Closed-forms of Kirchhoff elastic rods shape and sensitivity in the planar case

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Olivier Roussel[1], Marc Renaud[1] and Michel Taïx[1]*

Abstract

In this report we give closed-forms of Kirchhoff 3-D elastic rods curvature in terms of elliptic functions and, by treating planar rods as a special case, we show we can also obtain closed-forms of planar rods shape, sensitivity and total elastic energy.

1 General case of 3-D rods

Consider an inextensible, non-shearable and unit length linearly elastic rod. The shape of the rod traces a curve that we will describe by the mapping $q : [0, 1] \to SE(3)$. The position along the rod is parametrized by $t \in [0, 1]$ and we will name "base" and "tip" of the rod its extremity at $t = 0$ and $t = 1$ respectively. Let the mappings $u_1(t), u_2(t), u_3(t)$ such that $u_i : [0, 1] \to \mathbb{R}$ be axial and bending rod strains respectively, and c_1, c_2, c_3 be the constants that reflect its elasticity properties. As in [3], we say the elastic rod is in static equilibrium in the sense of Kirchhoff if it locally minimizes the elastic energy defined by

$$E_{el} = \frac{1}{2}\int_0^1 \sum_{i=1}^{3} c_i u_i^2 \, dt. \tag{1.1}$$

Without loss of generality, we will also assume that the base of the rod is held fixed at the origin, i.e. $q(0) = e$ where e is the identity element of $SE(3)$. Under these assumptions, we will denote by $\mathcal{B}$ the set of positions that the other extremity of the rod $q(1)$ can reach. As shown in [2], the problem of static equilibrium of such rods can be formulated as an optimal control problem by

$$\begin{aligned} \underset{q,u}{\text{minimize}} \quad & \frac{1}{2}\int_0^1 \sum_{i=1}^{3} c_i u_i^2 \, dt \\ \text{subject to} \quad & \dot{q} = q\left(\sum_{i=1}^{3} u_i X_i + X_4\right) \\ & q(0) = e, \quad q(1) = b \end{aligned} \tag{1.2}$$

for some $b \in \mathcal{B}$ and where

$$X_1 = \begin{bmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad X_5 = \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad X_6 = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$$

is a basis for $\mathfrak{se}(3)$, the Lie algebra of $SE(3)$. Note that when solving this optimal control problem, the rod tip position b is not an input.

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In these conditions, the Maximum Principle states that solutions to this optimal control problem are the projections of extremal curves defined on the cotangent bundle $T^*SE(3)$ onto $SE(3)$. Thanks to the Lie Group structure of $SE(3)$, the Hamiltonian can be reduced on the dual of the Lie algebra $\mathfrak{se}(3)^*$ and the corresponding (time-varying) Hamiltonian vector fields $\mu : [0,1] \to \mathfrak{se}(3)^*$ can be expressed by

$$\begin{cases} \dot{\mu_1} = \frac{\mu_3\mu_2}{c_3} - \frac{\mu_2\mu_3}{c_2} \\ \dot{\mu_2} = \mu_6 + \frac{\mu_1\mu_3}{c_1} - \frac{\mu_1\mu_3}{c_3} \\ \dot{\mu_3} = -\mu_5 + \frac{\mu_1\mu_2}{c_2} - \frac{\mu_1\mu_2}{c_1} \\ \dot{\mu_4} = \frac{\mu_3\mu_5}{c_3} - \frac{\mu_2\mu_6}{c_2} \\ \dot{\mu_5} = \frac{\mu_1\mu_6}{c_1} - \frac{\mu_3\mu_4}{c_3} \\ \dot{\mu_6} = \frac{\mu_2\mu_4}{c_2} - \frac{\mu_1\mu_5}{c_1} \end{cases} \tag{1.3}$$

where vector fields μ are related to controls u_i by $u_i = c_i^{-1}\mu_i$ for $i \in \{1,2,3\}$.

Let $\mathcal{A}$ be the set homeomorphic to $\mathbb{R}^6$ and $a \in \mathcal{A}$ such that $a_i \triangleq \mu_i(0), i \in \{1,\ldots,6\}$. It has been shown in [2] that coordinates in $\mathcal{A}$ offer a global parameterization to the set of static equilibrium configuration for the rod. In other words, we can describe configurations of quasi-static 3-D elastic rods using the 6-dimensional configuration space $\mathcal{A}$.

Assuming isotropy and normalized elasticity constants such that $c_i = 1$ for $i \in \{1,2,3\}$, we have from (1.3) $\dot{\mu_1} = 0$. Then μ_1 is a constant of motion with $\mu_1 = a_1$ and

$$\begin{cases} \dot{\mu_2} = \mu_6 \\ \dot{\mu_3} = -\mu_5 \\ \dot{\mu_4} = \mu_3\mu_5 - \mu_2\mu_6 \\ \dot{\mu_5} = a_1\mu_6 - \mu_3\mu_4 \\ \dot{\mu_6} = \mu_2\mu_4 - a_1\mu_5 \end{cases} \tag{1.4}$$

The signed curvature κ and the torsion τ of the curve can be expressed in terms of μ by

$$\kappa^2 = \mu_2^2 + \mu_3^2 \qquad \tau = \mu_1 - \frac{\mu_2\mu_5 + \mu_3\mu_6}{\mu_2^2 + \mu_3^2}$$

and, as mentioned in [2], the differential system (1.4) is equivalent to

$$2\ddot{\kappa} + \kappa^3 - 2\kappa(\tau - \lambda_1)^2 = \lambda_2\kappa \tag{1.5a}$$

$$\kappa^2(\tau - \lambda_1) = \lambda_3 \tag{1.5b}$$

where the constants of integration are given by

$$\begin{cases} \lambda_1 \triangleq \frac{a_1}{2} \\ \lambda_2 \triangleq a_2^2 + a_3^2 + 2a_4 - \frac{a_1^2}{2} \\ \lambda_3 \triangleq \frac{a_1}{2}(a_2^2 + a_3^2) - (a_2a_5 + a_3a_6). \end{cases}$$

Substituting (1.5b) into (1.5a) and integrating, we obtain

$$\dot{\kappa}^2 + \frac{1}{4}\kappa^4 + \lambda_3^2\kappa^{-2} - \frac{\lambda_2}{2}\kappa^2 = \lambda_4 \tag{1.6}$$

where the constant of integration λ_4 is given by

$$\lambda_4 \triangleq a_5^2 + a_6^2 - \frac{1}{4}(a_2^2 + a_3^2)^2 + \frac{1}{2}(a_2^2 + a_3^2)(a_1^2 - 2a_4) - a_1(a_2a_5 + a_3a_6)$$

