\mathsf{Id} - P_{\mathcal{I}})$ of invariant functions, it follows for $f \in \mathbf{L}^2(\Gamma)$ that

$$\int_\Gamma f d\lambda_{\mathcal{I}} = P_{\mathcal{I}} f, \quad \lambda - \text{a.s.}.$$

Setting $\chi_\xi(x) = e^{2i\pi\langle\xi,x\rangle}$ for $\xi \in \mathbb{Z}^N$ and $x \in \mathbb{R}^N$, one can check that the family $\{\chi_\xi : \xi \in \mathbb{Z}^N \cap E^\perp\}$ is an orthonormal basis of $\mathsf{ker}\ (\mathsf{Id} - P_{\mathcal{I}})$. Consequently, we get

$$P_{\mathcal{I}} f = \sum_{\xi\in\mathbb{Z}^N\cap E^\perp} \hat{f}_\xi \chi_\xi, \quad \text{where} \quad \hat{f}_\xi = \int_\Gamma f_\xi \overline{\chi_\xi} d\lambda.$$

Setting $f = \mathbf{1}_{\pi_\Gamma(A\times B)}$ a simple computation shows that

$$\lambda_{\mathcal{I}}(\pi_\Gamma(A \times B)) = \delta^d \sum_{\xi\in\mathbb{Z}^N} \chi_\xi \gamma_\xi \quad \text{with} \quad \gamma_\xi = \int_{E^\perp} \mathbf{1}_A(x) e^{-2i\pi\langle\xi,x\rangle} \mu_\perp(dx).$$

Also, the limit in the ergodic theorem is λ-a.s. zero if and only if $A \subset (\mathbb{Z}^N \cap E^\perp)^\perp$ which is excluded under Assumption 1. Moreover, the value of the limit is invariant on the orbit of the flow which implies the uniformity with respect to translation in E. □

Remark 3. Theorem 4.2 is stated for any p-norm but its proof only involves the 1-norm which is obviously sufficient since all the norms are equivalent.

4.2 Isoperimetric inequalities, reversible random walks

In this section, we fix $t \in E^\perp$ unambiguous and consider the cut-and-project graph $\mathbb{G}_t$ which shall be denoted $\mathbb{G}$ to keep notation light. Let us denote by $d_{\mathbb{G}}$ the usual *graph metric* on the cut-and-project graph $\mathbb{G}$ and by $B_{\mathbb{G}}(x, n)$ the ball of radius n centered at x, *i.e.*

$$B_{\mathbb{G}}(x, n) = \{y \in \mathbb{G}^0 : d_{\mathbb{G}}(x, y) \leq n\}.$$

The Cayley graph of $\mathbb{Z}^N$ (with standard generators) is naturally endowed with the graph metric d_Λ which is nothing but the metric induced by the 1-norm on $\mathbb{R}^N$ in the canonical basis which shall be denoted in the sequel $\|\cdot\|_\Lambda$. The following lemma compares the graph metrics $d_{\mathbb{G}}$ and d_Λ.

Lemma 4.3. *Under Assumption 1, for any $x, y \in \mathbb{G}^0$,*

$$d_\Lambda(\xi, \eta) = d_{\mathbb{G}}(x, y),$$

where $(\xi, \eta) \in \mathbb{Z}^N \times \mathbb{Z}^N$ is the unique pair in the strip $\mathscr{K}_t = K + E + t$, for $t \in E^\perp$, such that $\mathfrak{p}(\xi) = x$ and $\mathfrak{p}(\eta) = y$.

This lemma states that a geodesic path in the graph can not be the projection of a non geodesic path of the integer lattice $\mathbb{Z}^N$.

Proof. This lemma is a direct consequence of the fact, due to [14], that the tiling $\mathcal{T}_t$ is the projection of a unique d-dimensional faceted manifold entirely contained in the strip $\mathscr{K}_t$ (recall that $t \in E^\perp$ is chosen unambiguous). □

Theorem 4.2 together with Lemma 4.3 yields the following ball growth estimates.

Proposition 4.4. *Under Assumption 1, the following estimate is satisfied for all $x \in \mathbb{G}^0$*

$$\mathtt{k}^{-1} l^d \leq \mathsf{card}\ B_{\mathbb{G}}(x, l) \leq \mathtt{k} l^d, \quad d = \mathsf{dim}\ E,$$

for a constant $\mathtt{k} > 1$ independent of $x \in \mathbb{G}^0$.

