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CONTINUOUS FIELDS OF PROPERLY INFINITE C*-ALGEBRAS

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ABSTRACT. Any unital separable continuous C(X)-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space X has finite topological dimension. We study conditions under which this is still the case in the infinite dimensional case.

1. Introduction

One of the basic C*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C*-algebras through K-theoretical invariants is the Cuntz C*-algebra $O_{\infty}$ generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous C*-algebra $A$ the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra $D$ is a trivial C(X)-algebra provided the compact Hausdorff base space $X$ has finite topological dimension. Indeed, the strongly self-absorbing C*-algebra $D$ tensorially absorbs the Jiang-Su algebra $Z$ ([Win09]). Hence, this C*-algebra $D$ is $K_1$-injective ([Rør04]) and the C(X)-algebra $A$ satisfies $A \cong D \otimes C(X)$ ([DW08]).

But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product $\Pi_{n=0}^{\infty} S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2\textsuperscript{\infty} ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dăd09, §3] for all pair $(\Gamma_0, \Gamma_1)$ of countable abelian torsion groups a unital separable continuous C(X)-algebra $A$ such that

- the base space $X$ is the compact Hilbert cube $X = B_{\infty}$ of infinite dimension,
- all the fibres $A_x$ ($x \in B_{\infty}$) are isomorphic to the strongly self-absorbing Cuntz C*-algebra $O_{\infty}$ generated by two isometries $s_1, s_2$ satisfying $1_{O_{\infty}} = s_1 s_1^* + s_2 s_2^*$,
- $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $Y_0 \subset [0, 1]$ is the canonical Cantor set.

These $K$-theoretical conditions imply that the $C(B_{\infty})$-algebra $A$ is not a trivial one. But these arguments do not work anymore when the strongly self-absorbing algebra $D$ is the Cuntz algebra $O_{\infty}$ ([Cun77]), in so far as $K_0(O_{\infty}) = \mathbb{Z}$ is a torsion free group.

We describe in this article the link between several notions of proper infiniteness for C(X)-algebras which appeared during the recent years ([KR00], [BRR08], [CEI08], [RR11]). We then study whether certain unital continuous C(X)-algebras with fibres $O_{\infty}$, especially the Pimsner-Toeplitz algebra ([Pim95]) of Hilbert C(X)-modules with

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fibres of dimension greater than 2 and with compact base space $X$ of infinite topological dimension, are properly infinite C*-algebras.

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2. A FEW NOTATIONS

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0, 1, 2, \ldots \}$ the set of positive integers and we denote by $[S]$ the closed linear span of a subset $S$ in a Banach space.

**Definition 2.1.** ([Dix69], [Kas88], [Blan97]) Let $X$ be a compact Hausdorff space and let $C(X)$ be the C*-algebra of continuous function on $X$.

- A unital $C(X)$-algebra is a unital C*-algebra $A$ endowed with a unital morphism of C*-algebra from $C(X)$ to the centre of $A$.
- For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a|_F$ the image of $a$ in the quotient $A|_F := A/C_0(X \setminus F) \cdot A$. If $x \in X$ is a point in $X$, one calls fibre at $x$ the quotient $A_x := A_{\{x\}}$ and one write $a_x$ for $a_{\{x\}}$.
- The $C(X)$-algebra $A$ is said to be continuous if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$ is continuous for all $a \in A$.

**Remarks 2.2.**
a) ([Cun81], [BRR08]) For all integer $n \geq 2$, the C*-algebra $T_n := T(C^n)$ is the universal unital C*-algebra generated by $n$ isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1s_1^* + \ldots + s_ns_n^* \leq 1.$$  

b) A unital C*-algebra $A$ is said to be properly infinite if and only if one the following equivalent conditions holds true ([Cun77], [Rør03, Proposition 2.1]):

- the C*-algebra $A$ contains two isometries with mutually orthogonal range projections, i.e. $A$ unitally contains a copy of $T_2$,
- the C*-algebra $A$ contains a unital copy of the simple Cuntz C*-algebra $O_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges.

c) If $A$ is a C*-algebra and $E$ is a Hilbert $A$-module, one denotes by $\mathcal{L}(E)$ the set of adjointable $A$-linear operators acting on $E$ ([Kas88]).

