Averaged controllability of parameter dependent conservative semigroups

Jérôme Lohéac, Enrique Zuazua

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Jérôme Lohéac * Enrique Zuazua †‡§¶

Abstract

We consider the problem of averaged controllability for parameter depending (either in a discrete or continuous fashion) control systems, the aim being to find a control, independent of the unknown parameters, so that the average of the states is controlled. We do it in the context of conservative models, both in an abstract setting and also analysing the specific examples of the wave and Schrödinger equations.

Our first result is of perturbative nature. Assuming the averaging probability measure to be a small parameter-dependent perturbation (in a sense that we make precise) of an atomic measure given by a Dirac mass corresponding to a specific realisation of the system, we show that the averaged controllability property is achieved whenever the system corresponding to the support of the Dirac is controllable.

Similar tools can be employed to obtain averaged versions of the so-called *Ingham inequalities*.

Particular attention is devoted to the 1d wave equation in which the time-periodicity of solutions can be exploited to obtain more precise results, provided the parameters involved satisfy Diophantine conditions ensuring the lack of resonances.

keywords: Parameter dependent systems, Averaged control, Perturbation arguments, Ingham inequalities, Non-harmonic Fourier series, Wave equation, Schrödinger equation.
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1 Introduction and main results

1.1 Problem formulation

This paper is devoted to analyze the following question: *Given a system depending on a random variable, is it possible to find a control such that the average or expected value of the output of the system is controlled?*

The problem is addressed in the context of the abstract system

$$\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u\,, \tag{1.1a}$$

with the parameter dependent initial condition

$$y_\zeta(0) = \mathrm{y}^i_\zeta\,, \tag{1.1b}$$

where $\zeta \in \mathbb{R}$ is a random variable following a probability law η, A_ζ is an operator on X, the state space, B_ζ is a control operator, $y_\zeta(t) \in X$ is the parameter dependent state variable, and $u(t) \in U$ is the control variable (independent of the parameter ζ), U being the control space.

Given $T > 0$, the problem of *exact averaged controllability* consists in analysing whether, for every family of parameter dependent initial conditions $\mathrm{y}^i_\zeta \in X$ and every final target $\mathrm{y}^f \in X$, there exists a control $u \in L^2([0,T],U)$ (independent of the parameter ζ) such that:

$$\int_{\mathbb{R}} y_\zeta(T)\,\mathrm{d}\eta_\zeta = \mathrm{y}^f\,. \tag{1.2}$$

One can also address the weaker *approximate averaged control problem*, in which, for every $\varepsilon > 0$, one aims to find a control $u \in L^2([0,T],U)$ such that:

$$\left\| \int_{\mathbb{R}} y_\zeta(T)\,\mathrm{d}\eta_\zeta - \mathrm{y}^f \right\|_X^2 \leqslant \varepsilon\,. \tag{1.3}$$

In both (1.2) and (1.3), y_ζ is the solution of (1.1) with initial Cauchy condition y^i_ζ and control u.

As we shall see, the averaged controllability properties will significantly depend on the nature of the averaging measure η.

It is easy to see that averaged control problems cannot be handled by classical methods. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} y_\zeta(t)\,\mathrm{d}\eta_\zeta\right) = \int_{\mathbb{R}} A_\zeta y_\zeta\,\mathrm{d}\eta_\zeta + \left(\int_{\mathbb{R}} B_\zeta\,\mathrm{d}\eta_\zeta\right) u\,,$$

hence, the dynamics of the average is (in general) not governed by an abstract differential equation.

This paper is devoted to address these questions both in the abstract version (1.1), in which the generator of the semi-group A_ζ is skew-adjoint, and in some particular instances: the one-dimensional wave and Schrödinger equations. In particular, we will pay attention to the string equation with Dirichlet boundary control:

$$\begin{aligned}
&\ddot{y}_\zeta(t,x) = \zeta^2 \partial_x^2 y_\zeta(t,x) && ((t,x) \in \mathbb{R}_+^* \times (0,1))\,, && (1.4a)\\
&y_\zeta(t,0) = u(t) && (t \in \mathbb{R}_+^*)\,, && (1.4b)\\
&y_\zeta(t,1) = 0 && (t \in \mathbb{R}_+^*)\,, && (1.4c)
\end{aligned}$$

$$y_\zeta(0,x) = \mathrm{y}_\zeta^{i,0}(x) \quad \text{and} \quad \dot{y}_\zeta(0,x) = \mathrm{y}_\zeta^{i,1}(x) \qquad (x \in (0,1))\,. \tag{1.4d}$$

1.2 Main results

We address the problem of averaged control analyzing the equivalent one of averaged observability for the corresponding adjoint system. We do it in two complementary contexts that we briefly describe below. We first show the stability of the observability inequality under small enough perturbations, to later derive a much more specific result for Fourier series, using its periodicity properties.

Perturbation argument We focus on the case where the uncontrolled dynamics, i.e. the one associated with $u = 0$, is time-conservative. Our results apply also in a slightly larger context (for instance, involving bounded damping terms) but, for instance, cannot be applied directly for heat-like equations because of its time irreversibility.

In order to tackle the averaged controllability problem, we consider a probability measure of the form $\eta = (1-\theta)\delta_{\zeta_0} + \theta\tilde{\eta}$, where $\tilde{\eta}$ is a probability measure on $\mathbb{R}$ and $\theta \in [0,1]$ a small parameter so that, in practice, we deal with a small perturbation of an atomic measure concentrated at ζ_0. Our result ensures that, under suitable smallness conditions, averaged observability holds provided the realization of the system for $\zeta = \zeta_0$ is observable.

To be more precise, proving the exact averaged controllability in time $T > 0$ is equivalent to the averaged observability inequality:

$$\int_0^T \left\| \int_{\mathbb{R}} B_\zeta^* z_\zeta(t)\, \mathrm{d}\eta_\zeta \right\|_U^2 \mathrm{d}t \geqslant c(T) \|\mathrm{z}^f\|_X^2 \qquad (\mathrm{z}^f \in X)\,, \tag{1.5}$$

with $c(T) > 0$ independent of z^f, z_ζ being the solution of the adjoint system:

$$-\dot{z}_\zeta = A_\zeta^* z_\zeta\,, \qquad z_\zeta(T) = \mathrm{z}^f\,.$$

We assume that the system is exactly controllable/observable for the parameter value $\zeta = \zeta_0$, i.e. there exists $c_{\zeta_0}(T) > 0$ such that:

$$\int_0^T \left\| B_{\zeta_0}^* z_{\zeta_0}(t) \right\|_U^2 \mathrm{d}t \geqslant c_{\zeta_0}(T) \|\mathrm{z}^f\|_X^2 \qquad (\mathrm{z}^f \in X)\,.$$

With this assumption, we prove the existence of $\theta_0 \in (0,1]$ such that, for every $\theta \in [0,\theta_0)$, (1.5) holds, i.e. the parameter dependent system (1.1) is exactly controllable in average with respect to the probability measure η.

This result is the core of Theorem 3.1 and can be applied in many situations such as wave, Schrödinger or plate equations with internal or boundary control, see § 3.2. In particular, this result can be applied in the context of Ingham inequalities (see Proposition 3.3), an issue that we discuss now in more detail.

Averaged Ingham inequalities In the context of one-dimensional equations such as string or Schrödinger equations, the problem of averaged controllability can be reduced (by duality) to the analysis of averages of non-harmonic Fourier series and the recovery of its coefficients out of its $L^2(0,T)$-norm.

The results in section 4 are only valid for a system of the form

$$\dot{y}_\zeta = \varsigma(\zeta) A y_\zeta + B_\zeta u\,,$$

with $\varsigma \in \mathbb{R}^{\mathbb{R}}$, A, independent of ζ, a skew-adjoint and diagonalisable operator with eigenvalues $(i\lambda_n)_{n\in\mathbb{Z}}$ and B_ζ a boundary control operator.

To be more precise, the aim in section 4, is to obtain the *averaged observability* inequality,

$$\int_0^T \left| \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t}\, \mathrm{d}\eta_\zeta \right|^2 \mathrm{d}t \geqslant c(T) \sum_{n\in\mathbb{Z}} |a_n|^2\,, \tag{1.6}$$

with $c(T) > 0$ independent of $a = (a_n)_n$ or the *approximate averaged observability* result,

$$\int_0^T \left| \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t}\, \mathrm{d}\eta_\zeta \right|^2 \mathrm{d}t = 0 \qquad \Longrightarrow \qquad (a_n)_n = 0\,. \tag{1.7}$$

Here and in the sequel $(\lambda_n)_n$ is a sequence of real numbers (independent of ζ), $\varsigma(\zeta) \in \mathbb{R}$ and $L_\zeta \in \mathcal{L}(\ell^2)$. The linear map L_ζ is introduced here in order to tackle partial differential equations with second order derivative in time such as (1.4). For the particular example of the string equation considered in (1.4), we have (see Appendix A) $\varsigma(\zeta) = \zeta$ and L_ζ is affine with respect to ζ (see (A.2)). For first order in time systems, such as the Schrödinger equation, we will have $L_\zeta = \mathrm{Id}$.

The analysis of the well-posedness of the control system under consideration also requires the *averaged admissibility* inequality,

$$\int_0^T \left| \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t}\, \mathrm{d}\eta_\zeta \right|^2 \mathrm{d}t \leqslant C(T) \sum_{n\in\mathbb{Z}} |a_n|^2\,, \tag{1.8}$$

with $C(T) > 0$ independent of $a = (a_n)_n$.

In the case $\eta = \delta_{\zeta_0}$, these results are (provided some gap condition is satisfied for $(\lambda_n)_n$ and T is large enough) a consequence of Ingham inequality (see, for instance, the original paper of A. E. Ingham [14]), that has played an important role when dealing with one-dimensional control problems.

The inequality (1.6) can be achieved, as described above, by perturbation arguments (see Proposition 3.3 and Corollary 4.1). But in some specific situations, more precise results can be obtained, combining periodicity properties and classical Ingham inequalities.

To show how these arguments can be applied, we consider the particular cases in which :

1. η is a sum of Dirac masses located at points ζ_k;
2. the parameters $\varsigma(\zeta_k)$ satisfy a non-resonance condition guaranteeing the irrationality of one parameter $\varsigma(\zeta_{k_0})$ with respect to all other ones.
3. there exists $\gamma > 0$ such that $\lambda_n \in \gamma\mathbb{Z}$ for every n.

Notice that the 3$^{\text{rd}}$ condition is fulfilled for the string or one dimensional Schrödinger equation but is much stronger than the usual gap condition required for Ingham inequality to hold.

Under these conditions, in Theorem 4.1 we derive the unique continuation property (1.7) (see Corollary 4.2) and a weighted Ingham inequality (see corollaries 4.3 and 4.4) of the form:

$$\int_0^T \left| \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t} \, \mathrm{d}\eta_\zeta \right|^2 \mathrm{d}t \geqslant c(T) \sum_{n\in\mathbb{Z}} \rho_n |a_n|^2 \,, \tag{1.9}$$

where the weights $\rho_n > 0$ depend of the Diophantine properties of the parameters $\varsigma(\zeta_k)$. This weighted averaged Ingham inequality allows deriving averaged controllability results for 1d wave equation (1.4) in weighted spaces, see § 4.3.

1.3 Bibliographical comments

The notion of averaged controllability was introduced in [39] where necessary and sufficient rank conditions were given in the finite dimensional context.

The works of J.-S. Li et al. [26, 25] on *ensemble control* are also worth mentioning. The ensemble control notion is introduced to steer, with a control independent of the parameter, all the parameter dependent trajectories in an arbitrary small ball around a desired target.

In the PDE context, the problem of averaged control was considered in [24] for two different wave equations by means of a common interior control, using H-measure techniques. Other situations were also considered in [40], when, for instance, the solution of a given PDE is perturbed additively by the solution of another one. Furthermore, in [30], the authors considered one-parameter families of Schrödinger and heat equations in the multi-dimensional case, with controls distributed in some interior sub-domain, showing that, depending on the averaging measure, one can obtain either the controllability results corresponding to time-reversible or parabolic-like equations.

The present paper is the first contribution for PDEs depending on the unknown parameter in a rather general manner which are also of application in the context of boundary control.

The results we obtain for the string equation are related but different to previous ones on the simultaneous controllability (see [10, § 5.8.2]), a notion that was first introduced by D. L. Russell in [32] (see also [27, Chapter 5]) and extensively analyzed in the literature (see [10, 2, 1] and the references therein).

There are several other possible natural paths to extend the results of this paper. In particular, it would be natural to address similar issues for wave equations in networks. We refer to the book of R. Dáger and E. Zuazua [10] and to the papers of I. Joó [17] and J. Valein et al. [31, 37] for some of the main existing results on the control and stabilization of networks of 1d wave equations.

Averaged controllability can be seen also as a first step to achieve simultaneous controllability. Obviously, the later requires also the control of all possible parameter dependent states, and not only the control of their average. In the concluding section, we will show the link between these two notions via penalization, an issue that is treated in more detail in [29]. This procedure, quickly explained in the concluding remarks, is similar to the one implemented by J.-L. Lions in [28] in order to link optimal control and approximate controllability for the heat equation.

1.4 Structure of the paper

The core of this work is devoted to the obtention of averaged observability inequalities.

Basic notations and assumptions, ensuring that the averages considered in this paper are well defined, are given in section 2. The proof of the duality result, between averaged controllability and averaged observability, is given in § 2.2.

In section 3, we use a perturbation argument in order to derive some exact averaged observability results. These results are applied in § 3.2 (to be compared with [24]) to the averaged controllability of wave equations, with coefficients depending smoothly of the space variable and simply measurable with respect to the unknown parameter, and also in the context of non-harmonic Fourier series (see § 3.3). When analysing this last application, we deduce a first averaged Ingham inequality which is later used for the toy example (1.4) in § 3.4.

Other averaged Ingham inequalities are given in section 4, when the probability measure η is a finite sum of atomic masses. More precisely, in § 4.1, we apply the results of § 3.3 to this situation. In § 4.2 we prove an approximate averaged observability result, in the particular context of Fourier series expansions, and a weighted Ingham inequality of type (1.9), with weights depending on Diophantine approximation properties. The results obtained in § 3.3 and section 4 are applied to the string equation with Dirichlet boundary control (1.4) in § 4.3. Connections with simultaneous controllability are also discussed in § 4.3.

We conclude with some remarks and open questions in section 5.

The technical details related to (1.4) are given in Appendix A.

2 Averaged controllability

2.1 Functional setting and assumptions

In this paragraph, we present some basic notation, the abstract functional setting and the assumptions ensuring that averages are well defined.

