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A global uniqueness result for acoustic tomography of moving fluid

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We consider a model time-harmonic wave equation of acoustic tomography of moving fluid in an open bounded domain in dimension $d \geq 2$. We give global uniqueness theorems for related inverse boundary value problem at fixed frequency.

Keywords: inverse boundary value problems, time-harmonic wave equation, acoustic tomography

Subjects: partial differential equations, mathematical physics

AMS classification: 35R30 (Inverse problems), 35Q35 (PDEs in connection with fluid mechanics)

1 Introduction

Consider the operator

$$L_{A,V} = -\Delta - 2iA(x) \cdot \nabla + V(x),$$

where $\Delta$ is the standard Laplacian, $x \in D$, $A \in W^{1,\infty}(D, \mathbb{R}^d)$, $V \in L^\infty(D, \mathbb{R})$, $D$ is an open bounded domain in $\mathbb{R}^d$ ($d \geq 2$). In the present article we study an inverse boundary value problem for the equation $L_{A,V} \psi = 0$ in $D$.

As in [AN], [RW], [RE] we consider the equation $L_{A,V} \psi = 0$ as a model equation for a time-harmonic ($e^{-i\omega t}$) pressure $\psi$ in moving fluid. In this setting

$$A(x) = \frac{\omega}{c^2(x)} v(x), \quad V(x) = -\frac{\omega^2}{c^2(x)},$$

where $v$ is the fluid velocity vector, $c$ is the sound speed, $\omega$ is the frequency.

Suppose that $0$ is not a Dirichlet eigenvalue for operator $L_{A,V}$ in $D$. Then the Dirichlet problem

$$\begin{cases}
L_{A,V} \psi = 0 & \text{in } D, \\
\psi|_{\partial D} = f,
\end{cases}$$

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is uniquely solvable for $\psi \in H^1(D)$ for any $f \in H^{1/2}(\partial D)$. The Dirichlet-to-Neumann map $\Lambda_{A,V}$ sends $f \in H^{1/2}(\partial D)$ to $\Lambda_{A,V}f \in H^{-1/2}(\partial D)$ defined by the formula

$$\Lambda_{A,V}f = \frac{\partial u}{\partial \nu}|_{\partial D} + i(A \cdot \nu)f,$$  

(3)

where $\nu$ is the unit exterior normal to $\partial D$ and $\frac{\partial u}{\partial \nu}|_{\partial D} \in H^{-1/2}(\partial D)$ can be defined, in particular, by the following formula:

$$\langle \frac{\partial u}{\partial \nu}|_{\partial D}, u \rangle = \int_D \left( \nabla \psi(x) \nabla \tilde{u}(x) - 2i\tilde{u}(x)A(x)\nabla \psi(x) + \tilde{u}(x)V(x)\psi(x) \right) dx,$$  

(4)

for $u \in H^{1/2}(\partial D)$ and arbitrary $\tilde{u} \in H^1(D)$ with $\tilde{u}|_{\partial D} = u$. Note that since $\psi$ satisfies $L_{A,V}\psi = 0$, the right hand side of the above formula doesn't depend on the choice of $\tilde{u}$.

The inverse boundary value problem for equation $L_{A,V}\psi = 0$ in $D$ consists in finding $A, V$ from $\Lambda_{A,V}$. In the case when coefficients $A, V$ can be complex-valued there is an obstruction to the unique solvability of this problem caused by the gauge invariance of the map $\Lambda_{A,V}$ with respect to the gauge transformations

$$\begin{align*}
A &\rightarrow A + \nabla \varphi, \\
V &\rightarrow V - i\Delta \varphi + (\nabla \varphi)^2 + 2AV\varphi,
\end{align*}$$

where $\varphi \in W^{2,\infty}(D, \mathbb{C})$, $\varphi|_{\partial D} = 0$, see, e.g., [KU] ($d \geq 3$), [GT] ($d = 2$).

However, in the case of real-valued coefficients $A, V$ there is no gauge non-uniqueness as it was observed, for example, in [AN].

In addition, in general case, under some regularity assumptions on $\partial D$, $A$ and $V$, the Dirichlet-to-Neumann map $\Lambda_{A,V}$ uniquely determines the two-form $dA$ and the function $q$ in $D$ and the tangential component of $A$ on $\partial D$, where

$$dA = \sum_{1 \leq k < l \leq d} \left( \frac{\partial A^l}{\partial x_k} - \frac{\partial A^k}{\partial x_l} \right) dx_k \wedge dx_l,$$  

(5)

$$q = V + i\nabla \cdot A - A \cdot A,$$

and $A = (A^1, \ldots, A^d)$.

In particular, it was shown in [KU] that in dimension $d \geq 3$ the map $\Lambda_{A,V}$ uniquely determines $dA$ and $q$ in $D$ if $A \in L^\infty(D, \mathbb{C}^d)$ and $V \in L^\infty(D, \mathbb{C})$. And in dimension $d = 2$ it was shown in [GT] that if $D$ is a smooth Riemann surface with boundary (in particular, if $D$ is a planar domain with $\partial D \in C^\infty$) then the map $\Lambda_{A,V}$ uniquely determines $dA$ and $q$ provided that $A \in W^{2,p}(D, \mathbb{R}^d)$, $V \in W^{1,p}(D, \mathbb{C})$, $p > 2$.