3. GLOBAL PROPER INFINITENESS

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous $C(X)$-algebras with properly infinite fibres.

**Proposition 3.1.** Let $X$ be a second countable perfect compact Hausdorff space, i.e. without any isolated point, and let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres.

1) There exist:

(a) a finite integer $n \geq 1$,
(b) a covering $X = F_1^{\circ} \cup \ldots \cup F_n^{\circ}$ by the interiors of closed balls $F_1, \ldots, F_n$,
(c) unital embeddings of $C^*$-algebra $\sigma_k : \mathcal{O}_\infty \hookrightarrow A_{|F_k}$ ($1 \leq k \leq n$).

2) The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integer $p$.

**Proof.** 1) For all point $x \in X$, the semiprojectivity of the $C^*$-subalgebra $\mathcal{O}_\infty \hookrightarrow A_x$ ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood $F \subset X$ of the point $x$ and a unital embedding $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A_{|F}$ of $(C(F))$-algebra. The compactness of the topological space $X$ enables to conclude.

2) Proposition 2.7 of [BRR08] entails that the $C^*$-algebra $M_{2^{n-1}}(A)$ is properly infinite. Proposition 2.1 of [Rør97] implies that $M_p(A)$ for all integer $p \geq 2^{n-1}$.

**Remark 3.2.** If $X$ is a second countable compact Hausdorff space and $A$ is a separable unital continuous $C(X)$-algebra, then $\tilde{X} := X \times [0, 1]$ is a perfect compact space, $\tilde{A} := A \otimes C([0, 1])$ is a unital continuous $C(\tilde{X})$-algebra and every morphism of unital $C^*$-algebra $\mathcal{O}_\infty \hookrightarrow \tilde{A}$ induces a unital #homomorphism $\mathcal{O}_\infty \hookrightarrow \tilde{A}$ coming from the injection $x \in X \mapsto (x, 0) \in \tilde{X}$.

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the $C^*$-algebra $A$ is properly infinite ([HR98]). Indeed, there exists a unital $C^*$-algebra $A$ which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is a properly infinite $C^*$-algebra ([Rør03, Proposition 4.5]). The following corollary nevertheless holds true.

**Corollary 3.3.** Let $j_0, j_1$ denote the two canonical unital embeddings of the Cuntz extension $\mathcal{T}_2$ in the full unital free product $\mathcal{T}_2 \ast_\mathbb{C} \mathcal{T}_2$ and let $\tilde{u} \in U(\mathcal{T}_2 \ast_\mathbb{C} \mathcal{T}_2)$ be a $K_1$-trivial unitary satisfying $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$ ([BRR08, Lemma 2.4]).

The following assertions are equivalent:

(a) The full unital free product $\mathcal{T}_2 \ast_\mathbb{C} \mathcal{T}_2$ is $K_1$-injective.

(b) The unitary $\tilde{u}$ belongs to the connected component $U^0(\mathcal{T}_2 \ast_\mathbb{C} \mathcal{T}_2)$ of $1_{\mathcal{T}_2 \ast_\mathbb{C} \mathcal{T}_2}$.

(c) Every separable unital continuous $C(X)$-algebra $A$ with properly infinite fibres is a properly infinite $C^*$-algebra.

**Proof.** (a)$\Rightarrow$(b) A unital $C^*$-algebra $A$ is called $K_1$-injective if and only if all $K_1$-trivial unitaries $v \in U(A)$ are homotopic to the unit $1_A$ in $U(A)$ (see e.g. [Roh09]). Thus, (b) is a special case of (a) since $K_1(\mathcal{T}_2 \ast_\mathbb{C} \mathcal{T}_2) = \{1\}$ (see e.g. [Blan10, Lemma 4.4]).