Let us introduce two separable Hilbert spaces, namely the state space X and the control space U, each of them being identified with its dual.

For every $\zeta \in \mathbb{R}$, consider the operator A_ζ on X with domain $\mathcal{D}(A_\zeta)$ and assume that

(i) A_ζ has a non empty resolvent $\rho(A_\zeta)$;

(ii) A_ζ generates a strongly continuous group $\mathbb{T}_\zeta$ on X. Thus, $\mathcal{D}(A_\zeta)$ is a dense linear subspace of X;

(iii) For almost every $\zeta \in \mathbb{R}$ and every $t \in \mathbb{R}$, there exists a constant $\kappa > 0$ such that:
$$\|\mathbb{T}_\zeta(t)\|_{\mathcal{L}(X)} \leqslant \kappa \qquad (t \in \mathbb{R}\,,\ \zeta \in \mathbb{R})\,.$$

For every $\zeta \in \mathbb{R}$, define $X_{\zeta,1} = \mathcal{D}(A_\zeta)$, the Hilbert space endowed with the norm:
$$\|\mathrm{y}\|_{X_{\zeta,1}} = \|(\beta I - A_\zeta)\mathrm{y}\|_X \qquad (\mathrm{y} \in X_{\zeta,1})$$
and $X_{\zeta,-1}$ the completion of X with respect to the norm:
$$\|\mathrm{y}\|_{X_{\zeta,-1}} = \|(\beta I - A_\zeta)^{-1}\mathrm{y}\|_X \qquad (\mathrm{y} \in X)\,,$$
where, in the above, we have chosen $\beta \in \rho(A_\zeta)$. We refer to [35, § 2.10] for those definitions. Similarly, based on A_ζ^*, we define the spaces $X^d_{\zeta,1}$ and $X^d_{\zeta,-1}$.

In addition, [35, Propositions 2.10.1 and 2.10.2] ensure that the norms generated above for different β are equivalent and in particular, the $X_{\zeta,1}$-norm is equivalent to the graph norm, $\sqrt{\|\mathrm{y}\|_X^2 + \|A_\zeta \mathrm{y}\|_X^2}$. Moreover, $X^d_{\zeta,-1}$ is the dual of $X_{\zeta,1}$ with respect to the pivot space X (see [35, Remark 2.10.6]).

Let us also set A_ζ and $\mathbb{T}_\zeta$ the extensions of A_ζ and $\mathbb{T}_\zeta$ to $X_{\zeta,-1}$, see [35, Proposition 2.10.4].

Moreover, assume:

(iv) There exists an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of X such that for almost every $\zeta \in \mathbb{R}$ with respect to the measure η, we have $e_i \in \mathcal{D}\left((A_\zeta)^\infty\right)$ for every $i \in \mathbb{N}$.

(v) For every $i \in \mathbb{N}$, and every $n \in \mathbb{N}$, $\zeta \in \mathbb{R} \mapsto (A_\zeta)^n e_i \in X$ is measurable.

Let us now introduce the control operators. For every $\zeta \in \mathbb{R}$, we set $B_\zeta \in \mathcal{L}(U, X_{\zeta,-1})$ and consider the Cauchy problems (1.1) with $\zeta \in \mathbb{R}$ the random variable following a given probability law η.

For every $\zeta \in \mathbb{R}$, we introduce the *input to state map* $\Phi_t^\zeta \in \mathcal{L}\big(L^2(\mathbb{R},U), X_{\zeta,-1}\big)$, classically defined by:

$$\Phi_t^\zeta u = \int_0^t \mathbb{T}_\zeta(t-s) B_\zeta u(s)\,\mathrm{d}s \qquad (t>0\,,\ u \in L^2(\mathbb{R}_+,U))\,, \tag{2.1}$$

so that, the solution of (1.1) is:

$$y_\zeta(t) = \mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i + \Phi_t^\zeta u \qquad (t>0\,,\ u \in L^2(\mathbb{R}_+,U))\,. \tag{2.2}$$

Taking the average of (2.2) with respect to ζ, we obtain (formally):

$$\int_{\mathbb{R}} y_\zeta(t)\,\mathrm{d}\eta_\zeta = \int_{\mathbb{R}} \mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i\,\mathrm{d}\eta_\zeta + \mathbf{F}_t u \qquad (t>0\,,\ u \in L^2(\mathbb{R}_+,U))\,, \tag{2.3}$$

where we have defined the *averaged input to state map*:

$$\mathbf{F}_t u = \int_{\mathbb{R}} \Phi_t^\zeta u\,\mathrm{d}\eta_\zeta \qquad (t>0\,,\ u \in L^2(\mathbb{R}_+,U))\,. \tag{2.4}$$

Let us also classically define for every $\zeta \in \mathbb{R}$ the observability map $\psi_t^\zeta \in \mathcal{L}\big(X_{\zeta,1}^d, L^2(\mathbb{R},U)\big)$ by:

$$(\psi_t^\zeta \mathrm{z})(s) = \begin{cases} B_\zeta^* \mathbb{T}_\zeta^*(s)\mathrm{z} & \text{if } s \leqslant t\,, \\ 0 & \text{if } s > t \end{cases} \qquad (\mathrm{z} \in X_{\zeta,1}^d\,,\ t,s>0)\,. \tag{2.5}$$

We also define (formally) the *averaged observability map*:

$$\begin{aligned}(\Psi_t \mathrm{z})(s) &= \int_{\mathbb{R}} (\psi_t^\zeta \mathrm{z})(s)\,\mathrm{d}\eta_\zeta \\ &= \begin{cases} \displaystyle\int_{\mathbb{R}} B_\zeta^* \mathbb{T}_\zeta^*(s)\mathrm{z}\,\mathrm{d}\eta_\zeta & \text{if } s \leqslant t\,, \\ 0 & \text{if } s > t \end{cases} \qquad (\mathrm{z} \in X\,,\ t,s>0)\,. \end{aligned} \tag{2.6}$$

Let us set an assumption on B_ζ:

(vi) For almost every $\zeta \in \mathbb{R}$ with respect to the measure η, B_ζ is admissible for the semi-group $\mathbb{T}_\zeta$ generated by A_ζ.

Consequently, according to [35, Proposition 4.2.2 and Theorem 4.4.3], Φ_t^ζ (resp. ψ_t^ζ) can be seen as a linear bounded operator from $L^2([0,t],U)$ (resp. X) to X (resp. $L^2([0,t],U)$) for every $t>0$ and almost every ζ with respect to the measure η.

Finally, our last assumption is:

(vii) The parameter dependent initial condition $\zeta \in \mathbb{R} \mapsto \mathrm{y}_\zeta^i \in X$ is Bochner-integrable for the measure η.

The consequence of assumption (vii) is that the average of $\left(\mathbb{T}_\zeta(t)\mathrm{y}^i_\zeta\right)_\zeta$ is well defined in X.

Lemma 2.1. *If assumptions* (i)–(v) *and* (vii) *are satisfied, then,*

$$\int_{\mathbb{R}} \mathbb{T}_\zeta(t)\mathrm{y}^i_\zeta \,\mathrm{d}\eta_\zeta \,\in X \qquad (t \geqslant 0)\,.$$

Proof. According to Bochner Theorem, (see [34, Theorem 1.2, chapter III]), the fact that $\zeta \mapsto \mathrm{y}^i_\zeta$ is Bochner-integrable is equivalent to $\zeta \mapsto \mathrm{y}^i_\zeta$ being measurable and $\zeta \mapsto \|\mathrm{y}^i_\zeta\|_X$ integrable, i.e.,

$$\int_{\mathbb{R}} \left\|\mathrm{y}^i_\zeta\right\|_X \,\mathrm{d}\eta_\zeta < \infty\,. \tag{2.7}$$

We are going to prove here a stronger result which is the Bochner-integrability of $\zeta \mapsto \mathbb{T}_\zeta(t)\mathrm{y}^i_\zeta \in X$. Firstly, we will prove the measurability of $\zeta \mapsto \mathbb{T}_\zeta(t)^*\mathrm{z}$ for every $t \in \mathbb{R}$ and every $\mathrm{z} \in X$. To this end, let us notice that, according to [35, Proposition 2.4.2], we have

$$\mathbb{T}_\zeta(t)^*\mathrm{z} = \lim_{n\to\infty} \left(\mathrm{Id} - \frac{t}{n}A^*_\zeta\right)^{-n} \mathrm{z} \qquad (\zeta \in \mathbb{R}\,,\ t \in \mathbb{R}\,,\ \mathrm{z} \in X)\,.$$

Let us prove that $\zeta \mapsto \left(\mathrm{Id} - \frac{t}{n}A^*_\zeta\right)^{-n} \mathrm{z} \in X$ is measurable, for every $n \in \mathbb{N}$ and every $\mathrm{z} \in X$. To this end, we define:

$$L^{(n)}_\zeta \mathrm{x} = \left(\left\langle \mathrm{x}, \left(\mathrm{Id} - \frac{t}{n}A_\zeta\right)^n e_i \right\rangle_X\right)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N}) \qquad (\mathrm{x} \in X)\,,$$

with e_i defined in assumption (iv). So that, for almost every $\zeta \in \mathbb{R}$,

$$\left(\mathrm{Id} + \frac{t}{n}A^*_\zeta\right)^n \mathrm{x} = \sum_{i\in\mathbb{N}} \left[L^{(n)}_\zeta \mathrm{x}\right]_i e_i \qquad (t \in \mathbb{R}_+\,,\ n \in \mathbb{N}^*)\,.$$

According to the assumptions (ii) and (iii), together with [35, Proposition 2.3.1], we have $\mathbb{R}^*_+ \subset \rho(A^*_\zeta)$ for almost every ζ. Thus, for almost every ζ, every $t \in \mathbb{R}^*_+$ and every $n \in \mathbb{N}^*$, $\left(\mathrm{Id} + \frac{t}{n}A^*_\zeta\right)^n$ is invertible. Consequently, $L^{(n)}_\zeta$ is also invertible for almost every $\zeta \in \mathbb{R}$. Moreover, using assumption (v), we deduce that $\zeta \mapsto L^{(n)}_\zeta$ is measurable. Consequently, $\zeta \mapsto \left(L^{(n)}_\zeta\right)^{-1}$ is also measurable. But,

$$\left(\mathrm{Id} - \frac{t}{n}A^*_\zeta\right)^{-n} \mathrm{z} = \left(L^{(n)}_\zeta\right)^{-1} \left((\langle \mathrm{z}, e_i\rangle_X)_{i\in\mathbb{N}}\right)$$
$$(\mathrm{z} \in X\,,\ n \in \mathbb{N}^*\,,\ t \in \mathbb{R}^*\,,\ \zeta \in \mathbb{R})\,,$$

ensures that $\zeta \mapsto \left(\mathrm{Id} - \frac{t}{n} A_\zeta^*\right)^{-n} \mathrm{z} \in X$ is measurable. Finally, $\zeta \mapsto \mathbb{T}_\zeta(t)^* \mathrm{z}$ being the limit of measurable functions, is also measurable.

Since X is a separated Hilbert space, $\zeta \mapsto \mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i$ is almost everywhere separably valued. This map is also weakly measurable, that is to say, for every $\mathrm{z} \in X$, $\zeta \mapsto \langle \mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i, \mathrm{z}\rangle$ is measurable. In fact, $\langle \mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i, \mathrm{z}\rangle = \langle \mathrm{y}_\zeta^i, \mathbb{T}_\zeta(t)^*\mathrm{z}\rangle$ and it is easy to see that the scalar product of two measurable functions is measurable. Using Pettis Theorem (see [34, Theorem 1.1 of chapter III]), we conclude that $\zeta \mapsto \mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i$ is measurable.

Finally, using again assumption (iii), we have, $\|\mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i\|_X \leqslant \kappa \|\mathrm{y}_\zeta^i\|_X$. Consequently, assumption (vii) ensures that $\zeta \mapsto \|\mathbb{T}_\zeta(t)\mathrm{y}_\zeta^i\|_X$ is η-integrable.
The conclusion follows from Bochner Theorem. $\square$

Remark 2.1. *Given $t \in \mathbb{R}_+$ and $\mathrm{y} \in X$, Lemma 2.1 ensures the measurability of $\zeta \in \mathbb{R} \mapsto \mathbb{T}_\zeta(t)\mathrm{y} \in X$, under the assumptions* (iv) *and* (v)*. But there exist some situations for which these assumptions are not required.*

- *Assume that the support of η is a finite or countable set. In this case, we implicitly consider the probability set $(\Omega, \mathcal{P}(\Omega), \mu)$, with $\Omega = \operatorname{supp} \eta$ and for every $\zeta \in \Omega$, $\mu(\{\zeta\}) = \eta(\{\zeta\})$. Thus, the measurability of $\zeta \in \Omega \mapsto \mathbb{T}_\zeta(t)\mathrm{y} \in X$ is equivalent to:*

  $$\{\zeta \in \Omega\,,\ \mathbb{T}_\zeta(t)\mathrm{y} = \mathbb{T}_{\zeta_0}(t)\mathrm{y}\} \quad \textit{is measurable for every } \zeta_0 \in \Omega\,.$$

  This last property is obvious since, for every $\zeta_0 \in \Omega$, the above set is in $\mathcal{P}(\Omega)$.

- *With fewer restrictions on the measure η, if $\zeta \in \mathbb{R} \mapsto \mathbb{T}_\zeta(t)\mathrm{y}$ is continuous, then the measurability is obvious. We refer to [19] and [38] for general assumptions on the infinitesimal generators A_ζ ensuring the continuity of $\zeta \mapsto \mathbb{T}_\zeta$. More precisely, applying [38, Theorem 2.9] in our case, leads to:*

  Theorem 2.1. *[38, Theorem 2.9] Assume that*

  1. *The domain $X_{\zeta,1}$ of A_ζ is independent of ζ and set $X_1 = X_{\zeta,1}$;*
  2. *The operator $\zeta \in \mathbb{R} \mapsto A_\zeta \in \mathcal{L}(X_1, X)$ is continuous in the generalized sense on $\mathbb{R}$, that is to say:*

     $$\lim_{\zeta \to \zeta_0} \max\left\{\delta(A_\zeta, A_{\zeta_0}),\, \delta(A_{\zeta_0}, A_\zeta)\right\} = 0 \qquad (\zeta_0 \in \mathbb{R})\,,$$

     with δ defined by (2.8)*.*
  3. *The semigroup $\mathbb{T}_\zeta$ generated by A_ζ satisfies a locally stable condition in $\mathbb{R}$, that is to say, for every $\zeta_0 \in \mathbb{R}$, there exist $M_0 > 0$, $\omega_0 > 0$ and $\varepsilon_0 > 0$ (depending on ζ_0), such that if $|\zeta - \zeta_0| < \varepsilon_0$, then $\|\mathbb{T}_\zeta(t)\|_{\mathcal{L}(X)} < M_0 e^{\omega_0 t}$ for every $t \in \mathbb{R}_+$.*

Then, the semigroup $\mathbb{T}_\zeta$, generated by A_ζ is strongly continuous on $\mathbb{R}$. That is, for every $\zeta_0 \in \mathbb{R}$ and $\mathrm{y} \in X$,

$$\lim_{\zeta \to \zeta_0} \|\mathbb{T}_\zeta(t)\mathrm{y} - \mathbb{T}_{\zeta_0}(t)\mathrm{y}\|_X = 0\,,$$

and, in addition, this convergence is uniform with respect to $t \in [0, T]$, where $T \in \mathbb{R}_+^*$ is arbitrary.