By making the change of variable $\upsilon = \kappa^2$, (1.6) transforms to

$$\dot{\upsilon}^2 + \upsilon^3 - 2\lambda_2\upsilon^2 - 4\lambda_4\upsilon + 4\lambda_3^2 = 0 \tag{1.7}$$

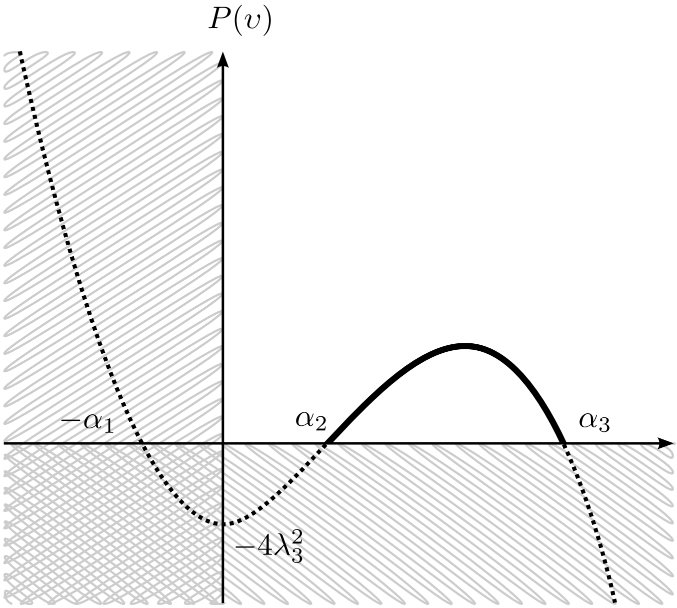

Figure 1: Plot of the cubic polynomial $P(\upsilon)$ with respect to the squared curvature υ. Hatched regions correspond to impossible values illustrating that the only valid range for υ is given by α_2 and α_3, the zeros of $P(\upsilon)$.

As already stated in [4], this equation is in the form $\dot{\upsilon}^2 = P(\upsilon)$ with P is the cubic polynomial

$$P(\upsilon) \triangleq -\upsilon^3 + 2\lambda_2\upsilon^2 + 4\lambda_4\upsilon - 4\lambda_3^2. \tag{1.8}$$

Let $-\alpha_1, \alpha_2, \alpha_3$ be the zeros of the polynomial $P(\upsilon)$ such that

$$-\alpha_1 \leqslant 0 \leqslant \alpha_2 \leqslant \alpha_3. \tag{1.9}$$

.

As $P(\pm\infty) = \mp\infty$ and $P(0) = -4\lambda_3^2 \leqslant 0$, $P(\upsilon)$ is in the form illustrated in figure 1.
Also, we have $\upsilon \geqslant 0$ and $P(\upsilon) \geqslant 0$ as they are both squares, so $\upsilon \in [\alpha_2, \alpha_3]$.
The polynomial $P(\upsilon)$ can be rewritten for its zeros by

$$P(\upsilon) = -(\upsilon + \alpha_1)(\upsilon - \alpha_2)(\upsilon - \alpha_3).$$

We can express the polynomial zeros $-\alpha_1, \alpha_2, \alpha_3$ from the constants of integrations λ_i by

$$\begin{aligned} \alpha_1 - \alpha_2 - \alpha_3 &= -2\lambda_2 \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_2\alpha_3 &= 4\lambda_4 \\ \alpha_1\alpha_2\alpha_3 &= 4\lambda_3^2. \end{aligned} \tag{1.10}$$

The squared curvature υ can be expressed in terms of elliptic functions by

$$\upsilon(t) = \alpha_3\Big(1 - n\,\mathrm{sn}^2\left(rt + \varphi|m\right)\Big) \tag{1.11}$$

the parameter m, the characteristic n and r can be expressed from the polynomial zeros by

$$m = \frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1} \qquad n = \frac{\alpha_3 - \alpha_2}{\alpha_3} \qquad r = \frac{1}{2}\sqrt{\alpha_3 + \alpha_1} \tag{1.12}$$

Given

$$\varpi \triangleq \sqrt{\frac{1}{n}\left(1-\frac{a_2^2+a_3^2}{\alpha_3}\right)} \tag{1.13}$$

the phase φ can be retrieved from $a_2^2+a_3^2=\alpha_3\big(1-n\,\mathrm{sn}^2(\varphi|m)\big)$, and is given by

$$\varphi = \mathrm{sgn}(a_3a_5-a_2a_6)\,\mathrm{arcsn}(\varpi|m) \tag{1.14}$$

where arcsn is the inverse of the Jacobi elliptic function sn.

Note that from (1.9), we have $0 \leqslant m \leqslant n \leqslant 1$.

As outlined in [3], it has been shown the Hamiltonian vector fields in (1.4) is integrable and we have proved it can be expressed in the following form

$$\begin{cases} \mu_2 = \kappa\sin\psi \\ \mu_3 = \kappa\cos\psi \\ \mu_4 = \frac{1}{2}(\lambda_2+\frac{a_1^2}{2}-\upsilon) \\ \mu_5 = -\dot{\kappa}\cos\psi+\kappa\dot{\psi}\sin\psi \\ \mu_6 = \dot{\kappa}\sin\psi+\kappa\dot{\psi}\cos\psi. \end{cases} \tag{1.15}$$

where

$$\psi(t) = \lambda_1 t - \frac{\lambda_3}{\alpha_3\,r}\Big(\Pi\Big(n,\mathrm{am}\left(rt+\varphi|m\right)|m\Big)-\Pi\Big(n,\mathrm{am}\left(\varphi|m\right)|m\Big)\Big)+\psi\,(0)$$

with $\Pi(n,u|m)$ the elliptic integral of the third kind and $\mathrm{am}(u|m)$ is the Jacobi amplitude.

2 Planar case

Although neither the curve $q(t)$ nor the rod sensitivity $\frac{\partial q(t)}{\partial a}$ can be explicitly expressed in the general 3-D case, we will show in this section that closed forms can be obtained in the planar case which can be treated as a particular case of the previously presented model.