(b)$\Rightarrow$(c) Let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ such that there exist unital embeddings $\sigma_k : \mathcal{T}_2 \rightarrow A_{|\mathcal{F}_k}$ for all $1 \leq k \leq n$. Set $G_k := \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k \subset X$ and let us construct by induction isometries $w_k \in A_{|G_k}$ such that the two projections $w_kw_k^*$ and $1_{|G_k} - w_kw_k^*$ are properly infinite and full in the restriction $A_{|G_k}$:

- If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

- If $k \in \{1, \ldots, n-1\}$ and the isometry $w_k \in A_{|G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exists a morphism of unital $C^*$-algebra $\pi_k :
\( T_2 \ast_C T_2 \to A_{|G_k \cap F_{k+1}} \) satisfying
\[
\begin{align*}
- \quad \pi_k(j_0(s_1)) &= w_k |G_k \cap F_{k+1}, \\
- \quad \pi_k(j_1(s_1)) &= \sigma_{k+1}(s_1)|G_k \cap F_{k+1} = \pi_k(\tilde{u}) \cdot w_k |G_k \cap F_{k+1}.
\end{align*}
\] (3.1)

If the unitary \( \tilde{u} \) belongs to the connected component \( U^0(T_2 \ast_C T_2) \), then \( \pi_k(\tilde{u}) \) is homotopic to \( 1_{A_{|G_k \cap F_{k+1}}} = \pi_k(1_{T_2 \ast_C T_2}) \) in \( U(A_{|G_k \cap F_{k+1}}) \), so that \( \pi_k(\tilde{u}) \) admits a unitary lifting \( z_{k+1} \) in \( U^0(A_{|F_{k+1}}) \) \( \) (see e.g. [LLR00, Lemma 2.1.7]). The only isometry \( w_{k+1} \in A_{|G_{k+1}} \) satisfying the two constraints
\[
\begin{align*}
- \quad w_{k+1}|G_k &= w_k, \\
- \quad w_{k+1}|F_{k+1} &= (z_{k+1})^* \cdot \sigma_{k+1}(s_1)
\end{align*}
\]
verifies that the two projections \( w_{k+1} w_{k+1}^* \) and \( 1_{G_{k+1}} - w_{k+1} w_{k+1}^* \) are properly infinite and full in \( A_{|G_{k+1}} \).

The proper infiniteness of the projection \( w_n^* w_n \) in \( A_{|G_n} = A \) implies that the unit \( 1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n \) is also a properly infinite projection in \( A \), i.e. the C*-algebra \( A \) is properly infinite.

(c)⇒(a) The C*-algebra \( \mathcal{D} := \{ f \in C([0,1], T_2 \ast_C T_2) : f(0) \in j_0(T_2) \text{ and } f(1) \in j_1(T_2) \} \) is a unital continuous \( C([0,1]) \)-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra \( \mathcal{D} \) is properly infinite, a statement which is equivalent to the \( \mathcal{K}_1 \)-injectivity of \( T_2 \ast_C T_2 \) \( \) ([Blan10, Proposition 4.2]). \( \square \)

\textbf{Remark 3.4.} The sum \( \tilde{u} \oplus 1 \) belongs to \( U^0(M_2(T_2 \ast_C T_2)) \) \( \) ([Blan10]).

\section{4. A question of proper infiniteness}

We describe in this section the link between the different notions of proper infiniteness which have been introduced during the last decades.

Recall first the link between properly infinite C*-algebras and properly infinite Hilbert C*-modules studied by K. T. Coward, G. Elliott and C. Ivanescu.

\textbf{Proposition 4.1.} ([CEI08]) Let \( A \) be a stable C*-algebra and let \( a \in A_+ \) be a positive element. The following assertions are equivalent:
\begin{itemize}
  \item[(a)] \( a \) is properly infinite in \( A \), i.e. \( a \oplus a \preceq a \) in \( \mathcal{K} \otimes A \) \( \) ([KR00, definition 3.2]).
  \item[(b)] There is an embedding of Hilbert \( A \)-module \( \ell^2(\mathbb{N}) \otimes A \hookrightarrow [aA] \) \( \) ([CEI08]).
\end{itemize}

\textbf{Proof.} (a)⇒(b) If there exists an infinite sequence \( \{ d_i \} \) in \( A \) such that \( a = d_i^* d_i \geq \sum_{j \in \mathbb{N}} d_j d_j^* \) for all \( i \in \mathbb{N} \), then \( [aA] \supset \sum_j [d_j A] \cong \ell^2(\mathbb{N}) \otimes A \).

(b)⇒(a) One has embeddings of Hilbert \( A \)-modules \([aA] \oplus [aA] \subset \ell^2(\mathbb{N}) \otimes A \subset [aA] \). \( \square \)

The following holds for continuous deformations of properly infinite C*-algebras.

