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Abstract

We study the asymptotic regime of strong electric fields that leads from the Vlasov–Poisson system to the Incompressible Euler equations. We also deal with the Vlasov–Poisson–Fokker–Planck system which induces dissipative effects. The originality consists in considering a situation with a finite total charge confined by a strong external field. In turn, the limiting equation is set in a bounded domain, the shape of which is determined by the external confining potential. The analysis extends to the situation where the limiting density is non–homogeneous and where the Euler equation is replaced by the Lake Equation, also called Anelastic Equation.

Key words. Plasma physics. Vlasov–Poisson system. Vlasov–Poisson–Fokker–Planck system. Incompressible Euler equations. Lake equations. Quasi–neutral regime. Modulated energy. Relative entropy.

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1 Introduction

1.1 The Vlasov-Poisson equation in a confining potential

We are interested in the behavior as ε tends to 0 of the solutions of the following Vlasov equation

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon - \left(\frac{1}{\varepsilon}\nabla_x\Phi_{\rm ext} + \nabla_x\Phi_\varepsilon\right)\cdot\nabla_v f_\varepsilon = 0, \tag{V}$$

where the potential Φ_ε is defined self-consistently by the Poisson equation

$$\Delta_x\Phi_\varepsilon = -\frac{1}{\varepsilon}\rho_\varepsilon, \qquad \rho_\varepsilon(t,x) = \int f_\varepsilon(t,x,v)\,\mathrm{d}v, \tag{P}$$

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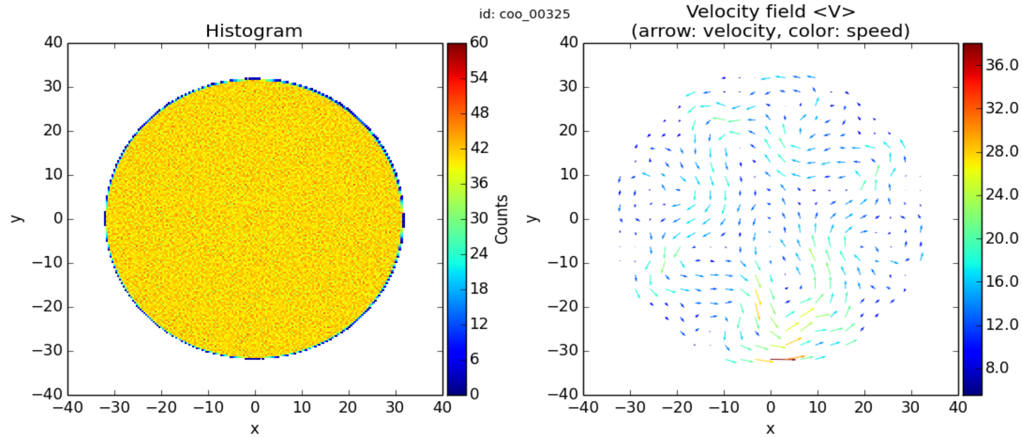

Figure 1: Snapshot of a 2D simulation of confined charged particles. Particles are subjected to the combination of a harmonic and isotropic external potential, a strong Coulomb repulsion, a friction and a noise. An external force has been added from the left to the right in the lower half of the cloud, in order to set the particles in motion. Left: instantaneous locally averaged density field. The density is almost uniform inside a ball, and almost zero outside. Right: instantaneous locally averaged velocity field. (By courtesy of A. Olivetti [34].)

and where $\varepsilon^{-1}\Phi_{\rm ext}$ is a strong external potential applied to the system. The problem holds in the entire space: $x \in \mathbb{R}^N$, $v \in \mathbb{R}^N$ and it is completed by an initial data with *finite* charge

$$f_\varepsilon\Big|_{t=0} = f_\varepsilon^{\rm init}, \qquad \iint f_\varepsilon^{\rm init}\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m} \in (0,\infty). \tag{1}$$

