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Abstract

The dynamical formulation of the optimal transport problem, introduced by J. D. Benamou and Y. Brenier [3], corresponds to the time-space search of a density and a momentum minimizing a transport energy between two densities. In order to solve this problem, an algorithm has been proposed to estimate a saddle point of a Lagrangian. We will study the convergence of this algorithm to a saddle point of the Lagrangian, in the most general conditions, particularly in cases where initial and final densities vanish on some areas of the transportation domain. The principal difficulty of our study will consist in the proof, under these conditions, of the existence of a saddle point, and especially in the uniqueness of the density-momentum component. Indeed, these conditions imply to have to deal with non-regular optimal transportation maps. For these reasons, a detailed study of the properties of the velocity field associated to an optimal transportation map in quadratic space is required.

1 Introduction

The optimal transport problem is generally formulated as follows: considering two sets of particles or probability measures, find the assignment between those discrete or continuous objects while minimizing a given cost. This is referred to as optimal transport or optimal assignment. Even if these two denominations describe the same problem, they reflect two different approaches. Indeed, while it was initially a problem of optimal displacement, the pioneer Gaspard Monge, acknowledging the fact that the optimal trajectory from one point to another was a straight line, reduced this problem to a simple assignment problem. The same holds for the formulation given later by Leonid Kantorovitch: his problem was also reduced to a single assignment (or allocation) problem of the elements of a given resource to be transported. As such, the trajectories are not involved in the transport cost, which only reflects the price to pay to move a mass from one point to another.

The reduction of the optimal transport problem to an assignment problem first makes it easier to tackle theoretically (see [18]). However, when it comes to describe more accurately the optimal assignment plan, this formulation is less efficient. Some approaches then choose to reintroduce the notion of displacement: this will be the case of the method that we will deal with here.

The first attempt to link the optimal assignment and optimal displacement problems was proposed by R. J. McCann [21]. The continuous displacements between two measures was

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considered to determine a continuum of intermediate measures between two measures μ and ν, so that the integral sum of local optimal costs (ie the distance of Wasserstein step by step) is equal to the global optimal allocation cost between μ and ν. The idea is that for infinitely close measures, there is no difference between optimal assignment and optimal displacement.

Relying on this continuous interpolation idea between the densities ρ_0 and ρ_1 by densities ρ_t of constant mass, Y. Brenier and J.-D. Benamou introduced a continuous formulation of the optimal transport problem [3], originating from fluid mechanics. This amounts to determine directly the evolution of the density $\rho = t \mapsto \rho_t$ as well as the velocity field v which advects this density, corresponding to the optimal map $\nabla\phi$. Such a pair (ρ, v) therefore verifies a continuity equation.

Thereafter, second-order algorithms have been proposed to solve the problem efficiently [4], using resolutions of the time-continuous Monge-Ampere equation induced in [21]. However, this type of approach is nevertheless limited to non-vanishing densities. As far as we know, the only numerical method handling the general case is based on the dynamic fluid mechanic method introduced in [3].

One of the main interest of the dynamic formulation is its expression in terms of fluid mechanics quantities. It makes the model very flexible, and allows generalizations to other physical constraints [20, 8] which are relevant for practical applications. The introduction of these physical constraints (anisotropy of the domain, constraints of free divergence or rigidity on the velocity field) in the dynamic problem have been a subject of study in [17].

The main interest advanced by the authors of [3] was the ability to introduce an algorithm exploiting this physical formulation of the optimal transport problem. For this purpose, they introduced the impulsion $m = \rho v$ and reduced the problem to the search for an optimal component $\mu = (\rho^*, m^*)$ rather than (ρ^*, v^*). The objective is therefore to find the optimal velocity field v^* on the support of the density ρ^* (the velocity field v can indeed vary outside the support of ρ without modifying the value of the travel energy $\mathcal{K}$). Such a formulation makes it possible to convexify the transport energy and to linearize the mass conservation equation. The problem then becomes the following: minimizing a convex, proper, lower semicontinuous energy.

The authors of [3] also reformulates this convex dynamic problem as the search for a saddle point (ψ, q, μ) of a Lagrangian $\mathbf{L}$, and then address this new problem with an augmented Lagrangian algorithm (more precisely ADMM) such as developed in [12]. This algorithm has been used since 1999 and works well in practice, although a number of theoretical questions remain unanswered: the existence of solutions in the context of this new dynamic formulation, the existence of a saddle point for $\mathbf{L}$, the convergence (weak or strong) of the algorithm, and the possible uniqueness of a solution, in the general setting of possibly vanishing densities.

Before going further, let us first introduce the dynamic formulation of the optimal transport problem proposed by Benamou and Brenier in [3].

2 Formulation of the problem and presentation of the algorithm

2.1 The problem of Monge-Kantorovich in $\mathbb{R}^d$ for a quadratic cost

We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$, for all $d \in \mathbb{N}$, and consider two nonnegative densities (ρ_0, ρ_1) on $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), with bounded supports and of the same mass. The problem of Monge-Kantorovich consists in finding an optimal transport plan T between $\rho_0\mathcal{L}^d$ and $\rho_1\mathcal{L}^d$ that minimizes

$$\int_{\mathbb{R}^d} d(x, T(x))^2 \rho_0(x) dx, \tag{2-1}$$

where $d(x, y)$ is a distance on Ω. We write $T\#(\rho_0 \mathcal{L}^d) = \rho_1 \mathcal{L}^d$ the push of $\rho_0 \mathcal{L}^d$ by T forward $\rho_1 \mathcal{L}^d$, i.e. such that for any bounded subset A of $\mathbb{R}^d$, $\int_A \rho_0 = \int_{T^{-1}(A)} \rho_1$. The quadratic Wasserstein distance $\mathcal{W}_2(\rho_0, \rho_1)$ is defined by:

$$\mathcal{W}_2(\rho_0, \rho_1)^2 = \inf_{T\#(\rho_0 \mathcal{L}^d)=\rho_1 \mathcal{L}^d} \int_\Omega d(x, T(x))^2 \rho_0(x) dx. \tag{2-2}$$

In the Euclidean case (where $d(x, y) = |x - y|$), there exists a unique transport map T between ρ_0 and ρ_1 that can be written as the gradient of a lower semi-continuous (l.s.c.) convex function ϕ (Brenier's Theorem [24] p.66) i.e.

$$\mathcal{W}_2(\rho_0, \rho_1)^2 = \int_{\mathbb{R}^d} |\nabla\phi(x) - x|^2 \rho_0(x)\, dx = \inf_{T\#(\rho_0 \mathcal{L}^d)=\rho_1 \mathcal{L}^d} \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x)\, dx. \tag{2-3}$$

This problem, in the dynamic formulation of the Monge problem introduced by J. -D. Benamou and Y. Brenier [3], can be reformulated as a minimization problem of a kinetic energy $\mathcal{K}$, depending on a mass ρ and a velocity field v, such that ρ is transported from ρ_0 to ρ_1, by v.

Let us begin by detailing this new optimization problem in a framework that will be convenient for its resolution by the augmented Lagrangian algorithm, and which will be the main object of our study.

2.2 Convex and augmented lagrangian formulation

We propose to study the following problem: let $\rho_0, \rho_1 \in L^2(\mathbb{R}^d)$ be two probability densities with bounded supports. The dynamic optimal transport formulation consists in increasing the dimension of the problem by adding a temporal variable $t \in [0, 1]$. Formally, we look for a couple (ρ, v), where ρ denotes a nonnegative density, and v a vector field with values in $\mathbb{R}^d$, both defined on $(0, 1) \times \Omega$, where Ω is a bounded open convex set of $\mathbb{R}^d$ containing $\mathrm{supp}(\rho_0)$ and $\mathrm{supp}(\rho_1)$. This couple is required to satisfy the continuity equation,

$$\partial_t \rho + \mathrm{div}(\rho v) = 0 \tag{2-4}$$

with homogeneous Neumann boundary conditions on ρv, and with initial and final conditions on ρ:

$$\rho(0, x) = \rho_0(x), \;\; \rho(1, x) = \rho_1(x). \tag{2-5}$$

Among all such couples $(\rho\ , v)$, we look for a minimizer of $\mathcal{K}(\rho, v) = (1/2) \int_0^1 \int_\Omega |v|^2 \rho\, dx\, dt$.

As $\mathcal{K}$ is not convex, and the constraint (2-4) is nonlinear, the authors of [3] proposed as a new variable $m = \rho v$ instead of v, and consider the transport cost:

$$\widetilde{\mathcal{K}}(\rho, m) = \int_0^1 \int_\Omega \frac{|m(t, x)|^2}{2\rho(t, x)}\, dx\, dt, \tag{2-6}$$

with the corresponding continuity constraint:

$$\partial_t \rho + \mathrm{div}_x\, m = 0; \tag{2-7}$$

that are subject to homogeneous Neumann boundary conditions on m and initial/final conditions (2-5). The nonnegativity constraint on ρ turns to $\{\rho > 0$ or $(\rho, m) = (0, 0)\}$ through the change of variable $m = \rho v$. By introducing a Lagrange multiplier ψ to handle the linear constraints (2-7) and (2-5), we can write a saddle-point formulation of the problem:

$$\inf_{(\rho,m)} \sup_\psi \left[\int_{(0,1)\times\Omega} \frac{|m|^2}{\rho} - \int_{(0,1)\times\Omega} (\partial_t \psi \rho - \nabla_x \psi \cdot m) + \int_\Omega (\psi(0, \cdot)\rho_0 - \psi(1, \cdot)\rho_1) \right]. \tag{2-8}$$

Another crucial idea in [3] is to encode the non-negativity constraint on ρ by introducing the Legendre transform of $(\rho, m) \mapsto |m|^2/(2\rho)$:

$$F(q) = F(a,b) = \sup_{(\rho,m)} \left(\rho a + \langle m, b \rangle - \frac{|m|^2}{2\rho} \right) \quad \Leftrightarrow \quad F(q) = \begin{cases} 0 \text{ if } q \in \mathcal{P} \\ +\infty \text{ otherwise} \end{cases}$$

with $q = (a, b) \in \mathbb{R} \times \mathbb{R}^d$ and $\mathcal{P} = \{(a, b) \in \mathbb{R} \times \mathbb{R}^d, a \leq -|b|^2/2\}$. Since the transport cost $\widetilde{\mathcal{K}}$ is now convex and l.s.c., it is equal to its biconjugate by the Legendre transform. We therefore have $|m|^2/(2\rho) = \sup_{(a,b)} (\rho a + m \cdot b - F(a, b))$. The problem is thus partially linearized with respect to the variables (ρ, m): the non-linear part (i.e. F) reduces to the indicator function of $\mathcal{P}$, which will be implemented as a projection on that convex subset.

By some manipulations as sup-integral or inf-sup inversions, and by setting $q = (a, b)$ and $\mu = (\rho, m)$, the saddle point problem (2-8) is reformulated as $\inf_{(\psi,q)} \sup_\mu L(\psi, p, \mu)$ where

$$\mathbf{L}(\psi, p, \mu) = F(q) + G(\psi) + \langle \mu, \nabla_{t,x}\psi - q \rangle_{L^2} \tag{2-9}$$

with $G(\psi) = \int_\Omega \psi(0, \cdot)\rho_0 \, dx - \int_\Omega \psi(1, \cdot)\rho_1 \, dx$ and $F = \chi_{\widetilde{\mathcal{P}}}$ (meaning $F(q) = 0$ if $q \in \widetilde{\mathcal{P}}$ and $F(q) = +\infty$ otherwize), where $\widetilde{\mathcal{P}} = \{(\widetilde{a}, \widetilde{b}) \in L^2(Q) \times L^2(Q)^d, \widetilde{a} \leq -|\widetilde{b}|^2/2\}$. In the following we will write $\mathcal{P}$ in place $\widetilde{\mathcal{P}}$. The augmented Langrangian formulation is finally given, for some parameter $r > 0$, by:

$$\mathbf{L}_r(\psi, q, \mu) = F(q) + G(\psi) + \langle \mu, \nabla_{t,x}\psi - q \rangle_{L^2} + \frac{r}{2}\|\nabla_{t,x}\psi - q\|^2_{L^2}. \tag{2-10}$$

2.3 Benamou-Brenier algorithm

To solve the saddle point problem associated to (2-10), the authors of [3] have proposed an algorithm based on a Uzawa method: the ALG2 algortihm introduced by M. Fortin and R. Glowinski in [12]. This consists in performing the following steps, starting from $(\psi^{n-1}, q^{n-1}, \mu^n)$:

1. Step A: Find the unique ψ^n such that $\mathbf{L}_r(\psi^n, q^{n-1}, \mu^n) \leq \mathbf{L}_r(\psi, q^{n-1}, \mu^n), \ \forall \psi$.
2. Step B: Find the unique $q^n = (a^n, b^n)$ such that $\mathbf{L}_r(\psi^n, q^n, \mu^n) \leq \mathbf{L}_r(\psi^n, q, \mu^n), \ \forall q$.
3. Step C: Update (ρ^{n+1}, m^{n+1}), setting $\mu^{n+1} = \mu^n + r(\nabla_{t,x}\psi^n - q^n)$,

More precisely, the algorithm breaks down as follows: Step A can be interpreted as a projection on gradient fields. We look for the unique $\psi^n \in H^1(Q)/\mathbb{R}$ such that:

$$\forall h \in H^1(Q), \ G(h) + \langle \mu^n, \nabla_{t,x}h \rangle + r\langle \nabla_{t,x}\psi^n - q^{n-1}, \nabla_{t,x}h \rangle = 0.$$

Formally, this corresponds to find ψ^n solution of $-r\Delta_{t,x}\psi^n = \mathrm{div}_{t,x}\left(\mu^n - rq^{n-1}\right)$, with as initial and final conditions:

$$r\partial_t\psi^n(0, \cdot) = \rho_0 - \rho^n(0, \cdot) + ra^{n-1}(0, \cdot), \text{ and } r\partial_t\psi^n(1, \cdot) = \rho_1 - \rho^n(1, \cdot) + ra^{n-1}(1, \cdot),$$

and homogeneous Neumann boundary conditions on $(0, 1) \times \partial\Omega$. This operation corresponds to a kind of Helmoltz decomposition.

Step B is an L^2 orthogonal projection on $\mathcal{P}$: $q^n = P_{\mathcal{P}}\left((1/r)\mu^n + \nabla_{t,x}\psi^n\right)$, that can be done pointwise.

Step C uses the computed gradient of step A to implement a projection on the affine space of constraints (2-4) and (2-5): $\mu^{n+1} = \mu^n + r(\nabla_{t,x}\psi^n - q^n)$.

Remark 2.1. *We chose to take the same parameter $r > 0$ for the Uzawa step C to ensure the positivity constraint of ρ^n and cancellation of m^n when ρ^n vanishes. Indeed step B can be rewritten as $\forall q' \in \mathcal{P}$ $\langle \mu^{n+1}, q' - q^n \rangle \leq 0$, which means that $\mu^{n+1} = (\rho^{n+1}, m^{n+1})$ is orthogonal to the paraboloid $\mathcal{P}$ at q^n. We deduce by the strict convexity of $\mathcal{P}$, that for all $(t, x) \in (0, 1) \times \Omega$, $\rho^n(t, x) > 0$ or $(\rho^n(t, x), m^n(t, x)) = 0$.*

2.4 Objectives and related existing works

The main object of this article is to propose a theoretical framework allowing to answer the three following points: existence of saddle points, uniqueness of saddle points and convergence of the considered algorithm. The proposed study will also be the opportunity to characterize rigorously some properties of the velocity field associated to an optimal transport plan.

A first study of the Benamou-Brenier algorithm was carried out in [14] for periodic in space boundary condition: $\Omega = \mathbb{T}^d$, where $\mathbb{T}^d$ denotes the torus in dimension d, i.e. $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The strongest assumption of this study is that the density ρ^*, solving the problem (2-6), has to be larger than a positive constant. This assumption imply in particular a regularity of the associated transport plan (discontinuity free). Indeed, under such conditions, the potential ϕ must be of class C^1 and with Lipschitz gradient. Caffarelli studied in [6] and [7] the regularity of an optimal transport plan on a convex domain with respect to the regularity of the initial and final densities ρ_0 and ρ_1, additionally assumed to be positive. A special case mentioned in [14] is ρ_0 and ρ_1 strictly positive on $\mathbb{T}^d$ and belonging to $C^{\alpha,l}(\mathbb{T}^d)$, for some $l \in (0,1)$, and $\alpha \in \mathbb{N}^d$. Following the work of Cordero-Erausqui in [10], these conditions imply that the optimal transport potential ϕ is of class $C^{\alpha,l+2}$ and, for any $t \in [0,1]$, the density $\widetilde{\rho}_t$ also has a $C^{\alpha,l}$ regularity on $\mathbb{T}^d$ and is bounded from below by a strictly positive constant independent of t.

Under the above assumptions, the author of [14] first shows the existence of a solution (ρ^*, m^*) for the dynamic formulation of optimal transportation; solution from which is proven the existence of a saddle point (ψ^*, q^*, μ^*) for the Lagrangian $\mathbf{L}$. However, there is no uniqueness result for the density-momentum couple $\mu^* = (\rho^*, m^*)$.

A convergence result of the Benamou-Brenier algorithm is also presented in [14]. Nevertheless, this does not explicitly give the strong or weak convergence of the main component of the triplet (ψ_n, q_n, μ_n). Indeed, considering the problem (2-6), the component of interest is the density-momentum component $\mu_n = (\rho_n, m_n)$. Moreover, only the strong convergences in $H^1/\mathbb{R}$ and L^2 of the components ψ_n and q_n are shown and the proof seems incomplete (see section 9).

In this article, we consider a more general framework. The open set Ω will here be assumed to be convex and bounded, with homogeneous Neumann boundary conditions on the momentum m, but more importantly, the density $\widetilde{\rho}$ **will not be assumed to be minored** by a strictly positive regular constant. We will simply assume that the densities ρ_0 and ρ_1 are non-negative elements of $L^2(\Omega)$ (thus potentially non-regular) in the neighborhood of $\partial\Omega$. We will show the existence of a saddle point for the Lagrangian $\mathbf{L}$, solution of the problem (2-6), as well as the uniqueness among the set of saddle points (ψ^*, q^*, μ^*) of the Lagrangian $\mathbf{L}$ of $\mu^* = (\rho^*, m^*)$, which shows that the density corresponds to the McCann interpolation. The uniqueness result established in this article only concerns the component μ, since, as we will see at the beginning of the section 6, **there is no uniqueness of the saddle points of $\mathbf{L}$**: in fact, the components ψ and q can vary outside the support of ρ.

These first two points (existence and uniqueness) will be established in parallel with a preliminary study on the regularity of the velocity field v associated to an optimal transport. We underline important new regularity results presented in Theorem 8.2 and Corollary 8.2: $\nabla_{t,x} v \in L^p(0,1; L^q_{loc}(\mathbb{R}^d))$ for all $1 < p \leq +\infty$ and $1 \leq q < +\infty$ such that $1/p + 1/q > 1$, and especially $v \in W^{1,p}_{loc}([0,1] \times \mathbb{R}^d)$ (i.e. $\forall \Omega \subset \mathbb{R}^d$, $v \in W^{1,p}((0,1) \times \Omega)$) for all $1 \leq p < 2$.

This study will also lead us to characterize accurately a velocity field inherent to an optimal transport in L^2. More precisely, we will determine sufficient assumptions on a velocity field $v \in L^\infty_{loc}([0,1] \times \mathbb{R}^d)$ so that a density transported by v is the McCann interpolation of an optimal (unique) transport (Theorem 7.1). These hypotheses will be reduced to the usual characteristics of optimal isotropic transport, in particular straight-line trajectories, at constant speed, and without crossing. Finally, we will study the convergence of the Benamou-Brenier algorithm.

2.5 Organization of the paper

Our study will be structured as follows. In section 3 we will start by developing the different challenges of our problem concerning the existence of the saddle points for $\mathbf{L}$, as well as the uniqueness properties. We will give a detailed statement of the various properties characterizing a saddle point. These properties will be exploited one by one in the continuation of our study in order to characterize the couple density-velocity field (ρ, v).

In section 4, we will carry out a preliminary study of the properties of the velocity field. It will give crucial materials for the following three sections, in which we will establish the existence of a saddle point (section 5), the uniqueness of the component $\mu = (\rho, m)$ (section 6), and finally characterize *a minima* a velocity field which represents an optimal transport (section 7).

In order to simplify the reading of this paper, some results on velocity fields used in sections 5, 6 and 7 will be latter proven in the section 8, which also contains the statements and proofs of the new and independent regularity results of Theorem 8.2 and Corollary 8.2.

Finally, in section 9, we will establish the weak and strong convergence of the Benamou-Brenier algorithm towards a saddle point of the Lagrangian $\mathbf{L}$, which can be interpreted as the search for a fixed point of a non-expansive operator. This method had been used to study the convergence of many "splitting algorithms" (see for instance [19, 9], or for an overview [2]).

3 Characterization of a saddle point and statement of the mains results

The main objective of this first part is to directly build a saddle point of $\mathbf{L}$ defined in (2-10). Let us therefore define the framework rigorously: let ρ_0 and ρ_1 be two probability densities (i.e. non-negative and of integral equal to 1) of $L^2(\mathbb{R}^d)$ with bounded supports, and Ω be a bounded convex open set of $\mathbb{R}^d$. We assume that Ω is piecewise of class C^1 and such that $\mathrm{supp}(\rho_0) \cup \mathrm{supp}(\rho_1) \subset \Omega$. In the remaining of this paper, we denote $Q = (0,1) \times \Omega$. For all $r > 0$, we write $\mathbf{L}_r^{ps}(\rho_0, \rho_1, \Omega)$ the set of Lagrangian $\mathbf{L}_r$'s saddle points which are elements of $S = H^1(Q)/\mathbb{R} \times L^2(Q)^{d+1} \times L^2(Q)^{d+1}$. Let us define the following three properties for a triplet $(\psi, q, \mu) \in S$ of $\mathbf{L}_r$:

Properties (I). *$(\psi, q, \mu) \in S$ verifies:*

(I)$_1$ $\forall q' \in \widetilde{\mathcal{P}},\ \langle \mu, q' - q\rangle \leq 0,$

(I)$_2$ $\forall h \in H^1(Q), \quad G(h) + \langle \mu, \nabla_{t,x} h\rangle = 0,$

(I)$_3$ $\nabla_{t,x}\psi = q.$

where the paraboloid $\widetilde{\mathcal{P}}$ is defined by

$$\widetilde{\mathcal{P}} = \left\{ (a,b) \in L^2(Q) \times L^2(Q)^d,\ a + \frac{|b|^2}{2} \leq 0 \right\}, \tag{3-11}$$

and the linear operator G by $G(h) = \displaystyle\int h(0,\cdot)\rho_0\, dx - \int h(1,\cdot)\rho_1\, dx$, for all $h \in H^1(Q)$.