\textbf{Proposition 4.2.} Let \( X \) be a second countable compact Hausdorff space, let \( H \) be a separable Hilbert \( C(X) \)-module with non-zero fibres and let \( a \in \mathcal{K}(H) \) be a strictly positive compact contraction. Consider the following assertions:
\begin{itemize}
  \item[(a)] All the Hilbert spaces \( H_x \) are properly infinite, i.e. the operator \( a_x \) is properly infinite in the C*-algebra \( \mathcal{K}(H)_x \cong \mathcal{K}(H_x) \) for all \( x \in X \).
\end{itemize}
(b) $H$ is a properly infinite Hilbert $C(X)$-module, i.e. $a$ is properly infinite in $\mathcal{K}(H)$.
(c) The multiplier $C^*$-algebra $\mathcal{L}(H) = \mathcal{M}(\mathcal{K}(H))$ is properly infinite.

Then (c) $\Rightarrow$ (b) $\Rightarrow$ (a). But (a) $\not\Rightarrow$ (b) and (b) $\not\Rightarrow$ (c).

Proof. (c)$\Rightarrow$(b) If $\sigma : T_2 = C^* < s_1, s_2 > \rightarrow \mathcal{L}(H)$ is a unital $*$-homomorphism, then the two elements $d_1 = \sigma(s_1) \cdot a^{1/2}$ and $d_2 = \sigma(s_2) \cdot a^{1/2}$ satisfy $d_1^*d_2 = \delta_{i=j} \cdot a$ in $\mathcal{K}(H)$.

(b)$\Rightarrow$(a) The relations $c_i^*c_j = \delta_{i=j} \cdot a$ between 3 operators $c_1, c_2, a$ in a $C(X)$-algebra $D$ entails that $(c_i)^*(c_j)x = \delta_{i=j} \cdot a_x$ in the quotient $D_x = D/C_0(X \setminus \{x\}) \cdot D$ for all $x \in X$.

(a)$\not\Rightarrow$(b) Let $B_3^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; x_1^2 + x_2^2 + x_3^2 \leq 1\}$ be the unit ball of dimension 3, let $B_3^+, B_3^- = \{(x_1, x_2, x_3) \in B_3 ; x_3 < \frac{1}{2}\}$ and let $S_2 = \{(x_1, x_2, x_3) \in B_3 ; x_1^2 + x_2^2 + x_3^2 = 1\} \subset B_3$ be the unit sphere of dimension 2.

The self-adjoint operator $f \in C(B_3) \otimes M_2(\mathbb{C}) \cong C(B_3, M_2(\mathbb{C}))$ given by

$$f(x_1, x_2, x_3) = \frac{1}{2} \left( \begin{array}{cc} 1 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & 1 - x_3 \end{array} \right)$$

is a positive contraction since each self-adjoint matrix $f(x_1, x_2, x_3) \in M_2(\mathbb{C})$ satisfies

$$f(x_1, x_2, x_3)^2 = f(x_1, x_2, x_3) + \frac{1}{4} (x_1^2 + x_2^2 + x_3^2 - 1) \cdot 1_{M_2(\mathbb{C})},$$

i.e. $(f(x_1, x_2, x_3) - \frac{1}{2} \cdot 1_{M_2(\mathbb{C})})^2 \leq \frac{1}{4}(x_1^2 + x_2^2 + x_3^2 - 1) \cdot 1_{M_2(\mathbb{C})} \leq \left(\frac{3}{2} \cdot 1_{M_2(\mathbb{C})}\right)^2$.

The nontrivial Hilbert $C(B_3)$-module $F := [f \cdot \left(\frac{C(B_3)}{C(B_3)}\right)]$ satisfies the two isomorphisms of Hilbert $C(B_3)$-module:

$$F \cdot C_0(B_3^+) \cong C_0(B_3^+) \oplus C_0(B_3^- \setminus S_2 \cap B_3^+)$$

$$F \cdot C_0(B_3^-) \cong C_0(B_3^-) \oplus C_0(B_3 \setminus S_2 \cap B_3^-).$$

The set $B_\infty := \{x \in \ell^2(\mathbb{N}) ; \sum_{p} |x_p|^2 \leq 1\}$ is a metric compact space called the complex Hilbert cube when equipped with the distance $d((x_p), (y_p)) = \sum_p 2^{-p-2} |x_p - y_p|$. Denote by $E_{DD}$ the non-trivial Hilbert $C(B_\infty)$-module with fibres $\ell^2(\mathbb{N})$ constructed by J. Dixmier and A. Douady ([DD63], §17], [BK04a, Proposition 3.6]).