In addition, concerning the identifiability of tangential components of $A$ on the boundary, it was proved in [BS] that if $\partial D \in C^1$ ($d \geq 3$) or $\partial D \in C^{1,\alpha}$, $\alpha \in (0, 1)$ ($d = 2$) and if $A \in C(D, \mathbb{C}^d)$, $V \in L^\infty(D, \mathbb{C})$ then $\Lambda_{A,V}$ uniquely determines $A - \nu(A \cdot \nu)$ on $\partial D$, where $\nu$ is the unit exterior normal field to $\partial D$.

In the present article we combine the aforementioned results in order to obtain the following global uniqueness results in the case when coefficients $A, V$ are real-valued.
Theorem 1. Let $D$ be a bounded simply connected domain with path connected boundary in $\mathbb{R}^d$ ($d \geq 3$) with $\partial D \in C^1$. Let $A_1, A_2 \in W^{1,\infty}(D, \mathbb{R}^d)$ and $V_1, V_2 \in L^\infty(D, \mathbb{R})$. If $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$, then $A_1 = A_2, V_1 = V_2$.

Theorem 2. Let $D$ be a bounded simply connected domain in $\mathbb{R}^2$ with $\partial D \in C^\infty$. Let $A_1, A_2 \in W^{2,p}(D, \mathbb{R}^d)$ and $V_1, V_2 \in W^{1,p}(D, \mathbb{R})$ with $p > 2$. If $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$ then $A_1 = A_2$ and $V_1 = V_2$.

Theorems 1 and 2 are proved in Section 3. In Section 2 we present formulas and equations for finding $A, V$ from $dA, q, A - \nu(A \cdot \nu)|_{\partial D}$.

## 2 Formulas and equations for finding $A, V$

In this section we suppose that $D$ is a bounded contractible domain with path connected $C^2$ boundary in $\mathbb{R}^d$ ($d \geq 2$). By contractibility we mean that there exists $F \in C^2(D \times [0,1], D)$ such that $F_0 \equiv \bar{x}$, $F_1 = \text{id}_D$, where $F_t(x) = F(x, t)$, $\bar{x}$ is some fixed point in $D$ and $\text{id}_D$ is the identity mapping on $D$. We also suppose that $A \in W^{2,\infty}(D, \mathbb{R}^d)$, $V \in L^\infty(D, \mathbb{R})$. Given $dA, q$ as in (5) in $D$ and $A - \nu(A \cdot \nu)$ on $\partial D$, we can find $A, V$ in the following way:

1. Define $\bar{A} = (\bar{A}^1, \ldots, \bar{A}^d) \in W^{1,\infty}(D, \mathbb{R}^d)$ by the formula
   \[
   \bar{A}^k = \sum_{i < j} \int_0^1 \left( \frac{\partial F^i}{\partial t} \frac{\partial F^j}{\partial x_k} - \frac{\partial F^j}{\partial t} \frac{\partial F^i}{\partial x_k} \right) \left( \frac{\partial A^j}{\partial x_i} \circ F_i - \frac{\partial A^i}{\partial x_j} \circ F_i \right) dt,
   \]
   with $k = 1, \ldots, d$; $A = (A^1, \ldots, A^d)$, $F_i = (F^i_1, \ldots, F^i_d)$ and $\circ$ denotes the composition of maps, i.e. $\frac{\partial A^i}{\partial x_j} \circ F_i(y) = \frac{\partial A^i}{\partial x_j}(F_i(y))$, $y \in D$.

2. Fix $x^0 \in \partial D$. Define $\varphi_0 \in C^1(\partial D)$ by the formula
   \[
   \varphi_0(x) = \sum_{k=1}^d \int_{x^0}^x (A^k_t(y) - \bar{A}_t^k(y)) dy_k, \quad x \in \partial D,
   \]
   where $A_t = A - \nu(A \cdot \nu)$, $\bar{A}_t = \bar{A} - \nu(\bar{A} \cdot \nu)$, $A_t = (A^1_t, \ldots, A^d_t)$, $\bar{A}_t = (\bar{A}^1_t, \ldots, \bar{A}^d_t)$, $\nu$ is the unit exterior normal field to $\partial D$ and integration is over an arbitrary $C^1$ curve on $\partial D$ linking $x^0$ to $x$.