Proposition 3.1. *A saddle point $(\psi^*, q^*, \mu^*) \in S$ of $\mathbf{L}_r$ is characterized by the properties (I), for all $r \geq 0$.*

Sketch of the proof: We first check that for a triplet $(\psi^*, q^*, \mu^*) \in S$ satisfying the properties (I), we have the relation $\mathbf{L}_r(\psi, q, \mu^*) \geq \mathbf{L}_r(\psi^*, q^*, \mu^*) \geq \mathbf{L}_r(\psi^*, q^*, \mu)$, for all $(\psi, q, \mu) \in S$, which characterizes a saddle point of $\mathbf{L}_r$. Conversely, for a saddle point $(\psi^*, q^*, \mu^*) \in S$ of $\mathbf{L}$, one verifies one by one the properties (I), first by fixing $\psi = \psi^*$ and $q = q^*$ $(\mathrm{I})_3$, then fixing $q = q^*$ and $\mu = \mu^* (\mathrm{I})_2$, and finally by fixing $\mu = \mu^*$ and $\psi = \psi^*$ $(\mathrm{I})_1$.

Since the saddle points of the Lagrangian $\mathbf{L}_r$ are independent of $r \geq 0$, we will only consider the Lagrangian $\mathbf{L}$. □

By setting $\mu = (\rho, m)$ and $q = (a, b)$ (with $a + \frac{|b|^2}{2} \leq 0$), we can reinterpret the properties $(\mathrm{I})_1$ and $(\mathrm{I})_2$. Property $(\mathrm{I})_1$ means that if $\mu(t, x)$ is nonzero then it is orthogonal to the paraboloid at the point $q(t, x)$, i.e. **co-linear** to the vector $(1, b(t, x))$. The property $(\mathrm{I})_1$ can be translated as follows: $\rho \geq 0$, $m = \rho b$, $\rho(a + |b|^2/2) = 0$. Next, $(\mathrm{I})_2$ corresponds to **the mass conservation equation** verified by ρ and b (i.e. $\partial_t \rho + \mathrm{div}_x(\rho v) = 0$, taking $v = b$) for the initial and final densities ρ_0 and ρ_1.

We now recall that according to Brenier's Theorem [24] (p. 66), there exists a convex potential ϕ verifying $\rho_1 \mathcal{L}^d = \nabla\phi \#(\rho_0 \mathcal{L}^d)$ from which we define the following quantities:

Definition 3.1. *For all $t \in (0, 1)$, we define:*

1. *The characteristic displacement at the instant t,*

$$X(t, \cdot) = (1 - t)\,\mathrm{id} + t\nabla_x \phi = \nabla_x \phi_t \quad \text{with } \phi_t = (1 - t)\frac{|\cdot|^2}{2} + t\phi, \tag{3-12}$$

2. *The associated velocity field v,*

$$v(t, \cdot) = \frac{\mathrm{id} - \nabla_x(\phi_t)^*}{t} \quad \text{with } \phi_t = (1 - t)\frac{|\cdot|^2}{2} + t\phi, \tag{3-13}$$

where $(\phi_t)^* = (\phi_t)^*$ denotes the Legendre transform of the potential ϕ_t.

3. *The density ρ, defined as the union for $t \in [0, 1]$ of the McCann interpolation densities ρ_t between $\rho_0 \mathcal{L}^d$ and $\rho_1 \mathcal{L}^d$ [21]:*

$$\rho(t, \cdot)\mathcal{L}^d = \rho_t \mathcal{L}^d = X(t, \cdot)\#(\rho_0 \mathcal{L}^d). \tag{3-14}$$

Let us also define the (Γ_1) property on the potential ϕ:

Hypothesis (Γ_1)**.** *ϕ and ϕ^* are convex, continuous and achieve a minimum on $\mathbb{R}^d$.*

Here ϕ^* always represents the Legendre transform of ϕ and we recall that a convex and continuous function on $\mathbb{R}^d$ is locally Lipschitz. For the purpose of our study, we complete the Brenier's Theorem ([24] p. 66) as follows:

Proposition 3.2. *Let ρ_0 be a probability density Lebesgue-measurable on $\mathbb{R}^d$ and μ_1 a probability measure on $\mathbb{R}^d$. There exists a potential $\phi : \mathbb{R}^d \to \mathbb{R}$, satisfying the property (Γ_1), such as $\nabla\phi \# \left(\rho_0 \mathcal{L}^d\right) = \mu_1$.*

Sketch of the proof: One can first show that the optimal transport potential ϕ given by the Brenier's Theorem (convex, lower semicontinuous and gradient bounded almost everywhere on $\mathrm{supp}(\rho_0 \mathcal{L}^d)$) is finite and with a bounded gradient on an open neighborhood of the support of $\rho_0 \mathcal{L}^d$. It is then possible to extend the restriction of ϕ to this neighborhood by $\overline{\phi}$, a finite convex function on $\mathbb{R}^d$ continuous, supralinear and sub-quadratic. The supralinearity of $\overline{\phi}$ implies the existence of a global minimum for the latter, and ensures that its Legendre transform $\overline{\phi}^*$ will also be finite (and thus continuous) on $\mathbb{R}^d$. The sub-quadratic character of $\overline{\phi}$ ensures the supralinearity of $\overline{\phi}^*$, and therefore the existence of a global minimum. □

We can now define the following set:

Definition 3.2. *Let ρ_0 be a probability density Lebesgue-measurable on $\mathbb{R}^d$ and μ_1 a probability measure defined on the tribe of Lebesgue in $\mathbb{R}^d$. We denote by $\Phi(\rho_0\mathcal{L}^d, \mu_1)$ the set of functions $\phi : \mathbb{R}^d \to \mathbb{R}$ satisfying the property (Γ_1) such that $\nabla\phi\#\left(\rho_0\mathcal{L}^d\right) = \mu_1$.*

3.1 Reformulation of the Properties (I)

Let us give an idea of the approach we will follow. We set $\mu = (\rho, \rho v)$ and $q = (-\frac{1}{2}|v|^2, v)$. To construct a saddle point, μ and q should satisfy the properties (I). It will be sufficent to verify that with ρ and v defined like in (3-14) and (3-13) satisfy the following properties:

Properties (I)$'$**.**

(I$'$)$_1$ *The density ρ is an element of $\rho \in L^2(Q)$.*

(I$'$)$_2$ *The velocity field v is an element of $L^\infty(Q)^d$.*

(I$'$)$_3$ *The velocity field v satisfies the Burgers equation in the sense of the distributions:*

$$\partial_t v + \frac{1}{2}\nabla_x |v|^2 = 0.$$

(I$'$)$_4$ *The potential (ρ, v) satisfies the mass conservation equation in the distributions sense for the initial and final conditions ρ_0 and ρ_1 and the homogeneous Neumann boundary conditions:*

$$\int_Q (\partial_t h + v.\nabla h)\,\rho + \int_\Omega h(0,x)\rho_0(x)dx - \int_\Omega h(1,x)\rho_1(x)dx = 0 \tag{3-15}$$

We will see that ρ and v satisfying the properties (I)$'$ is sufficient to build a triplet (ψ, q, μ) satisfying properties (I). However, having a triplet (ψ, q, μ) satisfying the properties (I) is not sufficient to build a density-velocity field pair (ρ, v) satisfying properties (I)$'$, and such that $\mu = (\rho, \rho v)$ and $q = (-\frac{1}{2}|v|^2, v)$. Indeed, the component q may not belong to the boundary of the paraboloid $(a, b) \to a + (1/2)|b|^2 \leq 0$ outside the support of μ.

The properties (I$'$)$_1$ and (I$'$)$_2$, respectively established in Lemma 5.1 and 4.2, ensure that the saddle point is in the correct space, i.e. in S. Indeed, we have

$$q \in L^2(Q)^{d+1} \;\Leftrightarrow\; v \in L^4(Q)^d \subset L^\infty(Q)^d,\;\; \mu \in L^2(Q)^{d+1} \;\Leftrightarrow\; \rho \in L^2(Q) \text{ and } \rho v \in L^2(Q)^d,$$

and, for the potential ψ, we have $W^{1,\infty}(Q) \subset H^1(Q)$.

The properties (I$'$)$_2$ and (I$'$)$_3$ involve the property (I)$_3$. Indeed, having q deriving from a space-time potential amounts to verifying, for a dimension $d \leq 2$, that $\mathrm{curl}_{t,x}(q) = 0$ (recalling that $q = (-\frac{1}{2}|v|^2, v)$) in the sense of distributions (see [13] Theorem 2.9 p.31), so that

$$\begin{cases} \partial_t v + \frac{1}{2}\nabla|v|^2 = 0, \\ \mathrm{curl}_x(v) = 0 \;\Leftrightarrow\; \exists\psi \in \mathcal{D}'(Q), v = \nabla_x\psi \end{cases}$$

From Definition of v in (3-13), we see that the velocity **derives from a potential in space** in the sense of the distributions, namely:

$$v(t,\cdot) = \nabla_x\left(\frac{1}{t}\left(\frac{1}{2}|\cdot|^2 - (\phi_t)^*\right)\right) \tag{3-16}$$

where the potential ϕ is an element of $L^1_{loc}(Q)$. According to Lemma 4.1, this proves the property $(\mathrm{I})_3$, provided that the field v is an element of $L^\infty(Q)^d$ and verifies the Burgers equation

$$\partial_t v + \frac{1}{2}\nabla_x |v|^2 = 0$$

in the sense of distributions.

Even if the notion of rotational is less easy to cope with in dimension $d > 2$, Lemma 4.1 allows us to state the property $(\mathrm{I})_3$ from $(\mathrm{I}')_2$ and $(\mathrm{I}')_3$, whatever the dimension d is, provided that we have $v \in L^\infty(Q)^d$.

Notice that the property $(\mathrm{I}')_4$ translates the property $(\mathrm{I})_2$ of (I). Indeed, we can easily extend by a density argument the relation to $h \in H^1(Q)$ once it is established for $h \in C^\infty(\overline{Q})$. Finally remark that with the above results, the Property $(\mathrm{I})_1$ is verified by setting $m = \rho v$.

3.2 Main results of existence, uniqueness and regularity

Let Ω be an open set of $\mathbb{R}^d$. We say that v satisfies properties (II) if and only if:

Properties (II)**.**

1. *There exists $\psi \in W^{1,\infty}_{loc}([0,1] \times \Omega)$ (i.e. $\psi \in W^{1,\infty}((0,1) \times \omega)$, for all bounded open set $\omega \subset \Omega$) such that $v = \nabla_x \psi$.*

2. *The velocity field v satisfies* ***the Burgers equation*** *in the sense of the distributions, namely the relation*
$$\partial_t v + \frac{1}{2}\nabla_x |v|^2 = 0. \tag{3-17}$$

According to Lemma 4.1 that will be stated below, the properties (II) are equivalent to the following ones:

1. $v \in L^\infty_{loc}([0,1] \times \Omega)^d$ i.e. $v \in L^\infty((0,1) \times \omega)^d$, for all bounded open set $\omega \subset \Omega$,

2. $\partial_t v + \frac{1}{2}\nabla_x |v|^2 = 0$ (in the sense of distributions),

3. there exists $\psi \in L^1_{loc}((0,1) \times \Omega)$, such that $v = \nabla_x \psi$,

which correspond to the properties $(\mathrm{I}')_2$ and $(\mathrm{I}')_3$.

The properties (II) contain the characteristics of an isotropic optimal transport for a quadratic cost: the first point (i.e. $v = \nabla_x \psi$) corresponds to the property of **non crossing trajectories** (recalling that in dimension less than 3 this property is equivalent to a rotational free velocity field v); and the second point (the Burgers equation) is in line with the property of **straight-line displacement**.

At the end of section 7, we will give a framework in which we can rigorously characterize an optimal transport-type mass displacement from these properties alone. The principal results of existence and uniqueness we show are the followings.

Theorem 3.1 (Existence of a saddle point)**.** *Let ρ_0 and ρ_1 be two probability densities of $L^2(\mathbb{R}^d)$ with bounded supports, and let Ω be a sufficiently smooth bounded open set of $\mathbb{R}^d$ such that* $\mathrm{supp}(\rho_0) \cup \mathrm{supp}(\rho_1) \subset \Omega$. *For all $\phi \in \Phi(\rho_0\mathcal{L}^d, \rho_1\mathcal{L}^d)$ (see relation 3.2), there is a non-negative density $\rho_\phi \in C^0\left([0,1], L^2(\Omega)\right)$, such that for all $[0,1] \times \mathbb{R}^d$, we have*

$$\rho_\phi(t,\cdot)\mathcal{L}^d = X_\phi(t,\cdot)\#(\rho_0\mathcal{L}^d), \text{ avec } X_\phi(t,\cdot) = \nabla_x \phi_t = (1-t)\,\mathrm{id} + t\nabla_x \phi,$$

and a velocity field v_ϕ, defined by

$$v_\phi(t,x) = \frac{x - \mathbf{p}_\phi(t,x)}{t}, \tag{3-18}$$

Then, setting $\mu_\phi = (\rho_\phi, \rho_\phi v_\phi)$ and $q_\phi = (-(1/2)|v_\phi|^2, v_\phi)$, there exists $\psi_\phi \in W^{1,\infty}_{loc}([0,1]\times\Omega)$ such that $q_\phi = \nabla_{t,x}\psi_\phi$ and such that $(\psi_\phi, q_\phi, \mu_\phi)$ (or at least its restriction on $(0,1)\times\Omega$) is a saddle point Lagrangian $\mathbf{L}$. In addition, v_ϕ is locally Lipschitz on the space $(0,1)\times\mathbb{R}^d$, satisfies the properties (II) and $\nabla_{t,x}v \in L^\infty(0,1;L^1(\Omega))$, with $v \in W^{1,p}((0,1)\times\Omega)$ for all $1 \le p < 2$.

As we will see in section 7, the fact that v satisfies the properties (II) is sufficient to characterize an optimal transport in L^2. On the other hand, the fact that v verifies $\nabla_{t,x}v \in L^\infty(0,1;L^1(\Omega))$ is a result: this property, although interesting in itself, will not be directly used to characterize an optimal transport velocity field in L^2. However, very close properties will be considered to show the different statements on the uniqueness of the component (ρ, m) of the saddle points of $\mathbf{L}$, and the results related to the characterization of an optimal transport-type velocity field.

Theorem 3.2 (Unicity of density and momentum)**.** *If the triplet (ψ^*, q^*, μ^*) is a saddle point of $\mathbf{L}$ (the assumptions on ρ_0, ρ_1 and Ω being the same as in Theorem 3.1), then for any potential $\phi \in \Phi(\rho_0\mathcal{L}^d, \rho_1\mathcal{L}^d)$, we have $\mu^* = (\rho^*, m^*) = (\rho_\phi, \rho_\phi v_\phi)$, with the velocity field v_ϕ defined with respect to ϕ as in (3-18), and*

$$\rho_\phi(t,\cdot)\mathcal{L}^d = X_\phi(t,\cdot)\#(\rho_0\mathcal{L}^d) \in C([0,1], L^2(\Omega)), \text{ with } X_\phi(t,\cdot) = \nabla_x\phi_t = (1-t)\,\mathrm{id} + t\nabla_x\phi.$$

In general, the set of saddle points (ψ, q, μ) of $\mathbf{L}$ **is not reduced** to a single element: only the component $\mu = (\rho, m)$ is unique. In other words, the set of points (ψ, q, μ) of $\mathbf{L}$ share the same component μ, i.e. there is uniqueness of the density ρ and the velocity field v on the support of ρ. The components q and ψ can indeed vary outside the support of ρ. For more details, see subsection 4.2.1 of [15].

To prove these two results, we will have to study in details some properties of a velocity field v_ϕ defined as in (3-18), for ϕ satisfying the property (Γ_1).

4 First velocity field properties

In this section, we define a velocity field associated to an optimal transport map using Brenier's Theorem, and give associated properties about it. The results stated in this section will constitute **the basis** of the existence and uniqueness results concerning the saddle points of $\mathbf{L}$, as well as the generalized results of the section 7. Let us begin by introducing the notion of infimal convolution, or **inf-convolution**:

Definition 4.1 (inf-convolution ([2] chapter 12))**.** *Let f and g be two functions from $\mathbb{R}^d$ to $]-\infty,+\infty]$. The inf-convolution of f and g, denoted by $f\square g$, is defined by*

$$f\square g : \mathbb{R}^d \to]-\infty,+\infty] : x \longmapsto \inf_{y\in\mathbb{R}^d}\{f(x-y)+g(y)\}. \tag{4-19}$$

In the remaining of our paper, we will need the following property, in conjunction with the Legendre transform ([2] chapter 13): for all functions f and g from $\mathbb{R}^d$ to $]-\infty,+\infty]$, we have

$$(f\square g)^* = f^* + g^*. \tag{4-20}$$

We also recall the definition of **the proximal operator**:

Definition 4.2 (The proximal operator *([2] chapter 12)*). *Let f be a function of $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) in $\mathbb{R}$ proper, l.s.c. and convex, and let $x \in \mathbb{R}^n$. The proximal operator of f in x, denoted by* $\mathrm{Prox}_f(x)$ *is the unique minimizer of $f + \frac{1}{2}|x - \cdot|^2$ in $\mathbb{R}^n$. In other words:*

$$\mathrm{Prox}_f(x) = \operatorname*{argmin}_{y \in \mathbb{R}^n} \left(f(y) + \frac{1}{2}|y - x|^2 \right). \tag{4-21}$$

The proximal operator can be characterized by the following relation:

$$y = \mathrm{Prox}_f(x) \;\Leftrightarrow\; x - y \in \partial f(y). \tag{4-22}$$

The operators Prox_f *and* $\mathrm{id} - \mathrm{Prox}_f$ *are non-expansive (1-Lipschitz). Let us recall the identity of Moreau (linking the proximal operator of f with that of its Legendre transform f^*):*

$$\mathrm{Prox}_{\gamma f^*} = \mathrm{id} - \gamma\, \mathrm{Prox}_{f/\gamma}(\cdot/\gamma) \tag{4-23}$$

We have here defined the operator on $\mathbb{R}^n$, but it can be defined on more general spaces (Hilbert spaces for example), with the same properties. Let us finally define the Moreau envelope:

Definition 4.3 (Moreau envelope *([2] chapter 12)*). *Let $f : \mathbb{R}^d \longrightarrow]-\infty, +\infty]$ convex, l.s.c. proper and let $\gamma > 0$. The Moreau envelope of f with parameter γ is defined by:*

$$^{\gamma}f = f \Box \left(\frac{1}{2\gamma} |\cdot|^2 \right). \tag{4-24}$$

By definition of the proximal operator, we can also characterize $^{\gamma}f$, for all $x \in \mathbb{R}^d$ and $\gamma > 0$, by:

$$^{\gamma}f(x) = f(\mathrm{Prox}_{\gamma f(x)}) + \frac{1}{2\gamma}|x - \mathrm{Prox}_{\gamma f}(x)|^2 \tag{4-25}$$

where $^{\gamma}f$ is convex and Fréchet-differentiable on $\mathbb{R}^d$. Using (4-23), its gradient reads:

$$\nabla(^{\gamma}f) = \gamma^{-1}(\mathrm{id} - \mathrm{Prox}_{\gamma f}) = \mathrm{Prox}_{f^*/\gamma}(\cdot/\gamma). \tag{4-26}$$

The mapping $\nabla(^{\gamma}f)$ is γ^{-1}-Lipschitz. Moreover, for every $x \in \mathbb{R}^d$, from (4-26) and (4-22), we have

$$\nabla(^{\gamma}f)(x) \in \partial f(\mathrm{Prox}_{\gamma f}(x)). \tag{4-27}$$

4.1 Definition and first properties of the velocity field

Thanks to Definition 4.3, the velocity field (3-13) of an optimal transport can be written as a proximal operator $\mathbf{p}$. This will allow us to deal more easily with the problems of "breaks" of the velocity field (which are not necessarily discontinuities). An interesting property of this proximal opertor is that it realizes a bijection in the regular areas of the velocity, while being able to close the potential "breaks" of the velocity.

Definition 4.4 (Operator $\mathbf{p}$). *Let $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying the property (Γ_1) (especially ϕ is convex and continuous at every point of $\mathbb{R}^d$, and admiting in each of these point a non-empty and compact sub-differential). The operator $\mathbf{p}$ is defined as*

$$\begin{array}{cccc} \mathbf{p}_\phi : & [0,1) \times \mathbb{R}^d & \to & \mathbb{R}^d \\ & (t,x) & \longmapsto & \mathrm{Prox}_{\frac{t}{1-t}\phi}\left(\frac{x}{1-t}\right) \end{array}$$

$\mathbf{p}_\phi$ *satisfies the following properties:*

1. *for all* $t \in [0,1)$, $\mathbf{p}_\phi(t,\cdot)$ *is* $1/(1-t)$*-Lipschitz,*

2. *if* $t \in (0,1)$, *by setting* $\phi_t = (1-t)|\cdot|^2/2 + t\phi$, *then* $(\phi_t)^*$ *is of class* C^1 *on* $\mathbb{R}^d$ *and we have:*
$$\mathbf{p}_\phi(t,\cdot) = \nabla_x(\phi_t)^*, \tag{4-28}$$

3. *for all* $t \in [0,1)$, $\mathbf{p}_\phi(t,\cdot)$ *is surjective on* $\mathbb{R}^d$ *and for all* $x, y \in \mathbb{R}^d$,
$$y = \mathbf{p}_\phi(t,x) \;\Leftrightarrow\; x \in (1-t)y + t\partial\phi(y). \tag{4-29}$$

4. *for all* $t \in (0,1)$ *and* $x \in \mathbb{R}^d$, *the velocity* v *introduced in* (3-13) *can be defined from* $\mathbf{p}_\phi$ *by:*
$$v_\phi(t,x) = \frac{x - \mathbf{p}_\phi(t,x)}{t}. \tag{4-30}$$

Remark 4.1. *When there is no ambiguity on* ϕ, *we will use* v *to denote the velocity field* v_ϕ.

Proof: The first point simply results from the non-expansiveness of the proximal operator (Definition 4.2). The second point can be established by observing that the Legendre transform $(\phi_t)^*$ of ϕ_t can be written in the form of a Moreau envelope. Indeed, through the property (4-20) of the inf-convolution we have $(\phi_t)^* = {}^{1-t}(t\phi)^*$. Using relation (4-26), we can then deduce that for all $t \in (0,1)$, $(\phi_t)^*$ is of class C^1 on $\mathbb{R}^d$, and
$$\nabla_x(\phi_t)^* = \frac{\mathrm{id} - \mathrm{Prox}_{(1-t)(t\phi)^*}}{1-t} = \mathrm{Prox}_{\frac{t}{1-t}\phi}\left(\frac{\cdot}{1-t}\right) = \mathbf{p}_\phi(t,\cdot). \tag{4-31}$$
The third point is immediately deduced from the characterization (4-22) of the proximal operator. Regarding this last point, we could also remark that we have
$$\forall x, y \in \mathbb{R}^d,\;\; y = \mathbf{p}_\phi(t,x) = \nabla_x(\phi_t)^*(x) \Leftrightarrow x \in \partial_x(\phi_t)(y) = (1-t)y + t\partial\phi(y).$$
Finally, the fourth point comes by combining relations (4-28) and (3-13). □

We now recall, for all $t \in (0,1)$, that the field of trajectories X is defined in (3-12) by
$$X(t,\cdot) = \nabla_x(\phi_t) = (1-t)\,\mathrm{id} + t\nabla\phi.$$
We thus observe that $\mathbf{p}_\phi(t,\cdot)$ formally represents the reciprocal of the characteristic traces $X(t,\cdot) = \nabla_x(\phi_t)$. It would have been really the case if ϕ had been of class C^1 with a Lipschitz gradient. But in the general case (i.e. with ϕ not C^1 and only verifying the property (Γ_1)), $\mathbf{p}_\phi(t,\cdot)$ is not injective on $\mathbb{R}^d$. The operator $\mathbf{p}_\phi(t,\cdot)$ thus repairs the "breaks" that can be generated by a transport plan. Indeed, $\mathbf{p}_\phi(t,\cdot)$ re-concentrates the areas generated by diffusion (by the characteristic trajectories $X(t,\cdot)$) of the break points on these same points. Thus $\mathbf{p}_\phi(t,\cdot)$ can be bijective only in the case where there are no "breaks" in the transport plan.