Finally, consider the product $X := B_\infty \times B_3$ and the Hilbert $C(X)$-module

$$H := E_{DD} \otimes C(B_3) \oplus C(B_\infty) \otimes F.$$

The two Hilbert $C(X)$-submodules $H \cdot C_0(B_\infty \times B_3^+)$ and $H \cdot C_0(B_\infty \times B_3^-)$ are properly infinite, i.e. there exist embeddings of Hilbert $C(X)$-module $\ell^2(\mathbb{N}) \otimes C_0(B_\infty \times B_3^+) \hookrightarrow H \cdot C_0(B_\infty \times B_3^+)$ and $\ell^2(\mathbb{N}) \otimes C_0(B_\infty \times B_3^-) \hookrightarrow H \cdot C_0(B_\infty \times B_3^-)$. Hence, all the fibres of the Hilbert $C(X)$-module $H$ are properly infinite Hilbert spaces, i.e. $\ell^2(\mathbb{N}) \hookrightarrow H_x$ for all point $x$ in the compact space $X = B_\infty \times B_3^+ \cup B_\infty \times B_3^-$. (The equality $\mathcal{C}(B_3) = C_0(B_3^+) + C_0(B_3^-)$ only implies that $\ell^2(\mathbb{N}) \otimes C(X) \hookrightarrow H \oplus H$.)

(b)$\not\Rightarrow$(c) There exists a continuous field $\tilde{H}$ of Hilbert spaces over the compact space $Y := B_\infty \times (B_3)^\infty$ such that $\tilde{H} = [a \cdot H]$ for some properly infinite contraction $a \in \mathcal{K}(\tilde{H})$.
and the C*-algebra $L(\tilde{H})$ is not properly infinite. Indeed, let $\eta \in \ell^\infty(B_\infty,\ell^2(\mathbb{N}) \oplus \mathbb{C})$ be the section $x \mapsto x \oplus \sqrt{1-\|x\|^2}$, let $\tilde{F}$ be the closed Hilbert $C(B_\infty)$-module $\tilde{F} := [C(B_\infty,\ell^2(\mathbb{N}) \oplus 0) + C(B_\infty) \cdot \eta]$; let $\theta_{\eta,\eta} \in L(\tilde{F})$ be the projection $\zeta \mapsto \eta(\eta,\zeta)$ and let $E_{DD} = (1-\theta_{\eta,\eta}) \cdot \tilde{F}$ be the Hilbert $C(B_\infty)$-submodule built in [DD63]. Define also the sequence of contractions $\tilde{f} = (\tilde{f}_n)$ in $\ell^\infty\left(\mathbb{N}, M_2(C((B_3)\infty))\right)$ by

$$x_n = (x_{n,k}) \in (B_3)\infty \mapsto \tilde{f}_n(x_n) := f(x_{n,n}) \in M_2(\mathbb{C}).$$

(4.5)

The Hilbert $C(Y)$-module

$$\tilde{H} := C(Y) \oplus E_{DD} \otimes C((B_3)\infty) \oplus C(B_\infty) \otimes [\tilde{f} \cdot \ell^2\left(\mathbb{N}, \left(\frac{C((B_3)\infty)}{C((B_3)\infty)}\right)\right)]$$

(4.6)

has the desired properties ([RR11, Example 9.13]).

**Question 4.3.** What can be written when the operator $a$ in Proposition 4.2 is a projection in a unital continuous $C(X)$-algebra?

- Is the full unital free product $T_2 +\mathbb{C} T_2$ a properly infinite C*-algebra which is not $K_1$-injective? (see the equivalence (a)$\iff$(c) in Corollary 3.3)

5. **The Pimsner-Toeplitz algebra of a Hilbert $C(X)$-module**

We look in this section at the proper infiniteness of the unital continuous $C(X)$-algebras with fibres $O_\infty$ corresponding to the Pimsner-Toeplitz $C(X)$-algebras of Hilbert $C(X)$-modules with infinite dimension fibres.

