Misunderstanding geographical distances: two errors and an issue in the interpretation of violations of triangle inequality

Alain L'Hostis

Misunderstanding geographical distances: about three errors of interpretation of violations of the triangle inequality

Alain L'Hostis

Université Paris-Est, Laboratoire Ville Mobilité Transport (LVMT), Institut Français des Sciences et des Technologies des Transports de l'Aménagement et des Réseaux (Iftsttar)

Abstract

We investigate the meaning of the mathematical properties of distances in the fields of geography and economy. The key property for spatiality is the triangle inequality (TI) as ensurring the optimality of distance. We identify three different situations where several authors identify violations of the TI. We consider all of them as errors of interpretation.

The first error consists in considering sub-optimal measurements as distances. Yet distances are necessarily optimal since they respect the TI.

The second set of error, which is the most widespread, involves a confusion between the Euclidean straight line and the minimum path. The errors consist in considering the presence of a detour as a violation of the TI, while this situation simply corresponds to a non-Euclidean distance.

The third error concerns the issue of additivity of distances. The commonplace situation in geographical space where a break is needed to provide the energy necessary to renew the movement, is considered by some authors as another violation of the TI. We argue that as these routes are optimal, the TI must hold. We finally introduce a distance function that allows considering sub-additivity and over-additivity of distances, and in the same time respect the TI.

Keywords

Distance; triangle inequality; geography;spatial economy; transport

Introduction

Distances in geographical space are complicated to the extent that the phenomenon of spatial inversion has been introduced in the 1960 by Tobler (1961, 106) and Bunge (1962, 172) in order to express the idea that some trips can involve portions of routes that are made in an opposite direction of the general origin destination layout. Most distances in geographical spaces are not straight line, which means that the Euclidean geometry is an exceptional case, while detour is the norm (L'Hostis 2014). In this context, we want to build a relevant framework for understanding distances.

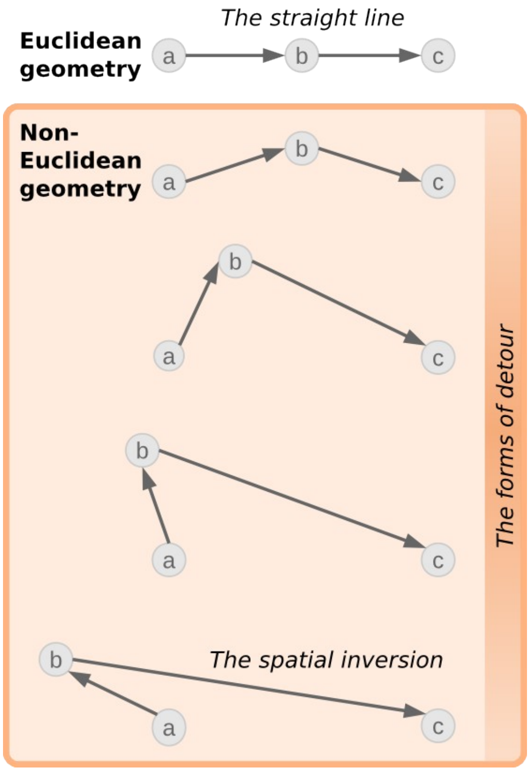

Figure 1: Forms of distance and corresponding geometry, from the straight line to the spatial inversion

Distance is a key concept for spatial analysis. And yet it has not received as much attention as one could expect for such a central concept. It is rarely the central focus of articles or books with a theoretical ambition, with noticeable exceptions of Deutsch and Isard (1961), Hall (1969), Gatrell (1983), the contributions of Smith and his co-authors (Huriot, Smith, and Thisse 1989; Tony E. Smith 1989), and a recent thematic issue of the revue Atala (2009). Distance in these sources by geographers and spatial economists is always linked to the mathematical concept. In this context, the study of the sense of the mathematical properties of distances for geographical or economical spaces is of central importance. In this paper we propose an examination of this link between spatial analysis and mathematics, applied to the concept of distance. During this study of the published literature we have discovered a series of errors in the interpretation of the triangular inequality never noticed before. Detailing these errors and elaborating correct interpretations for the issues raised in the literature is the core focus of the present paper.

Before entering in the subject of distance we recall its definition in mathematics as a function which, for any two given locations a and b in a given space, respects the following properties:

P1 (positivity) $d(a, b) \geq 0$

P2 (distinguishability) $d(a, b) = 0$ if and only if $a = b$

P3 (symmetry) $d(a, b) = d(b, a)$

P4 (triangle inequality) $d(a, c) \leq d(a, b) + d(b, c)$

We will now discuss the meaning for geographical and economical spaces of these mathematical properties.

It has been shown that the discussion around the first three properties does not lead to major considerations on spatiality (L'Hostis 1997, 114). Even symmetry, despite being almost unknown in empirical spaces, does not bring in the discussion a radically new way of considering or modeling space. It is only the discussion on the last property, the triangle inequality, that provide some truly deep indications on distances and spaces: this is the object of discussion we will develop now.