Next, we can deduce from (4-30) and the first property of Definition 4.4, that for $t \in (0,1)$ fixed, the velocity field $v_\phi(t,\cdot)$ is Lipschitz on $\mathbb{R}^d$. It is also possible to define a Lipschitz constant that is only time dependent so that it does not depend on ϕ. The Lipschitz constant
$$L_t = 2/t(1-t) \tag{4-32}$$
is for instance always valid on $\mathbb{R}^d$ (for the Euclidean norm $|\cdot|$), whatever ϕ is. The field of velocity v_ϕ is therefore continuous and Fréchet-differentiable almost everywhere in space (by Rademacher's Theorem 8.1), and thus $\|\nabla_x v_\phi(t,x)\|$ is additionally **uniformly bounded** by $L_t = 2/t(1-t)$ for almost all $x \in \mathbb{R}^d$, where $\|\cdot\|$ denotes the subordinate norm to $|\cdot|$.

Using the reformulation of v_ϕ in Definition 4.4, we finally deduce the following property on the velocity field.

Proposition 4.1. *We assume that ϕ satisfies the property* (Γ_1). *For every $t \in (0,1)$, and for every $y \in \mathbb{R}^d$ such that ϕ is a Fréchet-differentiable in y (for almost all $y \in \mathbb{R}^d$), we have :*

$$\nabla\phi(y) - y = \partial_t X(t,y) = v_\phi(t, X(t,y)), \tag{4-33}$$

with $X(t,\cdot) = \nabla\phi_t = \nabla[(1-t)\,\mathrm{id} + t\nabla\phi]$.

Proof: Let us take $y \in \mathbb{R}^d$ such that ϕ is differentiable at y (i.e. $\partial\phi(y) = \{\nabla\phi(y)\}$). Note that according to (4-29), we have $y = \mathbf{p}_\phi\left(t, (1-t)y + t\nabla\phi(y)\right) = \mathbf{p}_\phi(t, X(t,y))$. The equation (4-33) can be deduced immediately from (4-30). $\square$

The above proposition can also be reformulated as follows: given that $X(t,\cdot) = \nabla\phi_t$ and $\mathbf{p}_\phi(t,\cdot) = \nabla_x(\phi_t)^*$(from 4-28), then for any $t \in (0,1)$, and for all $x \in \mathbb{R}^d$ such that ϕ Frechet-differentiable in x, we have $\left(\nabla_x(\phi_t)^* \circ \nabla\phi_t\right)(x) = x$.

Remark 4.2. *The" break" points of the transport plan correspond to the points where the potential ϕ is not differentiable. Although Theorem 8.1 ensures that the set of such points is negligible, the diffusion of these breaks, and in particular the torsions/high variations of the velocity field at these points in $t = 0$ (or $t = 1$ if we consider the points of irregularity of ϕ^*) is not. Indeed, the torsions of the velocity field in the neighborhood of break points may* ***induce a loss of H^1 regularity of the velocity field at these points****. Notice that a H^1 regularity of the potential ϕ would have greatly simplified the study discussed in section 6 on the uniqueness of the saddle points of the Lagrangian* $\mathbf{L}$. *Unfortunately, such regularity can not be obtained in general.*

4.2 Velocity field control

In this subsection, we show some properties of the velocity field v defined in (4-30). In particular, we demonstrate that v is in the space $L^\infty_{loc}([0,1] \times \mathbb{R}^d)^d$ (Proposition 4.2), and therefore satisfies the property $(\mathrm{I}')_2$.

Lemma 4.1. *Let us consider a field of velocity $v \in L^\infty(Q)^d$ (property $(\mathrm{I}')_2$), satisfying the property $(\mathrm{I}')_3$ (the Burgers equation in the sense of distributions) for which there exists a potential $\psi \in L^1_{loc}(Q)$, such that $v = \nabla_x\psi$.*

Then there exists a potential $\psi^\star \in W^{1,\infty}(Q)$, satisfying in the sense of distributions the Hamilton-Jacobi equation $\partial_t\psi^\star + (1/2)|\nabla_x\psi^\star|^2 = 0$ and for which, by setting $q = (-(1/2)|v|^2, v)$, we have $q = \nabla_{t,x}\psi^\star$.

Proof: In the sense of distributions, we have

$$\partial_t v + \frac{1}{2}\nabla_x|v|^2 = 0 \Leftrightarrow \partial_t(\nabla_x\psi) + \frac{1}{2}\nabla_x|v|^2 = 0, \Leftrightarrow \nabla_x\left(\partial_t\psi + \frac{1}{2}|v|^2\right) = 0. \tag{4-34}$$

There exists a distribution T depending only on $t \in (0,1)$, such that $\partial_t\psi + \frac{1}{2}|v|^2 = T$ (see Theorem 2.16 in [25]). We set $\psi^\star = \psi - G$, where G is a primitive distribution of T on $(0,1)$ (and only depends on t). We then verify, in the sense of the distributions, that $\nabla_x\psi^\star = \nabla_x\psi = v \in L^\infty(Q)^d$, and $\partial_t\psi^\star = -(1/2)|v|^2 \in L^\infty(Q)$. We recall that the open set Ω is assumed to be regular. Since $\nabla_{t,x}\psi^\star \in L^\infty(Q)^{d+1}$, we then have $\psi^\star \in W^{1,\infty}(Q)$ (see Lemma 4.1-11 of [15]). $\square$

Proposition 4.2 (Property $(\mathrm{I}')_2$). *We assume that ϕ satisfies the property* (Γ_1). *Let a velocity field v be defined with respect to ϕ as in (4-30). We then have $v \in L^\infty_{loc}([0,1] \times \mathbb{R}^d)^d$. More precisely, for any bounded open set $\omega \subset \mathbb{R}^d$, if we define constants $M = \sup_{x\in\omega}|\partial\phi(x)|$ and $M^* = \sup_{x\in\omega}|\partial\phi^*(x)|$, we have*

$$\sup_{(t,x)\in(0,1)\times\omega} |v(t,x)| \leq 5\left(\max\{M, M^*\} + \sup(\omega)\right). \tag{4-35}$$

Moreover, there exists $\psi \in W^{1,\infty}_{loc}([0,1] \times \Omega)$*, such that* $v = \nabla_x \psi$*.*

Sketch of the proof: We show that $v(t,\cdot)$ is uniformly bounded on ω in the neighborhood of $t=0$. Take for example $t \in (0, 1/2]$ and $y \in \omega$, and let $x \in (1-t)y + t\partial\phi(y)$. According to (4-29), we have $\mathbf{p}_\phi(t,x) = y$, so $v(t,x) = (x-y)/t \in \partial\phi(y) - y$.

We have already seen in (4-32) that for the Euclidean norm $|\cdot|$, $v(t,\cdot)$ is $2/t(1-t)$-Lipschitz in space on $\mathbb{R}^d$ for $t \in (0,1)$, so

$$|v(t,y) - v(t,x)| \leq \frac{2}{t(1-t)}|x-y| \leq 4\left|\frac{x-y}{t}\right| = 4|v(t,x)|. \tag{4-36}$$

We have $|v(t,y)| \leq 5|v(t,x)| \leq 5\left(M + \sup(\omega)\right)$ (by (4-36)) for all $t \in (0,1/2]$ and all $y \in \omega$, with $M = \sup_{x\in\omega}|\partial\phi(x)|$. The same argument can be used in the neighborhood of $t=1$ on $[1/2, 1)$. For the second point, we recall that according to the last point of Definition 4.4, we have $v(t,\cdot) = (\mathrm{id} - \nabla_x(\phi_t)^*)/t$, with $\phi_t = (1-t)|\cdot|^2/2 + t\phi$. Then $v = \nabla_x\psi'$ in the sense of distributions, with $\psi'(t,x) = (1/t)\left(|\cdot|^2/2 - (\phi_t)^*\right)$, for all $t \in (0,1)$ and for all $x \in \mathbb{R}^d$. We thus deduce that $v \in L^\infty([0,1]\times\mathbb{R}^d)^d$. The existence of a potential $\psi \in W^{1,\infty}_{loc}([0,1]\times\mathbb{R}^d)$ such that $v = \nabla_x\psi$ can be obtained by applying Lemma 4.1. $\square$

In this case, M and M^* are finite. Indeed, as ϕ is assumed to satisfy the property (Γ_1), ϕ and ϕ^* are assumed to be finite and convex on $\mathbb{R}^d$ and therefore locally Lipschitz, in particular Lipschitz on ω. Thus, $\partial\phi$ and $\partial\phi^*$ are uniformly bounded on ω. Note also that if ϕ verifies the property (Γ_1), then, as $(\phi^*)^* = \phi$, ϕ^* also satisfies this property.

The difficulty in the proof of the latter proposition comes from the fact that the interpolated transport plans $\nabla\phi_t$ are not necessarily invertible (i.e. ϕ is not necessarily of class C^1 with Lipschitz gradient): otherwise the field v would have been extendable by continuity in $t=0$ and $t=1$ (see Proposition 8.1) and the result obvious. However, in the general case, a transport can induce a change in topology between the supports of the initial and final masses, that is to say admitting "breaks" and therefore points of non-regularity for the potential ϕ.

With respect to the initial saddle point problem, we have

$$q = \left(-(1/2)|v|^2, v\right) \in L^\infty(Q)^{d+1} \subset L^2(Q)^{d+1}.$$

As already stated in (4-32), for every $t \in (0,1)$, $v(t,\cdot)$ is continuous and Lipschitz on $\mathbb{R}^d$ (by providing $\mathbb{R}^d$ with Euclidean norm, one can take $2/t(1-t)$ as the Lipschitz constant). The field $v(t,\cdot)$ is therefore Lipschitz on $\mathbb{R}^d$, for a Lipschitz constant independent of t on any interval $[\alpha,\beta] \subset (0,1)$. One can for instance consider the constant $M_{\alpha,\beta} = \sup_{[\alpha,\beta]} 2/t(1-t)$. It is therefore possible to apply the Cauchy-Lipschitz Theorem on $[\alpha,\beta]$. Then, for every $x \in \mathbb{R}^d$ and $t \in (0,1)$, the Cauchy problem

$$\begin{cases} y'_{t,x} = v(\cdot, y_{t,x}) \\ y_{t,x}(t) = x, \end{cases} \tag{4-37}$$

admits a unique maximum solution over any interval (α,β), $0 < \alpha < t < \beta < 1$. We can then easily prove that there exists a unique solution defined on $(0,1)$ and that it can be written $y_{t,x}(s) = (s-t)v(t,x) + x$ for all $s \in (0,1)$. Indeed, such a solution satisfies $y_{t,x}(t) = x$, and $y'_{t,x}(s) = v(t,x) = v(s,(s-t)v(t,x)+x) = v(s, y_{t,x}(s))$, as stated in the next Proposition 4.3.

Proposition 4.3. *We assume that* ϕ *satisfies the property* (Γ_1)*. Then for all* $t,s \in (0,1)$*, and for all* $x \in \mathbb{R}^d$*,*

$$v(t,x) = v(s,(s-t)v(t,x)+x).$$

Sketch of the proof: It can be shown using the properties (4-29) and (4-30) of the operator $\mathbf{p}$. $\square$

In the above proof, the hypothesis (Γ_1) is only used for the conditions on ϕ, not ϕ^*, so that ϕ admits a non-empty and compact sub-differential at all points of $\mathbb{R}^d$.

It should also be noticed that the problem of Cauchy:

$$\begin{cases} y' = v(\cdot, y) \\ y(t) = X(t,x) = (1-t)x + t\nabla\phi(x), \end{cases} \tag{4-38}$$

has $y(s) = (s-t)v(t, X(t,x)) + X(t,x) = X(s,x)$ as unique solution on $(0,1)$. It is now important to show that the field v is locally Lipschitz on $(0,1)\times\mathbb{R}^d$, i.e. in **time-space** (and not just in space).

Proposition 4.4. *We suppose that ϕ satisfies the property (Γ_1). Then v is locally Lipschitz on the space $(0,1)\times\mathbb{R}^d$.*

Sketch of the proof: For all $(t_1,x_1),(t_2,x_2)\in[\alpha,\beta]\times\mathbb{R}^d$, we have, according to Proposition 4.3, the relation $v(t_1,x_1) = v\left(t_2,(t_2-t_1)v(t_1,x_1)+x_1\right)$. We can then conclude by the Lipschitz property (in space) of the field $v(t,\cdot)$ (we can deal with a same time t_2), with the Lipschitz constant $2/t(1-t)$ for the Euclidean norm (see (4-32)). □

We finally state the following proposition, proved in section 8, that ensures property $(\mathrm{I}')_3$.

Proposition 4.5 (Property $(\mathrm{I}')_3$)**.** *With the property (Γ_1), v satisfies the Burger's equation (3-17), that is to say, in the distribution sense:*

$$\partial_t v + \frac{1}{2}\nabla_x|v|^2 = 0,$$

which is a generalized form of $\partial_t v + v\cdot\nabla_x v = 0$.

5 Existence of a saddle point

In order to prove the existence of a saddle point for the Lagrangian $\mathbf{L}$, we have have built a couple density-velocity field (ρ, v) satisfying conditions $(\mathrm{I})'$. The velocity field $v = v_\phi$, defined in (4-30), immediately satisfies the properties $(\mathrm{I}')_2$ and $(\mathrm{I}')_3$, according respectively from Proposition 4.2 and Proposition 4.5. We now have to build a density $\rho = \rho_\phi$, satisfying the property $(\mathrm{I}')_1$ (i.e. $\rho\in L^2(Q)$), and such that the couple (ρ_ϕ, v_ϕ) satisfies the mass conservation in condition $(\mathrm{I}')_4$. The candidate densityis naturally the density (3-14) of the McCann interpolation between $\rho_0\mathcal{L}^d$ and $\rho_1\mathcal{L}^d$. Let us first define more accurately the notion of "pushforward measure".

Properties $(\mathcal{M})$. *Let Ω be an open set of $\mathbb{R}^d$, and let ν be a measure on the Lebesgue tribe of Ω. We said that ν satisfies the properties $(\mathcal{M})$ if and only if $\nu(K) < +\infty$ for any compact $K\subset\Omega$; $\nu(E) = \inf\{\nu(V),\ E\subset V,\ V \text{ open set}\}$ for every Lebesgue-measurable set E of Ω; $\nu(E) = \sup\{\nu(K),\ K\subset E,\ K \text{ compact}\}$ for any E open set and for any Lebesgue-measurable set E of Ω such that $\nu(E) < +\infty$.*

Proposition 5.1. *Let μ be a σ-finite positive measure on $\mathbb{R}^d$, and $T:\mathbb{R}^d\to\mathbb{R}^d$ measurable. We assume that the measure μ is finite. Then there exists a positive measure ν on $\mathbb{R}^d$, satisfying the properties $(\mathcal{M})$, such that*

$$\forall f\in C_c^0(\mathbb{R}^d),\qquad \int_{\mathbb{R}^d} f\,d\nu = \int_{\mathbb{R}^d} f(Tx)\,d\mu. \tag{5-39}$$

*Moreover, for every Lebesgue-measurable set $A\subset\mathbb{R}^d$, we have $\nu(A) = \mu\left(T^{-1}(A)\right)$. We then say that ν is **the pushforward measure of** μ by the operator T, denoted $\nu = T\#\mu$.*

The equation $\nu(A) = \mu\left(T^{-1}(A)\right)$ translates the fact that ν **conserves the mass** measured by μ: ν gives to any displaced, deformed, contracted or dilated area by the operator T the same mass than given by μ before applying the operator. The notion of pushforward measure thus translates a property of **mass conservative transport**. This is partly at the origin of the idea of a dynamic formulation of the optimal transport problem. As stated in the introduction, this dynamic formulation implies that we replace the "optimal conservative assignment" approach with that of an "optimal conservative displacement", where we study the evolution of a density ρ between ρ_0 and ρ_1 on a time scale $[0,1]$. The natural candidate density that we consider is therefore the one formed by the set of intermediate measurements between $\rho_0\mathcal{L}^d$ and $\rho_1\mathcal{L}^d = (\nabla\phi\#\rho_0\mathcal{L}^d)$, which can be assimilated to a series of "optimal micro-transports" along the time scale $[0,1]$. It corresponds to the interpolation density of McCann (3-14), defined at each instant t by the density $\rho_t = \rho_t^\phi$ of the measure

$$\rho_t\mathcal{L}^d = \rho_t^\phi\mathcal{L}^d = [(1-t)\,\mathrm{id} + t\nabla\phi]\#(\rho_0\mathcal{L}^d) = \nabla\phi_t\#(\rho_0\mathcal{L}^d). \tag{5-40}$$

The following proposition ensures that it is possible to choose the representatives of each of these densities ρ_t so that the density $(t,x)\mapsto\rho_t(x)$ is measurable and such that the weak formulation of the pushforward measure (5-39) remains valid for test functions which are only measurable. Indeed, the test functions involved in the weak formulation of the Benamou-Brenier algorithm are of type L^p.

Proposition 5.2. *Let $\rho_0 \in L^1(\mathbb{R}^d)$ be a compact support, such that $\rho_0 \geq 0$, and $\phi : \mathbb{R}^d \to \mathbb{R}$ satisfying the (Γ_1) property. Then, for all $t \in [0,1)$ there exists a positive measure ν_t on $\mathbb{R}^d$, with bounded support in $(t\nabla\phi + (1-t)\,\mathrm{id})\,(\mathrm{supp}(\rho_0))$, satisfying the properties $(\mathcal{M})$, and such that $\nu_t = (t\nabla\phi + (1-t)\,\mathrm{id})\#(\rho_0\mathcal{L}^d)$, i.e.*

$$\forall f \in C_c^0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f\,d\nu_t = \int_{\mathbb{R}^d} f(t\nabla\phi(x) + (1-t)x)\rho_0(x)\,d\mathcal{L}^d(x) \tag{5-41}$$

Moreover, there exists $\rho_t \in L^1(\mathbb{R}^d)$ such that $\nu_t = \rho_t\mathcal{L}^d$ ($\nu_t \ll \mathcal{L}^d$). It is also possible, for any $t \in [0,1)$, to choose a representative of ρ_t in such a way that $(t,x)\mapsto\rho_t(x)$ is measurable on $[0,1)\times\mathbb{R}^d$.

Finally, the following properties are satisfied:

1. *For all $h \in L^\infty_{loc}(\mathbb{R}^d)$ and $t \in [0,1)$, we have $h\circ(t\nabla\phi + (1-t)\,\mathrm{id}) \in L^\infty_{loc}(\mathbb{R}^d)$, and*

$$\int_{\mathbb{R}^d} h\,d\nu_t = \int_{\mathbb{R}^d} h(x)\rho_t(x)\,d\mathcal{L}^d(x) = \int_{\mathbb{R}^d} h(t\nabla\phi(x) + (1-t)x)\rho_0(x)\,d\mathcal{L}^d(x).$$

2. *For all $h \in L^\infty_{loc}([0,1)\times\mathbb{R}^d)$, the function $(t,x)\mapsto h\left(t, t\nabla\phi(x) + (1-t)x\right)$ is in the space $L^\infty_{loc}([0,1]\times\mathbb{R}^d)$, and*

$$\int_0^1\int_{\mathbb{R}^d} h(t,x)\rho_t(x)\,d\mathcal{L}^d(x)\,d\mathcal{L}(t) = \int_0^1\int_{\mathbb{R}^d} h\left(t, t\nabla\phi(x) + (1-t)x\right)\rho_0(x)\,d\mathcal{L}^d(x)\,d\mathcal{L}(t).$$

3. *For all $h \in C_c^0([0,1)\times\mathbb{R}^d)$, $t\mapsto\int_{\mathbb{R}^d} h(t,\cdot)\,d\nu_t = \int_{\mathbb{R}^d} h\,\rho_t\,d\mathcal{L}^d$ is continuous on $[0,1)$, in other words $t\mapsto\nu_t$ is continuous from $[0,1)$ to $\mathcal{D}'(\mathbb{R}^d)$.*

We do not present the technical proofs of the last two propositions, which are useless to the understanding of our purpose. However, we can find the proofs in the Appendix [16] associated with this article.

These statements are nevertheless important. Indeed, a property true "almost everywhere" for a measurable function, such as for instance the a.e.-boundness of a L^∞ function, does not necessarily still hold true when we compose this function on the right with another one. In our situation, we have to ensure that the image of a negligible set by $t\nabla\phi + (1-t)\,\mathrm{id}$ remains negligible. Similarly, the fact that for every $t \in [0,1)$, there exists a measurable spatial density ρ_t for $\mathcal{L}^d$ for a measure ν_t does not necessarily ensure the possibility to choose for each t a representative $\widetilde{\rho}_t$ of ρ_t such that the spatio-temporal density $(t,x) \mapsto \widetilde{\rho}_t(x)$ is measurable for $\mathcal{L}^{d+1}$. These propositions therefore justify the use of theorems such as Fubini.

Coming back to the problem of existence of a saddle point, we have to prove the property $(\mathrm{I}')_1$, which state that our candidate density, the McCann interpolation density between $\rho_0\mathcal{L}^d$ and $\rho_1\mathcal{L}^d$, is an element of $L^2(Q)$.