2.1 Curvature and internal wrenches

Considering only planar curves $q(t)$ in the xy-plane with $q=(0,0,\theta,x,y,0)^T$, Hamiltonian vector fields defined in (1.3) simplify to

$$\begin{cases} \dot{\mu_1} = 0 \\ \dot{\mu_2} = 0 \\ \dot{\mu_3} = -\mu_5 \\ \dot{\mu_4} = \mu_3\mu_5 \\ \dot{\mu_5} = -\mu_3\mu_4 \\ \dot{\mu_6} = 0 \end{cases} \tag{2.1}$$

Closed-forms of rod internal wrenches $\mu(t)$ defined in (1.15) reduce to

$$\begin{cases} \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = \kappa \\ \mu_4 = -\frac{1}{2}(\kappa^2+\lambda_2) \\ \mu_5 = -\dot{\kappa} \\ \mu_6 = 0 \end{cases} \tag{2.2}$$

And constants of integration defined in (1) simplify to

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = a_3^2 + 2a_4 \\ \lambda_3 = 0 \\ \lambda_4 = a_5^2 - a_3^2(\frac{1}{4}a_3^2 + a_4) \end{cases}$$

We retrieve the same results as we would have obtain by applying the same problem formulation on the Lie Group $SE(2)$ rather than $SE(3)$. Therefore, in the rest of this section we will restrict to solutions of (1.2) that are similar to trajectories on $SE(2)$, which are generated by the subset of initial conditions $\{a \in \mathcal{A} : (a_1, a_2, a_6) = (0, 0, 0)\}$.

In the following equations, when referring to an elliptic function pq, we will simplify the notation $\mathrm{pq}(u|m)$ to $\mathrm{pq}\, u$. Also, let us define

$$\Gamma(t) \triangleq rt + \varphi$$

and the following constants of motion that be needed in the following developments

$$\begin{aligned} \varepsilon &\triangleq \mathrm{sgn}(a_3) \\ \Gamma_0 &\triangleq \Gamma(0) \\ \delta &\triangleq \lambda_2^2 + 4\lambda_4 \end{aligned}$$

The expression of the phase φ given in (1.14) simplifies to

$$\varphi = \mathrm{sgn}(a_3 a_5)\, \mathrm{arcsn}(\varpi|m)$$

where ϖ given in (1.13) reduces to

$$\varpi = \sqrt{\frac{1}{n}\left(1 - \frac{a_3^2}{\alpha_3}\right)}.$$

2.1.1 Expression of the curvature

In the planar case, as $\lambda_3 = 0$, the polynomial $P(\upsilon)$ simplifies to

$$P(\upsilon) = -\upsilon^3 + 2\lambda_2\upsilon^2 + 4\lambda_4\upsilon$$

so $P(\upsilon)$ has one trivial zero at $\upsilon = 0$.
From (1.9), we can distinguish three cases as outlined in [4] and [5]:

- Case I: $\lambda_4 > 0$

  Using (1.10), we have that
  $$\lambda_1(\lambda_2 + \lambda_3) > \lambda_2\lambda_3.$$

  This imposes the choice for the zeros to

  $$\begin{cases} \alpha_1 = -\lambda_2 + \sqrt{\delta} \\ \alpha_2 = 0 \\ \alpha_3 = \lambda_2 + \sqrt{\delta}. \end{cases}$$

  From (1.12) we get $n = 1$, so the squared curvature formula in (1.11) simplifies to

  $$\begin{aligned} \upsilon(t) &= \alpha_3\left(1 - \mathrm{sn}^2\,\Gamma(t)\right) \\ &= \alpha_3\, \mathrm{cn}^2\,\Gamma(t). \end{aligned}$$

Then the signed curvature is given by

$$\kappa(t) = \varepsilon\sqrt{\alpha_3}\,\mathrm{cn}\,\Gamma(t). \tag{2.3}$$

The curvature $\kappa(t)$ oscillates between $\sqrt{\alpha_3}$ and $-\sqrt{\alpha_3}$ and the resulting curve $q(t)$ is called a "wavelike" elastica.

- Case II: $\lambda_4 < 0$ Using (1.10), we have

  $$\lambda_1(\lambda_2 + \lambda_3) < \lambda_2\lambda_3.$$

  This imposes the choice for the zeros to

  $$\begin{cases} \alpha_1 & = 0 \\ \alpha_2 & = \lambda_2 - \sqrt{\delta} \\ \alpha_3 & = \lambda_2 + \sqrt{\delta}. \end{cases}$$

  From (1.12) we get $n = m$, so the squared curvature formula in (1.11) simplifies to

  $$\begin{aligned} \upsilon(t) &= \alpha_3\left(1 - m\,\mathrm{sn}^2\,\Gamma(t)\right) \\ &= \alpha_3\,\mathrm{dn}^2\,\Gamma(t). \end{aligned}$$

  Then the signed curvature is given by

  $$\kappa(t) = \varepsilon\sqrt{\alpha_3}\,\mathrm{dn}\,\Gamma(t) \tag{2.4}$$

  The curvature $\kappa(t)$ is non-vanishing and the resulting curve $q(t)$ is called a "orbit-like" elastica.

- Case III: $\lambda_4 = 0$ This borderline case implies the polynomial $P(\upsilon)$ reduces to

  $$P(\upsilon) = -\upsilon^3 + 2\lambda_2\upsilon^2$$

  which has a double zero.

  Using first equation of (1.10), only one choice is possible for the zeros α_i:

  $$\begin{cases} \alpha_1 = \alpha_2 = 0 \\ \alpha_3 = |2\lambda_2|. \end{cases}$$

  which leads to the signed curvature

  $$\kappa(t) = \varepsilon\sqrt{\alpha_3}\,\mathrm{sech}\,\Gamma(t) \tag{2.5}$$

  This corresponds to the borderline case where the curvature is non-periodic.

2.1.2 Reduction to a unique formulation of the curvature

These cases can be reduced to a single formulation of the curvature by allowing the parameter m to be any positive or null real and applying the Jacobi's real transformation (see [1] §16.11). By relaxing the constraint on the zeros α_i given in (1.9), and keeping only one fixed choice on the zeros that we will denote by α' such that

$$\begin{cases} \alpha'_1 = 0 \\ \alpha'_2 = \lambda_2 - \sqrt{\delta} \\ \alpha'_3 = \lambda_2 + \sqrt{\delta}. \end{cases}$$

In this form, α'_3 is positive as $\sqrt{\delta} > |\lambda_2|$ but α'_2 can now be negative. Note that we still have $\alpha'_3 \geqslant \alpha'_2$.