In the above Theorem we have used the application $\delta : \mathcal{L}(X_1, X) \times \mathcal{L}(X_1, X) \to \mathbb{R}$ which is defined by:

$$\delta(A, \tilde{A}) = \sup_{(\mathrm{y}, A\mathrm{y}) \in S_A} \left(\inf_{(\tilde{\mathrm{y}}, \tilde{A}\tilde{\mathrm{y}}) \in S_{\tilde{A}}} \left(\|\mathrm{y} - \tilde{\mathrm{y}}\|_X^2 + \|A\mathrm{y} - \tilde{A}\tilde{\mathrm{y}}\|_X^2 \right) \right) \qquad (A, \tilde{A} \in \mathcal{L}(X_1, X))\,, \quad (2.8)$$

with $S_A = \big\{(\mathrm{y}, A\mathrm{y}) \in X_1 \times X\,,\ \|\mathrm{y}\|_X^2 + \|A\mathrm{y}\|_X^2 = 1\big\}$.

2.2 Averaged admissibility, controllability and observability

With the notations and assumptions introduced in the previous paragraph, we are now in position to define the averaged admissibility, controllability and observability concepts.

Definition 2.1 (Averaged admissibility)**.** *The family of control operators $(B_\zeta)_\zeta$ is said to be admissible in average for the family of semi-groups $(\mathbb{T}_\zeta)_\zeta$ if there exists a time $T > 0$ such that $\mathbf{F}_T \in \mathcal{L}\big(L^2([0, T], U), X\big)$, with $\mathbf{F}_T$ defined by* (2.4)*.*

Let us also introduce the following averaged controllability concepts.

Definition 2.2 (Exact/Approximate averaged controllability)**.** *Let $T > 0$. The family of pairs $(A_\zeta, B_\zeta)_\zeta$ is said to be exactly (resp. approximatively) controllable in average in time T if $\mathbf{F}_T\big(L^2([0, T], U)\big)$ is equal to (resp. dense in) X.*

As in classical control theory (see for instance [35, §4.4]), we have the following duality results:

Proposition 2.1. *Set $T > 0$, assume that* (vi) *is satisfied and, $\zeta \in \mathbb{R} \mapsto \psi_T^\zeta \in \mathcal{L}(X, L^2([0, T], U))$ is Bochner integrable for the measure η.*
Define the time reflection operator:

$$(\text{Я}_T f)(t) = f(T - t) \qquad (0 < t < T\,,\ f \text{ defined a.e. on } [0, T])\,.$$

Then, the admissibility inequalities hold, i.e. there exists a positive constant $C(T)$ such that:

$$\|\mathbf{F}_T u\|_X^2 \leqslant C(T)\, \|u\|_{L^2([0,T],U)}^2 \qquad (u \in L^2([0, T], U))\,, \qquad (2.9a)$$

$$\|\Psi_T \mathrm{z}\|_{L^2([0,T],U)}^2 \leqslant C(T)\, \|\mathrm{z}\|_X^2 \qquad (\mathrm{z} \in X)\,, \qquad (2.9b)$$

where $\mathbf{F}_T$ *and* Ψ_T *are defined by* (2.4) *and* (2.6) *and*

$$C(T) \leqslant \left(\int_{\mathbb{R}} \left\| \psi_T^{\zeta} \right\|_{\mathcal{L}(X, L^2([0,T],U))} \,\mathrm{d}\eta_\zeta \right)^2 .$$

In addition, we have,

$$\mathbf{F}_T^* = Я_T \Psi_T \,, \tag{2.10}$$

and

1. $\mathbf{F}_T\big(L^2([0,T],U)\big)$ *is dense in* X *if and only if* $\operatorname{Ker} \Psi_T = \{0\}$*;*
2. $\mathbf{F}_T\big(L^2([0,T],U)\big) = X$ *if and only if* $\Psi_T \in \mathcal{L}\big(X, L^2([0,T],U)\big)$ *is bounded from bellow.*

Proof. First of all, since for almost every ζ, B_ζ is admissible for the semigroup $\mathbb{T}_\zeta$, we have (see [35, Proposition 4.4.1]) $\Phi_T^\zeta \in \mathcal{L}(L^2([0,T],U),X)$, $\psi_T^\zeta \in \mathcal{L}(X, L^2([0,T],U))$ and $\left(\Phi_T^\zeta\right)^* = Я_T \psi_T^\zeta$.

Since $\zeta \mapsto \psi_T^\zeta \in \mathcal{L}(X, L^2([0,T],U))$ is Bochner-integrable, we have $\Psi_T \in \mathcal{L}(X, L^2([0,T],U))$ and for every $\mathrm{z} \in X$,

$$\begin{aligned} \|\Psi_T \mathrm{z}\|_{L^2([0,T],U)} = \left\| \int_{\mathbb{R}} \psi_T^\zeta \mathrm{z} \,\mathrm{d}\eta_\zeta \right\|_{L^2([0,T],U)} &\leqslant \int_{\mathbb{R}} \left\| \psi_T^\zeta \mathrm{z} \right\|_{L^2([0,T],U)} \mathrm{d}\eta_\zeta \\ &\leqslant \int_{\mathbb{R}} \left\| \psi_T^\zeta \right\|_{\mathcal{L}(X,L^2([0,T],U))} \mathrm{d}\eta_\zeta \, \|\mathrm{z}\|_X \,. \end{aligned}$$

Thus, there exists a constant $C(T) \leqslant \left(\int_{\mathbb{R}} \left\| \psi_T^\zeta \right\|_{\mathcal{L}(X,L^2([0,T],U))} \mathrm{d}\eta_\zeta \right)^2$ such that (2.9b) holds.

Since $\Phi_T^\zeta = \left(\psi_T^\zeta\right)^* Я_T^*$ and $Я_T \in \mathcal{L}(L^2([0,T],U))$ is a unitary operator, it is obvious that $\mathbf{F}_T \in \mathcal{L}(L^2([0,T],U),X)$, $\mathbf{F}_T$ is given by (2.10) and

$$\|\mathbf{F}_T\|_{\mathcal{L}(L^2([0,T],U),X)} = \|\Psi_T\|_{\mathcal{L}(X,L^2([0,T],U))} \,.$$

Let us finally prove the last two items.

1. According to [35, Remark 2.8.2], $\mathbf{F}_T\big(L^2([0,T],U)\big)$ is dense in X if and only if $\operatorname{Ker} \mathbf{F}_T^* = \{0\}$.
2. According to [35, Proposition 12.1.3], $\mathbf{F}_T$ is onto if and only if $\mathbf{F}_T^*$ is bounded from bellow.

We end the proof by considering (2.10), and noticing that $Я_T$ is isometric. □

A major difficulty in order to use Proposition 2.1 is to prove that $\zeta \mapsto \psi_T^\zeta$ is Bochner integrable. Let us give a simple case where this is easy to obtain.

Lemma 2.2. *Set η a probability measure and assume $B_\zeta = B \in \mathcal{L}(U, X)$ is a bounded control operator independent of ζ and assumptions* (i)–(v) *are fulfilled.*

Then, for every $T > 0$, $\zeta \in \mathbb{R} \mapsto \psi_T^\zeta \in \mathcal{L}(X, L^2([0,T], U))$ is Bochner integrable for the measure η.

Proof. First of all, since B is a bounded control operator, it is obvious that $\psi_T^\zeta \in \mathcal{L}(X, L^2([0,T], U))$ for every $\zeta \in \mathbb{R}$.
Using assumption (iii), we have that (see proof of Lemma 2.1) $\zeta \mapsto \mathbb{T}_\zeta(t)^*\mathrm{z} \in X$ measurable for every $\mathrm{z} \in X$ and every $t \in \mathbb{R}_+^*$. Consequently, since the control operator B is independent of ζ, we easily obtain that $\zeta \mapsto B^*\mathbb{T}_\zeta(t)^*\mathrm{z}$ is measurable. In addition, using again assumption (iii), we have $\|B^*\mathbb{T}_\zeta(t)^*\mathrm{z}\|_U \leqslant \kappa\|B^*\|_{\mathcal{L}(X,U)}\|\mathrm{z}\|_X$. This ends the proof. □

In the next paragraphs, following this general abstract path, we prove admissibility and exact averaged observability results for the corresponding adjoint systems.

3 An abstract perturbation result

Using the admissibility condition given in the previous section, one can easily develop a perturbation argument leading to averaged controllability.

3.1 Perturbation argument

Let us prove our general perturbation result.

Theorem 3.1. *Set $T > 0$, let $\tilde\eta$ be a probability measure and $\zeta_0 \in \mathbb{R}$. Assume that*

1. *Condition* (vi) *is satisfied, i.e., for almost every $\zeta \in \mathbb{R}$, there exists a positive constant $C_\zeta(T)$, $C_\zeta(T) = \left\|\psi_T^\zeta\right\|^2_{\mathcal{L}(X,L^2([0,T],U))}$, such that:*

$$\left\|\psi_T^\zeta \mathrm{z}\right\|^2_{L^2([0,T],U)} \leqslant C_\zeta(T)\|\mathrm{z}\|_X^2 \qquad (\mathrm{z} \in X)\,,$$

   with ψ_t^ζ defined by (2.5)*.*

2. *$\zeta \in \mathbb{R} \mapsto \psi_T^\zeta \in \mathcal{L}(X, L^2([0,T], U))$ is Bochner-integrable for the measure $\tilde\eta$.*

3. *The pair $(A_{\zeta_0}, B_{\zeta_0})$ is exactly controllable in time T, i.e., there exists $c_{\zeta_0}(T) > 0$ such that:*

$$c_{\zeta_0}(T)\|\mathrm{z}\|_X^2 \leqslant \left\|\psi_T^{\zeta_0}\mathrm{z}\right\|^2_{L^2([0,T],U)} \qquad (\mathrm{z} \in X)\,. \tag{3.1}$$

Set $\theta_0 = \left(1 + \int_{\mathbb{R}} \sqrt{\frac{\tilde C_\zeta(T)}{c_{\zeta_0}(T)}}\,\mathrm{d}\tilde\eta_\zeta\right)^{-1}$. Then for every $\theta \in [0, \theta_0)$, $(A_\zeta, B_\zeta)_\zeta$ is exactly controllable in average in time T with respect to the probability measure η given by:

$$\eta = (1-\theta)\delta_{\zeta_0} + \theta\tilde\eta\,. \tag{3.2}$$

In addition, for every $\theta \in [0, \theta_0)$, we have:

$$c^\theta(T)\|\mathrm{z}\|_X^2 \leqslant \|\Psi_T \mathrm{z}\|_{L^2([0,T],U)}^2 \leqslant C^\theta(T)\|\mathrm{z}\|_X^2 \qquad (\mathrm{z} \in X)\,, \tag{3.3}$$

with Ψ_T defined by (2.6)*, $C^\theta(T) > 0$ and*

$$c^\theta(T) = \left((1-\theta)\sqrt{c_{\zeta_0}(T)} - \theta \int_{\mathbb{R}} \sqrt{C_\zeta(T)}\,\mathrm{d}\tilde{\eta}_\zeta\right)^2 .$$

Remark 3.1. *1. This result can be applied in many examples such as wave, Schödinger and plate equations, etc. with boundary or internal controls of different nature.*

However, the proof, which is rather straightforward, is based on a smallness argument and, hence, it does not cover the results in [24] for the averaged controllability of two wave equations with internal control, or the ones in [40] for the additive superposition of wave and heat equations.

2. *In Theorem 3.1, we assume that the* perturbation measure *$\tilde{\eta}$ is a probability measure so that $\eta = (1-\theta)\delta_{\zeta_0} + \theta\tilde{\eta}$ is a probability measure for every $\theta \in [0, 1]$.*

   Of course, a similar study could have been performed with non probabilistic measures but we consider probability measures so to guarantee that we are dealing with "averages".

3. *In Theorem 3.1, the probability space under consideration is $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \theta\tilde{\eta} + (1-\theta)\delta_{\zeta_0})$. Thus we consider one-parameter dependent problems. But similar results could have been obtained for more general probability spaces $(\Omega, \mathcal{T}, \theta\tilde{\eta} + (1-\theta)\delta_{\zeta_0})$, with $\theta \in [0, 1]$ and $\zeta_0 \in \Omega$, assuming that the σ-algebra, $\mathcal{T}$, contains $\{\zeta_0\}$. For instance, one could think on d-parameter depedning problems, using the probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \theta\tilde{\eta} + (1-\theta)\delta_{\zeta_0})$, with $\theta \in [0, 1]$ and $\zeta_0 \in \mathbb{R}^d$.*

Proof of Theorem 3.1. Using Proposition 2.1, assumptions 1 and 2 ensure the averaged admissibility of $(B_\zeta)_\zeta$ for $(\mathbb{T}_\zeta)_\zeta$ with respect to the measure $\tilde{\eta}$. This together with the admissibility of B_{ζ_0} for $\mathbb{T}_{\zeta_0}$ (3^{rd} assumption), ensure the averaged admissibility of $(B_\zeta)_\zeta$ for $(\mathbb{T}_\zeta)_\zeta$ with respect to the measure η given by (3.2) for every $\theta \in [0, 1]$.