Lemma 5.1 (Property $(\mathrm{I}')_1$). *Let $1 < p < +\infty$. Let $\rho_0, \rho_1 \in L^p(\mathbb{R}^d)$ be two densities on $\mathbb{R}^d$ with bounded support, et let $\phi : \mathbb{R}^d \to \mathbb{R}$ convex such that $\rho_1\,\mathcal{L}^d = (\nabla\phi)\#(\rho_0\,\mathcal{L}^d)$. Let $t \mapsto \rho_t$ be the McCann interpolation between ρ_0 and ρ_1, defined in (5-40), which existence and space-time mesurability is justified in Proposition 5.2.*
*Then we have **strong** continuity $t \mapsto \rho_t \in C^0\left([0,1], L^p(\mathbb{R}^d)\right)$.*

Proof: For all $1 < p < +\infty$, we introduce the functional $\mathcal{F}_p : \mathcal{P}_p(\mathbb{R}^d) \to [0,+\infty]$ defined for all $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ as

$$\mathcal{F}_p(\mu) = \left\{ \begin{array}{l} \displaystyle\int_{\mathbb{R}^d} |f(x)|^p \, d\mathcal{L}^d(x) \text{ if } \mu = f.\mathcal{L}^d \in \mathcal{P}_p(\mathbb{R}^d), \\ +\infty \text{ else,} \end{array} \right. \tag{5-42}$$

where $\mathcal{P}_p(\mathbb{R}^d)$ is defined as the space of probability measures μ on $\mathbb{R}^d$ satisfying the condition $\int_{\mathbb{R}^d} |x|^p \, d\mu(x) < +\infty$. Such a functional has been classified in [21] under the term "internal energy" of the space $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$. It is "geodesically convex" on the space $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$, in other words it is convex along the geodesics of this space, which are the interpolations of McCann. Thus, the function $\Lambda_p : [0,1] \to [0,+\infty]$, defined for all $t \in [0,1]$ by $\Lambda_p(t) = \mathcal{F}_p(\rho_t\mathcal{L}^d)$, for $t \mapsto \rho_t$ defined in (5-40), is convex on $[0,1]$, and is moreover finite in $t = 0$ and $t = 1$, since $\rho_0, \rho_1 \in L^p(\mathbb{R}^d)$, and $\rho_0, \rho_1 \in \mathcal{P}_p(\mathbb{R}^d)$). Hence it is finite and bounded on the whole interval $[0,1]$. It is therefore clear, by definition of Λ_p, that $\rho_t \in L^p(\mathbb{R}^d)$ for every $t \in [0,1]$. Moreover, $t \mapsto \|\rho_t\|_{L^p(\mathbb{R}^d)}$ is bounded on $[0,1]$ by a constant M. From the Proposition 5.2, we thus see that $t \mapsto \rho_t$ is weakly continuous by $[0,1]$ in $L^p(\mathbb{R}^d)$. It is then sufficient to use the density of $C^0_c(\mathbb{R}^d)$ in $L^q(\mathbb{R}^d)$, for $q \in (1,+\infty)$ such that $1/p + 1/q = 1$. Since the function Λ_p is convex and finite on $[0,1]$, $t \mapsto \Lambda(t) = \|\rho_t\|^p_{L^p(\mathbb{R}^d)}$ is continuous on $(0,1)$ and admits a right limit in $t = 0$ and a left limit in $t = 1$. Thus, for any $t_0 \in (0,1)$, we have $\lim \|\rho_t\|^p_{L^p(\mathbb{R}^d)} = \|\rho_{t_0}\|^p_{L^p(\mathbb{R}^d)}$, and, with respect to the right limit of Λ in $t = 0$ and its left limit in $t = 1$, we have

$$\lim_{t\to 0^+} \Lambda(t) = \lim_{t\to 0^+} \|\rho_t\|^p_{L^p(\mathbb{R}^d)} \le \Lambda(0) = \|\rho_0\|^p_{L^p(\mathbb{R}^d)},$$

$$\text{and } \lim_{t\to 1^-} \Lambda(t) = \lim_{t\to 1^-} \|\rho_t\|^p_{L^p(\mathbb{R}^d)} \le \Lambda(1) = \|\rho_1\|^p_{L^p(\mathbb{R}^d)}.$$

For all $t_0 \in [0,1]$, we thus have $\limsup_{t\to t_0} \|\rho_t\|_{L^p(\mathbb{R}^d)} \le \|\rho_{t_0}\|_{L^p(\mathbb{R}^d)}$. Using Proposition 3.30 in [25], we can conclude that the application $t \mapsto \rho_t$ is strongly continuous from $[0,1]$ to $L^p(\mathbb{R}^d)$. $\square$

Conversely, one can rigorously characterize the McCann interpolation by the relation

$$\forall \varphi \in C^0_c([0,1]\times\mathbb{R}^d), \int_{(0,1)\times\mathbb{R}^d} h\,\rho\,dx \otimes dt = \int_0^1 \int_{\mathbb{R}^d} h\left(t, t\nabla\phi(x) + (1-t)x\right)\rho_0(x)\,dx\,dt. \tag{5-43}$$

Indeed, using Fubini's Theorem and Lemma 5.1 (for extreme bounds $t_{min} = 0$ and $t_{max} < 1$), it can be shown that for any density ρ verifying (5-43), there exists a family of density $(\rho_t)_{t\in[0,1)}$ as defined in Proposition 5.2, such that $t \mapsto \rho_t \in C^0\left([0,1), L^p(\mathbb{R}^d)\right)$ and such that $\rho(t,x) = \rho_t(x)$ for almost all $(t,x) \in [0,1) \times \mathbb{R}^d$ (see Lemma 4.1-5 of [15]).

Remark 5.1. *By Brenier's Theorem ([24] p. 66), we have* $\mathrm{supp}(\rho_1) = \overline{\nabla\phi(\mathrm{supp}(\rho_0))}$. *This property also holds for the potentials satisfying the property* (Γ_1) *considered in Proposition 3.2. Thus, for* Ω *a convex open set of* $\mathbb{R}^d$ *containing* $\mathrm{supp}(\rho_0)$ *and* $\mathrm{supp}(\rho_1)$, *we have the inclusion* $(t\nabla\phi + (1-t)\,\mathrm{id})(\mathrm{supp}(\rho_0)) \subset \Omega$, *for all* $t \in [0,1]$. *The weak formulation of the McCann interpolation* (5-43) *clearly shows that if a test function* h *has its support disjoint of* $\overline{Q}$, *then* $\int_0^1\int_{\mathbb{R}^d} h\rho\,dx\,dt = 0$: *the support of* $\rho : (t,x) \mapsto \rho_t(x)$ *is therefore included in the set* $[0,1]\times\Omega \subset \overline{Q}$.

We have proved that the candidate density $\rho : (t,x) \mapsto \rho_t(x)$, defined in (5-40), satisfies the condition $(\mathrm{I}')_1$. The above paragraph now ensures that the component $\mu = (\rho, \rho v)$ is zero in the neighborhood of the space boundary, and thus verifies the Neumann conditions implicitly included in the weak form of mass conservation $(\mathrm{I}')_4$. The following proposition aims to prove that the pair (ρ, v) satisfies the condition $(\mathrm{I}')_4$.

Proposition 5.3 (Property $(\mathrm{I}')_4$)**.** *Let* Ω *be a convex open set of* $\mathbb{R}^d$. *Let* ρ_0 *be a probability density and* μ_1 *a probability measure such that* $\mathrm{supp}(\rho_0), \mathrm{supp}(\mu_1) \subset \Omega$, *and such that there exists* $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ *satisfying the property* (Γ_1), *with* $\mu_1 = \nabla\phi\#(\rho_0\mathcal{L}^d)$. *Let* $\rho : (t,x) \mapsto \rho_t(x)$ *and* v *as defined in* (5-40) *and* (4-30). *Then* (ρ, v) *satisfies the mass conservation relation* (3-15)*:*

$$\forall h \in C^\infty(\overline{Q}),\ \int_Q (\partial_t h + v\cdot\nabla_x h)\,\rho\,dx\,dt + \int_\Omega h(0,\cdot)\rho_0\,dx - \int_\Omega h(1,\cdot)\,d\mu_1 = 0. \tag{5-44}$$

Proof: Remark 5.1: the inclusion $\mathrm{supp}(\rho) \subset [0,1]\times\Omega$, permits us to conclude for the homogeneous Neumann boundary conditions, i.e. to extend the space of test functions $C_c^\infty(Q)$ to the space $C^\infty(\overline{Q})$. We recall that by Proposition 4.1, for all $t \in (0,1)$, and for almost all $x \in \mathbb{R}^d$, we have $\partial_t X(t,x) = v(t, X(t,x))$, therefore for all $h \in C^\infty(\overline{Q})$,

$$\begin{aligned}\int_0^1\int_{\mathbb{R}^d} (\partial_t h + v\cdot\nabla_x h)\,\rho\,dx\,dt &= \int_0^1\int_{\mathbb{R}^d} (\partial_t h(t,X(t,\cdot)) + \partial_t X(t,\cdot)\cdot\nabla_x h(t,X(t,\cdot)))\,\rho_0\,dx\,dt\\ &= \int_0^1 \frac{d}{dt}\int_{\mathbb{R}^d} h(t,X(t,\cdot))\rho_0\,dx\,dt = \int_{\mathbb{R}^d} h(1,\nabla_x\phi(x))\rho_0(x)\,dx - \int_{\mathbb{R}^d} h(0,x)\rho_0(x)\,dx.\end{aligned} \tag{5-45}$$

Since $\mu_1 = \nabla\phi\#(\rho_0\mathcal{L}^d)$, we have $\int_{\mathbb{R}^d} h(1,\nabla_x\phi)\rho_0\,dx = \int_\Omega h(1,\cdot)\,d\mu_1$. The integrals are well defined as $v \in L^\infty(Q)$ from Proposition 4.2. □

Corollary 5.1. *Under the hypotheses of Proposition 5.3, we choose* $\mu_1 \ll \mathcal{L}^d$, *i.e.* $\mu_1 = \rho_1\mathcal{L}^d$, *and we assume that* $\rho_0, \rho_1 \in L^p(\mathbb{R}^d)$ *for* $p \geq 2$ *(we then have* $\rho \in L^p((0,1)\times\mathbb{R}^d)$*, according to Lemma 5.1).*

Then, by taking $q \leq 2$ *such that* $1/p + 1/q = 1$, *the weak relation* (5-44) *extends to the test functions* $h \in W^{1,q}(Q)$.

Proof: By density of $C^\infty(\overline{Q})$ in $W^{1,q}(Q)$. □

Since all the conditions $(\mathrm{I})'$ are now satisfied, we are able to show that they imply conditions (I) and thus prove Theorem 3.1 establishing **the existence of a saddle point** for the Lagrangian $\mathbf{L}$.

Proof of Theorem 3.1: Let us remember that an element (ψ, q, μ) of $\mathbf{L}^{ps}(\rho_0, \rho_1, \Omega)$ must satisfy $(\psi, q, \mu) \in S$, as well as the properties (I) and (II).

$(\mathrm{II})_1$ First, by Proposition 4.2 $(\mathrm{I}')_2$, we know that $v_\phi \in L^\infty(Q)^d$ and, by Lemma 5.1 $(\mathrm{I}')_1$, that $t \mapsto \rho_\phi(t,\cdot) = X_\phi(t,\cdot)\#\rho_0 \in C^0\left([0,1], L^2(\Omega)\right)$.

Then $\mu_\phi = (\rho_\phi, \rho_\phi v_\phi)$, $q_\phi = (-(1/2)|v_\phi|^2, v_\phi) \in L^2(Q)$, and $\mathrm{supp}(\mu_\phi) \subset \mathrm{supp}(\rho_\phi) \subset [0,1]\times\Omega$. The homogeneous Neumann conditions on the space edges of μ are thus verified.

Moreover, by setting $\mu_\phi = (\rho_\phi, \rho_\phi v_\phi)$ and $q_\phi = (-(1/2)|v_\phi|^2, v_\phi)$, the condition $(\mathrm{II})_1$ is naturally verified. Indeed, for all $q' = (a,b) \in \mathcal{P}$ (the paraboloid defined in (3-11)), we have

$$
\begin{aligned}
\langle \mu_\phi, q' - q_\phi\rangle &= \int_0^1 \int_\Omega (a\rho_\phi + b\cdot v_\phi \rho)\, dx\, dt - \int_0^1\int_\Omega \frac{1}{2}|v_\phi|^2 \rho_\phi\, dx\, dt \\
&\leq \int_0^1 \int_\Omega \left(a + \frac{1}{2}|b|^2 + \frac{1}{2}|v_\phi|^2\right)\rho_\phi\, dx\, dt - \int_0^1\int_\Omega \frac{1}{2}|v_\phi|^2\rho_\phi\, dx\, dt \leq 0.
\end{aligned}
$$

$(\mathrm{II})_2$ The condition $(\mathrm{II})_2$ results from Proposition 5.3 $(\mathrm{I}')_4$.

$(\mathrm{II})_3$ According to Proposition 4.5, v_ϕ satisfies the Burgers equation $(\mathrm{I}')_3$ in the sense of distributions. Moreover, by (3-16), we know that v_ϕ derives from a spatial potential. Hence, v_ϕ verifies the condition of Lemma 4.1, which gives us the existence of one $\psi_\phi \in W^{1,\infty}(Q)$ such that $q_\phi = \nabla_{t,x}\psi_\phi$ $(\mathrm{II})_3$.

Finally, from above, we obtain that the triplet $(\psi_\phi, q_\phi, \mu_\phi)$ is an element of

$$
W^{1,\infty}(Q)/\mathbb{R} \times L^\infty(Q)^{d+1} \times L^2(Q)^{d+1} \subset S.
$$

From the property $\psi_\phi \in W^{1,\infty}(Q)$ such that $q_\phi = \nabla_{t,x}\psi_\phi$, we have in particular $v_\phi = \nabla_x \psi_\phi$: the field v_ϕ then satisfies the properties (II).

The final regularity properties of v_ϕ given in Theorem 3.1 come from three other results. Proposition 4.4 states that the velocity field v_ϕ is locally Lipschitz on the space $(0,1)\times\mathbb{R}^d$. The property $\nabla_{t,x} v \in L^\infty(0,1; L^1(\Omega))$, and its corollary $v \in W^{1,p}((0,1)\times\Omega)$ for all $1 \leq p < 2$, are shown in the subsection 8.4. $\square$

6 Uniqueness properties of saddle points

6.1 Uniqueness of the velocity field on the density support

We start by studying the problem of the uniqueness of the velocity field on the support of the different candidate densities. More precisely, we show that for all the saddle points of $\mathbf{L}$, denoted by $(\psi^*, q^*, \mu^*) = (\psi^*, q^*, (\rho^*, m^*))$, the densities ρ^* are transported with the same velocity field v.

Lemma 6.1. *We consider Ω a bounded convex open set of $\mathbb{R}^d$, and $\rho_0, \rho_1 \in L^2(\mathbb{R}^d)$ two densities which supports are included in Ω. If the triplet (ψ^*, q^*, μ^*) is an element of $\mathbf{L}^{ps}(\rho_0, \rho_1, \Omega)$ such that $\mu^* = (\rho^*, m^*)$, then, for all $\phi \in \Phi(\rho_0\mathcal{L}^d, \rho_1\mathcal{L}^d)$, we have $m^* = \rho^* v_\phi$, with v_ϕ defined in* (4-30).

Sketch of the proof: Follwing Figure 1, we will give a "schematic" proof of the uniqueness of the velocity field on the union of supports of the candidate densities, which is based on the convexity of the set of saddle points and the strict convexity of the paraboloid

$$
\mathcal{P} = \{(a,b) \in \mathbb{R}\times\mathbb{R}^d,\ a + |b|^2/2 \leq 0\}.
$$

For a more rigorous proof we refer to [15] (chapter 4).

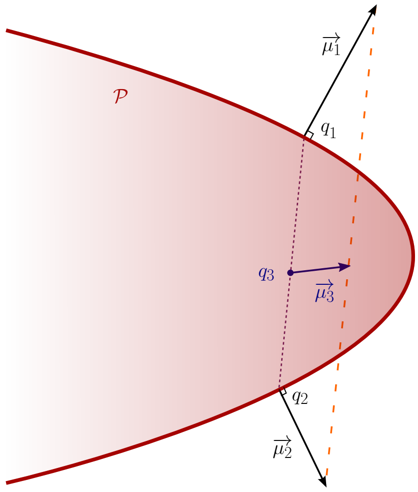

Figure 1: Illustration of the characterization of the saddle points of $\mathbf{L}$.

We assume (ψ_1, q_1, μ_1) and (ψ_2, q_2, μ_2) to be two saddle points of $\mathbf{L}$. The fields μ_1 and μ_2 are both orthogonal (in the sense of the canonical scalar product of L^2) to the hyperparaboloid defined by $\widetilde{\mathcal{P}} = \{(\widetilde{a}, \widetilde{b}) \in L^2(Q) \times L^2(Q)^d, \widetilde{a} + |\widetilde{b}|^2/2 \leq 0\}$ respectively at points q_1 and q_2. We will see later in subsection 9.2, that the set of saddle points of $\mathbf{L}$ is convex so that the $(1/2)[(\psi_1, q_1, \mu_1) + (\psi_2, q_2, \mu_2)]$ is also a saddle point of $\mathbf{L}$. The vector $(1/2)(\mu_1 + \mu_2)$ is also orthogonal to $\widetilde{\mathcal{P}}$ at point $(1/2)(q_1 + q_2)$.

Let $(t_0, x_0) \in [0, 1] \times \Omega$ be a point such that the vectors $\mu_1(t_0, x_0)$, $\mu_2(t_0, x_0)$ as well as the vector $(1/2)(\mu_1 + \mu_2)(t_0, x_0)$ are all orthogonal to the paraboloid $\mathcal{P}$ respectively at points $q_1(t_0, x_0)$, $q_2(t_0, x_0)$ and $(1/2)(q_1 + q_2)(t_0, x_0)$ and such that $\mu_1(t_0, x_0) \neq 0$ or $\mu_2(t_0, x_0) \neq 0$. From the orthogonality of vectors $\mu_1(t_0, x_0)$ and $\mu_2(t_0, x_0)$ at the paraboloid $\mathcal{P}$, then we have $(1/2)(\mu_1 + \mu_2)(t_0, x_0) \neq 0$.

If we have $q_1(t_0, x_0) \neq q_2(t_0, x_0)$, the point $(1/2)(q_1+q_2)(t_0, x_0)$ is strictly inside the paraboloid $\mathcal{P}$, because of its strict convexity. The vector $(1/2)(\mu_1+\mu_2)(t_0, x_0)$ is then necessarily zero, which contradicts the above assumption.

We thus have $q_1(t_0, x_0) = q_2(t_0, x_0)$. The vectors $\mu_1(t_0, x_0)$ and $\mu_2(t_0, x_0)$ are therefore both orthogonal to the paraboloid $\mathcal{P}$ at the same point $q_1(t_0, x_0)$ and also proportional to the vector $(1, b_1(t_0, x_0))$.

We then have $\mu_k(t_0, x_0) = (\rho_k(t_0, x_0), m_k(t_0, x_0)) = (\rho_k(t_0, x_0), \rho_k(t_0, x_0) b_1(t_0, x_0))$, for $k = 1$ or $k = 2$, and thus for $\mathcal{L}^d$-almost all $(t_0, x_0) \in \mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2)$. The fields μ_1 and μ_2 therefore share the same velocity field (i.e. the field b_1) on $\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2)$. $\square$

To sum up: for any fixed $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$ and for every saddle point (ψ^*, q^*, μ^*) of $\mathbf{L}$, ρ^* is associated with the same velocity field v_ϕ with $\partial_t \rho^* + \mathrm{div}_x(\rho^* v_\phi) = 0$ (because $m^* = \rho^* v_\phi$), with the initial and final conditions $\rho^*(0, \cdot) = \rho_0$ and $\rho^*(1, \cdot) = \rho_1$. In other words, for all $h \in H^1(Q)$, we have

$$\int_0^1 \int_\Omega \left(\partial_t h(t,x) + v_\phi(t,x).\nabla_x h(t,x)\right) \rho^*(t,x)\, dx\, dt + \int_\Omega h(0,\cdot)\rho_0 - \int_\Omega h(1,\cdot)\rho_1 = 0.$$

We now prove that there exists a unique ρ^* which satisfies these conditions, that is to say $\rho^*(t, \cdot) = X_\phi(t, \cdot)\#\rho_0$, for all $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$. We will use the method of the characteristics, based on Proposition 4.3.

6.2 Uniqueness of density in L^2

In the previous subsection, we have shown that the velocity field v corresponding to the displacement of the densities is unique on the union of the supports of the candidate densities, and can be written in the explicit form (4-30). Hence, the uniqueness of the density ρ will imply the uniqueness of the momentum $m = \rho v$. The next proposition is the main ingredient to show the uniqueness of the density.

Proposition 6.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ a convex potential satisfying the property (Γ_1), $\rho_0 \in L^2(\mathbb{R}^d)$ with a bounded support, and consider a velocity field $v = v_\phi$ defined from ϕ as in (4-30). If $\rho \in L^2((0,1) \times \mathbb{R}^d)$ is a density with bounded support in $[0,1] \times \mathbb{R}^d$, such that*

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) = 0, \\ \rho(0,\cdot) = \rho_0 \end{cases}$$

(in the distributions sense), i.e.

$$\forall h \in C_c^\infty([0,1) \times \mathbb{R}^d), \; \int_0^1 \int_{\mathbb{R}^d} (\partial_t h + v \cdot \nabla_x h)\, \rho \, dx \, dt + \int_{\mathbb{R}^d} h(0,\cdot)\rho_0 \, dx = 0. \tag{6-46}$$

then $\rho(t,\cdot) = \rho_\phi(t,\cdot) = (t\nabla\phi + (1-t)\,\mathrm{id})\#\rho_0$ for almost all $t \in [0,1]$. In other words:

$$\forall \varphi \in C_c^0((0,1) \times \mathbb{R}^d), \; \int_0^1 \int_{\mathbb{R}^d} \varphi\rho \, dx \, dt = \int_0^1 \int_{\mathbb{R}^d} \varphi\,(t, t\nabla\phi(x) + (1-t)x)\, \rho_0(x) \, dx \, dt. \tag{6-47}$$

Moreover $t \mapsto \rho(t,\cdot) \in C^0\left([0,1), L^2(\mathbb{R}^d)\right)$.