Using same forms as in (1.12), the elliptic parameter m' and r' by

$$m' = \frac{\alpha'_3 - \alpha'_2}{\alpha'_3} \qquad r' = \frac{1}{2}\sqrt{\alpha'_3}$$

but as mentioned before, the new elliptic parameter m' is only constrained to in $[0, \infty)$. Then, the signed curvature can be expressed by a unique expression by

$$\kappa(t) = \varepsilon\sqrt{\alpha'_3}\,\mathrm{dn}\left(r'\,(t+\phi)\middle|m'\right) \tag{2.6}$$

When $m' > 1$, the Jacobi's real transformation can be applied to reduce to a parameter m such that $0 \leqslant m \leqslant 1$ and we retrieve the previously described cases.

2.1.3 Explicit formulation of rod total elastic energy

Recall from (1.1) the total elastic energy of the rod is given by

$$\begin{aligned} E_{el} &= \frac{1}{2}\int_0^1 u_3(t)^2\,dt \\ &= \frac{1}{2}\int_0^1 \kappa(t)^2\,dt \end{aligned}$$

Using the unique formulation of the curvature $\kappa(t)$ given in (2.6) can be integrated to give an explicit formulation in terms of the elliptic integral of the second kind by

$$E_{el} = \frac{\alpha'_3}{2}\,\mathrm{E}\left(r'\,(t+\phi)\middle|m'\right)$$

2.2 Integration of the curve $q(t)$

From the differential system defined in (1.2), it follows that

$$\dot{\theta} = u_3 = \kappa \qquad \dot{x} = \cos\theta \qquad \dot{y} = \cos\theta.$$

Using (2.2), the integration of the curvature is given by

$$\cos\theta(t) = \beta_1(0)\beta_1(t) + 4\beta_2(0)\beta_2(t) \tag{2.7a}$$

$$\sin\theta(t) = 2\,\varepsilon\left(\beta_1(0)\beta_2(t) - \beta_2(0)\beta_1(t)\right) \tag{2.7b}$$

$$x(t) = \beta_1(0)\int\beta_1(t) + 4\beta_2(0)\int\beta_2(t) \tag{2.7c}$$

$$y(t) = 2\,\varepsilon\left(\beta_1(0)\int\beta_2(t) - \beta_2(0)\int\beta_1(t)\right). \tag{2.7d}$$

The functions $\beta_1(t)$ and $\beta_2(t)$ can be explicitly given using Jacobi elliptic functions and the elliptic integral of second kind $\mathrm{E}(u|m)$ in the three cases previously described as follows

- Case I: $\lambda_4 > 0$

  Integrating the curvature in (2.3) (see [1] §16.24) leads to

$$\theta(t) = 2\,\varepsilon\left(\arccos\left(\operatorname{dn}\Gamma(t)\right) - \arccos\left(\operatorname{dn}\Gamma_0\right)\right)$$

  Let $A(t) \triangleq \arccos\left(\operatorname{dn}\Gamma(t)\right)$ and $A(0) \triangleq \arccos\left(\operatorname{dn}\Gamma_0\right)$, then

$$\begin{aligned}\cos A(t) &= \operatorname{dn}\Gamma(t)\\ \sin A(t) &= \pm\sqrt{1-\operatorname{dn}^2\Gamma(t)}\\ &= \sqrt{m}\operatorname{sn}\Gamma(t)\end{aligned}$$

  Given that $\frac{\theta(t)}{2} = \varepsilon\left(A(t) - A(0)\right)$, we have

$$\begin{aligned}\cos\frac{\theta(t)}{2} &= \cos A(t)\cos A(0) + \sin A(t)\sin A(0)\\ &= \operatorname{dn}\Gamma(t)\operatorname{dn}\Gamma_0 + m\operatorname{sn}\Gamma(t)\operatorname{sn}\Gamma_0\\ \sin\frac{\theta(t)}{2} &= \sin A(t)\cos A(0) - \cos A(t)\sin A(0)\\ &= \varepsilon\sqrt{m}\left(\operatorname{sn}\Gamma(t)\operatorname{dn}\Gamma_0 - \operatorname{sn}\Gamma_0\operatorname{dn}\Gamma(t)\right)\end{aligned}$$

  Using half-angle formulas, we get

$$\begin{aligned}\cos\theta(t) &= \cos^2\frac{\theta(t)}{2} - \sin^2\frac{\theta(t)}{2}\\ &= \left(2\operatorname{dn}^2\Gamma_0 - 1\right)\left(2\operatorname{dn}^2\Gamma(t) - 1\right) + 4m\operatorname{dn}\Gamma(t)\operatorname{sn}\Gamma(t)\operatorname{dn}\Gamma_0\operatorname{sn}\Gamma_0\\ \sin\theta(t) &= 2\cos\frac{\theta(t)}{2}\sin\frac{\theta(t)}{2}\\ &= 2\,\varepsilon\sqrt{m}\left(\left(2\operatorname{dn}^2\Gamma_0 - 1\right)\operatorname{dn}\Gamma(t)\operatorname{sn}\Gamma(t) - \left(2\operatorname{dn}^2\Gamma(t) - 1\right)\operatorname{dn}\Gamma_0\operatorname{sn}\Gamma_0\right)\end{aligned}$$

  which is in the form (2.7) with β_1 and β_2 given by

$$\beta_1(t) \triangleq 2\operatorname{dn}^2\Gamma(t) - 1 \tag{2.8a}$$

$$\beta_2(t) \triangleq \sqrt{m}\operatorname{sn}\Gamma(t)\operatorname{dn}\Gamma(t) \tag{2.8b}$$

  and can be integrated to

$$\int\beta_1(t) = 2r^{-1}\left(\mathrm{E}(\operatorname{am}\Gamma(t)) - \mathrm{E}(\operatorname{am}\Gamma_0)\right) - t \tag{2.9a}$$

$$\int\beta_2(t) = -r^{-1}\left(\operatorname{cn}\Gamma(t) - \operatorname{cn}\Gamma_0\right) \tag{2.9b}$$

(2.9c)

- Case II: $\lambda_4 < 0$

  Integrating the curvature in (2.4) (see [1] §16.24) leads to

$$\theta(t) = 2\,\varepsilon\left(\arcsin\left(\operatorname{sn}\Gamma(t)\right) - \arcsin\left(\operatorname{sn}\Gamma_0\right)\right)$$

  Let $A(t) \triangleq \arcsin\left(\operatorname{sn}\Gamma(t)\right)$ and $A(0) \triangleq \arcsin\left(\operatorname{sn}\Gamma_0\right)$, then

$$\begin{aligned}\cos A(t) &= \pm\sqrt{1-\operatorname{sn}^2\Gamma(t)}\\ &= \operatorname{cn}\Gamma(t)\\ \sin A(t) &= \operatorname{sn}\Gamma(t)\end{aligned}$$