Proof. Let $x, y \in \mathbb{G}^0$ and let $(\xi, \eta) \in (\mathbb{Z}^N \cap \mathscr{K}_t)^2$ be the unique pair of points such that $\mathfrak{p}(\xi) = x$ and $\mathfrak{p}(\eta) = y$. On one hand, we obtain

$$d_2(x, y) \leq \|\mathfrak{p}\| d_\Lambda(\xi, \eta)$$

where $\|\mathfrak{p}\|$ is the matrix norm defined by

$$\|\mathfrak{p}\| = \sup_{y \in E \oplus E^\perp : \|y\|_\Lambda \leq 1} \frac{\|\mathfrak{p}(y)\|_2}{\|y\|_\Lambda}.$$

On the other hand, there exist $u, v \in \mathbb{X}_t$ such that $\xi = x + u$ and $\eta = y + u$ (and obviously these u, v are uniquely determined). Thus, we get the following obvious inequality :

$$d_\Lambda(\xi, \eta) = \|\xi - \eta\|_\Lambda \leq \|x - y\|_\Lambda + \|u - v\|_\Lambda \leq c_0 \|x - y\|_2 + \mathsf{diam}(\mathbb{X}_t).$$

Consequently, by Lemma 4.3, we get

$$\|\mathfrak{p}\|^{-1} d_2(x,y) \leq d_{\mathbb{G}}(x,y) \leq c_0 d_2(x,y) + \mathsf{diam}(\mathbb{X}_t).$$

Applying Theorem 4.2 and remarking that

$$B_2(x, c_0^{-1}(n - \mathsf{diam}(\mathbb{X}_t))) \subset B_{\mathbb{G}}(x,n) \subset B_2(x, n\|\mathfrak{p}\|),$$

we get the inequality of the proposition. □

If $\mathbb{G}$ is a graph, the k-fuzz of $\mathbb{G}$, denoted by $\mathsf{Fuzz}_k(\mathbb{G})$, has the same set of vertices as $\mathbb{G}$ and two vertices x and y are neighbors in $\mathsf{Fuzz}_k(\mathbb{G})$ if and only if $1 \leq d_{\mathbb{G}}(x,y) \leq k$. We note ρ the graph metric on $\mathsf{Fuzz}_k(\mathbb{G})$. It is well known that the balls in the two graphs can be compared as well as the spheres, namely

$$B_\rho(x,n) = B_{\mathbb{G}}(x,kn) \text{ and } \partial B_\rho(x,n) = \bigcup_{l=kn-k+1}^{kn} \partial B_{\mathbb{G}}(x,l).$$

Proposition 4.5. *Let $d = \mathsf{dim}\ E$. The k-fuzz $\mathsf{Fuzz}_k(\mathbb{G})$ satisfies a d-dimensional isoperimetric inequality for k large enough,* i.e.

$$\mathsf{card}\ B_\rho(x,n) \leq \mathtt{k}\ \mathsf{card}\ \partial B_\rho(x,n)^{d/(d-1)}$$

for some $\mathtt{k} > 0$.

Proof. According to Proposition 4.4 there exist $C_-, C_+ > 0$ such that for all $n \geq 1$

$$C_-(kn)^d \leq \mathsf{card}\ B_\rho(x,n) \leq C_+(kn)^d.$$

Hence, we need a lower bound of $\mathsf{card}\partial B_\rho(x,n)$, namely, we have to show that

$$\mathsf{card}\ \partial B_\rho(x,n) \geq \kappa(kn)^{d-1}.$$

By Lemma 4.3, and from the proof of 4.4, we get

$$\|\mathfrak{p}\|^{-1} d_2(x,y) \leq d_{\mathbb{G}}(x,y) \leq c_0 d_2(x,y) + \mathsf{diam}(\mathbb{X}_t).$$

Consequently, a point $y \in \partial B_\rho(x,n)$ satisfies

$$c_0^{-1}(kn - k + 1 - \mathsf{diam}(\mathbb{X}_t)) \leq d_2(x,y) \leq \|\mathfrak{p}\| kn,$$

and in terms of balls we get

$$B_2(x, \|\mathfrak{p}\|kn) \setminus B_2(x, c_0^{-1}(kn - k + 1 - \mathsf{diam}(\mathbb{X}_t))) \subset \partial B_\rho(x,n).$$

Since $c_0\|\mathfrak{p}\| \geq 1$, it is obvious that for any $k \geq 1$

$$B_2(x, c_0^{-1}kn) \setminus B_2(x, c_0^{-1}kn - c_0^{-1}(k - 1 + \mathsf{diam}(\mathbb{X}_t))) \subset \partial B_\rho(x,n).$$