Proof: Let us begin by explaining our proof. Let Ω a bounded open set of $\mathbb{R}^d$ such that $\mathrm{supp}(\rho_0)$ is included in Ω and $\mathrm{supp}(\rho) \subset [0,1] \times \Omega$, and let $Q = (0,1) \times \Omega$. Let (ψ, q, μ) be a saddle point of $\mathbf{L}$ as defined in (2-10), and let $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$ (thus satisfying the property (Γ_1) on $\mathbb{R}^d$). The triplet (ψ, q, μ) satisfies the properties (I), which implies in particular the weak mass conservation $G(h) + \langle \mu, \nabla_{t,x} h \rangle = 0$ for all $h \in H^1(Q)$, as well as the linear relation between momentum and density: $\mu = (\rho, m) = (\rho, \rho v) \in L^2(Q)$, with v defined as in (4-30) (see Lemma 6.1) and satisfying the properties (II) (see the subsection 3.2). From these properties, we deduce that for every $h \in H^1(Q)$:

$$\int_0^1 \int_\Omega (\partial_t h + v \cdot \nabla_x h)\, \rho \, dx \, dt = \int_\Omega h(1,\cdot)\rho_1 \, dx - \int_\Omega h(0,\cdot)\rho_0 \, dx. \tag{6-48}$$

Let $\varphi \in C_c^\infty\left((0,1) \times \mathbb{R}^d\right)$ such that $\mathrm{supp}(\varphi) \subset (0,1) \times \Omega \subset Q$. By solving the transport problem in v and φ with a characteristics method, we consider the function $\mathfrak{h}$ defined for any $(t,x) \in (0,1) \times \mathbb{R}^d$ by:

$$\mathfrak{h}(t,x) = -\int_t^1 \varphi(s, (s-t)v(t,x) + x) \, ds, \tag{6-49}$$

which satisfies

$$\partial_t \mathfrak{h} + v \cdot \nabla_x \mathfrak{h} = \varphi \quad \text{and} \quad \nabla_{t,x}\mathfrak{h} = {}^t(\nabla_{t,x} v)\, \alpha + \beta, \tag{6-50}$$

with $\alpha \in L^\infty(Q)^d$, and $\beta \in L^\infty_{loc}(Q)^{d+1}$ (since $\nabla_{t,x} v$ is of size $d \times (d+1)$). Moreover, we have

$$\mathfrak{h}(1,\cdot) = 0, \quad \text{and} \quad \mathfrak{h}(0,\cdot) = -\int_0^1 \varphi(t, X(t,\cdot)) \, dt, \; \text{with } X(t,\cdot) = t\nabla\phi + (1-t)\,\mathrm{id} \tag{6-51}$$

To solve our problem, it would be sufficient to introduce $\mathfrak{h}$ in (6-48), as a test function. Unfortunately, as we will see in the subsection 8.4, the velocity field also satisfies the properties $\nabla_{t,x}v \in L^\infty(0,1;L^1(\Omega))$ and $v \in W^{1,p}((0,1)\times\Omega)$ for all $1 \leq p < 2$, whence

$$v = v_\phi \in \mathcal{H} := \bigcap_{1\leq p<2} W^{1,p}(Q).$$

The function $\mathfrak{h}$ is therefore also an element of $\mathcal{H}$. Since we does not have a better integrability than L^2 on ρ, **we cannot extend the space of test functions** of (6-48) to a largest space than $H^1(Q)$ as $\mathcal{H}$.

The counter-example of Caffarelli: the strict division of the mass, show us that, in general, the field v is not an element of $H^1(Q)$.

Remark 6.1. *As we have already mentioned above, this default of H^1 regularity is partly due to the possible "breaks" in the continuous transport scheme in $t = 0$ (or conversely to possible connections in $t = 1$). Such breaks correspond to the points of non-differentiability of the potential ϕ and the connections in $t = 1$ are linked to the non-differentiability points of the Legendre transform ϕ^* of the potential ϕ. However, because of the symmetry of the problem, we will not need to deal with the connection problems in $t = 1$.*

Thus, we will cannot use directly the funtion $\mathfrak{h}$ as a test function in (6-48). We then choose to approach $\mathfrak{h}$ by approximating the velocity field v (associated to the transport plan $\nabla\phi$) with velocity fields $v_\gamma = v_{^\gamma\phi}$ associated to the regularized transport plans $\nabla^\gamma\phi$, where $^\gamma\phi$ denotes the γ-regularization by a Moreau envelope of the potential ϕ (see Definition 4.3). This regularization has the property of erasing the breaks of the transport plan, which are responsible for the fact that v does not have regularity H^1 in the neighborhood of $t = 0$. The neighborhood of $t = 1$ is not an issue, since $\mathfrak{h}$ is uniformly zero on this neighborhood by construction.

In summary, by a characteristics method, it is possible to construct some test functions $\mathfrak{h}_\gamma \in H^1(Q)$, uniformly zero in the neighborhood of $t = 1$ (independently of γ), such that:

$$\partial_t\mathfrak{h}_\gamma + v_\gamma \cdot \nabla_x\mathfrak{h}_\gamma = \varphi = \partial_t\mathfrak{h}_\gamma + v \cdot \nabla_x\mathfrak{h}_\gamma + (v_\gamma - v)\cdot\nabla_x\mathfrak{h}_\gamma,$$

and such that $\mathfrak{h}_\gamma(0,\cdot)$ converges to $-\int_0^1 \varphi[t,(1-t)\,\mathrm{id} + t\nabla_x\phi]\,dt$ in $L^2(\mathbb{R}^d)$ when γ tends to 0. By injecting such a function $\mathfrak{h}_\gamma$ in (6-48), one obtains

$$\int_0^1\int_{\mathbb{R}^d}\varphi\rho\,dx\,dt = \int_0^1\int_{\mathbb{R}^d}\varphi[t,(1-t)x + t\nabla_x\phi(x)]\rho_0(x)\,dx\,dt + R_\gamma(\varphi),\ \text{with} \tag{6-52}$$

$$R_\gamma(\varphi) = \int_0^1\int_{\mathbb{R}^d}(v_\gamma - v)\cdot\nabla_x\mathfrak{h}_\gamma\rho\,dx\,dt - \int_{\mathbb{R}^d}\mathfrak{h}_\gamma(0,\cdot)\rho_0\,dx - \int_0^1\int_{\mathbb{R}^d}\varphi[t,(1-t)\,\mathrm{id} + t\nabla_x\phi]\rho_0\,dx\,dt. \tag{6-53}$$

In order to prove Proposition 6.1, it is therefore necessary to show that $R_\gamma(\varphi)$ converges to 0 when γ tends to 0. This will make use of the results of the subsection 6.3.

The $\mathfrak{h}_\gamma$ are defined with respect to v_γ by (6-49). We can then prove by (6-50) that we have,

$$|\nabla_x\mathfrak{h}_\gamma(t,x)| \leq (|\nabla_x v_\gamma(t,x)| + 1)\|\nabla_x\varphi\|_{L^\infty([0,1]\times\mathbb{R}^d)},\ \text{for almost all } (t,x) \in (0,1)\times\mathbb{R}^d. \tag{6-54}$$

For more details, we refer to [15] (subsection 4.2.5), in which it is proven that $\mathfrak{h}_\gamma \in H^1(Q)$.

From Lemma 8.3, the potential $^\gamma\phi$ verifies the property (Γ_2) and according to (8-78), v_γ is extended by continuity in $t = 0$. It is the same for $\mathfrak{h}_\gamma$, which is thus continuous on $[0,1)\times\overline{\Omega}$, and for all $x \in \mathbb{R}^d$, one has:

$$\mathfrak{h}_\gamma(0,x) = -\int_0^1\varphi(s, s\,v_\gamma(0,x) + x)\,ds = -\int_0^1\varphi(s, s\nabla^\gamma\phi(x) + (1-s)x)\,ds \tag{6-55}$$

(which coincides with the trace L^2 of $\mathfrak{h}_\gamma$ in $t=0$). According to Lemma 8.4, $\nabla^\gamma\phi(x)$ converges for almost all $x \in \Omega$ to $\nabla\phi(x)$ (for all x where ϕ is differentiable).

In addition, for all $(s,x) \in (0,1)\times\Omega$, the term $\varphi(s, s\nabla^\gamma\phi(x)+(1-s)x)$ is uniformly bounded by $\|\varphi\|_{L^\infty}$. Thus, by dominated convergence, we have

$$r_\gamma(\varphi) = \int_\Omega \mathfrak{h}_\gamma(0,\cdot)\rho_0\,dx + \int_0^1\int_\Omega \varphi(s, s\nabla\phi(x)+(1-s)x)\rho_0(x)\,dx\,ds \underset{\gamma\to 0}{\longrightarrow} 0. \tag{6-56}$$

Let $t_m \in (0,1)$ such that $\operatorname{supp}(\varphi) \subset (0,t_m)\times\Omega$. From (6-54), we thus have

$$|R_\gamma(\varphi)| \le |r_\gamma(\varphi)| + \|\nabla_x\varphi\|_{L^\infty}\left(\int_0^{t_m}\int_\Omega |v-v_\gamma|\cdot|\nabla_x v_\gamma|\cdot|\rho|\,dx\,dt + \int_0^{t_m}\int_\Omega |v-v_\gamma|\cdot|\rho|\,dx\,dt\right). \tag{6-57}$$

Proposition 8.5 (see the subsection 8) tells us that $|v-v_\gamma|$ is uniformly bounded and simply converges to 0 on $(0,1)\times\Omega$ when γ tends to 0. Thus, since $\rho \in L^2((0,1)\times\Omega)$, we conclude via dominated convergence that the term $\int_0^{t_m}\int_\Omega |v-v_\gamma|\cdot|\rho|\,dx\,dt$ converges to 0.

Finally, to complete the proof of Proposition 6.1, we have to show that $\int_0^{t_m}\int_\Omega |v-v_\gamma|\cdot|\nabla_x v_\gamma|\cdot|\rho|\,dx\,dt$ converges to 0, which is the subject of the following Lemma 6.2. Therefore, the proof of Proposition **will be complete after the proof** of Lemma 6.2. □

Lemma 6.2. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a convex potential verifying the property (Γ_1). We consider a velocity field $v = v_\phi$ defined with respect to ϕ as in (4-30), and $0 < t_m < 1$. Let $\rho \in L^2((0,1)\times\mathbb{R}^d)$, with bounded support into $[0,1]\times\mathbb{R}^d$. Let Ω a bounded open set of $\mathbb{R}^d$ such that $\operatorname{supp}(\rho_0) \subset \Omega$ and $\operatorname{supp}(\rho) \subset [0,1]\times\Omega$ (and let $Q=(0,1)\times\Omega$). For any $\gamma > 0$, we define $v_\gamma = v_{^\gamma\phi}$, where $^\gamma\phi$ is the Moreau envelope of ϕ by the parameter γ (see Definition 4.3). Then we have the result of convergence:*

$$\int_0^{t_m}\int_\Omega |v-v_\gamma|\cdot|\nabla_x v_\gamma|\cdot|\rho|\,dx\,dt \underset{\gamma\to 0}{\longrightarrow} 0. \tag{6-58}$$

6.3 Some results for the proof of Lemma 6.2

The three results of this subsection concern the control of the regularized velocity fields v_γ. The first one is an important uniform regularity result for the velocity field and its regularization.

Proposition 6.2. *We assume that ϕ satisfies the property (Γ_1). Let $R' > R > 0$ and $a \in \mathbb{R}^d$ such that $\phi(a) = \inf_{\mathbb{R}^d}\phi$ and let $M = \sup_{x\in B(a,2(R+|a|))}|\partial\phi(x)|$.*

Then there exists an constant $C > 0$ – independent of ϕ, γ, a, R and R', such that for all $t_0 \in (0,1)$ satisfying the condition $t_0 < \min\{1/2, (R'-R)/(M+2|a|)\}$, and by setting $v_0 = v$, we have the property:

$$\forall\gamma \ge 0,\ \forall t \in (0,t_0],\ \int_{B(a,R)} |\nabla_x v_\gamma(t,x)|_1\,dx \le \frac{C}{t_0(1-t_0)}\mathcal{L}^d(B(a,R')). \tag{6-59}$$

Thus $\nabla_x v_\gamma \in L^\infty(0,t_0;L^1(B(a,R)))$, for all $\gamma \ge 0$.

This proposition will be shown in the subsection 8.3. Nevertheless, we prove immediately the two following results.

Corollary 6.1 (Corollary of Proposition 6.2)**.** *We suppose that ϕ satisfies the property (Γ_1). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $0 < t_m < 1$. Then there exists a constant $K > 0$ such that for every $\gamma > 0$ and any $t \in [0,t_m]$, we have*

$$\int_\Omega |\nabla_x v_\gamma(t,x)|_1\,dx \le K. \tag{6-60}$$

Proof: For $t < t_0$, we apply Proposition 6.2; and for $t_m \geq t > t_0$ we use the fact that the term $|\nabla_x v(t_0, \cdot)|_1$ is bounded by $c/t(1-t)$, for c a constant depending only of the chosen norm. $\square$

Lemma 6.3. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ convex verifying the property (Γ_1). Then there exists a constant c, independent of γ, such that for every $1 \geq \gamma > 0$, and $t \in (0,1)$,*

$$\|\nabla_x v_\gamma(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c}{\gamma(1-t)}.$$

Proof: The relation (4-26) of Definition 4.3, as well as the non-expansiveness of the operator $\mathrm{id} - \mathrm{Prox}_{\gamma f}$ (Definition 4.2), assert that there exists a constant C_0, independent of γ, such that $\left\|\mathrm{D}^2({}^\gamma\phi)\right\|_{L^\infty(\mathbb{R}^d)} \leq C_0/\gamma$. We then conclude by Lemma 8.2. $\square$

6.4 Proofs of Lemma 6.2 and Theorem 3.2

We will begin by proving Lemma 6.2 in the more restrictive case where $\rho \in L^p(Q)$, with $p > 2$. We formulate it in the following lemma:

Lemma 6.4. *Under the same assumptions as Lemma 6.2 and if in addition there exists $p > 2$ such that $\rho \in L^p(Q)$, we have the following convergence result:*

$$\int_0^{t_m} \int_\Omega |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| \, dx \, dt \underset{\gamma \to 0}{\longrightarrow} 0. \tag{6-61}$$

We will then be able to prove Lemma 6.2 by density of $\bigcap_{1 \leq p < 2} L^p(Q)$ in $L^2(Q)$. For this, we will also need the following lemma:

Lemma 6.5. *Under the same assumptions as Lemma 6.2, there exists a constant M such that:*

$$\forall \gamma \in (0,1], \quad \int_0^{t_m} \int_\Omega |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| \, dx \, dt \leq M \|\rho\|_{L^2(Q)}. \tag{6-62}$$

Proof of Lemma 6.4: Assuming that $\rho \in L^p(Q)$, with $p > 2$. By taking $1 < q < 2$ such that $1/p + 1/q = 1$, note that:

$$|\nabla_x v(t,x)| = |\nabla_x v(t,x)|^{\frac{1}{p} + \frac{1}{q}} \leq \left(\frac{c}{t(1-t)}\right)^{\frac{1}{p}} |\nabla_x v(t,x)|^{\frac{1}{q}}. \tag{6-63}$$

For all $0 < \alpha < 1$, we then have:

$$\begin{aligned}
\int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| \, dx \, dt &\leq C \int_0^{t_m} \left(\frac{\gamma}{t}\right)^\alpha \left(\frac{c}{t(1-t)}\right)^{\frac{1}{p}} \int_\Omega |\nabla_x v_\gamma|^{\frac{1}{q}} \cdot |\rho| \, dx \, dt \\
&\leq C \left(\frac{c}{1-t_m}\right)^{\frac{1}{p}} \gamma^\alpha \int_0^1 \frac{1}{t^{\alpha + 1/p}} \left(\int_\Omega |\nabla_x v_\gamma| \, dx\right)^{\frac{1}{q}} \cdot \left(\int_\Omega |\rho|^p \, dx\right)^{\frac{1}{p}} dt \\
&\leq C \left(\frac{c}{1-t_m}\right)^{\frac{1}{p}} K^{\frac{1}{q}} \gamma^\alpha \left(\int_0^1 \frac{1}{t^{q\alpha + q/p}} \, dt\right)^{\frac{1}{q}} \|\rho\|_{L^p(Q)}.
\end{aligned} \tag{6-64}$$

Now, we have $q < 2$, and it follows that by fixing $0 < \alpha < 1$ small enough, we can then have $q(\alpha + 1) < 2 \Leftrightarrow q\alpha + q/p = q\alpha + q - 1 < 1$: the term $1/t^{q\alpha + q/p}$ is thus integrable on $(0,1)$, and we then obtain the result of convergence (6-61). $\square$

Proof of Lemma 6.5: We will proceed to a Chasles division in time of type

$$\int_0^{t_m}\int_\Omega F(t,x)\,dx\,dt = \underbrace{\int_0^{\gamma}\int_\Omega F(t,x)\,dx\,dt}_{T_1} + \underbrace{\int_\gamma^{t_m}\int_\Omega F(t,x)\,dx\,dt}_{T_2},$$

with $F = |v-v_\gamma|\cdot|\nabla_x v_\gamma|\cdot|\rho|$. Recall that for $t\in(0,1)$, the fields $v_\gamma(t,\cdot)$ and $v(t,\cdot)$ are $c/(t(1-t))$-Lipschitz sur $\mathbb{R}^d$, with c independent of ϕ and γ. With Lemma 6.3, we have

$$\|\nabla_x v_\gamma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le c/(1-t)\min\{1/\gamma, 1/t\}.$$

In the first term T_1, we will thus have $t\le\gamma$, and we will use the bound

$$|\nabla_x v_\gamma| \le \sqrt{c}\,|\nabla_x v_\gamma|^{1/2}/(\sqrt{\gamma}\sqrt{1-t}).$$

In the second one T_2, we will have $t\ge\gamma$, and we will write

$$|\nabla_x v_\gamma| \le \sqrt{c}\,|\nabla_x v_\gamma|^{1/2}/(\sqrt{t}\sqrt{1-t}).$$

We will use the Corollaries 8.1 and 6.1, keeping the same notations for the constants and parameters involved in these utterances (the parameter α and the constants C and K).

For the term T_1, we choose $\alpha=0$. We have:

$$\begin{aligned}\int_0^{\gamma}\int_{\mathbb{R}^d}|v-v_\gamma|\cdot|\nabla_x v_\gamma|\cdot|\rho|\,dx\,dt &\le \frac{C\sqrt{c}}{\sqrt{\gamma}}\int_0^\gamma\frac{1}{\sqrt{1-t}}\int_\Omega|\nabla_x v_\gamma(t,\cdot)|^{\frac12}\cdot|\rho(t,\cdot)|\,dx\,dt\\ &\le \frac{C\sqrt{c}}{\sqrt{1-t_m}\sqrt{\gamma}}\int_0^\gamma\left(\int_\Omega|\nabla_x v_\gamma(t,\cdot)|\;dx\right)^{\frac12}\cdot\left(\int_\Omega|\rho|^2\,dx\right)^{\frac12}\,dt\\ &\le \frac{C\sqrt{Kc}}{\sqrt{1-t_m}\sqrt{\gamma}}\int_0^\gamma\left(\int_\Omega|\rho|^2\,dx\right)^{\frac12}\,dt \le \frac{M}{2\sqrt{\gamma}}\cdot\sqrt{\gamma}\|\rho\|_{L^2(Q)} = \frac{M}{2}\|\rho\|_{L^2(Q)},\end{aligned}\tag{6-65}$$

with $M=(2C\sqrt{Kc})/\sqrt{1-t_m}$. For the term T_2, we choose $\alpha=1/2$. We then have:

$$\begin{aligned}\int_\gamma^{t_m}\int_{\mathbb{R}^d}|v-v_\gamma|\cdot|\nabla_x v_\gamma|\cdot|\rho|\,dx\,dt &\le C\sqrt{c}\sqrt{\gamma}\int_\gamma^{t_m}\frac{1}{t\sqrt{1-t}}\int_\Omega|\nabla_x v_\gamma|^{\frac12}\cdot|\rho|\,dx\,dt\\ &\le \frac{C\sqrt{c}}{\sqrt{1-t_m}}\sqrt{\gamma}\int_\gamma^{t_m}\frac1t\left(\int_\Omega|\nabla_x v_\gamma(t,\cdot)|\;dx\right)^{\frac12}\cdot\left(\int_\Omega|\rho|^2\,dx\right)^{\frac12}\,dt\\ &\le \frac{C\sqrt{Kc}}{\sqrt{1-t_m}}\sqrt{\gamma}\left(\int_\gamma^{t_m}\frac{1}{t^2}\,dt\right)^{\frac12}\cdot\|\rho\|_{L^2(Q)} \le \frac{M}{2}\sqrt{\gamma}\sqrt{\frac1\gamma-\frac{1}{t_m}}\cdot\|\rho\|_{L^2(Q)} \le \frac{M}{2}\|\rho\|_{L^2(Q)},\end{aligned}\tag{6-66}$$

Thus, by summing (6-65) and (6-66), we obtain the inequality (6-62). □

Proof of Lemma 6.2: The intersection of the spaces $L^p(Q)$, for all $p>2$, is dense into $L^2(Q)$. Indeed, if $\rho\in L^2(Q)$, then for all $0<\lambda<1$, we have $2/\lambda>2$ and $|\rho|^\lambda\in L^{2/\lambda}(Q)$. Moreover, it is easy to show, by dominated convergence, that the family $(|\rho|^\lambda)_{\lambda\in(0,1)}$ converges to $|\rho|$ in $L^2(Q)$ when λ goes to 1.

Let $\epsilon > 0$. For all $\gamma > 0$ and all $0 < \lambda < 1$, we have the upper estimate:

$$\begin{aligned}\int_0^{t_m}\int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| \, dx \, dt \leq & \int_0^{t_m}\int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot \left||\rho| - |\rho|^\lambda\right| \, dx \, dt \\ & + \int_0^{t_m}\int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho|^\lambda \, dx \, dt.\end{aligned} \tag{6-67}$$

By fixing $\lambda \in (0,1)$, such that $\left\||\rho| - |\rho|^\lambda\right\|_{L^2(Q)} \leq \epsilon/M$, we thus have by Lemma 6.5:

$$\forall \gamma \in (0,1], \int_0^{t_m}\int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot \left||\rho| - |\rho|^\lambda\right| \, dx \, dt \leq M \left\||\rho| - |\rho|^\lambda\right\|_{L^2(Q)} \leq \epsilon. \tag{6-68}$$

By injecting the last inequality into (6-67), and from Lemma 6.4, we can then set a rank $\gamma_0 > 0$ such that for any $0 < \gamma \leq \gamma_0$,

$$\int_0^{t_m}\int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| \, dx \, dt \leq \epsilon + \int_0^{t_m}\int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho|^\lambda \, dx \, dt \leq 2\epsilon. \tag{6-69}$$

□

To finish to prove Theorem 3.2 which deals with the uniqueness of the component $\mu = (\rho, m)$ shared by the saddle points of $\mathbf{L}$, we only have to show that a density ρ^* associated with one of these saddle points (ψ^*, q^*, μ^*) verifies the conditions of application of Proposition 6.1.

Proof of Theorem 3.2: Let (ψ^*, q^*, μ^*) an element of $\mathbf{L}^{ps}(\rho_0, \rho_1, \Omega)$ (i.e. a saddle point). According to Proposition 3.1, we have $G(h) + \langle \mu^*, \nabla_{t,x} h\rangle = 0$, for all $h \in H^1(Q)$. Let $\phi \in \Phi(\rho_0\mathcal{L}^d, \rho_1\mathcal{L}^d)$ (thus verifying the property (Γ_1)). According to Lemma 6.1, by defining v_ϕ on $(0,1) \times \mathbb{R}^d$ as in (4-30), we have $m^* = \rho^* v_\phi$. In other words, for all $h \in H^1(Q)$:

$$\int_Q (\partial_t h + v_\phi \cdot \nabla_x h)\, \rho^* \, dx \, dt + \int_\Omega h(0,\cdot)\rho_0 \, dx - \int_\Omega h(1,\cdot)\rho_1 \, dx = 0. \tag{6-70}$$

Let $\overline{\rho^*} \in L^2((0,1) \times \mathbb{R}^d)$ be the extension in 0 of ρ^* on $(0,1) \times \mathbb{R}^d$. Noting that for all test funtion $h \in H^1_{loc}((0,1) \times \mathbb{R}^d)$, we have $h_Q \in H^1(Q)$ and $\nabla_{t,x} h_Q = (\nabla_{t,x} h)_{|Q}$, the relation (6-70) can be extended from Q to the entire space $(0,1) \times \mathbb{R}^d$.