For every $\theta \in [0, 1]$ and every $\mathrm{z} \in X$, we have:

$$\begin{aligned}\|\Psi_T \mathrm{z}\|_{L^2([0,T],U)} &= \left\|(1-\theta)\psi_T^{\zeta_0}\mathrm{z} + \theta\tilde{\Psi}_T\mathrm{z}\right\|_{L^2([0,T],U)} \\ &\geqslant (1-\theta)\left\|\psi_T^{\zeta_0}\mathrm{z}\right\|_{L^2([0,T],U)} - \theta\left\|\tilde{\Psi}_T\mathrm{z}\right\|_{L^2([0,T],U)} ,\end{aligned}$$

with Ψ_T given by (2.6) (with η given by (3.2)) and with $\tilde{\Psi}_T\mathrm{z} = \displaystyle\int_{\mathbb{R}} \psi_T^\zeta \mathrm{z}\,\mathrm{d}\tilde{\eta}_\zeta$.
Thus, from Proposition 2.1 and (3.1), we easily obtain:

$$\|\Psi_T\mathrm{z}\|_{L^2([0,T],U)} \geqslant \left((1-\theta)\sqrt{c_{\zeta_0}(T)} - \theta\int_{\mathbb{R}}\sqrt{C_\zeta(T)}\,\mathrm{d}\tilde{\eta}_\zeta\right)\|\mathrm{z}\|_X \qquad (\mathrm{z} \in X)\,.$$

This ends the proof. $\square$

3.2 Averaged control of parameter depending Schrödinger and wave systems

A Schrödinger system

For every $\zeta \in \mathbb{R}$, let us consider the controlled Schrödinger equation:

$$\dot{y}_\zeta = i \operatorname{div}(a_\zeta(x)\nabla y_\zeta) + \chi_\omega u \qquad \text{in } (0,T)\times\Omega\,, \tag{3.4a}$$

$$y_\zeta = 0 \qquad \text{on } (0,T)\times\partial\Omega\,, \tag{3.4b}$$

$$y_\zeta(0,x) = \mathrm{y}^i_\zeta(x) \qquad (x\in\Omega)\,, \tag{3.4c}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^d$, ω an open subset of Ω, $a_\zeta \in C^\infty(\overline{\Omega},\mathbb{R})$ is uniformly strictly positive and $\mathrm{y}^i_\zeta \in L^2(\Omega)$.

Let us explain why the parameter dependent control system (3.4) fits our abstract setting. To this end, define the control space $U = L^2(\omega,\mathbb{C})$ and the state space $X = L^2(\Omega,\mathbb{C})$, and for every ζ the operator A_ζ given by:

$$\mathcal{D}(A) = H^2(\Omega,\mathbb{C})\cap H^1_0(\Omega,\mathbb{C}) \qquad \text{and} \qquad A_\zeta \mathrm{y} = i\operatorname{div}\big(a_\zeta\nabla\mathrm{y}\big) \quad (\mathrm{y}\in\mathcal{D}(A))\,,$$

and define the control operator $B\in\mathcal{L}(U,X)$ by:

$$B\mathrm{v} = \chi_\omega v \qquad v\in U\,.$$

Notice that $\mathcal{D}(A_\zeta)$ is independent of ζ and set $X_1 = H^2(\Omega,\mathbb{C})\cap H^1_0(\Omega,\mathbb{C})$. It is well known that assumptions (i) and (ii) are satisfied. In addition, for every $\zeta\in\mathbb{R}$, it is classical that A_ζ is a skew adjoint operator generating a unitary group $\mathbb{T}_\zeta$ on X. Thus, assumptions (ii) and (iii) (with $\kappa = 1$) are fulfilled. Since $a_\zeta \in C^\infty(\overline{\Omega},\mathbb{R})$ and a_ζ do not vanish, it is obvious that $C^\infty_0(\Omega,\mathbb{C}) \subset \mathcal{D}\left((A_\zeta)^\infty\right)$. Thus, assumption (iv) is satisfied. In addition, assuming that $\zeta\mapsto a_\zeta \in C^\infty(\overline{\Omega},\mathbb{R})$ is measurable, for every $\mathrm{y}\in C^\infty_0(\Omega,\mathbb{C})$ and every $n\in\mathbb{N}$, we have that $\zeta\mapsto(A_\zeta)^n\mathrm{y}\in X$ is measurable, i.e. assumption (v) is satisfied.

Since B is a bounded control operator, it is obvious that assumption (vi) is satisfied and the Bochner integrability of $\zeta\in\mathbb{R}\mapsto\psi^\zeta_T\in\mathcal{L}(X,L^2([0,T],U))$ is given by Lemma 2.2.

Applying Theorem 3.1 to this system, we obtain the following:

Proposition 3.1. *Let $\tilde\eta$ be a probability measure and assume that*

1. $\zeta\in\mathbb{R}\mapsto a_\zeta\in C^\infty(\overline{\Omega},\mathbb{R})$ *is measurable and for every* $\zeta\in\mathbb{R}$, $\inf_{x\in\Omega} a_\zeta(x) > 0$*;*
2. *There exists $\tau>0$ such that the set $(0,\tau)\times\omega$ satisfies the geometric control condition (see [3]) for the wave equation* (3.5) *indexed by $\zeta = 1$.*

Then, for every $T>0$, there exists $\theta_0(T)\in(0,1]$ such that system (3.4) *fulfils the exact averaged control property, in time T, for every $\theta\in[0,\theta_0(T))$ with measure $\eta^\theta = (1-\theta)\delta_1 + \theta\tilde\eta$ and for the space $X = L^2(\Omega,\mathbb{C})$.*

Proof. According to the previous remarks, in order to apply Theorem 3.1, we only need to prove the observability inequality for $\zeta = 1$. This inequality is obtained by combining [3] and [35, Theorem 6.7.2]. $\square$

A wave system
Set a probability measure $\tilde{\eta}$ on $\mathbb{R}$. For every $\zeta \in \mathbb{R}$, let us consider the controlled wave equation:

$$\ddot{y}_\zeta = \operatorname{div}\left(a_\zeta(x)\nabla y_\zeta\right) + \chi_\omega u \qquad \text{in } (0,T)\times\Omega\,, \tag{3.5a}$$
$$y_\zeta = 0 \qquad \text{on } (0,T)\times\partial\Omega\,, \tag{3.5b}$$

$$y_\zeta(0,x) = \mathrm{y}_\zeta^{i,0}(x) \quad \text{and} \quad \dot{y}_\zeta(0,x) = \mathrm{y}_\zeta^{i,1}(x) \qquad (x\in\Omega)\,, \tag{3.5c}$$

where Ω is a smooth and bounded domain of $\mathbb{R}^d$, ω an open subset of Ω, $(\mathrm{y}_\zeta^{i,0}, \mathrm{y}_\zeta^{i,1}) \in H_0^1(\Omega)\times L^2(\Omega)$ and $a_\zeta \in C^\infty(\overline{\Omega})$ is uniformly bounded from below and above by positive constants independent of ζ and $\zeta\in\mathbb{R}\mapsto a_\zeta \in C^\infty(\overline{\Omega})$ measurable.

Let us briefly explain how the parameter dependent control system (3.5) fits in our abstract setting. To this end, we define the control space $U = L^2(\omega)$ and the state space $X = H_0^1(\Omega)\times L^2(\Omega)$ with the scalar product:

$$\left\langle \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix}, \begin{bmatrix} \mathrm{y}^0 \\ \mathrm{y}^1 \end{bmatrix} \right\rangle_X = \langle \nabla \mathrm{z}^0, \nabla \mathrm{y}^0\rangle_{L^2(\Omega)^d} + \langle \mathrm{z}^1, \mathrm{y}^1\rangle_{L^2(\Omega)}\,.$$

For every $\zeta\in\mathbb{R}$, let us define the operator $\mathcal{A}_\zeta$ on $L^2(\Omega)$ by:

$$\mathcal{D}(\mathcal{A}_\zeta) = H^2(\Omega)\cap H_0^1(\Omega) \qquad \text{and} \qquad \mathcal{A}_\zeta f = -\operatorname{div}\left(a_\zeta \nabla f\right) \quad (f\in\mathcal{D}(\mathcal{A}_\zeta))$$

and let us define the space $X_{\zeta,0} = X$ endowed with the scalar product:

$$\begin{aligned}\left\langle \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix}, \begin{bmatrix} \mathrm{y}^0 \\ \mathrm{y}^1 \end{bmatrix} \right\rangle_{X_{\zeta,0}} &= \langle \mathcal{A}_\zeta^{\frac{1}{2}} \mathrm{z}^0, \mathcal{A}_\zeta^{\frac{1}{2}} \mathrm{y}^0\rangle_{L^2(\Omega)} + \langle \mathrm{z}^1, \mathrm{y}^1\rangle_{L^2(\Omega)} \\ &= \left\langle \sqrt{a_\zeta}\,\nabla \mathrm{z}^0, \sqrt{a_\zeta}\,\nabla \mathrm{y}^0\right\rangle_{L^2(\Omega)^d} + \langle \mathrm{z}^1, \mathrm{y}^1\rangle_{L^2(\Omega)^d}\,.\end{aligned}$$

Since a_ζ is uniformly bounded from above and below, the X and $X_{\zeta,0}$-norms are equivalent.
For every $\zeta\in\mathbb{R}$, let us now define the operator A_ζ on X by:

$$\mathcal{D}(A_\zeta) = H_0^1(\Omega)\times L^2(\Omega) \qquad \text{and}$$
$$A_\zeta \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathcal{A}_\zeta & 0 \end{bmatrix} \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix} = \begin{bmatrix} \mathrm{z}^1 \\ \operatorname{div}\left(a_\zeta \nabla \mathrm{z}^0\right) \end{bmatrix} \qquad \left(\begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix} \in \mathcal{D}(A_\zeta)\right).$$

With these definitions, A_ζ is skew adjoint on $X_{\zeta,0}$ and generates a unitary group on $X_{\zeta,0}$. Since the X and $X_{\zeta,0}$-norms are equivalent, the assumptions (ii) and (iii) (with $\kappa = \max\left\{1, \sup_{\zeta\in\mathbb{R}} \|a_\zeta\|_{L^\infty(\Omega)}\right\}$) are satisfied.

Let us now define the bounded control operator $B\in\mathcal{L}(U,X)$, independent of ζ, by:

$$B\mathrm{v} = \begin{bmatrix} 0 \\ \chi_\omega v \end{bmatrix} \qquad (v\in U)\,.$$

All in all, the system (3.5) can be expressed in the condensed form:

$$\dot{w}_\zeta = A_\zeta w_\zeta + Bu\,, \qquad w_\zeta(0) = \begin{bmatrix} \mathrm{y}_\zeta^{i,0} \\ \mathrm{y}_\zeta^{i,1} \end{bmatrix} \in X\,,$$

where $w_\zeta(t) = \begin{bmatrix} y_\zeta(t) \\ \dot{y}_\zeta(t) \end{bmatrix}$.

Since we have assumed that $\zeta \mapsto a_\zeta \in C^\infty(\overline{\Omega})$ is measurable, a similar discussion to the one on (3.4) ensures that assumptions (iv) and (v) are satisfied and the Bochner integrability of $\zeta \in \mathbb{R} \mapsto \psi_T^\zeta \in \mathcal{L}(X, L^2([0,T], U))$ is guaranteed by Lemma 2.2.

Applying Theorem 3.1 to this system, we obtain the following:

Proposition 3.2. *Set $T > 0$ and $\tilde{\eta}$, a probability measure. Assume,*

1. *$\zeta \in \mathbb{R} \mapsto a_\zeta \in C^\infty(\overline{\Omega})$ is measurable, $0 < \inf\limits_{\substack{x\in\Omega\,, \\ \zeta\in\mathbb{R}}} a_\zeta(x)$ and $\sup\limits_{\substack{x\in\Omega\,, \\ \zeta\in\mathbb{R}}} a_\zeta(x) < \infty$;*

2. *$(0,T)\times\omega$ satisfies the geometric control condition (see [3]) for the equation (3.5) indexed by $\zeta = 1$.*

Then, there exists $\theta_0 \in (0,1]$ such that system (3.5) fulfils the exact averaged control property, in time T, for every $\theta \in [0, \theta_0)$ with measure $\eta^\theta = (1-\theta)\delta_1 + \theta\tilde{\eta}$ and for the space $X = H_0^1(\Omega) \times L^2(\Omega)$.

Proof. According to the previous remarks, in order to apply Theorem 3.1, we only need to prove the observability inequality for $\zeta = 1$. But, from [3], the geometric control condition for the control system indexed by $\zeta = 1$ ensures that this system is exactly controllable in time T. □

Remark 3.2. *This result holds in the particular case $\eta^\theta = (1-\theta)\delta_1 + \theta\delta_2$ where two wave equations with different velocities of propagation are averaged.*

This case was addressed in [24, Theorem 2.1] where it was proved that the system satisfies the averaged control property for every $\theta \in [0,1)$, assuming,

$$a_1(x) \neq a_2(x) \qquad (x \in \omega)\,. \tag{3.6}$$

The proof of this result is based on micro-local defect measures and the fact that the characteristic manifolds of the two wave equations involved are disjoint. This example shows that the smallness condition we impose on the perturbations is not sharp.

3.3 Perturbation of Ingham inequalities

In this paragraph, we apply our perturbation result to Ingham inequalities. To this end, let us first introduce some technical material.

Define the Hilbert space of square summable sequences:

$$\ell^2 = \left\{ (a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}\,,\ \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \right\}$$

and consider a real sequence $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$, which is assumed to satisfy the following gap condition: there exists $\gamma > 0$ such that

$$\inf_{\substack{(m,n)\in\mathbb{Z}^2\\ m\neq n}} |\lambda_m - \lambda_n| \geqslant \gamma\,. \tag{3.7}$$

When η is the atomic mass located in ζ_0, according to Ingham inequalities, (1.8) and (1.6) are valid for $T > 1/|\varsigma(\zeta_0)|\gamma$ (with $L_{\zeta_0} = \mathrm{Id}$ and $\varsigma(\zeta_0) \neq 0$). More precisely, for $T > 0$,

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t} \right|^2 \mathrm{d}t \leqslant C_{\varsigma(\zeta_0)}(T) \sum_{n\in\mathbb{Z}} |a_n|^2 \qquad ((a_n)_n \in \ell^2)\,, \tag{3.8a}$$

and for $T > \dfrac{1}{|\varsigma(\zeta_0)|\gamma}$,

$$c_{\varsigma(\zeta_0)}(T) \sum_{n\in\mathbb{Z}} |a_n|^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t} \right|^2 \mathrm{d}t \qquad ((a_n)_n \in \ell^2)\,, \tag{3.8b}$$

where for every $\xi \in \mathbb{R}^*$, we have defined:

$$c_\xi(T) = \frac{2}{\pi}\frac{(\xi\gamma T)^2 - 1}{(\xi\gamma T)^2} T \qquad \text{and} \qquad C_\xi(T) = \frac{10\,T}{\pi \min(1, 2|\xi|\gamma T)}\,. \tag{3.9}$$

This classical result can be found in the original paper by A. E. Ingham, [14, Theorems 1 and 2]. For its relation with control theory, we refer, for instance, to [15, 21, 22] and the books [10, 35].