In the sequel, we adapt the proof of [16, Proposition 2.1]. Setting $r = c_0^{-1}kn$ and $w = w(k) = c_0^{-1}(k - 1 + \mathsf{diam}(\mathbb{X}_t))$, and defining

$$N(r,w,x,\mathbb{X}_t) = \frac{\mathsf{card}(\mathbb{Z}^N \cap (B_2(x,r) \setminus B_2(x,r-w)) + \mathbb{X}_t))}{\mu(B_2(x,r) \setminus B_2(x,r-w))},$$

we want to show that the inequality

$$\alpha\mu_\perp(\mathbb{X}_t) \leq N(r,w,x,\mathbb{X}_t) \leq (1-\alpha)\mu_\perp(\mathbb{X}_t) \tag{11}$$

holds for some $\alpha > 0$. Obviously, we have,

$$\begin{aligned} N(r,w,x,\mathbb{X}_t) = \beta_r \frac{\mathsf{card}(\mathbb{Z}^N \cap (B_2(x,r) + \mathbb{X}_t))}{\mu(B_2(x,r))} \\ + (1-\beta_r)\frac{\mathsf{card}(\mathbb{Z}^N \cap (B_2(x,r-w) + \mathbb{X}_t))}{\mu(B_2(x,r-w))}, \end{aligned}$$

where $\beta_r = \frac{\mu(B_2(x,r))}{\mu(B_2(x,r)\setminus B_2(x,r-w))}$. Consequently, we can majorize

$$\begin{aligned} |N(r,w,x,\mathbb{X}_t) - \mu_\perp(\mathbb{X}_t)| \leq \beta_r \left| \frac{\mathsf{card}(\mathbb{Z}^N \cap (B_2(x,r) + \mathbb{X}_t))}{\mu(B_2(x,r))} - \mu_\perp(\mathbb{X}_t) \right| \\ + (\beta_r - 1)\left| \frac{\mathsf{card}(\mathbb{Z}^N \cap (B_2(x,r-w) + \mathbb{X}_t))}{\mu(B_2(x,r-w))} - \mu_\perp(\mathbb{X}_t) \right|. \end{aligned}$$

It follows from relations (3.20) and (3.23) in [16] that

$$\begin{aligned} |N(r,w,x,\mathbb{X}_t) - \mu_\perp(\mathbb{X}_t)| \leq \mu_\perp(\mathbb{X}_t) \left[\beta_r \frac{\mu\{B_2(x,r+\delta+\epsilon) \setminus B_2(x,r-\delta-\epsilon)\}}{\mu(B_2(x,r))} \right. \\ \left. + (\beta_r - 1)\frac{\mu\{B_2(x,r-w+\delta+\epsilon) \setminus B_2(x,r-w-\delta-\epsilon)\}}{\mu(B_2(x,r-w))} \right], \end{aligned}$$

for some $\delta, \epsilon > 0$. Obviously, for r large enough, there exists $\kappa_0 > 0$ such that $\beta_r \leq \kappa_0 \frac{r}{dw}$ and $\kappa_1 > 0$ such that

$$\beta_r \frac{\mu\{B_2(x,r+\delta+\epsilon) \setminus B_2(x,r-\delta-\epsilon)\}}{\mu(B_2(x,r))} \leq \kappa_1 \frac{\delta+\epsilon}{w(k)}.$$

Since w can be made arbitrarily large with $k \geq 1$, the quantity $\kappa_1 \frac{\delta+\epsilon}{w}$ can be made strictly smaller than 1, and we conclude that

$$|N(r,w,x,\mathbb{X}_t) - \mu_\perp(\mathbb{X}_t)| \leq (1-\alpha)\mu_\perp(\mathbb{X}_t),$$

for some $\alpha = \alpha(w) = \alpha(k) > 0$. Consequently, the following holds for large enough k

$$\mathsf{card}\ \partial B_\rho(x,n) \geq \mathsf{card}(\mathbb{Z}^N \cap (B_2(x,r) \setminus B_2(x,r-w)) + \mathbb{X}_t)) \geq \kappa(kn)^{d-1},$$

and the k-fuzz $\mathsf{Fuzz}_k(\mathbb{G})$ satisfies a d-dimensional isoperimetric inequality. □

Proof of Theorem 4.1. Proposition 4.4 together with [19, Theorem 14.22, p. 159], remarking that $1 \leq \mathsf{deg}(x) \leq 2N$, $x \in \mathbb{G}^0$, imply that

$$p^{(2n)}(x,x) \geq C_0(n\log n)^{-d/2},$$

for some constant $C_0 > 0$ and the lower bound in 4.1 follows.

For the upper bound, we assume $d \geq 3$. The k-fuzz graph $\mathbb{G}^k$ satisfies a d-dimensional isoperimetric inequality by Proposition 4.5. Obviously, the original graph satisfies, also, a d-dimensional isoperimetric inequality by [19, Theorem 4.7, p. 42] and a rough isometry argument. Applying, successively [19, Proposition 4.3, p. 40], [18, Proposition of section 3, p. 221] and [18, Theorem 1, p. 215], we get the expected estimate on the n-step transition probabilities :

$$p^{(n)}(x,y) \leq C_1 n^{-d/2},$$

for some $C_1 > 0$. □

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