Given that $\frac{\theta(t)}{2} = \varepsilon\left(A(t) - A(0)\right)$, we have

$$\begin{aligned}\cos\frac{\theta(t)}{2} &= \cos A(t)\cos A(0) + \sin A(t)\sin A(0)\\ &= \operatorname{cn}\Gamma(t)\operatorname{cn}\Gamma_0 + \operatorname{sn}\Gamma(t)\operatorname{sn}\Gamma_0\\ \sin\frac{\theta(t)}{2} &= \sin A(t)\cos A(0) - \cos A(t)\sin A(0)\\ &= \varepsilon\left(\operatorname{sn}\Gamma(t)\operatorname{cn}\Gamma_0 - \operatorname{cn}\Gamma_0\operatorname{sn}\Gamma(t)\right)\end{aligned}$$

Using half-angle formulas, we get

$$\begin{aligned}\cos\theta(t) &= \cos^2\frac{\theta(t)}{2} - \sin^2\frac{\theta(t)}{2}\\ &= \left(1 - 2\operatorname{dn}^2\Gamma(t)\right)\left(1 - 2\operatorname{dn}^2\Gamma_0\right) + 4\operatorname{cn}\Gamma(t)\operatorname{sn}\Gamma(t)\operatorname{cn}\Gamma_0\operatorname{sn}\Gamma_0\\ \sin\theta(t) &= 2\cos\frac{\theta(t)}{2}\sin\frac{\theta(t)}{2}\\ &= 2\varepsilon\left(\left(1 - 2\operatorname{sn}^2\Gamma_0\right)\operatorname{cn}\Gamma(t)\operatorname{sn}\Gamma(t) - \left(1 - 2\operatorname{dn}^2\Gamma(t)\right)\operatorname{cn}\Gamma_0\operatorname{sn}\Gamma_0\right)\end{aligned}$$

which is in the form (2.7) with β_1 and β_2 given by

$$\beta_1(t) \triangleq 1 - 2\operatorname{sn}^2\Gamma(t) \tag{2.10a}$$

$$\beta_2(t) \triangleq \operatorname{sn}\Gamma(t)\operatorname{cn}\Gamma(t) \tag{2.10b}$$

and can be integrated to

$$\int \beta_1(t) = m^{-1}\left(t\,(m-2) + 2r^{-1}\left(\operatorname{E}\left(\operatorname{am}\Gamma\left(t\right)\right) - \operatorname{E}\left(\operatorname{am}\Gamma_0\right)\right)\right) \tag{2.11a}$$

$$\int \beta_2(t) = -r^{-1}\left(\operatorname{cn}\Gamma(t) - \operatorname{cn}\Gamma_0\right) \tag{2.11b}$$

(2.11c)

- Case III: $\lambda_4 = 0$

Integrating the curvature in (2.5) (see [1] §16.24) leads to

$$\theta(t) = 2\,\varepsilon\left(\arctan\left(\sinh\Gamma(t)\right) - \arctan\left(\sinh\Gamma_0\right)\right)$$

Let $A(t) \triangleq \arctan\left(\sinh\Gamma(t)\right)$ and $A(0) \triangleq \arctan\left(\sinh\Gamma_0\right)$, then

$$\begin{aligned}\cos A(t) &= \pm\left(1 + \sinh^2\Gamma(t)\right)^{-\frac{1}{2}}\\ &= \operatorname{sech}\Gamma(t)\\ \sin A(t) &= \tanh\Gamma(t)\end{aligned}$$

Given that $\frac{\theta(t)}{2} = \varepsilon\left(A(t) - A(0)\right)$, we have

$$\begin{aligned}\cos\frac{\theta(t)}{2} &= \cos A(t)\cos A(0) + \sin A(t)\sin A(0)\\ &= \operatorname{sech}\Gamma(t)\operatorname{sech}\Gamma_0 + \tanh\Gamma(t)\tanh\Gamma_0\\ \sin\frac{\theta(t)}{2} &= \sin A(t)\cos A(0) - \cos A(t)\sin A(0)\\ &= \varepsilon\left(\tanh\Gamma(t)\operatorname{sech}\Gamma_0 - \tanh\Gamma_0\operatorname{sech}\Gamma(t)\right)\end{aligned}$$

Using half-angle formulas, we get

$$\begin{aligned}
\cos\theta(t) &= \cos^2\frac{\theta(t)}{2} - \sin^2\frac{\theta(t)}{2} \\
&= \left(2\,\mathrm{sech}^2\,\Gamma(t) - 1\right)\left(2\,\mathrm{sech}^2\,\Gamma_0 - 1\right) + 4\,\mathrm{sech}\,\Gamma(t)\tanh\Gamma(t)\,\mathrm{sech}\,\Gamma_0\tanh\Gamma_0 \\
\sin\theta(t) &= 2\cos\frac{\theta(t)}{2}\sin\frac{\theta(t)}{2} \\
&= 2\,\varepsilon\left(\left(2\,\mathrm{sech}^2\,\Gamma_0 - 1\right)\mathrm{sech}\,\Gamma(t)\tanh\Gamma(t) - \left(2\,\mathrm{sech}^2\,\Gamma(t) - 1\right)\mathrm{sech}\,\Gamma_0\tanh\Gamma_0\right)
\end{aligned}$$

which is in the form (2.7) with β_1 and β_2 given by

$$\beta_1(t) \triangleq 2\ \mathrm{sech}^2\,\Gamma(t) - 1 \tag{2.12a}$$

$$\beta_2(t) \triangleq \mathrm{sech}\,\Gamma(t)\tanh\Gamma(t) \tag{2.12b}$$

which integrate to

$$\int\beta_1(t) = 2r^{-1}\left(\tanh\Gamma(t) - \tanh\Gamma_0\right) - t \tag{2.13a}$$

$$\int\beta_2(t) = -r^{-1}\left(\mathrm{sech}\,\Gamma\,(t) - \mathrm{sech}\,\Gamma_0\right). \tag{2.13b}$$

2.3 Explicit formulation of elastic rod sensitivity

In the 3-D case, the elastic rod sensitivity is given by the 6-dimensional Jacobian matrix

$$\mathbf{J}(t,a) = \begin{pmatrix} \frac{\partial q_1}{\partial a_1} & \cdots & \frac{\partial q_1}{\partial a_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_6}{\partial a_1} & \cdots & \frac{\partial q_6}{\partial a_6} \end{pmatrix} \tag{2.14}$$

In the planar case, this simplifies to

$$\mathbf{J}(t,a) = \begin{pmatrix} \ast_{2,2} & 0_{2,3} & \ast_{2,1} \\ 0_{3,2} & \mathbf{J}^P_{3,3}(t,a) & 0_{3,1} \\ \ast_{1,2} & 0_{1,3} & \ast_{1,1} \end{pmatrix} \tag{2.15}$$

where $\ast$ represents indeterminate values.