Let us now apply our perturbation argument developed in § 3.1 to these non-harmonic Fourier series.

Proposition 3.3. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers satisfying the gap condition (3.7). Let $L_\zeta \in \mathcal{L}(\ell^2)$, $\tilde\eta$ a probability measure on $\mathbb{R}$, $\varsigma \in \mathbb{R}^{\mathbb{R}}$ and $\zeta_0 \in \mathbb{R}$. Assume $\varsigma(\zeta_0) \neq 0$ and let $T > \dfrac{1}{\gamma|\varsigma(\zeta_0)|}$.*
Assume in addition that $\zeta \mapsto L_\zeta$ and ς are measurable, $\varsigma(\zeta) \neq 0$ for almost every $\zeta \in \mathbb{R}$ with respect to the measure $\tilde\eta$, and

$$\int_{\mathbb{R}} \|L_\zeta\|_{\mathcal{L}(\ell^2)}\, \mathrm{d}\tilde\eta_\zeta < \infty\,, \qquad \int_{\mathbb{R}} \frac{\|L_\zeta\|_{\mathcal{L}(\ell^2)}}{\sqrt{|\varsigma(\zeta)|}}\, \mathrm{d}\tilde\eta_\zeta < \infty,$$

L_{ζ_0} being bounded from bellow, i.e. there exists $\Lambda_{\zeta_0} > 0$ such that:

$$\Lambda_{\zeta_0} \|a\|_{\ell^2} \leqslant \|L_{\zeta_0} a\|_{\ell^2} \qquad (a \in \ell^2)\,.$$

Set

$$\theta_0 = \left(1 + \frac{1}{\Lambda_{\zeta_0}}\sqrt{\frac{5(\gamma\varsigma(\zeta_0)T)^2}{(\gamma\varsigma(\zeta_0)T)^2 - 1}}\int_{\mathbb{R}} \frac{\|L_\zeta\|_{\mathcal{L}(\ell^2)}}{\min(1, \sqrt{2\gamma|\varsigma(\zeta)|T})}\,\mathrm{d}\tilde{\eta}_\zeta\right)^{-1}.$$

Then for every $\theta \in [0, \theta_0)$, there exists $c^\theta(T) > 0$ and $C^\theta(T) > 0$ such that:

$$c^\theta(T)\|a\|_{\ell^2}^2 \leqslant \int_0^T \left|\theta \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t}\,\mathrm{d}\tilde{\eta}_\zeta + (1-\theta)\sum_{n\in\mathbb{Z}}[L_{\zeta_0}a]_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t}\right|^2 \mathrm{d}t \leqslant C^\theta(T)\|a\|_{\ell^2}^2 \qquad (a \in \ell^2).$$

Proof. First of all, we have from (3.8):

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t}\right|^2 \mathrm{d}t \leqslant \|L_\zeta\|_{\mathcal{L}(\ell^2)}^2 C_{\varsigma(\zeta)}(T)\|a\|_{\ell^2}^2 \qquad (a \in \ell^2),$$

for every $\zeta \in \mathbb{R}$ and $T > 0$, and

$$\Lambda_{\zeta_0}^2 c_{\varsigma(\zeta_0)}(T)\|a\|_{\ell^2}^2 \leqslant \int_0^T \left|\sum_{n\in\mathbb{Z}}[L_{\zeta_0}a]_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t}\right|^2 \mathrm{d}t \qquad (a \in \ell^2),$$

for every $T > 1/(|\varsigma(\zeta_0)|\gamma)$ and with $c_{\varsigma(\zeta_0)}(T)$ and $C_{\varsigma(\zeta)}(T)$ defined by (3.9). We conclude as in the proof of Theorem 3.1. □

Remark 3.3. *The condition $T > 1/(|\varsigma(\zeta_0)|\gamma)$ is only required in view of the fact that we have employed the classical formulation of Ingham's inequality. But, for instance, if the sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ is nondecreasing and satisfies the asymptotic gap condition $\liminf_{n\to\infty} \lambda_{n+1} - \lambda_n = +\infty$ then, employing generalised versions of Ingham's inequalities (see [18]), our result can be shown to hold true for every $T > 0$.*

3.4 Averaged control of parameter depending string systems

Using Proposition 3.3 together with (A.3) and the duality result, given in Proposition 2.1, we obtain an averaged controllability result for the string equation (1.4).

Proposition 3.4. *Let $\zeta_0 \in \mathbb{R}^*$ and $\tilde{\eta}$ be a probability measure on $\mathbb{R}$ with:*

$$\int_{\mathbb{R}} |\zeta|^\alpha \,\mathrm{d}\tilde{\eta}_\zeta < \infty \qquad (\alpha \in [\tfrac{1}{2}, 2]).$$

Set $T > 2/|\zeta_0|$ and

$$\theta_0 = \begin{cases} \left(1 + \dfrac{\sqrt{5}T}{\zeta_0\sqrt{(\zeta_0 T)^2-4}} \displaystyle\int_{\mathbb{R}} \dfrac{|\zeta|\left|1+(\zeta^2-1)\mathbf{1}_{(1,\infty)}(|\zeta|)\right|^{\frac12}}{\min\left(1,\sqrt{|\zeta|T}\right)}\,\mathrm{d}\tilde{\eta}_\zeta\right)^{-1} \\ \hfill \text{if } 0<\zeta_0<1\,, \\ \left(1 + \dfrac{\sqrt{5}T}{\zeta_0^2\sqrt{(\zeta_0 T)^2-4}} \displaystyle\int_{\mathbb{R}} \dfrac{|\zeta|\left|1+(\zeta^2-1)\mathbf{1}_{(1,\infty)}(|\zeta|)\right|^{\frac12}}{\min\left(1,\sqrt{|\zeta|T}\right)}\,\mathrm{d}\tilde{\eta}_\zeta\right)^{-1} \\ \hfill \text{if } \zeta_0\geqslant 1\,. \end{cases}$$

Then, for every $\theta \in [0,\theta_0)$, every target $(\mathrm{y}^{f,0},\mathrm{y}^{f,1}) \in L^2(0,1)\times H^{-1}(0,1)$ and every Bochner-integrable initial conditions $\zeta \mapsto (\mathrm{y}^{i,0}_\zeta,\mathrm{y}^{i,1}_\zeta) \in L^2(0,1)\times H^{-1}(0,1)$, there exists a control $u \in L^2(0,T)$ so that:

$$(1-\theta)y_{\zeta_0}(T)+\theta\int_{\mathbb{R}} y_\zeta(T)\,\mathrm{d}\tilde{\eta}_\zeta = \mathrm{y}^{f,0} \quad \text{and} \quad (1-\theta)\dot{y}_{\zeta_0}(T)+\theta\int_{\mathbb{R}} \dot{y}_\zeta(T)\,\mathrm{d}\tilde{\eta}_\zeta = \mathrm{y}^{f,1}\,,$$

where, for every $\zeta \in \mathbb{R}$, y_ζ is solution of (1.4).

Moreover, there exists a positive constant $C^\theta(T)$, independent of the initial and final conditions, such that:

$$\begin{aligned} \|u\|^2_{L^2(0,T)} \leqslant C^\theta(T) \Bigg(&\left\|(1-\theta)\mathrm{y}^{i,0}_{\zeta_0}+\theta\int_{\mathbb{R}}\mathrm{y}^{i,0}_\zeta\,\mathrm{d}\tilde{\eta}_\zeta\right\|^2_{L^2(0,1)} \\ &+\left\|(1-\theta)\mathrm{y}^{i,1}_{\zeta_0}+\theta\int_{\mathbb{R}}\mathrm{y}^{i,1}_\zeta\,\mathrm{d}\tilde{\eta}_\zeta\right\|^2_{H^{-1}(0,1)} + \|\mathrm{y}^{f,0}\|^2_{L^2(0,1)} + \|\mathrm{y}^{f,1}\|^2_{H^{-1}(0,1)}\Bigg)\,. \end{aligned}$$

4 Discrete averages of Ingham inequalities

In § 3.3, we used a perturbation argument to prove, roughly speaking, the stability of Ingham inequalities when the measure η is a Dirac mass plus a small enough perturbation. Obviously, there are many other cases of interest that do not enter on that setting.

In this paragraph we consider another interesting particular case, in which a finite number of equations are involved. In other words, we address the case in which the unknown parameter varies on a finite set.

Of course, our perturbation argument can be applied in this case (see § 4.1), but this argument required some smallness assumptions. We will see in § 4.2 that some averaged Ingham inequalities are still valid without this smallness assumption. In order to handle this case and prove the needed averaged Ingham inequalities we use a different argument. Instead of arguing through a perturbation principle, we shall rather use a method inspired by [10] and [40] whose key tool is to use the fact the solutions of the model under consideration, for given values of the parameter, are annihilated by a given linear bounded operator commuting with all other equations. This is the case for the 1d wave and Schrödinger equations with Dirichlet boundary conditions, for which the solutions are time-periodic.

In order to present these cases we consider a sequence $(\lambda_n)_{n\in\mathbb{Z}}$ satisfying:

$$\lambda_m \neq \lambda_n\,, \text{ for } m \neq n \quad \text{and} \quad \lambda_n \in \gamma\mathbb{Z} \qquad (m, n \in \mathbb{Z})\,, \tag{4.1}$$

with $\gamma > 0$. Of course, in this case the Ingham gap condition (3.7) holds. In order to enforce (4.1), we set $\lambda_n = \mu_n \gamma$, with $\mu_n \in \mathbb{Z}$.

As in § 3.3, we consider an operator $L_\zeta \in \mathcal{L}(\ell^2)$ and a function $\varsigma \in \mathbb{R}^{\mathbb{R}}$, and define the function f by:

$$f(t) = \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n\gamma\varsigma(\zeta_k)t} \qquad (a \in \ell^2\,,\ t > 0)\,, \tag{4.2}$$

with $K > 0$, $\theta_k \in [0, 1)$ such that $\sum_{k=0}^{K} \theta_k = 1$, and $\zeta_k \in \mathbb{R}$.
Since we are averaging on a finite number of parameters, the inequality (3.8) ensures, for every $T > 0$, the existence of a constant $C(T) > 0$ such that:

$$\|f\|^2_{L^2(0,T)} \leqslant C(T)\|a\|^2_{\ell^2} \qquad (a \in \ell^2)\,.$$

4.1 Application of the perturbation result

Let us make precise the statement of Proposition 3.3 for discrete averages.

Corollary 4.1. *Let $(\mu_n)_{n\in\mathbb{Z}}$ be a sequence of integers and $\gamma > 0$.
Let $\varsigma \in \mathbb{R}^{\mathbb{R}}$, $K \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $k_0 \in \{0, \cdots, K\}$, and for every $k \in \{0, \cdots, K\}$, let $\theta_k \in [0, 1]$ be the weights (so that $\sum_{k=0}^{K} \theta_k = 1$), $\zeta_k \in \mathbb{R}$ and $L_{\zeta_k} \in \mathcal{L}(\ell^2)$.
Assume for every $k \in \{0, \cdots, K\}$, $\varsigma(\zeta_k) \neq 0$ and assume $L_{\zeta_{k_0}}$ is bounded from below, i.e. there exists $\Lambda_{\zeta_{k_0}} > 0$ such that:*

$$\Lambda_{\zeta_{k_0}} \|a\|_{\ell^2} \leqslant \|L_{\zeta_{k_0}} a\|_{\ell^2} \qquad (a \in \ell^2)\,. \tag{4.3}$$

Then, if

$$T > \frac{1}{\gamma|\varsigma(\zeta_{k_0})|} \qquad \text{and} \qquad \theta_{k_0} > \frac{\Theta_{k_0}}{1 + \Theta_{k_0}}\,,$$

with

$$\Theta_{k_0} = \frac{1}{\Lambda_{\zeta_{k_0}}} \sqrt{\frac{5(\gamma\varsigma(\zeta_{k_0})T)^2}{(\gamma\varsigma(\zeta_{k_0})T)^2 - 1}} \sum_{\substack{k=0\\k\neq k_0}}^{K} \theta_k \frac{\|L_{\zeta_k}\|_{\mathcal{L}(\ell^2)}}{\min(1, \sqrt{2\gamma|\varsigma(\zeta_k)|T})},$$

there exist two constants, $c_{k_0}(T) > 0$ and $C_{k_0}(T) > 0$, such that:

$$c_{k_0}(T)\|a\|^2_{\ell^2} \leqslant \int_0^T \left| \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\gamma\mu_n\varsigma(\zeta_k)t} \right|^2 \mathrm{d}t \leqslant C_{k_0}(T)\|a\|^2_{\ell^2}\,,$$

for every $a \in \ell^2$.

Remark 4.1. *We have shown that averaged versions of Ingham's inequalities hold true under a suitable smallness condition on the perturbing measures. On one hand, this is necessary in some sense as the example below shows but, on the other hand, with some more assumptions, this assumption is not needed as we will see in § 4.2.*

Consider the case $L_\zeta = \mathrm{Id}$, $\gamma = 1$, $\mu_n = n$, $\zeta_0 = 1$, $\zeta_1 = 2$, $\varsigma(\zeta) = \zeta$ *and the measure* $\eta = (1-\theta)\delta_{\zeta_0} + \theta\delta_{\zeta_1}$. *The assumption (4.3) of the above corollary holds for* $k_0 = 0$ *and* $k_0 = 1$. *Consequently, for every* $T > 1$, *there exist* $\theta_0^0, \theta_0^1 \in [0,1)$ *such that if* $\theta \in [0, \theta_0^0) \cup (1 - \theta_0^1, 1]$, *there exist two nonnegative constants* $c^\theta(T)$ *and* $C^\theta(T)$ *such that:*

$$c^\theta(T)\|a\|_{\ell^2}^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{Z}} a_n \left((1-\theta) e^{2i\pi nt} + \theta e^{4i\pi nt} \right) \right|^2 \mathrm{d}t \leqslant C^\theta(T)\|a\|_{\ell^2}^2 \qquad (a \in \ell^2)\,.$$

The closeness condition on θ, θ *close enough to* 0 *or close enough to* 1, *is necessary. In particular, for* $\theta = 1/2$, *no such* $c^{\frac{1}{2}}(T) > 0$ *exists. Indeed, setting:*

$$f(t) = \sum_{n\in\mathbb{Z}} a_n \left(\tfrac{1}{2} e^{2i\pi nt} + \tfrac{1}{2} e^{4i\pi nt} \right) \qquad (t \in [0,T])\,,$$
$$\text{with} \qquad a_n = \begin{cases} (-1)^k & \text{if } n = 2^k \text{ and } k \in \{0, \cdots, N\}\,, \\ 0 & \text{otherwise,} \end{cases} \qquad (n \in \mathbb{Z})\,,$$

where $N \in \mathbb{N}^*$ *is given, we obtain,*

$$f(t) = \frac{(-1)^N}{2} e^{4i\pi 2^{N+1} t} \qquad (t \in [0,T])\,.$$

Thus, for every $T > 0$, $\int_0^T |f(t)|^2 \,\mathrm{d}t = T/4$, *whereas,* $\|a\|_{\ell^2} = \sqrt{N+1}$. *Letting* N *go to infinity we see that no Ingham inequality can hold whatever* $T > 0$ *is.*

4.2 A time-periodicity argument

Let us start with an Ingham type inequality which is valid with little hypothesis on the coefficients $\varsigma(\zeta_k)$.