Questioning the mathematical properties of distances: the three errors of interpretation of the violation of the triangle inequality

A large part of the literature on distances in the spatial disciplines ignores the issue of its mathematical properties. The choice made by many authors not to insist on the mathematical properties of distances, or to leave open the issue of the respect or not, and in particular the property to the triangle inequality, is justified by two different postures.

A first reason is the fact that some want to elaborate upon a theoretical framework that covers spatial as well as non-spatial distances. This idea is clearly stated by Gatrell (Gatrell 1983) and is shared more generally by a majority of the *plastic space* community (Marchand 1973; R.G. Golledge and Hubert 1982), as we propose to name them in reference the expression of the geographer Forer (1978). Only exception in this community, Tobler who focuses on the foundations of cartography, with the problems of projections, and on the distances produced by the transport systems, is reluctant to tear his maps, to break their internal topology. When he nevertheless does so, he emphasizes the exceptional character of such a transformation. The other authors are more inclined to limit the mathematical constraints as much as possible, to the extent of taking the risk of sacrificing the properties linked to spatiality.

A second reason for this choice comes from an epistemological orientation towards the research of an analytical formula of the distance. It is particularly true of many researches on mental spaces (Tobler 1976; R.G. Golledge and Hubert 1982) or on transport surfaces (Tobler 1961; Marchand 1973; Ewing and Wolfe 1977; Hyman and Mayhew 2004). This approach seeks, by approximation, to establish a formula that represents the distance data coming from mental representations or from geographical measurement with the least error. If these authors discuss the mathematical properties of the analytical formulas that they propose, they do not focus on the metric or non-metric nature of the measures. The gap between observed and computed measures is a key element of discussion. The focus is on understanding the distance, and not on space directly, following the rationale that a better characterization of the first will lead to a better understanding of the latter.

These two postures apart, the most interesting debate bears on the questioning of the mathematical properties of distances; we will expose it here.

On the one side we dispose of observations and definitions of distance in the domains of geography, economy and spatial analysis, and on the other side we have abstract mathematical definitions with well established, verifiable and demonstrable properties. From the human and social sciences perspective, it is relevant to link the two domains with connections between definitions and properties. One seeks to determine whether the mathematical properties are verified by empirical measurement. When it is not the case, one may try to develop a mathematical framework adapted to the data, with enriched or impoverished properties set (L'Hostis 1997, 113).

When dealing with an intellectual construct such as distance, the scientific approach implies to raise the totality of possible issues. The questioning of existing conceptions is an integral part of the scientific approach. The new developments in knowledge consists in the most general case in evolutions, i.e. in the abandon of previous conceptions. Nevertheless, we want to affirm here that the approach of systematic questioning, while being necessary and fundamental, is not always fruitful in the advance of knowledge. In the domain of human and social sciences, if the questioning of a concept or a theoretical idea, such as distance, is not validated by facts, by empirical measures, then it is not relevant and remain a purely intellectual hypothesis, a conjecture in the sense of the common language.

In the scientific enterprise of questioning the mathematical foundations of distance, one of the privileged directions consists in testing the meaning of the lack of the four properties of distances for geographical analysis. L'Hostis shown that the discussion on the last property, the triangle inequality, is the most worthy of interest for geographical analysis (1997, 120). The triangle inequality property ensures the optimal character of the distance. Symmetry is patently never verified in geographical spaces (Brunet 2009, 16), but the epistemological implications of this observation are not as powerful as those linked to the triangle inequality.

A set of contributions in geography and in economy since the 1960, is influenced by the idea of plastic spaces and proposes a renewal of the conceptions of space with deformations as in the anamorphic cartography (Tobler 1963; Gatrell 1983; Cauvin 1984a; Rimbert 1990). These contributions have sought to illustrate the forms taken by the geographical space in function of relationships between locations (Pumain 2009, 37), in reaction to the rigidity of conventional cartography tensed in an effort to reduce inaccuracy and distortion, that can be seen as an extension of the scientific orientation for cartography fixed by Ptolemy as a quest for exactitude (Ptolémée 1828). Even if these ideas were latent in some cartographic advances in late XIXth and early XXth century (Letaconnoux 1907), plastic spaces have constituted a profound renewal of the geographical spaces representation. In the same time, the images produced by these researches imply and explicit or implicit questioning of previous conceptions of space and distance, essentially Euclidean.

No published source develops the geographical meaning of the triangle inequality. In the contrary, when this property is evoked, it is often rather superficially and regularly the object of errors in the interpretation. We will now study critically the academic literature on the definition of distances and on the property on triangular inequality. We have identified three errors of interpretation of the violations of this property made by geographers and economists. One of these errors is general while the two others constitute particular forms of this general error; the three errors are tightly linked.

The error of the optimality

The first series of errors consists in considering a sub-optimal measure of the separation between two locations as equivalent to a measure of distance.

