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The Erdős-Hajnal Conjecture for Paths and Antipaths

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Abstract

We prove that for every $k$, there exists $c_k > 0$ such that every graph $G$ on $n$ vertices with no induced path $P_k$ or its complement $\overline{P_k}$ contains a clique or a stable set of size $n^{c_k}$.

Keywords: Erdős-Hajnal, path, antipath, Ramsey

An $n$-graph is a graph on $n$ vertices. For every vertex $x$, $N(x)$ denotes the neighborhood of $x$, that is the set of vertices $y$ such that $xy$ is an edge. The degree $\text{deg}(x)$ is the size of $N(x)$. In this note, we only consider classes of graphs that are closed under induced subgraphs. Moreover a class $\mathcal{C}$ is strict if it does not contain all graphs. It is said to have the (weak) Erdős-Hajnal property if there exists some $c > 0$ such that every graph of $\mathcal{C}$ contains a clique or a stable set of size $n^c$ where $n$ is the size of $G$. The Erdős-Hajnal conjecture [8] asserts that every strict class of graphs has the Erdős-Hajnal property; see [3] for a survey.

This fascinating question is open even for graphs not inducing a cycle of length five. When excluding a single graph $H$, Alon, Pach and Solymosi showed in [2] that it suffices to consider prime $H$, namely graphs without nontrivial modules (a module is a subset $V'$ of vertices such that for every $x, y \in V'$, $N(x) \setminus V' = N(y) \setminus V'$).

A natural approach is then to study classes of graphs with intermediate difficulty, hoping to get a proof scheme which could be extended. A natural prime candidate to forbid is certainly the path. Unfortunately, even excluding the path on five vertices seems already hard. Chudnovsky and Zwols studied the class $\mathcal{C}_k$ of graphs not inducing the path $P_k$ on $k$ vertices or its complement $\overline{P_k}$. They proved the Erdős-Hajnal property for $P_5$ and $\overline{P_5}$-free graphs [7]. This was extended for $P_5$ and $\overline{P_7}$-free graphs by Chudnovsky and Seymour [6]. Moreover structural results have been provided for $\mathcal{C}_5$ [4, 5]. We show in this note that for every fixed $k$, the class $\mathcal{C}_k$ has the Erdős-Hajnal property. An $n$-graph is an $\varepsilon$-stable set if it has at most $\varepsilon(n^2)$ edges. The complement of an $\varepsilon$-stable set is an $\varepsilon$-clique. Fox and Sudakov [11] proved the following:

**Theorem 1** ([11]). For every positive integer $k$ and every $\varepsilon \in (0, 1/2)$, there exists $\delta > 0$ such that every $n$-graph $G$ satisfies one of the following:

- $G$ induces all graphs on $k$ vertices.
- $G$ contains an $\varepsilon$-stable set of size at least $\delta n$.
- $G$ contains an $\varepsilon$-clique of size at least $\delta n$.

Note that a stronger result was previously showed by Rödl [14] using Szemerédi’s regularity lemma, but Fox and Sudakov’s proof provides a much better quantitative estimate ($\delta = 2^{-ck(\log 1/\varepsilon)^2}$ for some constant $c$). They further conjecture that a polynomial estimate should hold, which would imply the Erdős-Hajnal conjecture.

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In a graph \( G \), a biclique of size \( t \) is a (not necessarily induced) complete bipartite subgraph \((X,Y)\) such that both \(|X|, |Y| \geq t\). Observe that it does not require any condition inside \( X \) or inside \( Y \). Erdős, Hajnal and Pach proved in \cite{erdos1995erdos} that for every strict class \( C \), there exists some \( c > 0 \) such that for every \( n \)-graph \( G \) in \( C \), \( G \) or its complement \( \overline{G} \) contains a biclique of size \( nc \). This "half" version of the conjecture was improved to a "three quarter" version by Fox and Sudakov \cite{fox2009erdos}, where they show the existence of a polynomial size stable set or biclique. Following the notations of \cite{neumann2010erdos}, a class \( C \) of graphs has the strong Erdős-Hajnal property if there exists a constant \( c \) such that for every \( n \)-graph \( G \) in \( C \), \( G \) or \( \overline{G} \) contains a biclique of size \( cn \). It was proved that having the strong Erdős-Hajnal property implies having the (weak) Erdős-Hajnal property:

**Theorem 2** (\cite{erdos1980erdos,neumann2010erdos}). If \( C \) is a class of graphs having the strong Erdős-Hajnal property, then \( C \) has the weak Erdős-Hajnal property.

**Proof.** (sketch) Let \( c \) be the constant of the strong Erdős-Hajnal property, meaning that for every \( n \)-graph \( G \) in \( C \), \( G \) or \( \overline{G} \) contains a biclique of size \( cn \). Let \( c' > 0 \) be such that \( c'c \geq 1/2 \). We prove by induction that every \( n \)-graph \( G \) in \( C \) induces a \( P_4 \)-free graph of size \( n^{c'} \). By our hypothesis on \( C \), there exists, say, a biclique \((X,Y)\) of size \( cn \) in \( G \). Applying the induction hypothesis inside both \( X \) and \( Y \), we form a \( P_4 \)-free graph on \( 2(cn)c' \geq n^{c'} \) vertices. The Erdős-Hajnal property of \( C \) follows from the fact that every \( P_4 \)-free \( n^{c'} \)-graph has a clique or a stable set of size at least \( n^{c'/2} \).