Notice that Φ_ε is of size ε^{-1} and we shall consider the applied potential $\frac{1}{\varepsilon}\Phi_{\rm ext}$ also of size ε^{-1}. The problem is motivated by the study of *non neutral plasmas* (see [12] for a review): these are collections of particles all with the same sign of charge, for instance pure electron, or pure ion plasmas. There are several methods to confine such a plasma, among which the Paul trap, which uses an oscillating electric field. The Penning trap, which uses a combination of static electric and magnetic fields, is also standard, but (V) is not directly relevant to this situation since there is no magnetic field in it. A non neutral plasma picture has also been used to describe trapped neutral atoms [31], in the regime where multiple diffusion of quasi resonant photons induces an effective interaction force between atoms which is formally similar to a Coulomb force [40]. In this case, the system is however dissipative; a standard way to take this effect into account is to add to (V) a Fokker-Planck operator acting on velocities [9]. We will also discuss this situation. In these physical examples, the small ε limit is indeed relevant in many experimental situations. Figure 1 corresponds to a numerical simulation of such an experiment. It strongly suggests the existence of a limiting fluid model where the density is nothing but the characteristic function of a ball. Our goal is to justify that, indeed, a simpler model, purely of hydrodynamic type, can be used to describe the particles in this asymptotic limit.

In fact, we shall see that the limiting model holds in a domain the shape of which depends on the external potential $\Phi_{\rm ext}$. But, to start with, we can consider a quadratic and isotropic potential, say:

$$\Phi_{\rm ext}(x) = \frac{1}{2N}|x|^2 \tag{2}$$

where we remind the reader that N stands for the space dimension. It corresponds to the case displayed in Figure 1. The confining potential $\varepsilon^{-1}\Phi_{\rm ext}$ tends to strongly localize in space the particle density. On the support of the limiting density ρ, the electric force $\varepsilon^{-1}\nabla_x\Phi_{\rm ext} + \nabla_x\Phi_\varepsilon$

should be of order one. By (P), this imposes that $\Delta\Phi_{\rm ext} + \varepsilon\Delta\Phi_\varepsilon = \varepsilon\nabla_x \cdot (\varepsilon^{-1}\nabla_x\Phi_{\rm ext} + \nabla_x\Phi_\varepsilon) = \Delta\Phi_{\rm ext} - \rho_\varepsilon = 1 - \rho_\varepsilon$ is of order $\mathscr{O}(\varepsilon)$ on the support of ρ for the potential (2). Clearly, due to the condition of finite charge (1), the limiting density cannot be constant uniformly on the whole space. The intuition is that the limiting density has the same radial symmetries as both the external potential (2) and the Poisson kernel, see (14) below. Actually, we shall prove some convergence of ρ_ε to

$$n_{\rm e}(x) = \mathbf{1}_{B(0,R)}(x), \tag{3}$$

where $\mathbf{1}_U$ denotes the characteristic function of the set U. The radius R depends on the total mass $\mathfrak{m}$ so that the charge constraint (1) is fulfilled. In order to find a hydrodynamic description of the particles, it is convenient to associate to the particle distribution function f_ε the following macroscopic quantities

$$\begin{array}{ll} \text{Current:} & J_\varepsilon(t,x) \overset{\rm def}{=} \displaystyle\int v\ f_\varepsilon(t,x,v)\,\mathrm{d}v, \\ \text{Kinetic pressure:} & \mathbb{P}_\varepsilon(t,x) \overset{\rm def}{=} \displaystyle\int v\otimes v\ f_\varepsilon(t,x,v)\,\mathrm{d}v. \end{array}$$