Thus, according to Proposition 6.1, we have the equivalence

$$\overline{\rho^*}(t,\cdot) = \rho_\phi(t,\cdot) = (t\nabla\phi + (1-t)\,\mathrm{id})\#\rho_0$$

for almost all $t \in [0,1]$, with in addition $t \mapsto \rho_\phi(t,\cdot) \in C^0\left([0,1), L^2(\mathbb{R}^d)\right)$. □

7 Characterization of an optimal transport velocity field

In this section, we present a generalization of our study about the uniqueness of the component density-momentum μ: we want to use this study to try to characterize less formally an optimal transport velocity field. The result will be roughly the following:

*Any density of L^2, with **bounded support**, and advected by a **locally bounded** velocity field, whose **trajectories are all straight lines that never intersect**, corresponds to an **optimal transport** (an interpolation of McCann) and is **the only solution** for such a displacement.*

These properties state that the velocity field have to satisfy correspond to the properties (II).

For a convex open set Ω of $\mathbb{R}^d$, we define the space ${}^bL^2_+((0,1)\times\Omega)$ of densities $\rho \in L^2((0,1)\times\Omega)$ which are non-negative and with compact supports into $[0,1] \times \Omega$.

Theorem 7.1. *Let Ω be a convex open set of $\mathbb{R}^d$, not necessarily bounded, and let $Q = (0,1)\times\Omega$. Let v^* be a velocity field on Ω satisfying the properties (II), and let $\rho_0 \in L^2(\Omega)$, with $\rho_0 \geq 0$ and such that $\mathrm{supp}(\rho_0)$ is bounded in Ω. Let $\rho^* \in {}^bL^2_+(Q)$ be a density solution, in the sense of the distributions, of*

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v^*) = 0, \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{7-71}$$

Then the density ρ^* is ***the unique solution*** *of the system (7-71) in the space ${}^bL^2_+(Q)$, and we have $\rho^* \in C^0\left([0,1), L^2(\Omega)\right)$ in addition. Moreover, there exists a unique non-negative measure ν_1 on Ω, which support is bounded in $\mathrm{supp}(\rho_0) \cup [\bigcup_{t\in[0,1]} \mathrm{supp}(\rho^*(t,\cdot))]$ and that satisfies the properties $(\mathcal{M})$. There also exists a convex function ϕ on $\mathbb{R}^d$ verifying the property (Γ_1), such as: $\nu_1 = \nabla\phi\#\left(\rho_0\,\mathcal{L}^d\right)$, and (link with McCann interpolation)*

$$\forall t \in [0,1),\ \ (\rho^*(t,\cdot)\mathcal{L}^d) = (\rho_\phi(t,\cdot)\mathcal{L}^d) = (t\nabla\phi + (1-t)\,\mathrm{id})\,\#(\rho_0\mathcal{L}^d). \tag{7-72}$$

The couple (ρ^*, v^*) is then solution of

$$\begin{cases} \partial_t \rho^* + \mathrm{div}_x(\rho^* v^*) = 0, \\ \rho^*(0,\cdot) = \rho_0,\ \ \rho^*(1,\cdot) = \nu_1. \end{cases} \tag{7-73}$$

which , reformulated in the weak sense, gives

$$\forall h \in C_c^\infty([0,1]\times\Omega),\ \int_0^1\int_\Omega (\partial_t h + v^*\cdot\nabla_x h)\,\rho^*\,dx\,dt + \int_\Omega h(0,\cdot)\rho_0\,dx - \int_\Omega h(1,\cdot)\,d\nu_1 = 0. \tag{7-74}$$

Finally, for all $(t,x) \in \mathrm{supp}(\rho^*)$, we have $v^*(t,x) = v_\phi(t,x)$ (still defined by (4-30)). The field v^* therefore satitisfies the properties of the velocity field v_ϕ on $\mathrm{supp}(\rho^*)$. In particular, the field v_ϕ is locally Lipschitz on the space $(0,1)\times\mathbb{R}^d$ and it satisfies $\nabla_{t,x}v \in L^\infty(0,1;L^1_{loc}(\mathbb{R}^d))$ and $v \in W^{1,p}_{loc}([0,1]\times\mathbb{R}^d)$ for all $1 \leq p < 2$.

Sketch of the proof: The proof gathers elements from sections 5 and 6. It is more technical, since the final measure (7-71) is no longer a density measure, of type $\rho_1\mathcal{L}^d$, but simply a finite measure ν_1 satisfying the properties $(\mathcal{M})$. We here only give the main steps of the proof and refer to [15] (section 4.3) for all the details.

The first step is simply to prove the existence, in the sense of the distributions, of the final measure ν_1, as defined in the statement of Theorem 7.1. For this purpose we use classical functional analysis tools [15] (in particular the Riesz Representation Theorem [23]). We also show that the weak formulation (7-74) is always valid for test functions taken from $W^{1,\infty}_c([0,1]\times\Omega)$.

Then we consider ϕ, an optimal transport potential between $\rho_0\mathcal{L}^p$, and ν_1 satisfying the property (Γ_1). We consider "a saddle point", i.e. a triplet $(\mu_\phi, q_\phi, \psi_\phi)$, as done in section 5 and built in Theorem 3.1 and Lemma 4.1. As Brenier's Theorem only assumes density for the initial density $\rho_0\mathcal{L}^d$, Proposition 3.2 is still valid. This is also the case all inductions we have done while building the velocity field v_ϕ. However, we can not obtain $\rho_\phi \in L^2(Q)$ via Lemma 5.1, since it requires $\nu_1 = \rho_1\mathcal{L}^d$ with $\rho_1 \in L^2(\Omega)$. Hence, we can not extend the test functions of the weak formulation of the mass conservation for the pair (ρ_ϕ, v_ϕ) to the space $H^1_{loc}(Q)$, as in Proposition 5.3, which only considers absolutely continuous initial measures. Extending these test functions to the space $W^{1,\infty}_{loc}(Q)$ is required since the potential ψ_ϕ necessarily belongs to this space.

We then construct a second saddle point from the pair (ρ^*, v^*), that is to say a triplet (μ^*, q^*, ψ^*), with $\mu^* = (\rho^*, \rho^* v^*)$, $q^* = (-(1/2)|v^*|^2, v^*)$ and $\nabla_{t,x}\psi^* = q^*$ (Lemma 4.1). We can then, as well as for Lemma 6.1, prove the uniqueness of the velocity field on the supports of ρ_ϕ and ρ^*, i.e. $\rho^* v^* = \rho^* v_\phi$. Although our triplets $(\mu_\phi, q_\phi, \psi_\phi)$ and (μ^*, q^*, ψ^*) are not necessarily in

L^2, and we can no longer speak of "projections" and "orthogonality" in the schematic proof of Lemma 6.1, the reasoning remains globally the same and we reach the same conclusion (Lemma 4.3-14 of [15]). Thus, according to Proposition 6.1, the density ρ^* verifies the relation (7-72), with $t \mapsto \rho^*(t,\cdot) \in C^0\left([0,1), L^2(\Omega)\right)$.

Now, let us show that ρ^* is the only solution with bounded support of the system (7-71) in the space ${}^bL^2_+(Q)$. Assume there exist two solutions ρ^1, ρ^2 of (7-71) in ${}^bL^2_+(Q)$. Then the density $\overline{\rho} = (\rho^1 + \rho^2)/2$ is still a solution in ${}^bL^2_+(Q)$. Therefore, there would exist a convex function $\overline{\phi}$ of $\mathbb{R}^d$ satisfying the property (Γ_1), such that, by defining $v_{\overline{\phi}}$ as in (4-30), we have $\overline{\rho}\, v^* = \overline{\rho}\, v_{\overline{\phi}}$, i.e. $(\rho^1 + \rho^2)\, v^* = (\rho^1 + \rho^2)\, v_{\overline{\phi}}$. The field v^* is then almost everywhere equal to the field $v_{\overline{\phi}}$ on $\operatorname{supp}(\rho^1) \cup \operatorname{supp}(\rho^2)$, thus $\rho^1\, v^* = \rho^1\, v_{\overline{\phi}}$ and $\rho^2\, v^* = \rho^2\, v_{\overline{\phi}}$. Therefore, ρ^1 and ρ^2 both satisfy the system (7-71), by replacing v^* by $v_{\overline{\phi}}$. According to Proposition 6.1, we thus have $\rho^1 = \rho^2 \,:\, t \mapsto \rho_{\overline{\phi}} = \left(t\nabla\overline{\phi} + (1-t)\,\mathrm{id}\right) \# \rho_0$ in $L^2((0,1)\times\mathbb{R}^d))$ (and then in $L^2(Q)$). $\square$

8 Burgers Equation and regularity results on the velocity field

The aim of this section is to prove Proposition 4.5 (Burgers equation satsified by the velocity field $v = v_\phi$) and Proposition 6.2 (uniform control of the gradients of fields $v = v_\phi$ and $v_\gamma = v_{\gamma\phi}$).

In subsection 8.1, we first consider an "ideal" framework, i.e. without breaks. For this purpose, we assume that the potential ϕ is regular and satisfies the following property (Γ_2).

Hypothesis (Γ_2)**.** *ϕ satisfies (Γ_1), is of class C^1, and $\nabla_x\phi$ is Lipschitz (i.e. $\phi \in C^{1,1}(\mathbb{R}^d)$).*

We recall that ϕ fulfills the property (Γ_1) if and only if ϕ and ϕ^* are convex, continuous and admit a minimum on $\mathbb{R}^d$.

Once the propositions will be established under the assumption (Γ_2), we will consider the general framework of the optimal transport for ϕ satisfying the property (Γ_1). To that end, a regularization of the potential ϕ with Moreau envelope will be considered in the subsection 8.2.

The final proofs (by compilation of previous results) of Propositions 4.5 and 6.2 will be stated in the subsection 8.3.

To finish, in the subsection 8.4, we will state additional results concerning the regularity of the velocity field, already mentioned in the proof of Proposition 6.1, for instance the fact that the velocity field v_ϕ is in all space $W^{1,p}(Q)$ for all $p < 2$). These complementary results are not directly necessary for the proofs of uniqueness that we have treated.

In what follows, we will often require the Rademacher's Theorem on the differentiability of locally Lipschitz functions.

Theorem 8.1 (The Rademacher's Theorem *([11] p.81))***.** *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a locally Lipschitz function. Then f is $\mathcal{L}^n$-almost everywhere Fréchet-differentiable (and its differential in the sense of Fréchet coincides with its differential in the sense of distributions).*

Let us give a useful example of the application of Rademacher's Theorem.

Example 8.1. *Proposition 4.4 is valid under the property (Γ_1), and therefore also under (Γ_2). We know that v is locally Lipschitz on $(0,1)\times\mathbb{R}^d$. Thus, according to Rademacher's Theorem 8.1, v is almost everywhere differentiable on $(0,1)\times\mathbb{R}^d$, and its differential corresponds to its derivative in the sense of distributions. In particular, we have*

$$v \cdot \nabla_x v = \frac{1}{2}\nabla_x |v|^2, \tag{8-75}$$

in the sense $\mathcal{L}^{d+1}$-almost everywhere, in the sense of distributions.

8.1 Regular case (Γ_2)

In this subsection, we will assume that ϕ satisfies the hypothesis (Γ_2). In this "ideal" case (without breaks), we want to show that the velocity field v satisfies the Burgers equation in the sense of distributions (3-17), as well as the control of the gradient of the velocity field: $\nabla_{t,x} v \in L^\infty_{loc}\left([0,1), L^1(\mathbb{R}^d)\right)$.

The operator $\mathbf{p}_\phi$ (Definition 4.4) can be interpreted as a spatial "inverse" of the operator $X(t,\cdot) = \nabla_x(\phi_t) = (1-t)\,\mathrm{id} + t\nabla\phi$. This is a rough interpretation since, apart from the assumption (Γ_2), $X(t,\cdot)$ is generally not invertible.

Proposition 8.1. *Under the property (Γ_2), $\mathbf{p}_\phi$ satisfies the following properties, for all $t \in [0,1)$*

1. *$\mathbf{p}_\phi(t,\cdot)$ is bijective on $\mathbb{R}^d$ and for all $x, y \in \mathbb{R}^d$,*
$$y = \mathbf{p}_\phi(t,x) \ \Leftrightarrow\ x = (1-t)y + t\nabla\phi(y). \tag{8-76}$$
2. *$\mathbf{p}_\phi(0,\cdot) = \mathrm{id}$.*
3. *the velocity v defined (3-13) can be defined from $\mathbf{p}_\phi$ by :*
$$v(t,x) = \nabla\phi(\mathbf{p}_\phi(t,x)) - \mathbf{p}_\phi(t,x) = \frac{x - \mathbf{p}_\phi(t,x)}{t}, \tag{8-77}$$
for all $(t,x) \in (0,1)\times\mathbb{R}^d$,
4. *v can be continuously extended on $[0,1)\times\mathbb{R}^d$ (i.e. in $t=0$) and for all $x\in\mathbb{R}^d$,*
$$v(0,x) = \nabla\phi(x) - x \tag{8-78}$$

Proof:

- The potential ϕ is assumed to be of class C^1. It is therefore differentiable at every point $x \in \mathbb{R}^d$, and by convexity we have $\partial\phi(x) = \{\nabla\phi(x)\}$.
- We immediately deduce the second (taking $t=0$) and third points with
$$x = (1-t)\,\mathbf{p}_\phi(t,x) + t\nabla\phi(\mathbf{p}_\phi(t,x)) \ \Leftrightarrow\ \nabla\phi(\mathbf{p}_\phi(t,x)) - \mathbf{p}_\phi(t,x) = \frac{x-\mathbf{p}_\phi(t,x)}{t} = v(t,x). \tag{8-79}$$
- Let $x \in \mathbb{R}^d$ and a $(t_n,x_n)_n \in Q^{\mathbb{N}}$ converging to $(0,x)$. For a a minimum of ϕ, we have $\mathbf{p}_\phi(t,(1-t)a) = a$ for all $t\in[0,1)$, since $\mathbf{p}_\phi$ is Lipschitz. By estimating the distance from $\mathbf{p}_\phi$ to a, we can show that the sequence $\left(\mathbf{p}_\phi(t_n,x_n)\right)_n$ is bounded. From the continuity of $\nabla\phi$, we also get that the sequence $\left(\nabla\phi(\mathbf{p}_\phi(t_n,x_n))\right)_n$ is bounded. Then, the right hand side of the equivalence (8-79) can be used to show that $\left(\mathbf{p}_\phi(t_n,x_n)\right)_n$ converges to x (by continuity of $\mathbf{p}_\phi$ at $(0,x)$). The left term of the equivalence allows us to conclude.

$\square$

Remark 8.1. *We note that $\phi_t = (1/2)(1-t)|\cdot|^2 + t\phi$ is of class C^1, strictly convex and superlinear, since ϕ is convex and $|\cdot|^2$ is strictly convex and superlinear. Thus, for all $t\in(0,1)$, $X(t,\cdot) = (1-t)\,\mathrm{id} + t\nabla\phi = \nabla\phi_t$ is bijective of inverse $\mathbf{p}_\phi(t,\cdot) = \nabla_x(\phi_t)^*$.*

Proposition 8.2. *Under the property* (Γ_2)*,* v *satisfies* (3-17)*, namely:*

$$\partial_t v + \frac{1}{2}\nabla_x |v|^2 = 0$$

in the sense of distributions.

Sketch of the proof: If potential ϕ is of class C^1, then by differentiating the advection relation $\nabla_x \phi - \mathrm{id} = \partial_t \nabla_x \phi_t = v(t, \nabla_x \phi_t)$, we obtain (see Example 8.1):

$$0 = \partial_{tt} \nabla_x \phi_t = (\partial_t v + v \cdot \nabla_x v)(\nabla_x \phi_t)$$

(recall: $\phi_t = (1-t)|\cdot|^2/2 + t\phi$). $\square$

In order to prove the second regularity result $\nabla_{t,x} v \in L^\infty_{loc}\big([0,1), L^1(\mathbb{R}^d)\big)$, we now present intermediate results on the potential ϕ.

Proposition 8.3. *We assume that* ϕ *satisfies the property* (Γ_2)*. Let* $R' > R > 0$ *and* $a \in \mathbb{R}^d$ *such that* $\phi(a) = \inf_{\mathbb{R}^d} \phi$*. Then there exists* $t_0 \in (0,1)$ *such that for all* $t \in [0, t_0]$*,*

$$\mathbf{p}_\phi(t, B(a,R)) \subset \mathbf{p}_\phi(t_0, B(a,R')). \tag{8-80}$$

Moreover, a sufficient condition to have property (8-80) *is:*

$$t_0 < \min\left\{\frac{1}{2}, \frac{R'-R}{M+2|a|}\right\} \quad \text{with} \quad M = \sup_{x \in B(a, 2(R+|a|))} |\partial\phi(x)|. \tag{8-81}$$

Sketch of the proof: Let $x \in B(a,R)$ and $t, t_0 \in [0,1)$ such that $t_0 > 0$ and $t \in [0, t_0]$. We have $\mathbf{p}_\phi(t,x) \in \mathbf{p}_\phi(t_0, B(a,R'))$ if and only if there exists $y \in B(a,R')$ such that $\mathbf{p}_\phi(t_0, y) = \mathbf{p}_\phi(t,x)$. According to the definition of $\mathbf{p}_\phi$ (by the equivalence (8-76)), for $y \in \mathbb{R}^d$ we have:

$$\mathbf{p}_\phi(t_0, y) = \mathbf{p}_\phi(t,x) \;\Leftrightarrow\; y = (1-t_0)\,\mathbf{p}_\phi(t,x) + t_0 \nabla\phi(\mathbf{p}_\phi(t,x)). \tag{8-82}$$

Let us take $y = (1-t_0)\,\mathbf{p}_\phi(t,x) + t_0 \nabla\phi(\mathbf{p}_\phi(t,x))$ and look for a sufficient condition on t_0 for $y \in B(a,R')$. From relation (4-22), we recall that $\mathbf{p}_\phi(t,(1-t)a) = a$ and therefore that for every $x \in B(a,R)$:

$$|\,\mathbf{p}_\phi(t,x) - a| = |\,\mathbf{p}_\phi(t,x) - \mathbf{p}_\phi(t,(1-t)a)| \le \frac{1}{1-t}|x-(1-t)a| \le \frac{1}{1-t_0}(R+|a|).$$

By taking $t_0 \le 1/2$, we thus have $\mathbf{p}_\phi(t,x) \in B(a, 2(R+|a|))$. Now, under the property (Γ_2), ϕ is of class C^1 on $\mathbb{R}^d$ and therefore locally Lipschitz, and thus Lipschitz on $B(a,2(R+|a|))$. We can therefore take

$$M = \sup_{x \in B(a,2(R+|a|))} |\partial\phi(x)| < +\infty.$$

For all $t \in [0,t_0]$, for $x \in B(a,R)$ and $y = (1-t_0)\,\mathbf{p}_\phi(t,x) + t_0\nabla\phi(\mathbf{p}_\phi(t,x))$, we get

$$\begin{aligned}
|y-a| &\le (1-t_0)|\,\mathbf{p}_\phi(t,x) - a| + t_0|\nabla\phi(\mathbf{p}_\phi(t,x)) - a| \\
&\le (1-t_0)|\,\mathbf{p}_\phi(t,x) - \mathbf{p}_\phi(t,(1-t)a)| + t_0\left(|\nabla\phi(\mathbf{p}_\phi(t,x))| + |a|\right) \\
&\le \frac{1-t_0}{1-t}|x-(1-t)a| + t_0\,(M+|a|) \le |x-a| + t|a| + t_0\,(M+|a|) \\
&\le R + t_0\,(M+2|a|)\,.
\end{aligned}$$

If we assume $t_0 < \min\{1/2, (R'-R)/(M+2|a|)\}$, then for all $t \in [0,t_0]$ and $x \in B(a,R)$, $y = (1-t_0)\,\mathbf{p}_\phi(t,x) + t_0\nabla\phi(\mathbf{p}_\phi(t,x)) \in B(a,R')$ i.e. $\mathbf{p}_\phi(t_0,y) = \mathbf{p}_\phi(t,x)$, and therefore we have the inclusion $\mathbf{p}_\phi(t,B(a,R)) \subset \mathbf{p}_\phi(t_0,B(a,R'))$. $\square$

Lemma 8.1. *We assume that ϕ satisfies the property (Γ_2). For every $t \in [0,1)$, $\mathbf{p}_\phi(t,\cdot)$ is differentiable almost everywhere on $\mathbb{R}^d$. Moreover, for almost every $x \in \mathbb{R}^d$, $\nabla\phi$ is differentiable in $\mathbf{p}_\phi(t,x)$ and $\mathrm{D}^2\phi$ is such that:*

$$\nabla_x \mathbf{p}_\phi(t,x) = (t\,\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x)) + (1-t)I)^{-1} \tag{8-83}$$

where $I \in \mathcal{M}_d(\mathbb{R})$ is the identity matrix.

Proof: Let $t \in [0,1)$. The operator $\mathbf{p}_\phi(t,\cdot)$ is Lipschitz and bijective, and from (8-76) its inverse is $\mathbf{p}_\phi(t,\cdot)^{-1} = t\nabla\phi + (1-t)\,\mathrm{id} = X(t,\cdot)$. Recall that by hypothesis $\nabla\phi$ is assumed to be Lipschitz. According to the Rademacher's Theorem 8.1, $\nabla\phi$ and $\mathbf{p}_\phi(t,\cdot)$ are differentiable almost everywhere on $\mathbb{R}^d$ and their gradients coincide with their derivatives in the sense of distributions. Thus the set F of points in $\mathbb{R}^d$ where $\nabla\phi$ is not differentiable is of zero Lebesgue measure. Since $\nabla\phi$ is assumed to be lipschitz, then so does $X(t,\cdot)$, which gives $\mathcal{L}^d(X(t,F)) = 0$ ([11] p. 75). As $X(t,F)$ is the set of points $x \in \mathbb{R}^d$ for which $\nabla\phi$ is not differentiable in $\mathbf{p}_\phi(t,x)$, this means that $\nabla\phi$ is differentiable in $\mathbf{p}_\phi(t,x)$ for almost all $x \in \mathbb{R}^d$. Hence $\mathbf{p}_\phi(t,\cdot)$ is differentiable at almost every $x \in \mathbb{R}^d$, $\nabla\phi$ is differentiableat $\mathbf{p}_\phi(t,x)$ and $I = (t\,\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x)) + (1-t)I)\nabla_x \mathbf{p}_\phi(t,x)$. The potential ϕ being convex, $\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x))$ is symmetric positive, and hence by coercivity, $t\,\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x)) + (1-t)I$ is symmetric positive definite and therefore invertible in $\mathcal{M}_2(\mathbb{R})$, which concludes the proof. □

Remark 8.2 (On the contribution of property (Γ_2) in the previous proof)**.** *Note that for every $t \in [0,1)$, the operator $\mathbf{p}_\phi(t,\cdot)$ is differentiable almost everywhere on $\mathbb{R}^d$, and the operator $\nabla_x \mathbf{p}_\phi(t,\cdot)$ is thus well defined. This property is satisfied regardless of the regularity of ϕ, since the proximal operator is always Lipschitz. The additional property brought here by the regularity C^1 of ϕ is in fact the bijectivity of the operator $\mathbf{p}_\phi(t,\cdot)$.*

*The fact that $\nabla\phi$ is locally lipschitz on $\mathbb{R}^d$ is crucial to ensure that $\mathrm{D}^2\phi$ is well defined almost everywhere. However, the fact that $\nabla\phi$ is **globally** lipschitz on $\mathbb{R}^d$ ensures that $\mathbf{p}_\phi(t,\cdot)$ does not send sets of $\mathbb{R}^d$ of positive measure to negligible sets (see [11] p. 75), such as sets where ϕ is not twice differentiable. Such global regularity ensures that the operator $\mathrm{D}^2\phi(\mathbf{p}_\phi(t,\cdot))$ is well defined almost everywhere.*

Then, the assumption of the property (Γ_2) allows us to consider $\nabla_x \mathbf{p}_\phi(t,\cdot)$ as a function of $\mathbf{p}_\phi(t,\cdot)$ almost everywhere.