Theorem 4.1. *Let* $(\mu_n)_{n\in\mathbb{Z}}$ *be a sequence of integers and* $\gamma > 0$. *Let* $\varsigma \in \mathbb{R}^\mathbb{R}$, $K \in \mathbb{N}^*$, *and for every* $k \in \{0, \cdots, K\}$, *let* $\theta_k \in [0,1]$ *be the weights (so that* $\sum_{k=0}^K \theta_k = 1$*) and* $\zeta_k \in \mathbb{R}$. *Assume for every* $k \in \{0, \cdots, K\}$, $\varsigma(\zeta_k) \neq 0$.

Then, if

$$T > \frac{1}{\gamma} \sum_{k=0}^K \frac{1}{|\varsigma(\zeta_k)|}\,,$$

there exists a constant $c(T) > 0$, *independent of the sequences* $(a_n^k)_n \in \ell^2$, *such* *that:*

$$\int_0^T \left| \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{Z}} a_n^k e^{2i\pi\mu_n \gamma\varsigma(\zeta_k)t} \right|^2 \,\mathrm{d}t$$
$$\geqslant \theta_0^2 c(T) \sum_{n\in\mathbb{Z}} |a_n^0|^2 \prod_{l=1}^{K} \sin^2\left(\pi\mu_n \frac{\varsigma(\zeta_0)}{\varsigma(\zeta_l)}\right) . \quad (4.4)$$

Remark 4.2. *Inequality* (4.4) *is similar to [10, (5.87), p. 139] which can be applied to the simultaneous control of finitely many strings (see § 5.8.2 of that book).*

Proof. First of all, by changing $\gamma\varsigma(\zeta_k)$ in ζ_k, we can assume without loss of generality that $\gamma = 1$ and $\varsigma(\zeta) = \zeta$.
Given $(a_n^k)_n \in \ell^2$ for each $k \in \{0, \cdots, K\}$, we define the function f by:

$$f(t) = \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{Z}} a_n^k e^{2i\pi\mu_n\zeta_k t} \qquad (t \in \mathbb{R}) .$$

We have:

$$f(t + |\zeta_K|^{-1}) - f(t) = \sum_{k=0}^{K-1} \theta_k \sum_{n\in\mathbb{Z}} a_n^k \left(e^{2i\pi\mu_n \frac{\zeta_k}{|\zeta_K|}} - 1 \right) e^{2i\pi\mu_n\zeta_k t} \qquad (t \in \mathbb{R}) .$$

Iterating this argument, we obtain:

$$F_0(t) = \theta_0 \sum_{n\in\mathbb{Z}} a_n^0 \prod_{l=0}^{K-1} \left(e^{2i\pi\mu_n \frac{\zeta_0}{|\zeta_{K-l}|}} - 1 \right) e^{2i\pi\mu_n\zeta_0 t} \qquad (t \in \mathbb{R}) ,$$

where F_0 is defined recursively by:

$$\begin{array}{rcll} F_K(t) & = & f(t) , & \\ F_{k-1}(t) & = & F_k(t + |\zeta_k|^{-1}) - F_k(t) & (1 \leqslant k \leqslant K) . \end{array} \qquad (t \in \mathbb{R}) . \quad (4.5)$$

For any $\tau > 0$, we can apply the classical Ingham inequality [14, Theorem 1] to deduce the existence of a constant c_τ (depending only on τ and ζ_0) such that:

$$\int_0^{\frac{1}{\zeta_0}+\tau} |F_0(t)|^2 \,\mathrm{d}t \geqslant c_\tau\, \theta_0^2 \sum_{n\in\mathbb{Z}} \left|a_n^0\right|^2 \prod_{l=0}^{K-1} \left| e^{2i\pi\mu_n \frac{\zeta_0}{|\zeta_{K-l}|}} - 1 \right|^2 . \quad (4.6)$$

Independently, we have:

$$\int_0^{\frac{1}{|\zeta_0|}+\tau} \left|F_0(t)\right|^2 \,\mathrm{d}t = \int_0^{\frac{1}{|\zeta_0|}+\tau} \left|F_1(t + |\zeta_1|^{-1}) - F_1(t)\right|^2 \,\mathrm{d}t$$
$$\leqslant 2 \int_0^{\frac{1}{|\zeta_0|}+\frac{1}{|\zeta_1|}+\tau} \left|F_1(t)\right|^2 \,\mathrm{d}t \qquad (\tau > 0)$$

and by iteration,

$$\|F_0\|^2_{L^2(0,\,\frac{1}{|\zeta_0|}+\tau)} \leqslant 2^K \|f\|^2_{L^2\left(0,\,\tau+\sum_{k=0}^K |\zeta_k|^{-1}\right)} \qquad (\tau > 0)\,. \tag{4.7}$$

We end the proof by combining (4.6) and (4.7). □

With some more conditions on the parameters $\varsigma(\zeta_k)$, the following unique continuation property can be easily obtained from (4.4).

Corollary 4.2. *Let $(\mu_n)_{n\in\mathbb{Z}}$ be a sequence of integers and $\gamma > 0$.*
Let $\varsigma \in \mathbb{R}^{\mathbb{R}}$, $K \in \mathbb{N}^*$, and for every $k \in \{0, \cdots, K\}$, let $\theta_k \in [0, 1]$ be the weights (so that $\sum_{k=0}^K \theta_k = 1$), $(a_n)_n \in \ell^2$, $\zeta_k \in \mathbb{R}$ and $L_{\zeta_k} \in \mathcal{L}(\ell^2)$.
Assume $\theta_0 \neq 0$, L_{ζ_0} is bounded from below, $\varsigma(\zeta_0) \neq 0$ and

$$\varsigma(\zeta_0)^{-1}\varsigma(\zeta_k) \notin \mathbb{Q} \qquad (k \in \{1, \cdots, K\})\,. \tag{4.8}$$

Then, for every $T > \dfrac{1}{\gamma}\displaystyle\sum_{k=0}^K \frac{1}{|\varsigma(\zeta_k)|}$,

$$\int_0^T \left|\sum_{k=0}^K \theta_k \sum_{n\in\mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n\gamma\varsigma(\zeta_k)t}\right|^2 \mathrm{d}t = 0 \qquad \Longleftrightarrow \qquad \forall n \in \mathbb{Z}\,,\ a_n = 0.$$

Proof. Set $a \in \ell^2$ and assume

$$\int_0^T \left|\sum_{k=0}^K \theta_k \sum_{n\in\mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n\gamma\varsigma(\zeta_k)t}\right|^2 \mathrm{d}t = 0.$$

Set $a^k = L_{\zeta_k} a$. We have from Theorem 4.1, (4.4):

$$0 = \theta_0^2 c(T) \sum_{n\in\mathbb{Z}} |[L_{\zeta_0} a]_n|^2 \prod_{l=1}^K \sin^2\left(\pi\mu_n \frac{\varsigma(\zeta_0)}{\varsigma(\zeta_l)}\right)\,,$$

with $c(T) > 0$. Since $\theta_0 \neq 0$ and $\varsigma(\zeta_0)^{-1}\varsigma(\zeta_k) \notin \mathbb{Q}$ for every $k \in \{1, \cdots, K\}$, we obtain $[L_{\zeta_0} a]_n = 0$ for every $n \in \mathbb{Z}$. We conclude with L_{ζ_0} bounded from below. □

In Corollary 4.2, we have presented a unique continuation result. However, with some more restrictive conditions on the parameters $\varsigma(\zeta_k)$, we can obtain an observability inequality.

Corollary 4.3. *Assume that the conditions of Corollary 4.2 are satisfied and, let $\varepsilon > 0$ such that, in addition, for every $\alpha > 0$, there exists $\Lambda_{\zeta_0,\alpha} > 0$ such that:*

$$\sum_{n\in\mathbb{Z}} \frac{|[L_{\zeta_0} a]_n|^2}{|\mu_n|^{2\alpha}} \geqslant \Lambda^2_{\zeta_0,\alpha} \sum_{n\in\mathbb{Z}} \frac{|a_n|^2}{|\mu_n|^{2\alpha}} \qquad (a \in \ell^2) \tag{4.9}$$

and assume, $\varsigma(\zeta_0)^{-1}\varsigma(\zeta_k) \in \mathbf{B}_\varepsilon$ *for every* $k \in \{1, \cdots, K\}$*, with* $\mathbf{B}_\varepsilon$ *defined, as in [6, p. 120] (see also [10, Proposition A.5] or Proposition 4.1).*

Then, for every $T > \dfrac{1}{\gamma}\displaystyle\sum_{k=0}^{K}\frac{1}{|\varsigma(\zeta_k)|}$*, there exists a constant* $C_\varepsilon(T) > 0$ *such that:*

$$\int_0^T \left|\sum_{k=0}^{K}\theta_k \sum_{n\in\mathbb{Z}}[L_{\zeta_k}a]_n e^{2i\pi\mu_n\gamma\varsigma(\zeta_k)t}\right|^2 \mathrm{d}t \geqslant C_\varepsilon(T)\sum_{n\in\mathbb{Z}}\frac{|a_n|^2}{|\mu_n|^{2K(1+\varepsilon)}} \qquad (a \in \ell^2)\,.$$

Proof. The proof follows directly from (4.4) and [6, p. 120]. $\square$

Proposition 4.1 (J. W. S. Cassles [6])**.** *For every* $\varepsilon > 0$ *there exists a set* $\mathbf{B}_\varepsilon \subset \mathbb{R}$ *such that the Lebesgue measure of* $\mathbb{R} \setminus \mathbf{B}_\varepsilon$ *is equal to zero, and a constant* $\rho_\varepsilon > 0$ *for which, if* $\zeta \in \mathbf{B}_\varepsilon$ *then,*

$$\min_{r\in\mathbb{Z}}|r - m\zeta| \geqslant \frac{\rho_\varepsilon}{m^{1+\varepsilon}} \qquad (m \in \mathbb{N}^*)\,.$$

Corollary 4.4. *Assume that the conditions of Corollary 4.2 hold and, furthermore, that for every* $\alpha > 0$*, there exists* $\Lambda_{\zeta_0,\alpha} > 0$ *such that* (4.9) *holds. Assume in addition that* $\varsigma(\zeta_0)^{-1}\varsigma(\zeta_1), \cdots, \varsigma(\zeta_0)^{-1}\varsigma(\zeta_K)$ *are algebraic and* $\varsigma(\zeta_0), \cdots, \varsigma(\zeta_K)$ *are* $\mathbb{Q}$*-linearly independent.*

Then for every

$$T > \frac{1}{\gamma}\sum_{k=0}^{K}\frac{1}{|\varsigma(\zeta_k)|}$$

and every $\varepsilon > 0$*, there exists* $C_\varepsilon(T) > 0$ *such that:*

$$\int_0^T \left|\sum_{k=0}^{K}\theta_k \sum_{n\in\mathbb{Z}}[L_{\zeta_k}a]_n e^{2i\pi\mu_n\gamma\varsigma(\zeta_k)t}\right|^2 \mathrm{d}t \geqslant C_\varepsilon(T)\sum_{n\in\mathbb{Z}}\frac{|a_n|^2}{|\mu_n|^{2(1+\varepsilon)}} \qquad (a \in \ell^2)\,.$$

Proof. The proof follows directly from (4.4) and [33, Theorem 1] (see also [10, Theorem A.7 and p. 209]). $\square$

Remark 4.3. *Let us compare Corollary 4.1 with the Corollaries 4.3 and 4.4. In Corollary 4.1, no irrationality condition is needed and the observability inequality holds in the classical* ℓ^2*-norm whereas in Corollaries 4.3 and 4.4, an irrationality condition is required and the observation inequality is valid only for coefficients which are in some subspace of* ℓ^2*. In addition, the minimal observation time required for the observability inequality in Corollaries 4.3 and 4.4 is larger than the one required in Corollary 4.1. But to obtain the observation inequality of Corollary 4.1, we need some weight* θ_{k_0} *to be close enough to* 1*.*

In conclusion, Corollaries 4.3 and 4.4, become relevant when none of the weights θ_k *is close enough to* 1*.*

4.3 Application to the string equation

Using the technical results in Appendix A together with the results of this section, we obtain the following consequences for the average controllability of finite combinations of the string equation (1.4).

From Corollary 4.2 we can derive an approximate averaged controllability results.

Proposition 4.2. *Let $K \in \mathbb{N}^*$, and for every $k \in \{0, \cdots, K\}$, define the weight $\theta_k \in (0,1)$ (so that $\sum_{k=0}^{K} \theta_k = 1$) and the parameter $\zeta_k \in \mathbb{R}_*$ and assume:*

$$\zeta_0^{-1}\zeta_k \notin \mathbb{Q} \qquad (k \in \{1, \cdots, K\}) .$$

Then for every $T > 2\sum_{k=0}^{K} \frac{1}{|\zeta_k|}$, every $\varepsilon > 0$, every target $(\mathrm{y}^{f,0}, \mathrm{y}^{f,1}) \in L^2(0,1) \times H^{-1}(0,1)$ and every initial conditions $(\mathrm{y}^{i,0}_{\zeta_k}, \mathrm{y}^{i,1}_{\zeta_k}) \in L^2(0,1) \times H^{-1}(0,1)$, there exists a control $u \in L^2(0,T)$ for which we have:

$$\left\| \mathrm{y}^{f,0} - \sum_{k=0}^{K} \theta_k y_{\zeta_k}(T) \right\|^2_{L^2(0,1)} \leqslant \varepsilon \qquad \text{and} \qquad \left\| \mathrm{y}^{f,1} - \sum_{k=0}^{K} \theta_k \dot{y}_{\zeta_k}(T) \right\|^2_{H^{-1}(0,1)} \leqslant \varepsilon \,,$$

where for every $\zeta \in \mathbb{R}^*$, y_ζ solves (1.4).