The geographer Haggett presents the issue of representing correctly the position of a set of cities p, q, r and s with given measures of separation (Haggett 2001, 341). This illustration reproduces the data of an example proposed earlier by the same author (Haggett, Cliff, and Frey 1977, 326). It can be seen on this example that the measures proposed by Haggett do not respect the triangle inequality: between q and s the value is 6 hours, while it exist a route through r which produces a value of 5 hours. This means that the measure indicated by Haggett is not the shortest measure observed in the route between cities. In the mathematical sense, the measure proposed by Haggett is a separation, i.e. a measure respecting the positivity (P1) and distinguishability (P2) but with no other property (Huriot, Smith, and Thisse 1989, 296), and hence is not a metric; this is not a distance because it can produce some non optimum measures. It is true that Haggett has remained prudent in his formulation, using the word separation, but the measures he represents cannot form a distance, and this is where, in our view, the error resides.

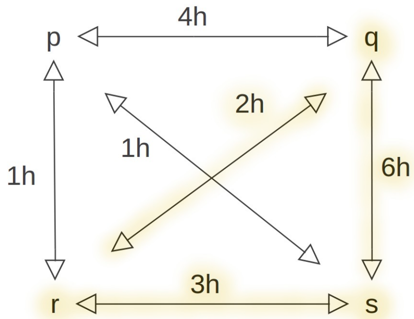

Figure 2. Four cities with non-optimal separation measures, violating the triangle inequality (Haggett 2001, 341) *underlined by the author*

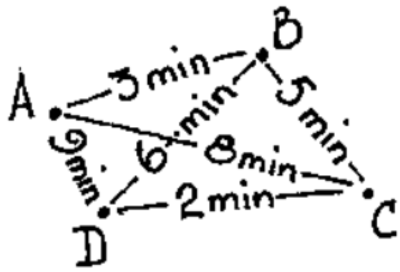

Figure 3. Four cities separated by distances, i.e. optimal measures respecting the triangle inequality (Lynch 1981)

The figures in Haggett's diagram seems having been chosen randomly. Yet this is a geographical space as using cities as nodes of the graph suggests. Hence, we would expect finding the properties of spaces and of networks embedded in the data he presents. Two routes can have different length or cost, but if a route is shorter it should correspond to the distance; if it exists a shorter route than the one which is indicated, then this is not a distance in the mathematical sense and this is not the observable distance. The shorter or minimum cost route should be privileged to represent the distance between two locations.

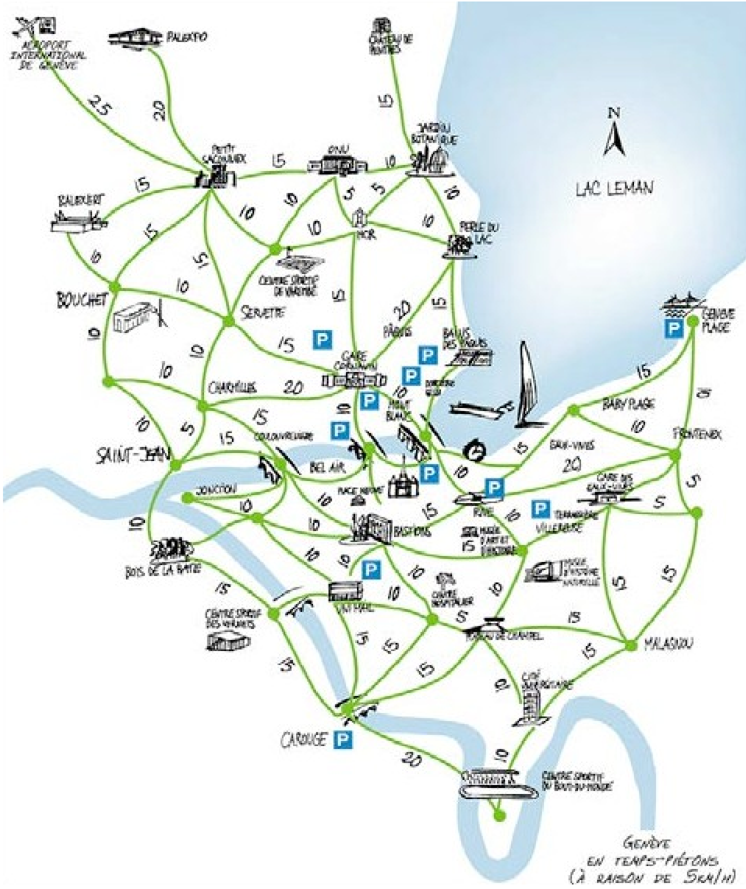

Figure 4. Time distances of the pedestrian in Geneva in 2000 (Lavadino 2011, 433)

Haggett's representation is the only one found in this literature analysis that exhibits sub-optimal measures but it is representative of a conception of distance as an abstraction without any particular property which can be found in the literature (Gatrell 1983; Dumolard 2011). Conversely, an illustration of the same type proposed by Lynch shows optimal measures of duration between four points (Lynch 1971, 191). Here on figure 2, we can see measures that are coherent with the properties of distance.

Among the representations proposed to the public, the Geneva experience since the 2000's provides some very rich illustrations (Lavadinho 2011, 433). The map shows a cartography of pedestrian distances in the city of Geneva. One can check that the measures indicated are distances in the mathematical sense, and that they all respect the triangle inequality. In order to verify the property, one has to examine, inside each face of the graph the values of distance attached to each edge; the eventual violations are then apparent as the one highlighted on figure 1. A sub-optimal measure on the map would be meaningless for the reader who seeks to identify a route and evaluate the time needed to complete it. This illustration is another example of the respect of the triangle inequality.