As we can only obtain closed-forms of the rod shape and thus of rod sensitivity in this special, we will focus in this section on the 3-dimensional block $\mathbf{J}^P(t,a)$ of $\mathbf{J}(t,a)$ for $i,j \in \{3,4,5\}$. Differentiating the general form of the curve $q(t)$ in (2.7)leads to

$$\frac{\partial\cos(\theta(t))}{\partial a} = \beta_1(0)\frac{\partial\beta_1(t)}{\partial a} + \frac{\partial\beta_1(0)}{\partial a}\beta_1(t) + 4\left(\beta_2(0)\frac{\partial\beta_2(t)}{\partial a} + \frac{\partial\beta_2(0)}{\partial a}\beta_2(t)\right) \tag{2.16a}$$

$$\frac{\partial\sin(\theta(t))}{\partial a} = 2\,\varepsilon\left(\beta_1(0)\frac{\partial\beta_2(t)}{\partial a} + \beta_2(t)\frac{\partial\beta_1(0)}{\partial a} - \beta_2(0)\frac{\partial\beta_1(t)}{\partial a} - \beta_1(t)\frac{\partial\beta_2(0)}{\partial a}\right) \tag{2.16b}$$

$$\frac{\partial x(t)}{\partial a} = \beta_1(0)\frac{\partial\int\beta_1(t)}{\partial a} + \int\beta_1(t)\frac{\partial\beta_1(0)}{\partial a} + 4\left(\beta_2(0)\frac{\partial\int\beta_2(t)}{\partial a} + \int\beta_2(t)\frac{\partial\int\beta_2(0)}{\partial a}\right) \tag{2.16c}$$

$$\frac{\partial y(t)}{\partial a} = 2\,\varepsilon\left(\beta_1(0)\frac{\partial\int\beta_2(t)}{\partial a} + \int\beta_2(t)\frac{\partial\beta_1(0)}{\partial a} - \beta_2(0)\frac{\partial\int\beta_1(t)}{\partial a} - \int\beta_1(t)\frac{\partial\int\beta_2(0)}{\partial a}\right) \tag{2.16d}$$

Regardless the three cases of curve elastica, we can derivate with respect to a the following forms:

- The elliptic parameters m, n and r

$$\frac{\partial m}{\partial a} = \frac{1}{(\alpha_3+\alpha_1)^2}\left(\left(\frac{\partial \alpha_3}{\partial a} - \frac{\partial \alpha_2}{\partial a}\right)(\alpha_3+\alpha_1) - \left(\frac{\partial \alpha_3}{\partial a} + \frac{\partial \alpha_1}{\partial a}\right)(\alpha_3-\alpha_2)\right) \tag{2.17a}$$

$$\frac{\partial r}{\partial a} = \frac{1}{4\sqrt{\alpha_3+\alpha_1}}\left(\frac{\partial \alpha_3}{\partial a} + \frac{\partial \alpha_1}{\partial a}\right) \tag{2.17b}$$

$$\frac{\partial n}{\partial a} = \frac{1}{\alpha_3^2}\left(\frac{\partial \alpha_3}{\partial a}\alpha_2 - \frac{\partial \alpha_2}{\partial a}\alpha_3\right) \tag{2.17c}$$

- The phase φ

Given

$$\frac{\partial \varpi}{\partial a} = \frac{1}{2n\varpi}\left(\frac{a_3^2}{\alpha_3^2}\begin{pmatrix} \frac{\partial \alpha_3}{\partial a_3} - 2\frac{\alpha_3}{a_3} \\ \frac{\partial \alpha_3}{\partial a_4} \\ \frac{\partial \alpha_3}{\partial a_5} \end{pmatrix} - \frac{1}{n}\left(1 - \frac{a_3^2}{\alpha_3}\right)\frac{\partial n}{\partial a}\right) \tag{2.18}$$

and the first order derivatives of the function $\operatorname{arcsn}(z|m)$

$$\frac{\partial \operatorname{arcsn}(z|m)}{\partial z} = \frac{1}{\sqrt{1-z^2}\sqrt{1-mz^2}}$$
$$\frac{\partial \operatorname{arcsn}(z|m)}{\partial m} = \frac{1}{2\,(m-1)\,m}\left(\frac{m\sqrt{1-z^2}z}{\sqrt{1-mz^2}} - \mathrm{E}(\arcsin z|m) - (m-1)\,\mathrm{F}(\arcsin z|m)\right)$$

with $\operatorname{cd}(z|m)$ is the Jacobi elliptic function defined by

$$\operatorname{cd} z = \frac{\operatorname{cn} z}{\operatorname{dn} z},$$

we can express the derivative of the function $\operatorname{arcsn}(\varpi|m)$ with respect to a using the chain rule

$$\begin{aligned}\frac{\partial \operatorname{arcsn}\varpi}{\partial a} &= \frac{\partial \operatorname{arcsn}\varpi}{\partial \varpi}\frac{\partial \varpi}{\partial a} + \frac{\partial \operatorname{arcsn}\varpi}{\partial m}\frac{\partial m}{\partial a} \\ &= \frac{1}{\sqrt{1-mz^2}}\left(\frac{1}{\sqrt{1-z^2}}\frac{\partial \varpi}{\partial a} + \frac{mz\sqrt{1-z^2} - \mathrm{E}(\arcsin\varpi) - (m-1)\,\mathrm{F}(\arcsin\varpi)}{2(m-1)m}\frac{\partial m}{\partial a}\right)\end{aligned}$$

Then, the general expression of the derivative of the phase φ with respect to a is

$$\frac{\partial \varphi}{\partial a} = \operatorname{sgn}(a_3 a_5)\frac{\partial \operatorname{arcsn}(\varpi|m)}{\partial a}$$

- The function $\Gamma(t)$

$$\frac{\partial \Gamma(t)}{\partial a} = t\frac{\partial r}{\partial a} + \frac{\partial \varphi}{\partial a}$$