We now have all the elements to state the following proposition, which is one of the main results of this section, concerning the control of the gradient of the velocity field v. This result is namely required to control the solutions of the transport problem generated by the field v in the uniqueness results of section 6.

For convenience, we will use the norm $|\cdot|_1$ on $\mathcal{M}_d(\mathbb{R})$, defined by $|A|_1 = \sum_{i,j}|a_{ij}|$, instead of the operator norm associated to the euclidean norm on $\mathbb{R}^d$.

Proposition 8.4. *We assume that ϕ satisfies the property (Γ_2). Let $R' > R > 0$ and $a \in \mathbb{R}^d$ such that $\phi(a) = \inf_{\mathbb{R}^d}\phi$. Then there exist constants C and C' (independent of ϕ, a, R and R') such that for all $t_0 \in (0,1)$ satisfying the condition (8-81), we have the property:*

$$\forall t \in (0,t_0],\ \int_{B(a,R)} |\nabla_x v(t,x)|_1\,dx \le C\int_{B(a,R')} |\nabla_x v(t_0,x)|_1\,dx \le \frac{C'}{t_0(1-t_0)}\mathcal{L}^2(B(a,R')) \tag{8-84}$$

so that $\nabla_x v \in L^\infty([0,t_0], L^1(B(a,R)))$.

Proof: Let $t \in (0,1)$. Remember that for all $x \in \mathbb{R}^d$, $v(t,x) = \nabla\phi(\mathbf{p}_\phi(t,x)) - \mathbf{p}_\phi(t,x)$. According to Lemma 8.1, for almost all $x \in \mathbb{R}^d$, $v(t,\cdot)$ is differentiable on x and

$$\begin{aligned} \nabla_x v(t,x) &= \nabla_x \mathbf{p}_\phi(t,x)(\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x)) - I) \\ &= (t\,\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x)) + (1-t)I)^{-1}(\mathrm{D}^2\phi(\mathbf{p}_\phi(t,x)) - I). \end{aligned} \tag{8-85}$$

Since, by (4-32), $v(t,\cdot)$ is Lipschitz with constant $2/t(1-t)$ for euclidean norm $|\cdot|$, it implies that $|\nabla_x v(t,\cdot)|_1$ is bounded by $c/t(1-t)$, where c a constant depending on the considered norm. Thus $|\nabla_x v(t,\cdot)|_1 \in L^\infty(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$. Since the function $X(t,\cdot) = t\nabla\phi + (1-t)\,\mathrm{id}$ is Lipschitz and bijective (and $\mathbf{p}_\phi(t,\cdot)$ is its inverse), we therefore can apply the generalized Change of Variable Theorem ([11] p. 117) and obtain:

$$\begin{aligned} \int_{B(a,R)} |\nabla_x v(t,x)|_1\,dx = \int_{\mathbf{p}_\phi(t,B(a,R))} &|\det(t\,\mathrm{D}^2\phi(y) + (1-t)I)| \\ &\times |(t\,\mathrm{D}^2\phi(y) + (1-t)I)^{-1}(\mathrm{D}^2\phi(y) - I)|_1\,dy. \end{aligned} \tag{8-86}$$

For every $y \in \mathbb{R}^d$ where $\nabla\phi$ is differentiable, the matrix $\mathrm{D}^2\phi$ is symmetric positive and can therefore be diagonalized in an orthonormal basis. We will consider $\lambda_1(y), \cdots, \lambda_d(y) \geq 0$ the eigenvalues associated with $\mathrm{D}^2\phi$. From the equivalence between the $|\cdot|_1$ norm and the Frobenius norm, we obtain the following relation:

$$\begin{aligned} C_1|(t\,\mathrm{D}^2\phi(y) + (1-t)I)^{-1}(\mathrm{D}^2\phi(y) - I)|_1 &\leq \sum_{i=1}^d \left|\frac{\lambda_i(y) - 1}{t\lambda_i(y) + (1-t)}\right|_1 \\ &\leq C_2|(t\,\mathrm{D}^2\phi(y) + (1-t)I)^{-1}(\mathrm{D}^2\phi(y) - I)|_1, \end{aligned} \tag{8-87}$$

where the constants C_1 and C_2 only depend on the constants of equivalence between the $|\cdot|_1$ norm and the Frobenius norm. Moreover, by invariance of the determinant through similarity transformations, we get for all $t \in [0,1)$ and almost all $y \in \mathbb{R}^d$:

$$\det(t\,\mathrm{D}^2\phi(y) + (1-t)I) = (t\lambda_1(y) + (1-t)) \cdots (t\lambda_d(y) + (1-t)). \tag{8-88}$$

By injecting (8-87) and (8-88) into (8-86), we obtain, for every $t \in (0,1)$, and every $R > 0$:

$$\begin{aligned} C_1 \int_{B(a,R)} |\nabla_x v(t,x)|_1\,dx &\leq \sum_{j=1}^d \int_{\mathbf{p}_\phi(t,B(a,R))} |\lambda_j(y) - 1| \prod_{\substack{i=1 \\ i\neq j}}^d (t\lambda_i(y) + (1-t)) \\ &\leq C_2 \int_{B(a,R)} |\nabla_x v(t,x)|_1\,dx. \end{aligned} \tag{8-89}$$

The constants C_1 and C_2 are independent of t and R. Let us take $t_0 \in (0,1)$ verifying the condition (8-81) of Proposition 8.3. Hence, we have $\mathbf{p}_\phi(t,B(a,R)) \subset \mathbf{p}_\phi(t_0,B(a,R'))$. The condition (8-81) also gives $t_0 \leq 1/2$, and so for any $t \in (0,t_0]$, we have $1-t \leq 1 \leq 2(1-t_0)$. Since $\lambda_i(y) \geq 0$ $(i = 1,\cdots,d)$ for almost all $y \in \mathbb{R}^d$, we have $0 \leq t\lambda_i(y) + (1-t) \leq 2(t_0\lambda_i(y) + (1-t_0))$.

Thus, thanks to the inequality (8-89), we can conclude:

$$\begin{aligned}\int_{B(a,R)} |\nabla_x v(t,x)|_1\,dx &\leq \frac{1}{C_1}\sum_{j=1}^{d}\int_{\mathbf{p}_\phi(t,B(a,R))} |\lambda_j(y)-1|\prod_{\substack{i=1\\i\neq j}}^{d}(t\lambda_i(y)+(1-t))\\ &\leq \frac{2^{d-1}}{C_1}\sum_{j=1}^{d}\int_{\mathbf{p}_\phi(t_0,B(a,R'))} |\lambda_j(y)-1|\prod_{\substack{i=1\\i\neq j}}^{d}(t_0\lambda_i(y)+(1-t_0))\\ &\leq \frac{2^{d-1}C_2}{C_1}\int_{B(a,R')} |\nabla_x v(t_0,x)|_1\,dx.\end{aligned} \tag{8-90}$$

Finally, as already mentioned at the beginning of this proof, there exists a constant $c>0$ such that $|\nabla_x v(t_0,\cdot)|_1$ is bounded by $c/t_0(1-t_0)$, which completes the proof of the Proposition. □

The proof of this proposition will be useful for another lemma concerning the control of v. Gathering the different results of this section concerning the control of the gradient of the field v , will allow us to control the solutions of the transport problem and thus obtain uniqueness results.

Lemma 8.2. *We assume that ϕ satisfies the property (Γ_2). Then there exists a constant $C>0$ such that for all $t\in(0,1)$,*

$$\|\nabla_x v(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{1-t}\left(\|\mathrm{D}^2\phi\|_{L^\infty(\mathbb{R}^d)}+d\right)$$

Proof: By taking the inequality (8-87) and the equivalence between $|\cdot|_1$ and Frobenius norms, as $\lambda_i(y)\geq 0$, we obtain, for every $y\in\mathbb{R}^d$ where $\nabla\phi$ is differentiable:

$$\begin{aligned}C_1|(t\,\mathrm{D}^2\phi(y)+(1-t)I)^{-1}(\mathrm{D}^2\phi(y)-I)|_1 &\leq \sum_{i=1}^{d}\left|\frac{\lambda_i(y)-1}{t\lambda_i(y)+(1-t)}\right|_1 \leq \frac{1}{1-t}\left(d+\sum_{i=1}^{d}\lambda_i(y)\right)\\ &\leq \frac{C_2}{1-t}\left(|\mathrm{D}^2\phi(y)|_1+d\right).\end{aligned} \tag{8-91}$$

We can then conclude by injecting the equation (8-85) into this last inequality. □

8.2 General case (Γ_1)

In the context of our optimal transport problem, we have proved at Proposition 3.2 that we can assume that the potential ϕ satisfies the property (Γ_1).

We are now able to extend the results of the previous subsection to potentials ϕ which satisfy (Γ_1), so that ϕ may have non-differentiability points **causing breaks in the transport plan**. More precisely we are now able to extend Proposition 8.2 to the case where ϕ only satisfies (Γ_1) (Proposition 4.5), and also to extend Proposition 8.4 (see proof of Proposition 6.2). In particular we will show that $\nabla_x v$ is uniformly integrable in the neighborhood of $t=0$, and in Theorem 8.2 we will argue symmetrically to show that it is the same in the neighborhood of $t=1$, and then on all $[0,1]$.

To that end we consider the regularization ${}^\gamma\phi$ of ϕ with the Moreau envelope defined for all $x\in\mathbb{R}^d$ and for all $\gamma>0$ as

$${}^\gamma\phi(x)=\inf_{y\in\mathbb{R}^d}\frac{1}{2\gamma}|x-y|^2+\phi(y)=\frac{1}{2\gamma}|x-\mathrm{Prox}_{\gamma\phi}(x)|^2+\phi(\mathrm{Prox}_{\gamma\phi}(x)). \tag{8-92}$$

For all $\gamma > 0$, we also define the velocity field v_γ for all $t \in (0,1)$ and $x \in \mathbb{R}^d$ by

$$v_\gamma(t,x) = v_{\gamma\phi}(t,x) = \nabla\phi(\mathbf{p}_{\gamma\phi}(t,x)) - \mathbf{p}_{\gamma\phi}(t,x) = \frac{x - \mathbf{p}_{\gamma\phi}(t,x)}{t}, \tag{8-93}$$

where

$$\mathbf{p}_{\gamma\phi}(t,x) = \mathrm{Prox}_{\frac{t}{1-t}{}^\gamma\phi}\left(\frac{x}{1-t}\right).$$

We also recall that ${}^\gamma\phi$ is of class C^1 and that for all $x \in \mathbb{R}^d$,

$$\nabla({}^\gamma\phi)(x) = \frac{x - \mathrm{Prox}_{\gamma\phi}(x)}{\gamma} \in \partial\phi\left(\mathrm{Prox}_{\gamma\phi}(x)\right) \quad \text{(by (4-22))}. \tag{8-94}$$

The potential ${}^\gamma\phi$ is then γ^{-1}-Lipschitz.

We now show that if ϕ satisfies the property (Γ_1), then ${}^\gamma\phi$ satisfies the (Γ_2) property. The results of the subsection 8.1 will then be applied on ϕ_γ, that corresponds to transport plans without breaks.

Lemma 8.3. *If ϕ satisfies the (Γ_1) property, then ${}^\gamma\phi$ statifies the property (Γ_2) for all $\gamma > 0$*

Sketch of the proof: As $({}^\gamma f)^* = f^* + \frac{\gamma}{2}|\cdot|^2$, then if f^* is in a Hölder space $C^{1,1}$ and admits a minimum on $\mathbb{R}^d$, the same holds for $({}^\gamma f)^*$. Notice on the other hand that the functions ϕ and ${}^\gamma\phi$ have the same minima on $\mathbb{R}^d$. □

Lemma 8.4. *We suppose that ϕ satisfies the property (Γ_1), then*

1. *the family $(\nabla{}^\gamma\phi)_{\gamma>0}$ is locally bounded, uniformly with respect to $\gamma > 0$.*
2. *for all $x \in \mathbb{R}^d$, the set of adherence values of the family $(\nabla{}^\gamma\phi(x))_\gamma$, when $\gamma > 0$ tends to 0 is included in $\partial\phi(x)$.*
3. *if ϕ is differentiable in x, then $\nabla{}^\gamma\phi(x)$ converges to $\nabla\phi(x)$ when $\gamma > 0$ tends to 0. The functions $\nabla{}^\gamma\phi$ converge simply almost everywhere to $\nabla\phi$ when γ tends to 0.*

Proof:

1. We can show that for every minimum a of ϕ on $\mathbb{R}^d$ and $r > 0$, we have the inclusion $\nabla({}^\gamma\phi)(B(a,r)) \subset \partial\phi\left(B(a,r)\right)$. To that end, it is sufficient to consider the relation (4-27) and the inclusion $\mathrm{Prox}_{\gamma\phi}(B(a,r)) \subset B(a,r)$, coming from the non-expansiveness of the operator $\mathrm{Prox}_{\gamma\phi}$ and the fact that a is a fixed point for these operators (see (4-22)). The union of the subdifferentials of ϕ on $B(a,r)$, denoted by $\partial\phi\left(B(a,r)\right)$, is bounded in $\mathbb{R}^d$, since the potential ϕ is convex and locally Lipschitz.
2. This can be obtained by applying the definition of the subdifferential to any sequence $[\nabla{}^{\gamma_n}\phi(x)]_n$ converging in $\mathbb{R}^d$ $(\gamma_n \to 0)$.
3. With the first point, we know that the family $(\nabla{}^\gamma\phi(x))_{\gamma>0}$ is bounded in $\mathbb{R}^d$, for all $x \in \mathbb{R}^d$. Hence, if ϕ is differentiable in x, we can extract a subsequence converging to $\nabla\phi(x)$ from any subsequence of $[\nabla{}^{\gamma_n}\phi(x)]_n$ (with $\gamma_n \to 0$). The last point can thus be proved by contradiction.

□

Proposition 8.5. *We assume that ϕ satisfies the property (Γ_1). The family of fields $(v_\gamma)_{\gamma>0}$ (with $v_\gamma = v_{\gamma\phi}$) satisfies the following convergence properties:*

1. *For every bounded open set ω in $\mathbb{R}^d$ and any $\gamma_0 > 0$, $(v_\gamma)_{\gamma_0 \geq \gamma > 0}$ is uniformly bounded on $(0,1) \times \omega$ by a constant independent of $\gamma \in (0, \gamma_0]$.*

2. *For every bounded open set ω in $\mathbb{R}^d$ and any $\gamma_0 > 0$, there exists a constant M independent of $\gamma \in (0, \gamma_0]$, such that*

$$\forall \gamma \in (0, \gamma_0],\ \forall t \in (0,1),\ \|v(t,\cdot) - v_\gamma(t,\cdot)\|_{L^\infty(\omega)} \leq M\frac{\gamma}{t}. \tag{8-95}$$

   We can deduce from (8-95) that v_γ pointwise converges to v on $(0,1) \times \mathbb{R}^d$ when γ goes to 0.

3. *For every $1 < p < +\infty$ and every bounded open set ω in $\mathbb{R}^d$, v_γ converges to the field v in $L^p((0,1) \times \omega)$ when γ goes to 0.*

4. *For every $1 < p < +\infty$, any bounded open set ω in $\mathbb{R}^d$ and for all $t \in (0,1)$, $v_\gamma(t,\cdot)$ converges weakly in $W^{1,p}(\omega)$ to $v(t,\cdot)$ when γ goes to 0.*

Sketch of the proof:

1. According to Proposition 4.2, we have $\|v_\gamma\|_{L^\infty((0,1)\times\omega)} \leq 5\left(\max\{M_\gamma, M_\gamma^*\} + \sup(\omega)\right)$, with $M_\gamma = \|\nabla^\gamma\phi\|_{L^\infty(\omega)}$ and $M_\gamma^* = \|\partial(^\gamma\phi)^*\|_{L^\infty(\omega)}$. According to Lemma 8.4, the constants M_γ are uniformly bounded with respect to $\gamma > 0$. The same holds for the constants M_γ^* with respect to $\gamma \in (0, \gamma_0]$ by noticing, according to the Property (4-20) of inf-convolution, that $\partial\,(^\gamma\phi)^* = \partial\phi^* + \gamma\,\mathrm{id}$.

2. From relation (4-31), we have the relation $\nabla_x(\phi_t)^* = (\mathrm{id} - \mathrm{Prox}_{(1-t)(t\phi)^*})/(1-t)$ for all $t \in (0,1)$. In addition, by applying twice the fundamental relation of the proximal operator (4-22) and the property (4-20), we have

$$\mathrm{Prox}_{(1-t)(t^\gamma\phi)^*} = \mathrm{Prox}_{(1-t)(t\phi)^* + (1-t)\frac{\gamma}{2t}|\cdot|^2} = \mathrm{Prox}_{(1-t)(t\phi)^*}\left(\mathrm{id} - \gamma(1-t)[\mathrm{id} + (1-t)v_\gamma(t,\cdot)]\right).$$

   Thus, according to (4-28) and (4-30), and by non-expansiveness of the proximal operator:

$$|v_\gamma(t,\cdot) - v(t,\cdot)| = \frac{1}{t(1-t)}\left|\mathrm{Prox}_{(1-t)(t^\gamma\phi)^*} - \mathrm{Prox}_{(1-t)(t\phi)^*}\right| \leq \frac{\gamma}{t}\left|\mathrm{id} + (1-t)v_\gamma(t,\cdot)\right|.$$

   Since the fields v_γ are uniformly bounded on $(0,1)\times\omega$ independently of $\gamma \in (0,\gamma_0]$ we thus get the relation (8-95).

3. This property is immediately deduced from the two previous ones by dominated convergence.

4. We also use dominated convergence to prove that $v_\gamma(t,\cdot)$ converges to $v(t,\cdot)$ in $L^p(\omega)$ as γ goes to 0. Note that the first two properties are valid on **all** $(0,1) \times \mathbb{R}^d$ and not only almost everywhere. Thus, in the sense of the distributions, any subsequence of $(v_\gamma(t,\cdot))_\gamma$ which weakly converges in $W^{1,p}(\omega)$ admits $v(t,\cdot)$ as limit. Since $v_\gamma(t,\cdot)$ and $v(t,\cdot)$ are $c/t(1-t)$-Lipschitz, where c is a constant depending only on the norm chosen on $\mathbb{R}^d$, we have $\|\nabla_x v_\gamma(t,\cdot)\|_{L^\infty(\omega)} \leq c/t(1-t)$, for all $\gamma > 0$. We can then extract a subsequence of $(v_\gamma(t,\cdot))_\gamma$ converging weakly to $v(t,\cdot)$ in $W^{1,p}(\omega)$, which is a reflexive space if $1 < p < +\infty$. We then conclude by contradiction: if the whole sequence is not converging, we could extract a subsequence such that no subsequence converge to $v(t,\cdot)$. By extracting a sub-subsequence that converges to $v(t,\cdot)$, we get a contradiction.

□

The first two points of ths latter Proposition imply the following corollary.

Corollary 8.1. *We assume that ϕ satisfies the property (Γ_1). For every bounded open set ω in $\mathbb{R}^d$ and any $\gamma_0 > 0$, there exists a constant C, independent of γ, such that the family $(v_\gamma)_{\gamma>0}$ satisfies:*

$$\forall \alpha \in [0,1], \forall \gamma \in (0,\gamma_0], \forall t \in (0,1), \ \|v(t,\cdot) - v_\gamma(t,\cdot)\|_{L^\infty(\omega)} \leq C\left(\frac{\gamma}{t}\right)^\alpha. \tag{8-96}$$

Proof: According to the first point of Proposition 8.5, there exists a constant K, independent of $\gamma \in (0,\gamma_0]$, such that $\|v(t,\cdot) - v_\gamma\|_{L^\infty((0,1)\times\omega)} \leq K$. According to (8-95), we then have for all $\alpha \in [0,1]$,

$$\|v(t,\cdot) - v_\gamma(t,\cdot)\|_{L^\infty(\omega)} \leq M^\alpha K^{1-\alpha}\left(\frac{\gamma}{t}\right)^\alpha \leq \max\{M,K\}\left(\frac{\gamma}{t}\right)^\alpha. \tag{8-97}$$

□

8.3 Finalization of the proof of Propositions 4.5 and 6.2

We now have enough elements to prove Proposition 4.5 (i.e. to prove that the field $v = v_\phi$ satisfies the Burger's equation in the general case) and Proposition 6.2.

Sketch of proof of Proposition 4.5: When ϕ satisfies the property (Γ_1), a transport plan may admit "breaks". To make the proof, we merely have to apply Proposition 8.5 in the weak formulation of the Burgers equation, and to involve Proposition 8.2. □

Sketch of proof of Proposition 6.2: As already mentioned in Lemma 8.4, we have the inclusion $\nabla({}^\gamma\phi)(B(a,r)) \subset \partial\phi(B(a,r))$, and thus, by setting $r = 2(R + |a|)$, we have $M_\gamma = \sup_{B(a,r)} |\nabla^\gamma\phi| \leq M$. Hence, a time t_0 verifying the hypothesis of the statement (independent of γ), also satisfies the hypothesis of Proposition 8.4 for all $\gamma > 0$. We can therefore apply this last proposition for such a t_0 for all v_γ and we get (6-59). For the case $\gamma = 0$ (i.e. $v_0 = v$), it is enough to apply the fourth convergence result of Proposition 8.5: we can conclude by weak lower semi-continuity of the L^1 norm on $B(a,r)$. □

8.4 An independent but notable regularity result

We now show that Proposition 6.2 can be generalized to $\nabla_{t,x}v \in L^\infty(0,1;L^1(\Omega))$ for every bounded open set Ω. This property is not required in the presented results on existence and uniqueness. It nevertheless gives a new insight of the regularity and the control of the velocity field of an isotropic optimal transport for the L^2 distance.