More recently, the Spanish city of Pontevedra has introduced a pedestrians map with distances indicated by path, duration and length between a set of significant urban locations. In addition to direct path between the major locations, the map shows also distances between remote locations, for instance between the urban center and a peripheral location like the university campus. In this example the respect of the triangle inequality is patent.

Figure 5. Time-distances in the pedestrian's map of the city of Pontevedra, Spain (website www.pontevedra.es browsed in august 2014)

The economist Smith attributes to the mathematician Fréchet (1906; 1918), who was the first to formalize the distance and its four properties the demonstration of the fact that the triangle inequality is the fundamental property of metrics (T. E. Smith 1989, 5). For Fréchet, the general form of the function indicating a measure of the separation between two points is a spread (*écart*) and becomes a distance only if it respects the triangle inequality (Fréchet 1918, 55).

In the domain of spatial economy Smith has shown that any measure based on minimum path respects the triangle inequality (T. E. Smith 1989, 15). This means that developing spaces that violates the triangle inequality implies creating linkages between locations that be not minimum path; as an example of distance violating the triangle inequality, Smith introduces the case of the discrimination distance between objects seen by a radar (T. E. Smith 1989, 7). In the same spirit, Gatrell also introduces a measure that violates the triangle inequality, with a non-spatial index of dissimilarity (Gatrell 1983, 38), and Felsenstein discusses the possibility of non-metric spaces in the domain of separation of living species (Felsenstein 1986). As we see this discussion takes us away from distances of transport and geography. This is a key observation because the layout of a transport network may include direct routes, close to the straight line, but sub-optimal. In this case the layout of the network creates some confusion for the reader: the direct route outdated by the fast transport system remain a strong reference for the travel plan.

Tobler introduced several methods to build mathematical spaces from empirical data obtained from measures on the transport system (Tobler 1997). From the length of routes between cities in the mountainous western Colorado he builds distances approximated by several methods derived from bidimensional regression. For Tobler if the measures of separation of adjacent cities are minimum, then the distance that will be produced will respect the triangle inequality (Tobler 1997). In his thesis, Tobler associates the violations of the triangle inequality to spatial inversions in stating that “a place located at two hours cannot be closer than a place situated at one hour” (Tobler 1961, 120). In a geographical context, for Tobler, violations of the triangle inequality are spatial aberrations. We can add that this principle applies equally in a cost space.

This discussion on proved or supposed violations of the triangle inequality allows stating that distances in geographical space are always optimum.

The confusion between the straight line and the shortest path

A second series of errors, which is the most frequently observed, is related to a confusion between the straight line and the shortest path.

We start with Müller, a geographer specialized in cartography and geographic information, who in 1982 gave the example of a driver making a detour to avoid congestion as an evidence of a violation of the triangular inequality (Müller 1982, 191). In this situation, the driver seeks to minimize the duration of his trip. In an economic perspective, the distance is defined as a minimum path. Therefore, here, the distance will be measured along an itinerary that minimizes the total duration of the trip, and consequently, the triangle inequality will be preserved. In the example given by Müller it is the law of Archimedes, i.e. the fact that the shortest way is the straight line[1], which is violated, and not the triangle inequality. Referring to contributions of his own research team in the 1970's (Rivizzigno 1976), Golledge describes a situation where the triangle inequality is violated, at some moments, in the urban cognitive space (Reginald George Golledge 1999, 8). This situation is very similar to the case of the congestion used by Müller.

The same erroneous interpretation is developed by the geographer Cauvin in her thesis in 1984. She compared the time duration of a route along two different itineraries, along a

1 It is not Euclide, third century before Christ, but rather Archimedes a century later, who is the first to express the fact that the straight line is optimum in stating that« the straight line is the shortest of all the lines having the same extremities » ((Heath 1897, 193)).

secondary road close to the straight line and along a motorway that makes a detour (Cauvin 1984b, 62). For Cauvin “the driver will need less time to effectuate the distance [pq + qr] than the distance [pr], particularly if he has a powerful car” (Cauvin 1984b, 62). From this observation she deduces that “in time units the triangle inequality is violated”. In this case, as in the example given by Müller, the straight line is not the shortest path, but the triangle inequality is not violated. The problem comes from a confusion between the Euclidean metric and an economic metric measured along a minimum cost route. All these authors conclude on a violation of the triangle inequality in situations where the straight line is not the shortest path.

The geographers Ahmed and Miller point the possibility for a matrix of trip durations to violate the triangle inequality in cases where “indirect routes are shorter than direct routes” (Ahmed and Miller 2007, 4). Here also the reference is made to a route close to the straight line, opposing it to a route with detours. It is implicitly a reference to the Euclidean straight line, as in the previous examples.

The confusion is also present in Lévy's work when he states that the triangle inequality is a characteristic of the Euclidean distance (Lévy 2009, 181). He considers that this property is not observed in the case of a route using fast transport networks with “low connexity” (Lévy 2009, 181). Another geographer, Dumolard, presents the triangle inequality as the fact that all routes differing from the straight line are longer than the straight line (Dumolard 2011, 190). Once again this is a reference to the Euclidean space; and we know that most geographical distances are non-Euclidean.