We now prove our main result. The key lemma is an adaptation of Gyárfás’ proof of the \( \chi \)-boundedness of \( P_k \)-free graphs, see \cite{gyarfas1986some}.

**Lemma 3.** For every \( k \geq 2 \), there exists \( \varepsilon_k > 0 \) and \( c_k \) (with \( 0 < c_k \leq 1/2 \)) such that every connected \( n \)-graph \( G \) with \( n \geq 2 \) satisfies one of the following:

- There exists a vertex of degree more than \( \varepsilon_k n \).
- For every vertex \( v \), \( G \) contains an induced \( P_k \) starting at \( v \).
- The complement \( \overline{G} \) of \( G \) contains a biclique of size \( c_k n \).

**Proof.** We proceed by induction on \( k \). For \( k = 2 \), since \( G \) is connected, every vertex is the endpoint of an edge (that is, a \( P_2 \)). Thus we can arbitrarily define \( \varepsilon_2 = c_2 = 1/2 \).

If \( k > 2 \), let \( \varepsilon_k = \frac{\varepsilon_{k-1}}{(2+\varepsilon_{k-1})} \) and \( c_k = \frac{c_{k-1}(1-\varepsilon_k)}{2} \). Let us assume that the first item is false. We will show that the second or the third item is true. Let \( v_1 \) be any vertex and \( S = V(G) \setminus (N(v_1) \cup \{v_1\}) \). The size \( s \) of \( S \) is at least \( (1-\varepsilon_k)n - 1 \). If \( S \) has only small connected components, meaning of size at most \( s/2 \), then one can divide the connected components into two parts with at least \( (s+1)/4 \) vertices each, and no edges between both parts. This gives in \( G \) a biclique of size \( (s+1)/4 \geq (1-\varepsilon_k)n \), thus of size at least \( c_k n \) since \( c_k \leq \frac{1-\delta}{4} \). Otherwise, \( S \) has a giant connected component \( S' \), meaning of size \( s' \) more than \( s/2 \). Let \( v_2 \) be a vertex adjacent both to \( v_1 \) and to some vertex in \( S' \). Observe that \( v_2 \) exists since \( G \) is connected. Consider now the graph \( G_2 \) induced by \( S' \cup \{v_2\} \). The maximum degree in \( G_2 \) is still at most \( \varepsilon_k n = \varepsilon_{k-1}(1-\varepsilon_k)n/2 \leq \varepsilon_{k-1}(s'+1) \). By the induction hypothesis, either the second or the third item is true for \( G_2 \) with parameter \( k-1 \). The second item gives an induced \( P_{k-1} \) in \( G_2 \) starting at \( v_2 \), thus an induced \( P_k \) in \( G \) starting at \( v_1 \). The third item gives a biclique of size \( c_{k-1} |G_2| \) in \( \overline{G_2} \). Since \( |G_2| = s' + 1 \geq \frac{1-\delta}{4} n \), this gives a biclique of size at least \( \frac{c_{k-1}(1-\varepsilon_{k-1})}{2} n = c_k n \) and concludes the proof.

**Theorem 4.** For every \( k \geq 2 \), \( C_k \) has the strong Erdős-Hajnal property. Thus, by Theorem 2, the class \( C_k \) has the (weak) Erdős-Hajnal property.

**Proof.** Let \( \varepsilon_k \) be as defined in Lemma 3 and \( \varepsilon = \varepsilon_k/8 > 0 \). By Theorem 1, there exists \( \delta > 0 \) such that every graph \( G \) not inducing \( P_k \) or \( P_k \) does contain an \( \varepsilon \)-stable set or an \( \varepsilon \)-clique of size at least \( \delta n \). Free to consider the complement of \( G \), we can assume that \( G \) contains an \( \varepsilon \)-stable set \( S_0 \) of size \( \delta n \). We start by deleting in \( S_0 \) all the vertices with degree in \( S_0 \) at least \( 2\varepsilon s_0 \) where \( s_0 \) is the size of \( S_0 \). Since the average degree in \( S_0 \)
is at most $\varepsilon s_0$, we do not delete more than half of the vertices. We call $S$ the remaining subgraph which is a $4\varepsilon$-stable set of size $s \geq \delta n/2$ with maximum degree less than $4\varepsilon s$.

Let $G_S$ be the graph induced by $S$. Our goal is to find a constant $c$ such that $G_S$ have a biclique of size $cs$, which gives a biclique in $\overline{G}$ of size at least $c\delta n/2$ and concludes the proof. Assume first that $G_S$ only has small connected components, meaning of size less than $s/2$. Then one can partition the connected components of $G_S$ in order to get a biclique in $\overline{G_S}$ of size $s/4$. Otherwise, $G_S$ has a connected component $S'$ of size $s' \geq s/2$. The degree of every vertex in $S'$ is at most $8\varepsilon s' = \varepsilon k s'$, and $S'$ does not contain any induced $P_k$ since $G$ does not. By Lemma 3, there exists a biclique of size $c_k s' \geq c_k s'/2$ in the complement of the graph induced by $S'$, thus in $\overline{G_S}$.