- The Jacobi elliptic function $\operatorname{sn}(\Gamma(t)|m)$

Given the first order derivatives of the function $\operatorname{sn}(z|m)$

$$\frac{\partial \operatorname{sn}(z|m)}{\partial z} = \operatorname{cn}(u|m)\operatorname{dn}(u|m)$$
$$\frac{\partial \operatorname{sn}(z|m)}{\partial m} = \frac{\operatorname{dn}(z|m)\operatorname{cn}(z|m)\left((1-m)\,z - \mathrm{E}\left(\operatorname{am}(z|m)|m\right) + m\operatorname{cd}(z|m)\operatorname{sn}(z|m)\right)}{2m(1-m)}$$

we can compute directly

$$\begin{aligned}\frac{\partial \operatorname{sn}\Gamma(t)}{\partial a} &= \frac{\partial \operatorname{sn}\Gamma(t)}{\partial \Gamma(t)}\frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \operatorname{sn}\Gamma(t)}{\partial m}\frac{\partial m}{\partial a} \\ &= \operatorname{cn}\Gamma(t)\operatorname{dn}\Gamma(t)\left(\frac{\partial \Gamma(t)}{\partial a} + \frac{(m-1)\Gamma(t) + \mathrm{E}(\operatorname{am}\Gamma(t)) - m\operatorname{cd}\Gamma(t)\operatorname{sn}\Gamma(t)}{2m(m-1)}\frac{\partial m}{\partial a}\right).\end{aligned}$$

- The Jacobi elliptic function $\operatorname{cn}(\Gamma(t)|m)$

  Given the first order derivatives of the function $\operatorname{cn}(z|m)$

$$\begin{aligned}\frac{\partial \operatorname{cn}(z|m)}{\partial z} &= -\operatorname{sn}(u|m)\operatorname{dn}(u|m) \\ \frac{\partial \operatorname{cn}(z|m)}{\partial m} &= \frac{\operatorname{dn}(z|m)\operatorname{sn}(z|m)\left((m-1)\,z + \mathrm{E}\left(\operatorname{am}(z|m)|m\right) - m\operatorname{cd}(z|m)\operatorname{sn}(z|m)\right)}{2m(1-m)}\end{aligned}$$

  we can compute directly

$$\begin{aligned}\frac{\partial \operatorname{cn}\Gamma(t)}{\partial a} &= \frac{\partial \operatorname{cn}\Gamma(t)}{\partial \Gamma(t)}\frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \operatorname{cn}\Gamma(t)}{\partial m}\frac{\partial m}{\partial a} \\ &= -\operatorname{sn}\Gamma(t)\operatorname{dn}\Gamma(t)\left(\frac{\partial \Gamma(t)}{\partial a} + \frac{(m-1)\Gamma(t) + \mathrm{E}(\operatorname{am}\Gamma(t)) - m\operatorname{cd}\Gamma(t)\operatorname{sn}\Gamma(t)}{2m(m-1)}\frac{\partial m}{\partial a}\right).\end{aligned}$$

- The Jacobi elliptic function $\operatorname{dn}(\Gamma(t)|m)$

  Given the first order derivatives of the function $\operatorname{dn}(z|m)$

$$\begin{aligned}\frac{\partial \operatorname{dn}(z|m)}{\partial z} &= -m\operatorname{cn}(u|m)\operatorname{sn}(u|m) \\ \frac{\partial \operatorname{dn}(z|m)}{\partial m} &= \frac{\operatorname{sn}(z|m)\operatorname{cn}(z|m)\left((m-1)\,z + m\,\mathrm{E}\left(\operatorname{am}(z|m)|m\right) - m\operatorname{dn}(z|m)\operatorname{sc}(z|m)\right)}{2(1-m)}\end{aligned}$$

  with $\operatorname{sc}(z|m)$ is the Jacobi elliptic function defined by

$$\operatorname{sc} z = \frac{\operatorname{sn} z}{\operatorname{cn} z},$$

  we can compute directly

$$\begin{aligned}\frac{\partial \operatorname{dn}\Gamma(t)}{\partial a} &= \frac{\partial \operatorname{dn}\Gamma(t)}{\partial \Gamma(t)}\frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \operatorname{dn}\Gamma(t)}{\partial m}\frac{\partial m}{\partial a} \\ &= -m\operatorname{cn}\Gamma(t)\operatorname{sn}\Gamma(t)\left(\frac{\partial \Gamma(t)}{\partial a}\right. \\ &\qquad \left. + \frac{(m-1)\Gamma(t) + m\,\mathrm{E}(\operatorname{am}\Gamma(t)) - m\operatorname{dn}\Gamma(t)\operatorname{sc}\Gamma(t)}{2m(m-1)}\frac{\partial m}{\partial a}\right).\end{aligned}$$

- The elliptic integral of the second kind $\mathrm{E}\left(\operatorname{am}(\Gamma(t)|m)|m\right)$

  We first need to express the derivative of the amplitude $\operatorname{am}(\Gamma(t)|m)$ with respect to a. Given first derivatives of the Jacobi amplitude $\operatorname{am}(z|m)$

$$\begin{aligned}\frac{\partial \operatorname{am}(z|m)}{\partial z} &= \operatorname{dn}(u|m) \\ \frac{\partial \operatorname{am}(z|m)}{\partial m} &= \frac{\operatorname{dn}(z|m)\left((m-1)\,z + \mathrm{E}\left(\operatorname{am}(z|m)|m\right)\right) - m\operatorname{cn}(z|m)\operatorname{sn}(z|m)}{2m(1-m)}\end{aligned}$$

we get the derivative of $\mathrm{am}(\Gamma(t)|m)$ with respect to a by applying the chain rule as usual

$$
\begin{aligned}
\frac{\partial \,\mathrm{am}\,\Gamma(t)}{\partial a} &= \frac{\partial \,\mathrm{am}\,\Gamma(t)}{\partial \Gamma(t)}\frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \,\mathrm{am}\,\Gamma(t)}{\partial m}\frac{\partial m}{\partial a} \\
&= \mathrm{dn}\,\Gamma(t)\left(\frac{\partial \Gamma(t)}{\partial a} + \frac{(m-1)\Gamma(t) + \mathrm{E}(\mathrm{am}\,\Gamma(t)) - m\,\mathrm{cd}\,\Gamma(t)\,\mathrm{sn}\,\Gamma(t)}{2m(m-1)}\frac{\partial m}{\partial a}\right).
\end{aligned}
$$

Then, given first derivatives of the elliptic integral of the second kind $\mathrm{E}(z|m)$

$$
\begin{aligned}
\frac{\partial \,\mathrm{E}(z|m)}{\partial z} &= \sqrt{1 - m\sin^2 z} \\
\frac{\partial \,\mathrm{E}(z|m)}{\partial m} &= \frac{\mathrm{E}(z|m) - \mathrm{F}(z|m)}{2m}
\end{aligned}
$$