It turns out that the current looks like

$$J_\varepsilon(t,x) = \rho_\varepsilon(t,x)V_\varepsilon(t,x) \xrightarrow[\varepsilon\to 0]{} n_{\rm e}(x)V(t,x) = \mathbf{1}_{B(0,R)}(x)V(t,x), \tag{4}$$

where V solves the Incompressible Euler system in $B(0,R)$:

$$\left\{ \begin{array}{l} \partial_t V + \nabla_x \cdot (V\otimes V) + \nabla_x p = 0, \\ \nabla_x \cdot V = 0, \end{array} \right. \tag{IE}$$

with an appropriate initial condition, and no flux boundary condition on $\partial B(0,R)$. In (IE), the pressure p appears as the Lagrange multiplier associated with the constraint that V is divergence free. This incompressibility condition comes from charge conservation: integrating (V) with respect to the velocity variable v, we get

$$\partial_t\rho_\varepsilon + \nabla_x \cdot J_\varepsilon = 0. \tag{5}$$

Letting ε go to 0, with (3) and (4), we deduce that V is solenoidal. Obtaining the evolution equation for V is more intricate.

The analysis of such asymptotic problems goes back to [5], where a specific modulated energy method was introduced. It has been revisited in [29], still by using a modulated energy method, but which is able to account for oscillations present within the system. Accordingly, more general initial data can be dealt with in [29]. However, these results hold either on the torus $\mathbb{T}^N$, or in the whole space with data having infinite charge, that is $\iint f(x,v)\,\mathrm{d}v\,\mathrm{d}x = \infty$. A case with finite charge, but a different Poisson equation which leads to a compressible hydrodynamic limit, has been considered in [19], again with a modulated energy. Our goal in this article is twofold:

- To prove the convergence to (IE) in the case of a trapped system, with finite charge. Even though our proof also relies on a modulated energy functional, there are new difficulties: the shape of the domain on which the limiting equation (IE) holds is determined by the external potential $\Phi_{\rm ext}$, and a careful treatment of the boundary is needed.
- To prove the convergence to the analog of (IE) in the case of a trapped dissipative system.

Both improvements are relevant for experiments on non neutral plasmas or large magneto-optical traps.

1.2 Statement of the results

In what follows we shall deal with a smooth solution $(t,x) \mapsto V(t,x) \in \mathbb{R}^N$ (possibly defined on a small enough time interval $[0,T]$) of the incompressible Euler equation (IE) set on the ball $B(0,R)$, completed with no-flux boundary condition

$$V(t,x)\cdot\nu(x)\Big|_{|x|=R}=0, \tag{6}$$

where $\nu(x)$ denotes the outward unit vector at $x\in\partial B(0,R)$ (namely $\nu(x)=x/|x|$). We work with solutions V that belongs to $L^\infty(0,T;H^s(B(0,R)))$, for a certain $s>0$ large enough.

Theorem 1.1 ([38]-[39]) *Let $V^{\mathrm{init}}:B(0,R)\to\mathbb{R}^N$ be a divergence free vector field in H^s, with $s>1+N/2$, satisfying the no flux condition $V^{\mathrm{init}}\cdot\nu=0$ on $\partial B(0,R)$. There exists $T>0$ and a unique solution $V\in L^\infty(0,T;H^s(B(0,R)))$ of* (IE) *with the no flux condition* (6)*. Moreover, we have*

$$\sup_{0\le t\le T}\Big(\|V(t)\|_{H^s}+\|\partial_t V(t)\|_{H^{s-1}}+\|\nabla_x p(t)\|_{H^s}+\|\partial_t\nabla_x p(t)\|_{H^{s-1}}\Big)\le C(T)$$

for some positive constant $C(T)$ depending on T and the initial datum.

If $N=1$, the only divergence free vector field V^{init} satisfying (6) is $V^{\mathrm{init}}\equiv 0$ and then the solution given in Theorem 1.1 is $V\equiv 0$.

For further purposes, we need to consider an extension $\mathscr{V}$ of the solution V to (IE) with (6), defined on the whole space and compactly supported. Namely we require $\mathscr{V}\in L^\infty(0,T;H^s(\mathbb{R}^N))$ to satisfy

$$\mathscr{V}\Big|_{B(0,R)}=V,\qquad \mathscr{V}\Big|_{\mathbb{R}^N\setminus B(0,2R)}=0,\qquad \mathscr{V}(t,x)\cdot\nu(x)\Big|_{|x|=R}=0. \tag{7}$$

For the construction of such an extension, we refer to [27, Chapter I: Theorem 2.1 p. 17 & Theorem 8.1 p. 42]. For an extension which is in addition divergence–free, see Lemma B.1 in the appendix.