The same interpretation is established, in spatial economy, by Perreur in 1989, concerning the minimum path measured according to the law of refraction in optics (Perreur 1989, 133). Perreur states that the systematic violation of the triangle inequality, in reality the fact that the straight line is rarely the optimum route, expresses the idea that “the economic space is not metric” (Perreur 1989, 133). This interpretation is not present in the article he wrote with Huriot in 1990 (Huriot and Perreur 1990, 227). The authors refer then to the non-Euclidean nature of movement in the observation that “ the route departs from the straight line”. This last quote shifts the debate on the shape of the distance and it is not anymore an issue related to some violations of the triangle inequality.

In spatial economy also the work of Rouget mentions the possibility on a short trip, a frequent case in cities, for the triangle inequality to be violated by the “effective distance expressed in transport time" (Rouget 1975, 203). This analysis refers to the non-Euclidean nature of short urban trips.

The same erroneous interpretation is present in the preface of an interdisciplinary book on proximity written by Lamure, researcher on information science, when he explains the four axioms of distance by "the triangle inequality implies that the shortest path from one point to another is the straight line" (Lamure 1998, 12). A few lines later he writes that this idea is obvious.

The reader will probably grow weary of this litany of errors of interpretation. Nevertheless, the accumulation reveals the deeply rooted nature of the issue, and demonstrates the need to address it and to propose a correct interpretation. The fact that the present paper is the first published source to identify the error is also revealing of the confusion of most the literature on this crucial point. Most of these confusions come from the idea that one considers the length of a route minimizing time as a measure of distance. Yet this is an error because this length is not the only element to be optimized in the case of a network with different speeds, which is the most general case in geographical spaces. Moreover, those spaces are not Euclidean which implies that checking the triangle inequality can not use straight lines.

The error of the additivity of distances

The third of error comes from an economic debate on the issue of the additivity of pieces of routes to obtain complete paths.

The *minimum cost distance* was proposed by Huriot, Smith and Thisse to address the problem of *sub-additivity* of sections of routes composing an optimal path (Huriot, Smith, and Thisse 1989). The aerial transport of persons is the best illustration of this principle of sub-additivity by proposing routes through hubs that are often cheaper for the traveler than the sum of the costs of each flight considered individually. Conversely, for the authors it exists many situations where the minimum cost distance between locations is greater than the sum of the minimum cost distance of segments of the same route. Going back to the aerial example, and focusing strictly on a spatio-temporal approach, then the waiting time in the hub is an additional cost to the sum of the duration of the flights (Huriot, Smith, and Thisse 1989, 313). If we follow the rationale of the authors, who associate this situation to a violation of the triangular inequality, this observation indicates a form of sub-optimality of the route for the individual. More generally, the authors consider the presence of "road side rests stops and motels" as an illustration of this issue of additivity in the measure of distance (Huriot, Smith, and Thisse 1989, 313). We will now elaborate from this observation. If the rest stop is located in B then the duration of the trip from A to C needing a pause in B will exceed the sum of the durations of the trips from A to B and from B to C. This situation constitutes an apparent direct violation of the triangle inequality, in the case of a cost function based on time, not continuous and involving a threshold beyond a given amount of time spent in the travel. Then in this case, in terms of travel time:

$$d(A, C) > d(A, B) + d(B, C)$$

This case occurs because the direct route includes a break time in B not counted in the two segments AB and BC. From an economic point of view this situation constitutes an aberration (Huriot, Smith, and Thisse 1989, 300) because if these portions of routes would represent exchangeable goods, it should be possible to add the cost of the two parts to account for the complete route. The geography of terrestrial hinterlands of ports admits the existence of a discontinuous function of distance with social rules for the travel time of truck drivers (Chapelon 2006; Kok 2010). This situation is omnipresent in the economic and geographic spaces: it refers to the limited travel times of drivers, to the connection times in timetable transport systems, but also to the need to fill in the fuel tank of cars and more generally to the supply of energy of vehicles and the satisfaction of the needs of the travelers. The English language provide expressions for all these situations with the lunch or coffee break, the relaxing break, not forgetting the bathroom break.

From the identification of this error of interpretation of the triangle inequality related to the issue of the additivity of distances we will now elaborate a suitable framework.

Introducing a new distance function

To deal with this configuration of a route with a break, let's suppose a route from A to C that is optimal through the location B where a needed break takes place. It does not exist a faster route. We can state that because the route AC needs time for a break in B, the duration of the complete trip AC is not limited to the sum of the two portions AB and BC that would exclude the time spent for the break in B which is indispensable to perform the whole journey. The duration of the trip from A to C passing through B is superior to the sum of the two trips AB and BC. Yet this trip is optimal in time; its optimality is not in question. It is not the triangle inequality, ensuring the optimality, which is challenged, but rather the additivity of distances. In time-distance one can write:

$$d(A, C) = d(A, B) + t_a + d(B, C) \qquad \text{with } t_a \text{ for the time needed of the break}$$

Yet this equation is in direct violation with the triangle inequality (P4).