**Definition 5.1.** ([Pim95]) Let $X$ be a compact Hausdorff space and let $E$ be a full Hilbert $C(X)$-module, i.e. without any zero fibre.

a) The full Fock Hilbert $C(X)$-module $F(E)$ of $E$ is the direct sum

$$F(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)})m},$$

(5.1)

where $E^{(\otimes_{C(X)})m} := \begin{cases} C(X) & \text{if } m = 0; \\ E \otimes_{C(X)} \cdots \otimes_{C(X)} E \text{ (m terms)} & \text{if } m \geq 1. \end{cases}$

b) The Pimsner-Toeplitz $C(X)$-algebra $T(E)$ of $E$ is the unital subalgebra of the $C(X)$-algebra $L(F(E))$ of adjointable $C(X)$-linear operators acting on $F(E)$ generated by the creation operators $\ell(\zeta) (\zeta \in E)$, where

$$- \ell(\zeta) (f \cdot \hat{1}_{C(X)}) := f \cdot \zeta = \zeta \cdot f \quad \text{for } f \in C(X) \quad \text{and}$$

$$- \ell(\zeta_1 \otimes \cdots \otimes \zeta_k) := \zeta_1 \otimes \zeta_1 \otimes \cdots \otimes \zeta_k \quad \text{for } \zeta_1, \ldots, \zeta_k \in E \quad \text{if } k \geq 1.$$  

(5.2)

c) Let $(C^*(\mathbb{Z}), \Delta)$ be the abelian compact quantum group generated by a unitary $u$ with spectrum the unit circle and with coproduct $\Delta(u) = u \otimes u$. Then, there is a unique coaction $\alpha$ of the Hopf C*-algebra $(C^*(\mathbb{Z}), \Delta)$ on the Pimsner-Toeplitz $C(X)$-algebra $T(E)$ such that $\alpha(\ell(\zeta)) = \ell(\zeta) \otimes u$ for all $\zeta \in E$, i.e.

$$\alpha : T(E) \to T(E) \otimes C^*(\mathbb{Z}) = C(T, T(E))$$

$$\ell(\zeta) \mapsto \ell(\zeta) \otimes u = (z \mapsto \ell(z\zeta))$$

(5.3)

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The fixed point $C(X)$-subalgebra $\mathcal{T}(E)^a = \{a \in \mathcal{T}(E) : a(a) = a \otimes 1\}$ under this coaction is the closed linear span

$$\mathcal{T}(E)^a = \left[ C(X) \cdot 1_{\mathcal{T}(E)} + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^* \right].$$

(5.4)

Besides, the projection $P \in \mathcal{L}(\mathcal{F}(E))$ onto the submodule $E$ induces a quotient morphism of $C(X)$-algebra $a \in \mathcal{T}(E)^a \mapsto \overline{a}(a) := P \cdot a \cdot P \in \mathcal{K}(E) + C(X) \cdot 1_{\mathcal{L}(E)} \subset \mathcal{L}(E)$.

**Proposition 5.2.** Let $X$ be a second countable compact Hausdorff perfect space and let $E$ be a separable Hilbert $C(X)$-module with infinite dimensional fibres.

1) There exist a covering $X = F_1 \cup \ldots \cup F_m$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ norm 1 sections $\xi_1, \ldots, \xi_m$ in $E$ such that $\mathcal{T}(E) = C(X) \cdot 1_{\mathcal{T}(E)^a} \cdot (\ell(\xi_1))^* \cdot \ldots \cdot (\ell(\xi_m))^*$, $(\ell(\xi_k), (\ell(\xi_k)^*)_{|F_k})$ and $(1 - \ell(\xi_k), (\ell(\xi_k)^*)_{|F_k})$ are properly infinite projections in $\mathcal{T}(E)|_{F_k}$ for all index $k \in \{1, \ldots, m\}$.