Theorem 8.2. *Assume that ϕ satisfies the property (Γ_1). Let Ω be a bounded open set of $\mathbb{R}^d$. Then $\nabla_{t,x}v \in L^\infty(0,1;L^1(\Omega))$, in other words there exists a constant $K > 0$ such that for all $t \in (0,1)$,*

$$\int_\Omega |\nabla_{t,x}v(t,x)|_1\,dx \leq K. \tag{8-98}$$

Sketch of the proof: We symmetrize the result of Proposition 6.2 (case $\gamma = 0$) respectively in the neighborhood of $t = 0$ and $t = 1$, and apply the upper bound $c/t(1-t)$ in the middle. □

Corollary 8.2. *We suppose that ϕ satifies the property (Γ_1). Let Ω be a bounded open set. For all $p, q \geq 1$ such that $1/p + 1/q > 1$, we have $\nabla_{t,x}v \in L^p(0,1;L^q(\Omega))$. And in particular $v \in W^{1,p}((0,1)\times\Omega)$ for all $1 \leq p < 2$.*

Sketch of the proof: We jointly use the result of Theorem 8.2 with the estimate by $c/t(1-t)$. In other words, we partially bound from above $|\nabla_x v(t_0,\cdot)|_1$ in order to be able to apply Theorem 8.2. For the particular case $W^{1,p}$, it is sufficient to take $p = q < 2$. $\square$

Theorem 8.2 and its Corollary 8.2 give the most consistent regularity that one can have for general velocity field v. Indeed, according to Corollary 8.2, a velocity field v defined with respect to a potential ϕ (verifying the property (Γ_1)) for all $t \in (0,1)$ by

$$v(t,\cdot) = v_\phi(t,\cdot) = \frac{\mathrm{id} - \mathbf{p}_\phi(t,\cdot)}{t} = \frac{1}{t}\left(\mathrm{id} - (t\phi + (1-t)\,\mathrm{id})^*\right), \tag{8-99}$$

is an element of $W^{1,p}_{loc}([0,1]\times\mathbb{R}^d)$. Hence the restriction of v to any bounded open set Ω of $\mathbb{R}^d$ is an element of $W^{1,p}((0,1)\times\Omega)$, for all $p < 2$. The question that arises naturally is whether $v = v_\phi$ could not be an element of $H^1_{loc}([0,1]\times\mathbb{R}^d)$. In general, this is not the case (see for instance Caffarelli's counter-example on mass splitting).

9 Convergence of the algorithm

The aim of this section is to demonstrate the weak convergence of the Benamou-Brenier algorithm, as well as the strong convergence of a relaxed version of the algorithm towards a saddle point of the Lagrangian $\mathbf{L}$. For this purpose, we will reformulate the problem of convergence of the algorithm towards one of these saddle points, into a more generic problem of convergence to a fixed point of non-expansive operator. We start by identifying the saddle points of $\mathbf{L}$ at the fixed points of an "iteration of the algorithm" operator.

Proposition 9.1. *There is an equivalence between a saddle point of* $\mathbf{L}$ *(and thus of* $\mathbf{L}_r$ *for all* $r > 0$*) and a fixed point of the algorithm (defined in the subsection 2.3):* (ψ, q, μ) *is a saddle point of* $\mathbf{L}$ *if and only if it remains invariant for the Benamou-Brenier algorithm.*

Proof: Let (ψ, q, μ) be a saddle point of $\mathbf{L}$. We denote by (ψ', q', μ') the new triplet obtained after one iteration of the algorithm. Let us show that $(\psi', q', \mu') = (\psi, q, \mu)$ in S. Fom property $(\mathrm{II})_2$ of (I), and taking $h = \psi' - \psi$, we obtain $\|\nabla_{t,x}(\psi' - \psi)\|^2 = 0$ in step A. According to the Poincaré inequality , we get $\psi' = \psi$ in $H^1((0,1)\times\Omega)/\mathbb{R}$. In step B, we look for the unique q' verifying $\langle \mu + \nabla\psi' - q', p - q'\rangle \leq 0$, for all $p \in \widetilde{\mathcal{P}}$. From Properties $(\mathrm{II})_1$ and $(\mathrm{II})_3$ which characterize a saddle point, we get $\psi = \psi'$. q is thus a good candidate and therefore the only one, hence $q' = q$. Finally, $\nabla_{t,x}\psi' = \nabla_{t,x}\psi = q = q'$ and we finally have $\mu' = \mu$ in step C.

Finally, let (ψ, q, μ) be a fixed point of the algorithm. Let us show that it is a saddle point of $\mathbf{L}$. Step C gives immediately $\nabla_{t,x}\psi = q$, and consequently step B gives $\langle \mu, p - q\rangle \leq 0$ for all $p \in \widetilde{\mathcal{P}}$, and step A gives $G(h) + \langle \mu, \nabla_{t,x} h\rangle = 0$ for all $h \in H^1(Q)$. Since the three properties (I) are verified, (ψ, q, μ) is therefore a saddle point of $\mathbf{L}$. $\square$

It is now possible, according to Proposition 9.1, to redefine our problem of convergence of the algorithm to a saddle point of Lagrangian $\mathbf{L}$, to a problem of convergence of a sequence of type $x_{n+1} = \mathbf{M}x_n$ to a fixed point of the operator $\mathbf{M}$. In subsection 9.2, we will see that it is possible to characterize such an operator $\mathbf{M}$ as a non-expansive type operator in an appropriate Hilbert space,.

General results on non-expansive operators are first recalled in the next subsection.

9.1 Some convergence results for non-expansive operators

For any non-expansive operator $\mathbf{M}$, we will state in this section a series of results allowing to obtain both **weak** convergence to a fixed point of $\mathbf{M}$ of the iterative algorithm $x_{n+1} = \mathbf{M}x_n$, and the **strong** convergence of a relaxed version of this algorithm.

We begin by recalling some useful standard definitions for the sequel. Let $(H, \langle . , . \rangle)$ an Hilbert space and let $\mathbf{M} : H \to H$.

Definition 9.1 (Non-expansive Operator)**.** *The operator* $\mathbf{M}$ *is called* ***non-expansive*** *if and only if it is* 1*-Lipschitz, and* ***firmly non-expansive*** *(implies non-expansive) if and only if we have* $\|\mathbf{M}x - \mathbf{M}y\|^2 \leq \langle x - y, \mathbf{M}x - \mathbf{M}y \rangle$*, for all* $x, y \in H$*.*

The operator $\mathbf{M}$ *is also called* ***quasi-firmly non-expansive*** *on a subset* A *of* H*, containing the set of fixed points of* $\mathbf{M}$*, if and only if, for any fixed point* x^* *of* $\mathbf{M}$*, we have*

$$\|x - x^*\|^2 - \|\mathbf{M}x - x^*\|^2 \geq \|x - \mathbf{M}x\|^2$$

for all $x \in A$*.*

We nos recall two main convergence results.

Theorem 9.1. *Let* $\mathbf{M}$ *be a non-expansive operator on* H*, and quasi-firmly non-expansive on* $\mathbf{M}(H)$ *(the image of* H *by* $\mathbf{M}$*). Assume that the set* $\mathrm{Fix}(\mathbf{M})$ *of the fixed points of* $\mathbf{M}$ *is non-empty. Let* $(x_n)_n$ *be a sequence of elements of* H *satisfying for every* $n \in \mathbb{N}$ *the estimate:* $\|\mathbf{M}(x_n) - x_{n+1}\| \leq \epsilon_n$*, where* $(\epsilon_n)_n$ *is a non-negative real sequence satisfying* $\sum\limits_n \epsilon_n < +\infty$*. Then* $(x_n)_n$ *weakly converges in* H *to a fixed point of* $\mathbf{M}$*.*

Theorem 9.2 (H. Bauschke [1])**.** *Let* $\mathbf{M}$ *be a non-expansive operator and assume that the set* $\mathrm{Fix}(\mathbf{M})$ *of the fixed points of* $\mathbf{M}$ *is non-empty, and let* $(\lambda_n)_{n \geq 0}$ *be a sequence of parameters of* $[0, 1)$*, converging to* 0*, and satisfying:* $\sum_n \lambda_n = +\infty$ *and* $\sum_n |\lambda_{n+1} - \lambda_n| < +\infty$*. Let* H *be a Hilbert space, and* $\mathbf{M}$ *a non-expansive operator over* H*. Given* a *and* x_0 *in* H*, we define the sequence* (x_n) *by the recurrence* $x_{n+1} = \lambda_n a + (1 - \lambda_n)\mathbf{M}_n x_n$ *(*$\forall n \geq 0$*), where for all* $n \in \mathbb{N}$ *is verified the estimate* $\|\mathbf{M}_n x_n - \mathbf{M}x_n\| \leq \epsilon_n$*, with* $\sum_n \epsilon_n < +\infty$*.*

Then the sequence $(x_n)_n$ *converges strongly to* $P_F a$ *(where* $F = \mathrm{Fix}(\mathbf{M})$ *is the closed convex set of fixed points of* $\mathbf{M}$*.*

The sequence $(\epsilon_n)_n$ here represents the inevitable numerical errors inherent in the implementation of such an algorithm. It is assumed here that these errors are highly controlled, which is not realistic in practice.

Theorem 9.1 can be shown using classical functional analysis tools (see for instance [19]) such as the Opial's Lemma [22]. The second theorem (Theorem 9.2) is a result due to H. Bauschke in [1]. The detailed proofs of those two theorems can be found in [15] (section 2.3 and appendix B). Notice that in Theorem 9.2, it is easy to prove that the set of fixed points of a non-expansive operator is a closed convex set (see Lemma 2.3-7 of [15]).

9.2 Formulation of the Benamou-Brenier algorithm in terms of non-expansive operator

We are now able to apply the convergence results sof the previous subsection in the context of the Benamou-Brenier algorithm. To that end, it is sufficient to show that an iteration of the algorithm can be considered as the iteration of a certain non-expansive operator.

We consider the space $H = L^2(Q)^{d+1} \times L^2(Q)^{d+1}$, provided with the scalar product

$$\langle (\mu_1, q_1), (\mu_2, q_2) \rangle_H = \langle \mu_1, \mu_2 \rangle_{L^2} + r^2 \langle q_1, q_2 \rangle_{L^2},$$

so that $(H, \langle .,. \rangle)$ is an Hilbert space.

Let $\mathbf{B} : H \to H$ be the operator which associate to (μ, q) the product (μ', q') of the last two steps (B and C) of the algorithm Benamou-Brenier. Here ψ is an auxiliary variable just use for intermediate computations: indeed, if (μ^*, q^*, ψ^*) is a saddle point of the Lagrangian $\mathbf{L}$ (and thus $\mathbf{L}_r$) defined in (2-9), then $(\mu^*, q^*) = \mathbf{B}(\mu^*, q^*)$. Conversely if (μ^*, q^*) is a fixed point of $\mathbf{B}$ then (μ^*, q^*, ψ^*) is a saddle point of the Lagrangian, where ψ^* is the unique element of S which satisfies $q^* = \nabla\psi^*$. The potential ψ is therefore only required for computational purposes.

Proposition 9.2. *The operator* $\mathbf{B}$ *is non-expansive on* H*, and quasi-firmly non-expansive on* $\mathbf{B}(H)$*.*

Proof: (μ_1, q_1) and (μ_2, q_2) being given, we determine $(\mu_1', q_1') = \mathbf{B}(\mu_1, q_1)$ and $(\mu_2', q_2') = \mathbf{B}(\mu_2, q_2)$ by the following iteration (see subsection 2.3). For $i = 1, 2$, we look for:

- Step A : The unique $\psi_i' \in (H^1/\mathbb{R})(Q)$ such that $G(h) + \langle \mu_i, \nabla h \rangle_{L^2} + r\langle \nabla\psi_i' - q_i, \nabla h \rangle_{L^2} = 0$, for all $h \in (H^1/\mathbb{R})(Q)$.
- Step B : The unique q_i' such that $\langle \mu_i + r(\nabla\psi_i' - q_i'), p - q_i' \rangle_{L^2} \leq 0$, for all $p \in \widetilde{\mathcal{P}}$.
- Step C : We define μ_i' by $\mu_i' = \mu_i + r(\nabla_{t,x}\psi_i' - q_i')$.

Let us start by studying the non-expansiveness of $\mathbf{B}$. Note that by injecting the equation of step C into step A and step B (for $i = 1$ or $i = 2$), we obtain the two new equations:

$$\forall h \in H^1(Q)/\mathbb{R},\ \ G(h) + \langle \mu_i', \nabla h \rangle_{L^2} + r\langle q_i' - q_i, \nabla h \rangle_{L^2} = 0,\ \text{et}\ p \in \widetilde{\mathcal{P}},\ \langle \mu_i', p - q_i' \rangle_{L^2} \leq 0. \quad (9\text{-}100)$$

Let us call them $(9\text{-}100)^1$ and $(9\text{-}100)^2$. We set:

$$\overline{\mu}^{(')} = \mu_2^{(')} - \mu_1^{(')},\ \overline{q}^{(')} = q_2^{(')} - q_1^{(')} \text{ and } \overline{\psi}^{(')} = \psi_2^{(')} - \psi_1^{(')}.$$

Respectively, by taking $(9\text{-}100)^1_{i=2} - (9\text{-}100)^1_{i=1}$ and $h = \overline{\psi}'$, and by summing $(\text{step B})_{i=2}$ with $p = q_1'$, and $(\text{step B})_{i=1}$ with $p = q_2'$, we obtain respectively the two following relations:

$$\langle \overline{\mu}', \nabla\overline{\psi}' \rangle + r\langle \overline{q}' - \overline{q}, \nabla\overline{\psi}' \rangle = 0 \quad \text{and} \quad \langle \overline{\mu}', \overline{q}' \rangle \geq 0. \quad (9\text{-}101)$$

denoted as $(9\text{-}101)^1$ and $(9\text{-}101)^2$.

By summing these two relations, we then have $\langle \overline{\mu}', \nabla\overline{\psi}' - \overline{q}' \rangle + r\langle \overline{q}' - \overline{q}, \nabla\overline{\psi}' \rangle \leq 0$. By factoring the term $|\overline{\mu}|^2 - |\overline{\mu}'|^2$, we then obtain

$$\begin{aligned} |\overline{\mu}|^2 - |\overline{\mu}'|^2 &= \langle \overline{\mu} - \overline{\mu}', \overline{\mu} + \overline{\mu}' \rangle = -r\langle \nabla\overline{\psi}' - \overline{q}', 2(\overline{\mu} + r(\nabla\overline{\psi}' - \overline{q}')) - r(\nabla\overline{\psi}' - \overline{q}') \rangle \\ &= -2r\langle \overline{\mu}', \nabla\overline{\psi}' - \overline{q}' \rangle + r^2|\nabla\overline{\psi}' - \overline{q}'|^2 \geq 2r^2\langle \nabla\overline{\psi}', \overline{q}' - \overline{q} \rangle + r^2|\nabla\overline{\psi}' - \overline{q}'|^2. \end{aligned} \quad (9\text{-}102)$$

Moreover, we have
$$\begin{aligned} \langle \nabla\overline{\psi}', \overline{q}' - \overline{q} \rangle &= \langle \nabla\overline{\psi}' - \overline{q}, \overline{q}' - \overline{q} \rangle + \langle \overline{q}, \overline{q}' - \overline{q} \rangle \\ &= (1/2)\left(|\overline{q}'|^2 - |\overline{q}|^2 + |\overline{q}' - \overline{q}|^2\right) + \langle \nabla\overline{\psi}' - \overline{q}, \overline{q}' - \overline{q} \rangle - |\overline{q}' - \overline{q}|^2 \\ &= (1/2)\left(|\overline{q}'|^2 - |\overline{q}|^2 + |\overline{q}' - \overline{q}|^2\right) + \langle \nabla\overline{\psi}' - \overline{q}', \overline{q}' - \overline{q} \rangle. \end{aligned} \quad (9\text{-}103)$$

By re-injecting (9-103) in (9-102), we get:

$$\begin{aligned} |\overline{\mu}|^2 - |\overline{\mu}'|^2 &\geq r^2\left((|\overline{q}'|^2 - |\overline{q}|^2) + \left(|\overline{q}' - \overline{q}|^2 + 2\langle \nabla\overline{\psi}' - \overline{q}', \overline{q}' - \overline{q} \rangle + |\nabla\overline{\psi}' - \overline{q}'|^2\right)\right) \\ &\geq r^2(|\overline{q}'|^2 - |\overline{q}|^2) + r^2|\nabla\overline{\psi}' - \overline{q}|^2, \end{aligned} \quad (9\text{-}104)$$

$$\text{i.e.} \quad (|\overline{\mu}|^2 + r^2|\overline{q}|^2) - (|\overline{\mu}'|^2 + r^2|\overline{q}'|^2) \geq r^2|\nabla\overline{\psi}' - \overline{q}|^2 \geq 0. \quad (9\text{-}105)$$

Now, we have

$$\|(\mu_1,q_1)-(\mu_2,q_2)\|_H=|\overline{\mu}|^2+r^2|\overline{q}|^2 \text{ and } \|\mathbf{B}(\mu_1,q_1)-\mathbf{B}(\mu_2,q_2)\|_H=|\overline{\mu}'|^2+r^2|\overline{q}'|^2 :$$

the operator $\mathbf{B}$ is therefore non-expansive.

We now demonstrate the quasi-firmly non-expansiveness of $\mathbf{B}$ on $\mathbf{B}(H)$. Let (μ^*,q^*) a fixed point of $\mathbf{B}$ (hence include in $\mathbf{B}(H)$), $x=(\mu,q)\in\mathbf{B}(H)$ let us define $(\mu',q')=\mathbf{B}(\mu,q)=\mathbf{B}x$. By using equation (9-105) with

$$(\overline{\psi},\overline{q},\overline{\mu})=(\psi-\psi^*,q-q^*,\mu-\mu^*)\text{and }(\overline{\psi}',\overline{q}',\overline{\mu}')=(\psi'-\psi^*,q'-q^*,\mu'-\mu^*),$$

we obtain:

$$(|\mu-\mu^*|^2+r^2|q-q^*|^2)-(|\mu'-\mu^*|^2+r^2|q'-q^*|^2)\geq r^2|\nabla(\psi'-\psi^*)-(q-q^*)|^2. \quad \text{(9-106)}$$

For the second term, we get

$$\begin{aligned}r^2|\nabla(\psi'-\psi^*)-(q-q^*)|^2&=r^2|\nabla\psi'-q|^2=|(\mu'-\mu)+r(q'-q)|^2\\&=|\mu'-\mu|^2+2r\langle\mu'-\mu,q'-q\rangle+r^2|q'-q|^2,\end{aligned}$$

with $\langle\mu'-\mu,q'-q\rangle\geq 0$, according to the relation $(9\text{-}101)^2$ (recalling that $(\mu,q)\in\mathbf{B}(H)$). In the same way as in [12], we thus obtain:

$$(|\mu-\mu^*|^2+r^2|q-q^*|^2)-(|\mu'-\mu^*|^2+r^2|q'-q^*|^2)\geq|\mu'-\mu|^2+r^2|q'-q|^2,$$

that is to say, $\|(\mu,q)-(\mu^*,q^*)\|_H^2-\|(\mu',q')-(\mu^*,q^*)\|_H^2\geq\|\mathbf{B}(\mu,q)-(\mu,q)\|_H^2$. □

By applying Theorem 9.1 to the operator $\mathbf{B}$, we are then able to show the weak-L^2 convergence of the Benamou-Brenier algorithm to a fixed point of $\mathbf{B}$ (i.e. to a saddle point of $\mathbf{L}$) whose existence is justified by Theorem 3.1. Similarly, by applying Theorem 9.2, we can easily define a relaxed version of the algorithm with strong-L^2 convergence. This non-expansive operator approach has recently been used to show the convergence, weak or strong, of proximal splitting algorithms [5].

Proposition 9.2 also justifies the **convexity** (and closure) of the set of saddle points of $\mathbf{L}$ (property mentioned in the schematic proof of Lemma 6.1). The set of fixed points of a non-expansive operator is indeed a closed convex set. The operator $\mathbf{B}$ immediately gives us this convexity for the components μ and q. The characteristic $(\text{II})_3$ of the properties (I) as well as the linearity of the gradient operator $\nabla_{t,x}$ transfer this convexity to the component ψ, and therefore to the set of saddle points of $\mathbf{L}$.

10 Conclusion and perspectives

The starting point of our work is the study of the Lagrangian augmented algorithm of Benamou-Brenier [3]. We have shown in the section 9 the weak convergence of this algorithm to a saddle point of the Lagrangian $\mathbf{L}$, which models the dynamic formulation of the optimal transport problem. This proof of convergence is based on a reformulation of the algorithm as an iterative sequence of a non-expansive operator $\mathbf{B}$, whose fixed points are equivalent to the saddle points of $\mathbf{L}$. This formulation allows us to exploit the literature associated with the theory of non-expansive operators, from which we were able to propose a relaxed version of the algorithm that strongly converge to a saddle point of $\mathbf{L}$. However, numerical experiments did not reveal any real speed or accuracy improvement of the relaxed algorithm with respect to original one.

The convergence of the algorithm and its relaxed version is conditioned to the existence of a saddle point for the Lagrangian $\mathbf{L}$, and therefore of a fixed point of the operator $\mathbf{B}$. Hence in sections 4 and 5, we have tackled the problem of existence of such saddle points. In section 6, we have then shown the uniqueness of the evolution of the density and the momentum resulting from such transport. Our study has been carried out in the most general conditions, especially in cases where the starting and arrival densities ρ_0 and ρ_1 could vanish on some subset of the transport domain. As far as we know, this is the first mathematical convergence proof of the Benamou-Brenier algorithm for vanishing densities in L^2.

Such conditions imply to take into account the case where the number of connected components of the support of the densities ρ_0 and ρ_1 are not the same. Such cases generally exhibit non-regular optimal transport plans. This is the reason why a large part of our work (namely sections 4 and 8) has involved an in-depth study of the regularity and behavior of a velocity field associated with transport plans.

This study also gave us two opportunities to obtain parallel results. Firstly, in subsection 8.4, we have stated new results about regularity and control of a velocity field associated to an optimal transport. Conversely, in section 7, we have proposed to state minimal properties that a velocity field has to satisfy in order to be associated to an optimal transport in L^2 (see Theorem 7.1).

In forthcoming works, we would like to analyze the convergence properties of general dynamic optimal transport algorithms: stopping or distance criterias with respect to the solution, theoretical information on the speed of convergence, etc.

Another perspective concerns the studies realized in sections 4, 5 and 6, on the existence and uniqueness of solutions of the dynamic optimal transport problem in L^2. We would like to extend such results to cases of dynamic optimal transport operating in less classical environments, especially non-isotropic domains (see [17]), or more generally within Riemannian manifolds.

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