In order to solve this issue, we could introduce a non-zero term of distance between B and B to express the time or cost spent in this place. But if the distance function gives a non-zero term for a place to itself, this will lead to the violation of the second property of distances, i.e. the distinguishability or separation (P2). This property states that a distance has a zero value if and only if it is measured from a location to itself. This means that two different places are always separated by a non-zero distance, and that a zero value corresponds to a lack of spatial extent. In our case this would mean the introduction of a non-zero distance for a link without spatial extent, which would imply to detach the measure of distance from the fundamental properties of spaces. Rather than challenging this property strongly related to spatiality, which moreover would take us into the non-metric domain (T. E. Smith 1989, 5), we will explore a different formulation that avoids bending the properties of distances.

To deal with this situation another way consists in introducing a mathematical operator inspired from the approach developed by Huriot, Thisse and Smith. In order to give the possibility of chaining optimal portions of routes to produce complete trips with potential sub-additivity, the three authors introduced a *trip-chained operator* (Huriot, Smith, and Thisse 1989, 300). This operator could also produce measures of distance with costs exceeding the sum of that of parts of the route. This new operator will replace the sum used in the properties of distances. As we gave seen previously, some functions of distance contain a discontinuity that generates problems of additivity. The operator will produce a distance corresponding to the trip chaining that may differ from the sum of the costs of each portion considered individually. The total cost of chained trips may be lower, equal or higher than the sum of the cost of each individual trips. This new operator extends the operator proposed by Huriot, Thisse and Smith by considering the *sub-additivity* of trips as in the example of the air hubs, as well as the *over-additivity* of trips that necessitate a break.

As by construction d(A, C) is minimum, one must never find a difficulty in the addition of the costs of portions of trips that form the complete route. In this aim we introduce the operator ‡ that produces the measure of the chaining of two elements of the same itinerary:

$$d(A, C) = d(A, B) \ddagger d(B, C)$$

$$d(A, C) = d(A, B) + t_b + d(B, C) \quad \text{with } t_b \text{ for the time or cost of the break}$$

With this new formulation, the sum of the length or time or cost produced by two portions of a route will generate an additional cost associated to the crossing of an eventual threshold or of a discontinuity of the cost function. We can then reformulate the triangle inequality as follows:

$$d(A, C) \leq d(A, B) \ddagger d(B, C)$$

$$d(A, C) \leq d(A, B) + t_b + d(B, C)b$$

This formulation states that, whatever B, the route between A and C passing through B has a cost, a length or a duration that is higher or equal to that of the minimum route between A and C. With this new formulation it becomes possible to consider two cases of the additivity of distances: the sub-additivity associated with the optimization by the transport operators, and the over-additivity associated with the discontinuities of the cost function due to the necessary reloading of the energy consumed by the effort of the displacement.

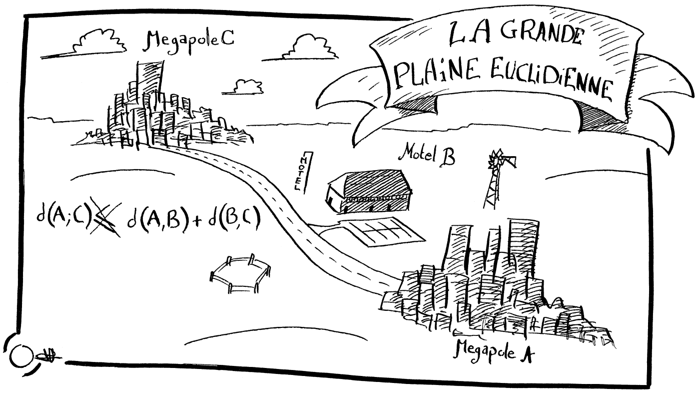

Figure 6. The great Euclidean plain, illustrating the respect of the triangular inequality in the case of a necessary break on a long route (drawing by Jules L'Hostis 2014)

We illustrate this formulation of the distance with an imaginary situation. We consider a large plain with long rectilinear roads between large urban entities. From the city A to the city C the trip takes 16 hours driving on an endless road, to the point that a night of rest is needed halfway at the motel B. The two portions of route from the city A to the motel B, and from B to the city C, both need 8 hours driving. The simple sum of both values give a total of 16, but with the indispensable night spent at the motel one attain the amount of 24 hours instead of 16. As the road is rectilinear, the distance has a Euclidean form; there is no chance of taking a wrong way, or considering a faster itinerary. Yet in this situation, the route ABC is optimum in time: it involves a break in B and the total duration is not the sum of the two legs of the trip, but rather the measure produced by the operator ‡ that we introduced and that adds the break time necessary in the optimum itinerary.

Let us precise that this new formulation of distance takes us out of the Euclidean domain because the measure of the distance cannot be produced by the formula indicating the length of a straight line; however we remain in the metric domain.

This investigation shows that, apart from errors and confusions, the possibility of errors in the triangle inequality is pure speculation, according to an intellectual construct that does not match any observable fact: many sources mention the existence of violations, but no scientific source exhibits economical or geographical spaces where violations of the triangle inequality can be observed.

This conclusion is important because the triangle inequality is at the heart of the definition of geographical space: its violation implies a spatial break or the irruption of sub-optimality in the distance.