2) Set $G_l := F_1 \cup \ldots \cup F_l$ for all integer $k \in \{1, \ldots, m\}$ and $\overline{G}_1 := G_l \cap F_{l+1}$ for all integer $l \in \{1, \ldots, m - 1\}$. If $\eta(l) \in E|_{\overline{G}_l}$ is a section such that $\|\eta(l)\| = 1$ for all $y \in \overline{G}_l$, then there is a unitary $w_l \in \mathcal{T}(E)|_{\overline{G}_l}$ such that

(a) $w_l \cdot (\ell(\eta(l)))_{|\overline{G}_l} = (\ell(\eta(l+1)))_{|\overline{G}_l}$,

(b) $w_l + 1_{|\overline{G}_l}$ is homotopic to $1_{|\overline{G}_l} + 1_{|\overline{G}_l}$ among the unitaries in $\mathcal{M}_2(\mathcal{T}(E)|_{\overline{G}_l})$.

3) If for all $K_1$-trivial unitary $w_k \in \mathcal{T}(E)|_{\overline{G}_k}$ there is a unitary $w_{k+1} \in \mathcal{T}(E)^a|_{\overline{G}_{k+1}}$ such that $(\varphi_k)_{|\overline{G}_k} = w_k (1 \leq k \leq m - 1)$, then there is a section $\xi \in E$ satisfying

$$\forall x \in X, \quad \|\xi_x\| = 1,$$

(5.5)

so that Lemma 6.1 of [Bran13] implies that the $C^*$-algebra $\mathcal{T}(E)$ is properly infinite.

**Proof.** 1) Given a point $x \in X$ and a unit vector $\xi \in E_x$, let $\xi_1, \xi_2, \xi_3$ be three norm 1 sections in $E$ such that $(\eta(1)) = \xi$ and the matrix $a = [\xi_1, \xi_2, \xi_3] \in M_3(C(X))$ satisfies $a_x = 1_3 \in M_3(\mathbb{C})$. Let $F \subset X$ be a closed neighbourhood of $x$ such that $\|a_y - 1_3\| \leq 1/2$ for all $y \in F$. Define the sections $\xi'_1, \xi'_2, \xi'_3$ in $E|_F$ by

$$\begin{pmatrix} \xi'_1_v \\ \xi'_2_v \\ \xi'_3_v \end{pmatrix} = \begin{pmatrix} \xi_1_v \\ \xi_2_v \\ \xi_3_v \end{pmatrix} \cdot \begin{pmatrix} a_1^* a_2 \\ a_2^* a_3 \\ a_3^* a_1 \end{pmatrix}^{-1/2} \quad \text{for all } y \in F.$$

(5.6)

One has $[\xi'_i, \xi'_j] = (a_{1_F} a_{1_F})^{-1/2} \cdot (a_{1_F} a_{1_F})^{-1/2} = 1$ in $M_3(C(F))$. Hence, $\ell(\xi'_1) \ell(\xi'_i)^* = \ell(\xi'_2) \ell(\xi'_3)^*$ and $q := 1_{|\overline{F}} - \ell(\xi'_1) \ell(\xi'_1)^* + \ell(\xi'_2) \ell(\xi'_2)^* = 1_{|\overline{F}}$ in $M_3(C(F))$. Hence, so projected infinite projections in $\mathcal{T}(E)|_{\overline{F}}$ since

$$-\ell(\xi'_1) \ell(\xi'_i)^* \geq \ell(\xi'_2) \ell(\xi'_2)^* \geq \ell(\xi'_3) \ell(\xi'_3)^* \geq \ell(\xi'_1) q \ell(\xi'_1)^* \ell(\xi'_1)^* \ell(\xi'_1)^* + \ell(\xi'_1) q^2 \ell(\xi'_2)^* \ell(\xi'_2)^* \ell(\xi'_3)^* \ell(\xi'_3)^* \ell(\xi'_3)^*,$$

(5.7)

so that there exist unital $*$-homomorphisms from $\mathcal{T}_2$ to $(1 - q) \cdot \mathcal{T}(E)|_{\overline{F}} \cdot (1 - q)$ and $q \cdot \mathcal{T}(E)|_{\overline{F}} \cdot q$ given by $s_1 \mapsto \ell(\xi'_1) \ell(\xi'_1)^* \ell(\xi'_1)^*$ and $s_i \mapsto \ell(\xi'_i) q^i$ for $i = 1, 2$.

The compactness of the space $X$ enables to end the proof of this first assertion.