Moreover, there exists a constant $C_\varepsilon(T) > 0$ independent of the initial and final conditions such that:

$$\|u\|^2_{L^2(0,T)} \leqslant C_\varepsilon(T) \left(\left\| \sum_{k=0}^{K} \theta_k \mathrm{y}^{i,0}_{\zeta_k} \right\|^2_{L^2(0,1)} + \left\| \sum_{k=0}^{K} \theta_k \mathrm{y}^{i,1}_{\zeta_k} \right\|^2_{H^{-1}(0,1)} \right.$$
$$\left. + \|\mathrm{y}^{f,0}\|^2_{L^2(0,1)} + \|\mathrm{y}^{f,1}\|^2_{H^{-1}(0,1)} \right) .$$

Now using Diophantine approximations, see Corollary 4.4, we obtain:

Proposition 4.3. *Let $\varepsilon > 0$ and let $K \in \mathbb{N}^*$, θ_k and ζ_k be defined by Proposition 4.2 and assume, $\zeta_0, \cdots, \zeta_K$ are $\mathbb{Q}$-linearly independent and*

$$\zeta_0^{-1}\zeta_1, \cdots, \zeta_0^{-1}\zeta_K \text{ are algebraic.} \tag{4.10}$$

For every $\alpha \in \mathbb{R}$, set:

$$X_\alpha = \left\{ \varphi : x \in (0,1) \mapsto \sum_{n=1}^{\infty} a_n \sin(n\pi x) \,, \ \sum_{n \in \mathbb{N}^*} n^{2\alpha} |a_n|^2 < \infty \right\} , \tag{4.11}$$

Then, if $(\mathrm{y}^{i,0}_{\zeta_0}, \mathrm{y}^{i,1}_{\zeta_0}), \cdots, (\mathrm{y}^{i,0}_{\zeta_K}, \mathrm{y}^{i,1}_{\zeta_K}), (\mathrm{y}^{f,0}, \mathrm{y}^{f,1}) \in X_{1+\varepsilon} \times X_\varepsilon$, for every $T > 2\sum_{k=0}^{K} \frac{1}{|\zeta_k|}$ there exists a control $u \in L^2(0,T)$ such that for every $k \in \{1, \cdots, K\}$, the solution y_{ζ_k} of (1.4) (with parameter $\zeta = \zeta_k$) satisfy:

$$\sum_{k=0}^{K} \theta_k y_{\zeta_k}(T) = \mathrm{y}^{f,0} \quad \text{and} \quad \sum_{k=0}^{K} \theta_k \dot{y}_k(T) = \mathrm{y}^{f,1} .$$

A similar result could have been obtained from Corollary 4.3.

Let us also notice that applying directly [10, Corollary 5.43], we obtain the following exact simultaneous controllability result:

Proposition 4.4. *Let $\varepsilon > 0$, $K \in \mathbb{N}^*$ and $\zeta_k \in \mathbb{R}^*$ for every $k \in \{0, \cdots, K\}$ and assume, $\zeta_0, \cdots, \zeta_K$ are $\mathbb{Q}$-linearly independent and*

$$\zeta_k^{-1}\zeta_l \text{ is algebraic for every } k, l \in \{0, \cdots, K\} . \tag{4.12}$$

Let $(\mathrm{y}^{i,0}_{\zeta_k}, \mathrm{y}^{i,1}_{\zeta_k})$ and choose final conditions $(\mathrm{y}^{f,0}, \mathrm{y}^{f,1})$ satisfying the assumption given in Proposition 4.3.

Then, for every $T \geqslant 2 \sum_{k=0}^{K} \frac{1}{|\zeta_k|}$ there exists a control $u \in L^2(0,T)$ such that:

$$y_{\zeta_k}(T) = \mathrm{y}^{f,0} \quad \text{and} \quad \dot{y}_{\zeta_k}(T) = \mathrm{y}^{f,1} \qquad (k \in \{0, \cdots, K\}) .$$

This result ensures that all the parameter dependent trajectories, and, consequently, their average, can be steered to a prescribed target with an input independent of the parameter.

Remark 4.4. *As expected, the assumption* (4.10) *needed to obtain averaged controllability is weaker than* (4.12)*, the assumption needed for simultaneous controllability.*

5 Concluding remarks

The aim of this article was to give a systematic result, based on perturbation arguments, on the averaged controllability and observability of parameter-dependent families of equations, mainly in the context of time-reversible groups of isometries.

Let us summarise the main results on the averaged control for the system (1.4), with two string equations, one parametrised by $\zeta_0 = 1$ and the other one by $\zeta_1 = \sqrt{2}$, with averaging measure:

$$\eta^\theta = (1-\theta)\delta_{\zeta_0} + \theta\delta_{\zeta_1} . \tag{5.1}$$

The following holds:

- From Corollary 4.1 (see also Proposition 3.4),
  - if $T > 2$, the system (1.4) is controllable in average with averaging measure η^θ for $\theta \in \left[0, \left(1 + \frac{2\sqrt{5}T}{\sqrt{T^2-4}}\right)^{-1}\right)$;
  - if $T > \sqrt{2}$, the system (1.4) is controllable in average with averaging measure η^θ for $\theta \in \left(1 - \left(1 + \frac{\sqrt{5}T}{2\sqrt{2T^2-4}}\right)^{-1}, 1\right]$;

- From Proposition 4.3, if $T > 2\left(1 + \frac{\sqrt{2}}{2}\right)$, the system (1.4) is controllable in average with averaging measure η^θ in some weighted space for $\theta \in [0, 1]$.

This leads to the time-dependent set of parameters θ for which we have averaged controllability, see Figure 1 below.

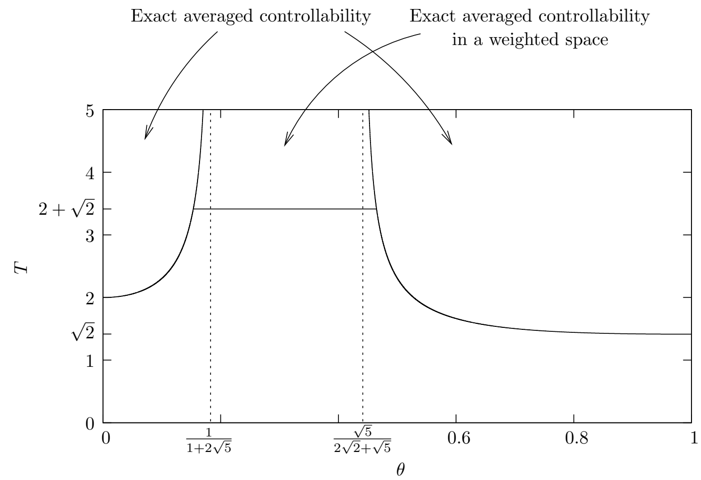

Figure 1: Time dependent set of parameters θ for which averaged controllability holds, for two strings driven by the system (1.4) with parameters $\zeta_0 = 1$ and $\zeta_1 = \sqrt{2}$ and averaging measure η^θ given by (5.1).

There are several interesting open problems that arise in the context of averaged controllability. This is so even for the one dimensional case, where Fourier series representations can be used. Let us point out some of them:

- In § 3.1, we gave an averaged observability inequality. However, this result only holds when the measure is the sum of a Dirac mass and a small enough perturbation measure. But it would be natural to consider more general cases as well.

  In the case $L_\zeta = \mathrm{Id}$ and $\varsigma(\zeta) = \zeta$, the issues we discussed in the previous paragraph on the averages of non-harmonic Fourier series can be recast in terms of the property of Riesz sequence stability of the family $\{t \mapsto \hat{\eta}(-\lambda_n t)\}_n$ ($\hat{\eta}$ being the Fourier-Stieltjes transform of the density of probability η), in the closed subspace of $L^2(0, T)$ they generate. This is so since the observation map is:

  $$\int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n \zeta t} \, \mathrm{d}\eta_\zeta = \sum_{n \in \mathbb{Z}} a_n \hat{\eta}(-\lambda_n t) \qquad (t \in \mathbb{R}) \, .$$

  This problem is related to frame theory. However, even if the literature on this subject is huge (see for instance [16, 4, 9, 11, 12, 8, 5, 23]), the results we needed did not seem to be available.

One of the simplest case to be considered is when η^ε is given by $\mathrm{d}\eta^\varepsilon_\zeta = \frac{1}{2\varepsilon}\mathbf{1}_{[1-\varepsilon,1+\varepsilon]}(\zeta)\,\mathrm{d}\zeta$ for $\varepsilon > 0$. Then, $(\eta^\varepsilon)_{\varepsilon>0}$ converges in the sense of measures to the Dirac mass δ_1 when ε goes to 0. Assuming that the sequence $(\lambda_n)_n$ satisfies the Ingham gap condition (3.7) for some $\gamma > 0$, we know from [14] that a constant $c(T) > 0$ such that:

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n t}\right|^2 \mathrm{d}t \geqslant c(T)\sum_{n\in\mathbb{Z}}|a_n|^2 \qquad (T > \frac{1}{\gamma}\,,\ (a_n)_{n\in\mathbb{Z}} \in \ell^2)$$

exists. It is then natural to wonder if there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$, we have:

$$\int_0^T \left|\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n\zeta t}\,\mathrm{d}\eta^\varepsilon_\zeta\right|^2 \mathrm{d}t \geqslant c_\varepsilon(T)\sum_{n\in\mathbb{Z}}|a_n|^2 \qquad (T > \frac{1}{\gamma}\,,\ (a_n)_{n\in\mathbb{Z}} \in \ell^2) \quad (5.2)$$

and if it is so, whether $c_\varepsilon(T)$ converges to $c(T)$ as ε tends to 0?

A way to prove this result is to bound the quantity:

$$\left|\int_0^T \left(\left|\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n\zeta t}\,\mathrm{d}\eta^\varepsilon_\zeta\right|^2 - \left|\sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n t}\right|^2\right) \mathrm{d}t\right|.$$

One can get the upper bound:

$$\varepsilon T^2 C\sqrt{\sum_{n\in\mathbb{Z}}|\lambda_n||a_n|^2}\sqrt{\sum_{n\in\mathbb{Z}}|a_n|^2}\,,$$

which goes to 0 as ε tends to 0, but does not ensure inequality (5.2) to hold.

One can also proceed with a direct computation and, in this case, from section 4 one can derive a weighted averaged Ingham inequality when the eigenvalues λ_n satisfy (4.1). Indeed, let us consider a measure η given by $\mathrm{d}\eta_\zeta = \frac{1}{\zeta_1-\zeta_0}\mathbf{1}_{[\zeta_0,\zeta_1]}(\zeta)\,\mathrm{d}\zeta$ for $\zeta_0 < \zeta_1$ and $\zeta_0, \zeta_1 \neq 0$. Writing $\lambda_n = \gamma\mu_n$ with $\mu_n \in \mathbb{Z}^*$, we obtain:

$$\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n\zeta t}\,\mathrm{d}\eta_\zeta = \frac{1}{2i\pi\gamma(\zeta_1-\zeta_0)t}\sum_{n\in\mathbb{Z}}\frac{a_n}{\mu_n}\left(e^{2i\pi\gamma\zeta_1\mu_n t} - e^{2i\pi\gamma\zeta_0\mu_n t}\right),$$

from which we deduce:

$$\int_0^T \left|\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} a_n e^{2i\pi\lambda_n\zeta t}\,\mathrm{d}\eta_\zeta\right|^2 \mathrm{d}t \geqslant \frac{1}{(2\pi\gamma)^2(\zeta_1-\zeta_0)^2T^2}\int_0^T \left|\sum_{n\in\mathbb{Z}}\frac{a_n}{\mu_n}\left(e^{2i\pi\gamma\zeta_1\mu_n t} - e^{2i\pi\gamma\zeta_0\mu_n t}\right)\right|^2 \mathrm{d}t\,.$$

Now, assuming $T > \frac{1}{\gamma}\left(\frac{1}{\zeta_0}+\frac{1}{\zeta_1}\right)$ and $\zeta_0^{-1}\zeta_1 \notin \mathbb{Q}$, we obtain the unique continuation from Corollary 4.2 and, under the assumption of Corollary 4.3 or 4.4 on ζ_0 and ζ_1, we end up with a weighted Ingham inequality.

- When dealing with the control system (3.5), in [24], the condition (3.6) was required to ensure averaged controllability. However, according to Proposition 3.2 (see Remark 3.2), this condition is not needed under a suitable smallness assumption on the averaging measure.
  The optimality of assumption (3.6) without smallness assumptions needs further clarification.

- The results derived in section 4 need $\lambda_n \in \gamma\mathbb{Z}$ to be a sequence of integers. But the unique continuation property, Corollary 4.2, could have been obtained directly from [13, Corollary 2.3.5]. This result still holds in the general case where $(\lambda_n)_n$ satisfies (3.7) and assuming that the values $\varsigma(\zeta_k)\lambda_n \neq \varsigma(\zeta_l)\lambda_m$ for $k \neq l$ or $n \neq m$.
  In addition, results similar to corollaries 4.3 and 4.4 could have been obtained from [20]. More precisely, assume that the sequence $(\lambda_n)_n$ satisfies the Ingham gap condition (3.7), and that $\varsigma(\zeta_k)\lambda_n \neq \varsigma(\zeta_l)\lambda_m$ for $k \neq l$ or $n \neq m$. Let us now consider the increasing sequence $(\Lambda_n)_n$ such that $\{\Lambda_n\,,\ n \in \mathbb{Z}\} = \{\varsigma(\zeta_k)\lambda_m\,,\ m \in \mathbb{Z}\,,\ k \in \{0,\cdots,K\}\}$. Then for every $n \in \mathbb{Z}$, we have $\Lambda_{n+K+1} - \Lambda_n \geqslant \gamma \min\{|\varsigma(\zeta_k)|\,,\ k \in \{0,\cdots,K\}\}$. Thus, [20, Theorem 4] applies and leads to a weighted averaged Ingham inequality valid for every
  $$T > \frac{K+1}{\gamma\min\{|\varsigma(\zeta_0)|,\cdots,|\varsigma(\zeta_K)|\}}.$$
  Notice that this minimal time is greater than $\sum_{k=0}^{K}\frac{1}{\gamma|\varsigma(\zeta_k)|}$, the one obtained in corollaries 4.3 and 4.4, but under stronger assumptions on the sequence $(\lambda_n)_n$.
  In addition, the results given in [13] and [20] ensure simultaneous observability. Thus, it would be interesting to see how the assumption given in these two works could be weakened in order to only ensure averaged observability.

  In the proof of Theorem 4.1, we strongly need that the sequence $(\lambda_n)_n$ satisfies (4.1) and, even for $\lambda_n = n + \varepsilon(n)$ with $\varepsilon(n) = o(1)$, the technique of proof fails. It would be worth exploring whether some improvements could be obtained with a perturbation argument, combined with the ideas of [7] and in particular with Ulrich's result [36].