3. Find the unique generalized solution $\varphi \in W^{2,\infty}(D, \mathbb{R})$ to
   \[
   \begin{cases}
   \Delta \varphi = \text{Im} \, q - \nabla \cdot \bar{A} & \text{in } D, \\
   \varphi|_{\partial D} = \varphi_0.
   \end{cases}
   \]

4. Coefficients $A, V$ are given by the following formulas:
   \[
   \begin{aligned}
   A &= \bar{A} + \nabla \varphi, \\
   V &= q - i\Delta \varphi - i\nabla \cdot \bar{A} + \bar{A} \cdot \bar{A} + 2\bar{A} \cdot \nabla \varphi + (\nabla \varphi)^2.
   \end{aligned}
   \]

This algorithm will be justified in Section 4.
3 Proofs of Theorems 1, 2

We will prove Theorems 1 and 2 simultaneously. Let $D, A_1, A_2, V_1, V_2$ satisfy the conditions of Theorem 1 (resp. Theorem 2) and suppose that $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$.

Using Theorem 1.1 of [BS] we obtain that
\[(A_1 - \nu(A_1 \cdot \nu))|_{\partial D} = (A_2 - \nu(A_2 \cdot \nu))|_{\partial D}, \tag{9}\]
where $\nu$ is the unit exterior normal field to $\partial D$. Using Theorem 1.1 of [KU] (resp. Theorem 1.1 of [GT]) we get
\[dA_1 = dA_2 \text{ in } D, \tag{10}\]
\[q_1 = q_2 \text{ in } D, \tag{11}\]
where
\[dA_j = \sum_{1 \leq k < l \leq d} \left( \frac{\partial A^l_j}{\partial x_k} - \frac{\partial A^k_j}{\partial x_l} \right) dx_k \wedge dx_l, \]
\[q_j = V_j + i \nabla \cdot A_j - A_j \cdot A_j, \]
where $A_j = (A^1_j, \ldots, A^d_j), j = 1, 2$.

Since the domain $D$ is simply connected it follows from (10) that there exists $\varphi \in W^{2,\infty}(D, \mathbb{R})$ such that
\[A_1 - A_2 = \nabla \varphi \text{ in } D. \tag{12}\]

In dimension $d = 2$ it follows from simple connectedness of $D$ and from smoothness of $\partial D$ that $\partial D$ is path connected. Formulas (9), (12) and path connectedness of $\partial D$ imply that $\varphi$ is constant on $\partial D$.

Using (11), (12) we obtain that
\[V_1 - V_2 = -i \Delta \varphi - (\nabla \varphi)^2 + 2A_1 \nabla \varphi \text{ in } D. \]
Taking the imaginary part of this equation we obtain the equation $\Delta \varphi = 0$ in $D$. Since $\varphi$ is constant on $\partial D$ and $\varphi \in W^{2,\infty}(D)$ it follows that $\varphi$ is constant in $D$. Hence $A_1 = A_2$ and $V_1 = V_2$. Theorems 1 and 2 are proved.

4 Justification of the algorithm of Section 2

It follows from formula (6) that $d\tilde{A} = dA$. More precisely, if we denote by $F^*dA$ the pullback of the form $dA$ by the map $F$ and by $\iota_{\partial_t}$ we denote the interior product with the vector field $\frac{\partial}{\partial t}$ on $\{(x, t) \in D \times [0, 1]\}$, then
\[\sum_{k=1}^d \tilde{A}^k dx_k = \int_0^1 (\iota_{\partial_t} F^*dA) \, dt, \]
and the equality \( d\tilde{A} = dA \) follows from the Cartan magic formula \( \mathcal{L}_{\partial_t} = d \circ \iota_{\partial_t} + \iota_{\partial_t} \circ d \), where \( \mathcal{L}_{\partial_t} \) is the Lie derivative along \( \partial_t \), \( d \) is the exterior derivative on \( \{(x, t) \in D \times [0, 1]\} \) and \( \circ \) denotes the composition of maps.

Hence we can define \( \varphi \in W^{2,\infty}(D, \mathbb{R}) \) by the formula
\[
\varphi(x) = \int_{\tilde{x}}^x \sum_{k=1}^d (A_k - \tilde{A}_k) dx_k, \quad x \in \overline{D},
\]
where \( \tilde{x} \in D \) is some fixed point and integration is over an arbitrary \( C^1 \) curve in \( D \) linking \( \tilde{x} \) to \( x \). Then \( \nabla \varphi = A - \tilde{A} \) in \( \overline{D} \) and this implies that \( \varphi|_{\partial D} \) differs by constant from \( \varphi_0 \) defined in (7). We also obtain from (5) the equation
\[
V = q - i\Delta \varphi - i\nabla \cdot \tilde{A} + \tilde{A} \cdot \tilde{A} + 2\tilde{A} \cdot \nabla \varphi + (\nabla \varphi)^2 \quad \text{in} \quad D.
\]
Taking into account that \( V \) is real-valued and separating the imaginary part in the latter equation we obtain (8). Since \( \text{Im} \ q \) and \( \nabla \cdot \tilde{A} \) belong to \( L^\infty(D, \mathbb{R}) \), the problem (8) is uniquely solvable for \( \varphi \in W^{2,\infty}(D, \mathbb{R}) \). Thus, the algorithm of Section 2 is justified.

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6  References