2) Let $v_l \in \mathcal{T}(E)|_{\overline{G}_l}$ be the partial isometry $v_l := (\ell(\eta_{l+1}))_{|\overline{G}_l} \cdot (\ell(\eta(l)))_{|\overline{G}_l}$. There exists by Lemma 2.4 of [BRRR08] a $K_1$-trivial unitary $w_l$ in the properly infinite unital $C^*$-algebra $\mathcal{T}(E)|_{\overline{G}_l}$ which has the two requested properties (a) and (b).
3) One constructs inductively the restrictions $\xi_{[G_k]}$ in $E_{[G_k]}$. Set $\xi_{[G_1]} := \xi_1$ and assume $\xi_{[G_k]}$ already constructed. As $\ell(\xi_{[G_k]})|_{G_{k+1}} = z_{k+1} \cdot \ell(\xi_{[G_k]}|_{G_{k+1}})$, the only extension $\xi_{[G_{k+1}]} \in E_{[G_{k+1}]}$ such that $(\xi_{[G_{k+1}]}|_{G_k}) = \xi_{G_k}$ and $(\xi_{[G_{k+1}]}|_{F_{k+1}}) = \bar{q}(z_{k+1})^* \cdot (\xi_{[k+1]}|_{F_{k+1}})$ satisfies $\|\xi_{[G_{k+1}]}\| = 1$ for all point $x \in G_{k+1}$.  

Remarks 5.3. a) The nontrivial separable Hilbert $C(B_{\infty})$-module $E_{DD}$ f constructed by J. Dixmier and A. Douady ([DD63]) has infinite dimensional fibres and every section $\zeta \in E_{DD}$ satisfies $\zeta_{x} = 0$ for at least one point $x \in B_{\infty}$. Thus, it cannot satisfy the assumptions for the assertion 3) of Proposition 5.2. There are some $k \in \{1, \ldots, m-1\}$ and a unitary $a_{k+1} \in U^0(M_2(\mathcal{T}(E_{DD})|_{F_{k+1}}))$ such that 

\[
(a_{k+1})|_{G_k} = w_k \oplus 1|_{G_k}
\]

and either $a_{k+1} \notin \mathcal{T}(E_{DD})|_{F_{k+1}} \oplus C(F_{k+1})$ or $\alpha(a_{k+1}) \neq a_{k+1} \otimes 1$.

b) If each of the $K_{1}$-trivial unitaries $w_l$ introduced in assertion 2) of Proposition 5.2 satisfies $\alpha(w_l) = w_l \otimes 1$ and $w_l \sim_k 1|_{G_k}$ in $U(\mathcal{T}(E))^*|_{G_k}$, then there exist by [LLR00, Lemma 2.1.7] $m - 1$ unitaries $z_{l+1} \in \mathcal{T}(E)^*|_{F_{l+1}}$ such that $(z_{l+1})|_{G_k} = w_l$, so that there exists a section $\xi \in E$ with $\xi_x = 0$ for all $x \in X$.

c) Let $A$ be a separable unital continuous $C(B_{\infty})$-algebra with fibres isomorphic to $O_{2}$ such that $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $(\Gamma_0, \Gamma_1)$ is a pair of countable abelian torsion groups ([Dâd09, §3]). Let $\varphi$ be a continuous field of faithful states on $A$. Then the $C(B_{\infty})$-algebra $A' \subset \mathcal{L}(L^2(A, \varphi))$ generated by $\pi_\varphi(A)$ and the algebra of compact operators $\mathcal{K}(L^2(A, \varphi))$ is a continuous $C(B_{\infty})$-algebra since both the ideal $\mathcal{K}(L^2(A, \varphi))$ and the quotient $A \cong A' / \mathcal{K}(L^2(A, \varphi))$ are continuous (see e.g. [Blan09, Lemma 4.2]). All the fibres of $A'$ are isomorphic to the Cuntz extension $\mathcal{T}_2$. But $A'$ is not a trivial $C(B_{\infty})$-algebra since $K_0(A') = C(Y_0, \Gamma_0) \oplus \mathbb{Z}$ and $K_1(A') = C(Y_0, \Gamma_1)$.

Question 5.4. The Pimsner-Toeplitz algebra $\mathcal{T}(E_{DD})$ is locally purely infinite ([BK04b, Definition 1.3]) since all its simple quotients are isomorphic to the Cuntz algebra $O_{\infty}$ ([BK04b, Proposition 5.1]). But is $\mathcal{T}(E_{DD})$ properly infinite?

REFERENCES


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