Noting the simplification

$$
\begin{aligned}
\frac{\partial \,\mathrm{E}\left(\mathrm{am}\,\Gamma(t)\right)}{\partial \,\mathrm{am}\,\Gamma(t)} &= \sqrt{1 - m\sin^2\left(\mathrm{am}\,\Gamma(t)\right)} \\
&= \sqrt{1 - \mathrm{sn}^2\,\Gamma(t)} \\
&= \mathrm{dn}\,\Gamma(t),
\end{aligned}
$$

and that

$$\mathrm{F}(\mathrm{am}\,\Gamma(t)) = \Gamma(t),$$

we finally have all the expressions to compute the derivative of $\mathrm{E}\left(\mathrm{am}\,\Gamma(t)\right)$ with respect to a by applying the chain rule

$$
\begin{aligned}
\frac{\partial \,\mathrm{E}\left(\mathrm{am}\,\Gamma(t)\right)}{\partial a} &= \frac{\partial \,\mathrm{E}\left(\mathrm{am}\,\Gamma(t)\right)}{\partial \,\mathrm{am}\,\Gamma(t)}\frac{\partial \,\mathrm{am}\,\Gamma(t)}{\partial a} + \frac{\partial \,\mathrm{E}\left(\mathrm{am}\,\Gamma(t)\right)}{\partial m}\frac{\partial m}{\partial a} \\
&= \mathrm{dn}^2\,\Gamma(t)\left(\frac{\partial \Gamma(t)}{\partial a} + \frac{\mathrm{E}(\mathrm{am}\,\Gamma(t)) - \mathrm{cd}\,\Gamma(t)\,\mathrm{sn}\,\Gamma(t)}{2(m-1)}\frac{\partial m}{\partial a}\right)
\end{aligned}
$$

Then, most of these forms simplify in the three previously introduced cases and we can give the explicit forms of the derivatives of functions $\beta_1(t)$ and $\beta_2(t)$ (and their respective integrals) with respect to a:

- Case I: $\lambda_4 > 0$

  From (2.8) and (2.9), we get

$$
\begin{aligned}
\frac{\partial \beta_1(t)}{\partial a} &= 4\,\mathrm{dn}\,\Gamma(t)\frac{\partial \,\mathrm{dn}\,\Gamma(t)}{\partial a} \\
\frac{\partial \beta_2(t)}{\partial a} &= \mathrm{dn}\,\Gamma(t)\frac{\partial \,\mathrm{sn}\,\Gamma(t)}{\partial a} + \mathrm{sn}\,\Gamma(t)\frac{\partial \,\mathrm{dn}\,\Gamma(t)}{\partial a} \\
\frac{\partial \int \beta_1(t)}{\partial a} &= \frac{2}{r}\left(\left(\frac{\partial \,\mathrm{E}(\Gamma(t))}{\partial a} - \frac{\partial \,\mathrm{E}(\Gamma_0)}{\partial a}\right) - \frac{1}{r}\Big(\mathrm{E}(\Gamma(t)) - \mathrm{E}(\Gamma_0)\Big)\frac{\partial r}{\partial a}\right) \\
\frac{\partial \int \beta_2(t)}{\partial a} &= \frac{1}{r}\left(\frac{1}{r}\left(\mathrm{cn}\,\Gamma(t) - \mathrm{cn}\,\Gamma_0\right)\frac{\partial r}{\partial a} - \left(\frac{\partial \,\mathrm{cn}\,\Gamma(t)}{\partial a} - \frac{\partial \,\mathrm{cn}\,\Gamma_0}{\partial a}\right)\right)
\end{aligned}
$$

- Case II: $\lambda_4 < 0$

From (2.10) and (2.11), we get

$$
\begin{aligned}
\frac{\partial \beta_1(t)}{\partial a} &= -4 \operatorname{sn}\Gamma(t) \frac{\partial \operatorname{sn}\Gamma(t)}{\partial a} \\
\frac{\partial \beta_2(t)}{\partial a} &= \operatorname{cn}\Gamma(t) \frac{\partial \operatorname{sn}\Gamma(t)}{\partial a} + \operatorname{sn}\Gamma(t) \frac{\partial \operatorname{cn}\Gamma(t)}{\partial a} \\
\frac{\partial \int \beta_1(t)}{\partial a} &= \frac{1}{m} \left(\frac{t(1-m)}{m} \frac{\partial m}{\partial a} + \frac{2}{r} \left(\mathrm{E}(\operatorname{am}\Gamma(t)) - \mathrm{E}(\operatorname{am}\Gamma_0) \right) \left(\frac{1}{r} \frac{\partial r}{\partial a} - \frac{1}{m} \frac{\partial m}{\partial a} \right) \right. \\
&\quad \left. + \frac{2}{r} \left(\frac{\partial \, \mathrm{E}(\Gamma(t))}{\partial a} - \frac{\partial \, \mathrm{E}(\Gamma_0)}{\partial a} \right) \right) \\
\frac{\partial \int \beta_2(t)}{\partial a} &= \frac{1}{rm} \left((\operatorname{dn}\Gamma(t) - \operatorname{dn}\Gamma_0) \left(\frac{1}{m} \frac{\partial m}{\partial a} + \frac{1}{r} \frac{\partial r}{\partial a} \right) - \left(\frac{\partial \operatorname{dn}\Gamma(t)}{\partial a} - \frac{\partial \operatorname{dn}\Gamma_0}{\partial a} \right) \right)
\end{aligned}
$$

- Case III: $\lambda_4 = 0$

  From (2.12) and (2.13), we get

$$
\begin{aligned}
\frac{\partial \beta_1(t)}{\partial a} &= -2 \operatorname{sech}^2 \Gamma(t) \tanh \Gamma(t) \\
\frac{\partial \beta_2(t)}{\partial a} &= \operatorname{sech}\Gamma(t) \left(1 - 2 \tanh^2 \Gamma(t) \right) \\
\frac{\partial \int \beta_1(t)}{\partial a} &= \frac{2}{r} \left(\frac{1}{r} \frac{\partial r}{\partial a} (\tanh \Gamma(t) - \tanh \Gamma_0) - \left(\tanh^2 \Gamma(t) - \tanh^2 \Gamma_0 \right) \right) \\
\frac{\partial \int \beta_2(t)}{\partial a} &= \frac{1}{r} \left(\tanh \Gamma(t) \operatorname{sech}\Gamma(t) - \tanh \Gamma_0 \operatorname{sech}\Gamma_0 - \frac{1}{r} \left(\operatorname{sech}\Gamma(t) - \operatorname{sech}\Gamma_0 \right) \right)
\end{aligned}
$$

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