This critical review of the economic and geographic literature illustrates the difficulty to interpreting the property of the triangle inequality, and demonstrates the weight of the Euclidean model of distance. In other words, our analysis unveils the Euclidean model of space in the implicit conceptions and highlights the contradictions between this implicit and the observed properties of geographical spaces.

Conclusion

We proposed a discussion on distances that represent a central element for spatial analysis in geography and economy, but that is comparatively with other concepts like space or accessibility, rather under studied. The article or books on distance itself with a theoretical ambition are not numerous. So there is still room for investigation in this direction.

In the perspective of a theoretical discussion on distance and geographical space, we have investigated in this article the meaning of the mathematical properties in the fields of geography and economy. In particular we have discussed the questioning of the mathematical properties of distance. Among the four properties, the last, the triangle inequality is the most worthy of interest for geographical analysis.

We have identified three different situations where some authors identify violations of the triangle inequality. We consider them as errors of interpretation; in each case it is possible whether to demonstrate that the triangle inequality is respected concerning the distances of transport, whether to propose a framework in which the triangular inequality holds.

The first error that we found consists in considering sub-optimal measurements as distances. Sub-optimal measures introduce direct violations of the triangle inequality, but they cannot be observed in the empirical measures in geographical spaces. This discussion on proved or supposed violations of the triangle inequality allows stating that distances in geographical space are always optimum.

The second set of error is the most frequently observed in the literature. It involves a confusion between the Euclidean straight line and the minimum path. The errors consist in considering the presence of a detour as a violation of the triangle inequality, while this situation simply corresponds to a non-Euclidean distance.

The third error concerns the issue of additivity of distances. In the literature some theoretical developments have been introduced to deal with sub-additivity of distances as exemplified by the cost of an aerial trip through a hub lower than the sum ob the costs of the two legs of the route. But the same source raises the issue of over-additivity, which means a situation where the sum of legs is lower than the total length of a route. This situation is commonplace in geographical space with the need for a break in order to provide energy to renew the movement. Presented by the authors as a violation of the triangle inequality, we argued that concerning an optimal route, the triangle inequality must hold, and that another formulation should be established.

From the identification of this error of interpretation of the triangle inequality related to the issue of the additivity of distances, we built a suitable framework. Our proposal is a distance function that allows considering sub-additivity and over-additivity of distances, and in the same time respect the triangle inequality.

These developments provide some new and in part rather counter-intuitive views on three elements of spatiality and movement: the optimality of distances and the role of detours and of breaks as contributing to this optimality. We will elaborate further from these elements in order to build an analytical framework in the domains of geography, of transport and of urban and spatial planning.

Finally, with this article we aim at proposing a better understanding of distances, and we wish that these errors of interpretation of violations of the triangle inequality of distances will not be reproduced in subsequent literature.

Acknoledgements

I thank T. E. Smith for his positive feedback on my proposal of an operator for chaining portions of a route and introducing eventual threshold.

Bibliography

Ahmed, Nobbir, and Harvey J. Miller. 2007. “Time–space Transformations of Geographic Space for