  The analysis of all these examples could contribute to achieve sharp results for the averaged controllability of finitely many string equations.

- Let us conclude this paper with a general remark linking averaged controllability and simultaneous controllability. The aim is to find controls independent of the parameter performing well for all values of the parameters.

With this goal, a first and natural choice was to control the average of the parameter dependent outputs. Of course, the best we could expect is a control, independent of the values of the unknown parameters, steering all parameter dependent trajectories to a common fixed target, i.e. looking to simultaneous controllability. But this is unfeasible in general.

There exists a natural link between the control of the average and the stronger notion of simultaneous control. This link can be made through penalisation and optimal control.

More precisely, for every $\kappa \geqslant 0$, let us consider the following optimal control problem:

$$\min \quad J_\kappa(u) := \frac{1}{2}\|u\|^2_{L^2([0,T],U)} + \kappa \int_{\mathbb{R}} \left\| y_\zeta(T) - \mathrm{y}^f \right\|_X^2 \,\mathrm{d}\eta_\zeta$$
$$\left| \begin{array}{l} \displaystyle\int_{\mathbb{R}} y_\zeta(T)\,\mathrm{d}\eta_\zeta = \mathrm{y}^f\,, \\ \dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u\,, \qquad y_\zeta(0) = \mathrm{y}^i_\zeta\,. \end{array} \right.$$

Under the property of averaged controllability, the minimiser u_κ exists for every $\kappa > 0$.

For $\kappa = 0$ this leads the averaged control of minimal norm as we considered here. But, as κ increases, the control, other than ensuring the averaged controllability property, also forces the reduction of the variance of the output.

It can also be proved that, if, in addition, $J_\kappa(u_\kappa)$ is uniformly bounded, then up to a subsequence, $(u_\kappa)_\kappa$ is weakly convergent to a simultaneous control u_∞ solution of:

$$\left| \begin{array}{l} y_\zeta(T) = \mathrm{y}^f \qquad (\zeta \in \mathbb{R} \quad \eta\text{-a.e.})\,, \\ \dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u\,, \qquad y_\zeta(0) = \mathrm{y}^i\,. \end{array} \right.$$

This issue is analysed in [29], where this idea is discussed in detail in the finite-dimensional control context.

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Appendix A Averaged controllability for a parameter dependent string equation

Let us briefly describe how the string equation with Dirichlet boundary control (1.4) enters in the abstract formalism introduced in section 2.

The PDE under consideration being second order in time, it is more convenient to assume that the parameter ζ enters quadratically in (1.4a). In addition, one can assume $\zeta \in \mathbb{R}_+$ or equivalently, that the averaging measure η satisfies $\operatorname{supp} \eta \subset \mathbb{R}_+$.

For the abstract tools introduced here, we refer to [35, Sections 10.9 and 11.6] for further details.

Let us introduce the one dimensional Dirichlet-Laplacian operator, A_0:

$$\mathcal{D}(A_0) = H^2(0,1) \cap H^1_0(0,1) \quad \text{and} \quad A_0 f = -\partial_x^2 f \qquad (f \in \mathcal{D}(A_0))$$

and the Hilbert spaces $H = L^2(0,1)$, $H_1 = \mathcal{D}(A)$, $H_{\frac{1}{2}} = H^1_0(0,1)$ and H_{-1} (resp. $H_{-\frac{1}{2}}$) the dual space of H_1 (resp. $H_{\frac{1}{2}}$) with respect to the pivot space H. Then A_0 can be seen as a unitary operator from H_1 to H, $H_{1/2}$ to $H_{-1/2}$ and H to H_{-1}.

We remind that the Dirichlet-Laplacian operator A_0 can be diagonalized in an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}^*}$ of $L^2(0,1)$. More precisely, we have:

$$\varphi_n(x) = \sqrt{2}\sin(n\pi x) \quad \text{and} \quad A_0\varphi_n = (n\pi)^2\varphi_n \qquad (x \in [0,1]\,,\ n \in \mathbb{N}^*)\,. \quad \text{(A.1)}$$

Define the state space $X = H \times H_{-\frac{1}{2}}$, the control space $U = \mathbb{R}$, the operator $A = \begin{bmatrix} 0 & \mathrm{Id} \\ -A_0 & 0 \end{bmatrix}$ with domain $\mathcal{D}(A) = H_{\frac{1}{2}} \times H := X_1$ and the control operator $B = \begin{bmatrix} 0 \\ A_0 D \end{bmatrix} \in \mathcal{L}(U, X_{-1})$, with D the Dirichlet map, see [35, Proposition 10.6.1]. We have $A^* = -A$ and $B^* \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix} = \partial_x \left(A_0^{-1}\mathrm{z}^1\right)(0)$. Denote by $\mathbb{T}$ the semi-group generated by A. It is classical that B is an admissible control operator for $\mathbb{T}$.

Now, define $f_\zeta(s,x) = y_\zeta(\frac{s}{\zeta}, x)$. The function f_ζ is solution of:

$$\begin{aligned}
&\ddot{f}_\zeta(s,x) = \partial_x^2 f_\zeta(s,x) && ((s,x) \in \mathbb{R}_+^* \times (0,1))\,,\\
&f_\zeta(s,0) = u(\tfrac{s}{\zeta}) && (s \in \mathbb{R}_+^*)\,,\\
&f_\zeta(s,1) = 0 && (t \in \mathbb{R}_+^*)\,,
\end{aligned}$$

$$f_\zeta(0,x) = \mathrm{y}_\zeta^{i,0}(x) \quad \text{and} \quad \dot{f}_\zeta(0,x) = \tfrac{1}{\zeta}\mathrm{y}_\zeta^{i,1}(x) \qquad (x \in (0,1))\,.$$

Setting $I_\zeta = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \zeta\mathrm{Id} \end{bmatrix}$ and $F_\zeta(s) = \begin{bmatrix} f_\zeta(s) \\ \dot{f}_\zeta(s) \end{bmatrix} = I_\zeta^{-1} \begin{bmatrix} y_\zeta(\frac{s}{\zeta}) \\ \dot{y}_\zeta(\frac{s}{\zeta}) \end{bmatrix}$, F_ζ is solution of:

$$\dot{F}_\zeta = AF_\zeta + Bu(\tfrac{\cdot}{\zeta})\,, \qquad F_\zeta(0) = I_\zeta^{-1} \begin{bmatrix} \mathrm{y}_\zeta^{i,0} \\ \mathrm{y}_\zeta^{i,1} \end{bmatrix}\,.$$

Using Duhamel formula, we obtain:

$$
\begin{aligned}
F_\zeta(\zeta T) &= \mathbb{T}(\zeta T) I_\zeta^{-1} \begin{bmatrix} \mathrm{y}_\zeta^{i,0} \\ \mathrm{y}_\zeta^{i,1} \end{bmatrix} + \int_0^{\zeta T} \mathbb{T}(\zeta T - s) B u(\tfrac{s}{\zeta})\, \mathrm{d}s \\
&= \mathbb{T}(\zeta T) I_\zeta^{-1} \begin{bmatrix} \mathrm{y}_\zeta^{i,0} \\ \mathrm{y}_\zeta^{i,1} \end{bmatrix} + \int_0^{T} \mathbb{T}(\zeta(T - t))\, \zeta B u(t)\, \mathrm{d}t
\end{aligned}
$$

and hence,

$$
\begin{bmatrix} y_\zeta(T) \\ \dot{y}_\zeta(T) \end{bmatrix} = I_\zeta \mathbb{T}(\zeta T) I_\zeta^{-1} \begin{bmatrix} \mathrm{y}_\zeta^{i,0} \\ \mathrm{y}_\zeta^{i,1} \end{bmatrix} + \int_0^T I_\zeta \mathbb{T}(\zeta(T-t)) I_\zeta^{-1}\, \zeta^2 B u(t)\, \mathrm{d}t\,.
$$

Consequently, the averaged input to state map is defined by:

$$
\mathbf{F}_T u = \int_{\mathbb{R}} \int_0^T I_\zeta \mathbb{T}(\zeta(T-t)) I_\zeta^{-1}\, \zeta^2 B u(t)\, \mathrm{d}t\, \mathrm{d}\eta_\zeta \qquad (u \in L^2(0,T))
$$

and the averaged observability map is:

$$
\begin{aligned}
\left(\Psi_T \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix}\right)(t) &= \int_{\mathbb{R}} \zeta^2 B^* I_\zeta^{-1} \mathbb{T}(-\zeta t) I_\zeta \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix} \mathrm{d}\eta_\zeta \\
&= \int_{\mathbb{R}} \zeta B^* \mathbb{T}(-\zeta t) \begin{bmatrix} \mathrm{z}^0 \\ \zeta \mathrm{z}^1 \end{bmatrix} \mathrm{d}\eta_\zeta \qquad (\begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix} \in X_1\,,\ t \in (0,T))\,.
\end{aligned}
$$

Let z_ζ be the solution of:

$$
\begin{aligned}
&\ddot{z}_\zeta = \zeta^2 \partial_x^2 z_\zeta\,, \\
&0 = z_\zeta(t,0) = z_\zeta(t,1) \qquad\qquad (t \geqslant 0)
\end{aligned}
$$

with initial conditions:

$$
z_\zeta(0,\cdot) = \mathrm{z}^0 \quad \text{and} \quad \dot{z}_\zeta(0,\cdot) = -\zeta^2 \mathrm{z}^1\,.
$$

Thus, $\mathbb{T}(-\zeta t) \begin{bmatrix} \mathrm{z}^0 \\ \zeta \mathrm{z}^1 \end{bmatrix} = \begin{bmatrix} z_\zeta(t) \\ \frac{-1}{\zeta} \dot{z}_\zeta(t) \end{bmatrix}$ and hence,

$$
\left(\Psi_T \begin{bmatrix} \mathrm{z}^0 \\ \mathrm{z}^1 \end{bmatrix}\right)(t) = -\int_{\mathbb{R}} \partial_x \left(A_0^{-1} \dot{z}_\zeta(t,\cdot)\right)(0)\, \zeta \mathrm{d}\eta_\zeta\,,
$$

Expanding the initial conditions $\mathrm{z}^0 = \sum \alpha_n \varphi_n$ and $\mathrm{z}^1 = \sum \beta_n \varphi_n$ on the eigenvector basis $\{\varphi_n\}_n$ of A_0 defined by (A.1) leads to:

$$
z_\zeta(t,x) = \sum_{n \in \mathbb{N}^*} \left(\alpha_n \cos(n\pi\zeta t) - \zeta \frac{\beta_n}{n\pi} \sin(n\pi\zeta t) \right) \varphi_n(x)\,.
$$

Thus, the observation operator is:

$$\left(\Psi_T \begin{bmatrix} z^0 \\ z^1 \end{bmatrix}\right)(t) = -\sqrt{2}\int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\alpha_n \sin(n\pi\zeta t) + \zeta \frac{\beta_n}{n\pi} \cos(n\pi\zeta t)\right) \zeta \mathrm{d}\eta_\zeta$$
$$= \frac{-\sqrt{2}}{2} \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{N}^*} \left(-i\alpha_n + \zeta\frac{\beta_n}{n\pi}\right) e^{in\pi\zeta t} + \sum_{n\in\mathbb{N}^*} \left(i\alpha_n + \zeta\frac{\beta_n}{n\pi}\right) e^{-in\pi\zeta t}\right) \zeta \mathrm{d}\eta_\zeta \,.$$

Notice that $\left\| \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right\|_X^2 = \sum_{n\in\mathbb{N}^*} \left(\alpha_n^2 + \frac{\beta_n^2}{(n\pi)^2}\right)$. Finally, setting for every $n \in \mathbb{Z}^*$,

$$\lambda_n = \frac{n}{2}\,, \qquad a_n = \begin{cases} \alpha_n & \text{if } n > 0\,, \\ \frac{\beta_{-n}}{n\pi} & \text{if } n < 0 \end{cases} \qquad \text{and}$$

$$[L_\zeta a]_n = \begin{cases} \frac{-\sqrt{2}}{2}\left(-ia_n + \zeta a_{-n}\right)\zeta & \text{if } n > 0\,, \\ \frac{-\sqrt{2}}{2}\left(ia_{-n} + \zeta a_n\right)\zeta & \text{if } n < 0\,, \end{cases} \qquad (a \in \ell^2(\mathbb{Z}^*)\,,\ \zeta \in \mathbb{R})\,, \quad \text{(A.2)}$$

the observation operator is $\displaystyle\int_{\mathbb{R}} \sum_{n\in\mathbb{Z}^*} [L_\zeta a]_n e^{2i\pi\lambda_n\zeta t}\, \mathrm{d}\eta_\zeta$.

Let us notice that:

$$\zeta^2 \left(1 + (\zeta^2 - 1)\mathbf{1}_{[0,1]}(|\zeta|)\right) \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{|n|^{2\alpha}} \leqslant \sum_{n\in\mathbb{Z}^*} \frac{|[L_\zeta a]_n|^2}{|n|^{2\alpha}} \leqslant \zeta^2 \left(1 + (\zeta^2 - 1)\mathbf{1}_{(1,\infty)}(|\zeta|)\right) \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{|n|^{2\alpha}}$$
$$(\zeta \in \mathbb{R}\,,\ \alpha \in \mathbb{R})\,, \quad \text{(A.3)}$$

These last inequalities ensure that for every $\zeta \in \mathbb{R}^*$, and every $\alpha > 0$, L_ζ is a linear continuous operator bounded from below in any of the spaces

$$\ell^2_{-\alpha}(\mathbb{Z}^*) = \left\{ a \in \mathbb{R}^{\mathbb{Z}^*}\,,\ \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{|n|^{2\alpha}} < \infty \right\} \qquad (\alpha \in \mathbb{R})\,.$$

The reason for introducing this real parameter α and these spaces $\ell^2_{-\alpha}$, will make sense in corollaries 4.3 and 4.4 and in Proposition 4.3, in particular, regarding the definition of the spaces X_α, given by relation (4.11). More precisely, for $\alpha > 0$, X_α corresponds to *α-differentiable functions* and this space can be identified to the subspace $\ell^2_\alpha(\mathbb{Z}^*)$ of $\ell^2(\mathbb{Z}^*)$. In order to prove the controllability in X_α, we will prove the observability in $X_{-\alpha}$, the dual space of X_α with pivot space $L^2(0,1)$. Finally, the space $X_{-\alpha}$ can be identified to the space $\ell^2_{-\alpha}$ introduced here.

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