Exploring, Analyzing and Visualizing Transportation Systems." *Journal of Transport Geography* 15 (1): 2–17. doi:10.1016/j.jtrangeo.2005.11.004.
Brunet, Roger. 2009. "Les Sens de La Distance." *Atala* 12: 13–32.
Bunge, William. 1962. *Theoretical Geography*. seconde éd. augmentée 1966. Lund: Gleerup.
Cauvin, C. 1984a. "Une Méthode de Comparaison de Données Localisées: La Régression Bidimensionnelle." *Espace Géographique*, no. 2: 94–109.
———. 1984b. "Espaces Cognitifs et Transformations Cartographiques."
Chapelon, Laurent. 2006. "L'accessibilité, marqueur des inégalités de rayonnement des villes portuaires en Europe." *Cybergeo : European Journal of Geography*. doi:10.4000/cybergeo.2463.
Deutsch, Karl W., and Walter Isard. 1961. "A Note on a Generalized Concept of Effective Distance." *Behavioral Science* 6 (4): 308–11.
Dumolard, Pierre. 2011. "Distances, Accessibility and Spatial Diffusion." In *Modelling Urban Dynamics*, edited by Marius Thériault and François des Rosiers, 189–204. Geographical Inforation Systems Series. ISTE/Wiley.
Ewing, G.O., and R. Wolfe. 1977. "Surface Feature Interpolation on Two-Dimensional Time-Space Maps." *Environment and Planning A.* 9: 419–37.
Felsenstein, Joseph. 1986. "Distance Methods: A Reply to Farris." *Cladistics* 2 (2): 130–43.
Forer, Pip. 1978. "A Place for Plastic Space." *Progress in Human Geography* 2 (2): 230–67.
Fréchet, Maurice. 1906. "Sur Quelques Points Du Calcul Fonctionnel." *Rendiconti Del Circolo Matematico Di Palermo (1884-1940)* 22 (1): 1–72.
———. 1918. "Relations Entre Les Notions de Limite et de Distance." *Transactions of the American Mathematical Society* 19 (1): 53–65.
Gatrell, Anthony C. 1983. *Distance and Space: A Geographical Perspective*. Clarendon Press Oxford. http://www.getcited.org/pub/102287206.
Golledge, Reginald George. 1999. *Wayfinding Behavior: Cognitive Mapping and Other Spatial Processes*. JHU Press.
Golledge, R.G., and L.J. Hubert. 1982. "Some Comments on Non-Euclidean Mental Maps." *Environment and Planning A.*, no. 14: 107–18.
Haggett, Peter. 2001. *Geography, a Global Synthesis*. Harlow: Prentice Hall.
Haggett, Peter, Andrew D. Cliff, and Allan E. Frey. 1977. *Locational Methods in Human Geography*. London: Edward Arnold.
Hall, Edward T. 1969. *The Hidden Dimension*. Anchor Books. New York.
Heath, T. L. 1897. *The Works Of Archimedes*. Cambridge University Press. http://archive.org/details/worksofarchimede029517mbp.
Huriot, Jean-Marie, and Jacques Perreur. 1990. "Distances, Espaces et Représentations." *RERU*, no. 2: 197–237.
Huriot, Jean-Marie, T. E Smith, and Jacques François Thisse. 1989. "Minimum-Cost Distances in Spatial Analysis." *Geographical Analysis* 21 (4): 294–315. doi:10.1111/j.1538-4632.1989.tb00898.x.
Hyman, G.M., and L. Mayhew. 2004. "Advances in Travel Geometry and Urban Modelling." *GeoJournal*, no. 59: 191–207.
Kok, Adrianus Leendert. 2010. *Congestion Avoidance and Break Scheduling within Vehicle Routing*. University of Twente. http://doc.utwente.nl/70636/.
Lamure, Michel. 1998. "Proximité (s), Voisinage et Distance." In *Approches Multiformes de La Proximité*, edited by Michel Bellet, Thierry Kirat, and Christine Largeron, 9–12. Interdisciplinarité et Nouveaux Outils. Hermès.
Lavadinho, Sonia. 2011. "Le Renouveau de La Marche Urbaine: Terrains, Acteurs et Politiques." Ecole normale supérieure de lyon-ENS LYON. http://hal.archives-ouvertes.fr/tel-00737160/.
Letaconnoux, Joseph. 1907. "Note Comparative Sur La Distance En Temps Entre L'intérieur de La

Bretagne et La Mer, Au XVIIIe, XIXe et XXe Siècle.” *Annales de Bretagne* 23 (3): 305–21. doi:10.3406/abpo.1907.1272.
Lévy, Jacques. 2009. “Entre Contact et Écart: La Distance Au Coeur de La Réflexion.” *Atala* 12: 175–85.
L’Hostis, Alain. 1997. “Images de Synthèse Pour L’aménagement Du Territoire: La Déformation de L’espace Par Les Réseaux de Transport Rapide.” Géographie et Aménagement, Tours, Aménagement de l’espace.
———. 2014. “Le Détour, La Pause et L’optimalité, Essai Sur La Distance et Ses Apports Au Transport et À L’urbanisme.” Université Paris-Est. https://hal.archives-ouvertes.fr/tel-01081570/.
Lynch, Kevin. 1971. *L’Image de La Cité*. Paris: Dunod.
Marchand, Bernard. 1973. “Deformation of a Transportation Surface.” *Annals of the Association of American Geographers* 63: 507–21.
Müller, Jean-Claude. 1982. “Non-Euclidean Geographic Spaces: Mapping Functional Distances.” *Geographical Analysis* 14: 189–203.
Perreur, Jacques. 1989. “L’Evolution Des Représentations de La Distance et l’Aménagement Du Territoire.” *RERU*, no. 1.
Ptolémée, Claude. 1828. *Traité de Géographie de Claude Ptolémée*. Translated by Abbé Halma. Paris: Eberhart.
Pumain, Denise. 2009. “Essai Sur La Distance et L’espace Géographique.” *Atala* 12: 33–49.
Rimbert, Sylvie. 1990. *Carto-Graphies*. Paris: Hermès.
Rivizzigno, Victoria Lynne. 1976. “Cognitive Representations of an Urban Area.” Ohio State University.
Rouget, Bernard. 1975. “L’Analyse Spatiale En Économie Urbaine: Essai Méthodologique.”
Smith, Tony E. 1989. “Shortest-Path Distances: An Axiomatic Approach.” *Geographical Analysis* 21 (1): 1–31.
Tobler, Waldo Rudolph. 1961. “Map Transformation of Geographic Space.” Geography, Washington: University of Washington, Geography.
———. 1963. “Geographic Area and Map Projections.” *Geographical Review* 53: 59–78.
———. 1976. “The Geometry of Mental Maps.” In *Spatial Choice and Spatial Behavior*, 69–81. Columbus: Ohio State University. http://www.dpi.inpe.br/sil/CST310/cst310_2010/Aula12_TOBLER/refs/Tobler/Tobler_Geometry_of_Mental_Maps.pdf.
———. 1997. “Visualizing the Impact of Transportation on Spatial Relations.” In *Western Regional Science Association Meeting*, 7. Hawaii.