In order to state our first result, we need to introduce an auxiliary potential function Φ_{e}. Suppose that (3) indeed holds true. Then, by using (P) and $\Delta\Phi_{\mathrm{ext}}=1$ for the potential (2), we infer, for $\varepsilon\to 0$,

$$\Delta(\Phi_{\mathrm{ext}}+\varepsilon\Phi_\varepsilon)=\Delta\Phi_{\mathrm{ext}}-\rho_\varepsilon\to\Delta\Phi_{\mathrm{ext}}-\mathbf{1}_{B(0,R)}=\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}.$$

Moreover, since we want the electric force $\varepsilon^{-1}\nabla_x\Phi_{\mathrm{ext}}+\nabla_x\Phi_\varepsilon=\varepsilon^{-1}(\nabla_x\Phi_{\mathrm{ext}}+\varepsilon\nabla_x\Phi_\varepsilon)$ to be of order one on the ball $B(0,R)$, this imposes $\Phi_{\mathrm{ext}}+\varepsilon\Phi_\varepsilon$ to be close to a constant, say zero, on the ball $B(0,R)$. It is therefore natural to look for a solution Φ_{e} to the Poisson problem

$$\Delta\Phi_{\mathrm{e}}(x)=1-n_{\mathrm{e}}(x)=\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)},\qquad \Phi_{\mathrm{e}}=0\text{ in }B(0,R). \tag{8}$$

In this specific case, we can find an explicit radially symmetric solution:

$$\Phi_{\mathrm{e}}(x)=\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}\times\begin{cases}\dfrac{|x|^2}{2N}+\dfrac{R^N}{N(N-2)|x|^{N-2}}-\dfrac{R^2}{2(N-2)} & \text{if } N>2,\\[2mm] \dfrac{|x|^2-R^2}{4}-\dfrac{R^2}{2}\ln(|x|/R) & \text{if } N=2,\\[2mm] \dfrac{1}{2}(|x|-R)^2 & \text{if } N=1.\end{cases} \tag{9}$$

With Φ_{e} and n_{e} in hand, we split the Poisson equation (P) as follows, where n_{e} is defined in (3),

$$\Delta_x\Phi_\varepsilon(t,x)=\frac{1-n_{\mathrm{e}}(x)}{\varepsilon}+\frac{n_{\mathrm{e}}(x)-\rho_\varepsilon(t,x)}{\varepsilon}-\frac{1}{\varepsilon}\Delta\Phi_{\mathrm{ext}}=\frac{1}{\varepsilon}\Delta_x\Phi_{\mathrm{e}}(x)+\frac{1}{\sqrt{\varepsilon}}\Delta_x\Psi_\varepsilon(t,x)-\frac{1}{\varepsilon}\Delta\Phi_{\mathrm{ext}},$$

namely, we have

$$\Phi_\varepsilon(x)+\frac{1}{\varepsilon}\Phi_{\mathrm{ext}}=\frac{1}{\varepsilon}\Phi_{\mathrm{e}}(x)+\frac{1}{\sqrt{\varepsilon}}\Psi_\varepsilon(t,x),\qquad \Delta_x\Psi_\varepsilon(t,x)=\frac{1}{\sqrt{\varepsilon}}(n_{\mathrm{e}}(x)-\rho_\varepsilon(t,x)), \tag{10}$$

where Ψ_ε represents the fluctuations of the potential. According to [5], we introduce a modulated energy:

$$\mathscr{H}_{\mathscr{V},\varepsilon} \stackrel{\text{def}}{=} \frac{1}{2}\iint |v-\mathscr{V}|^2\ f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int |\nabla_x \Psi_\varepsilon|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\iint \Phi_{\mathrm{e}}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x.$$

When the external potential is given by (2), we shall establish the following statement[1].

Theorem 1.2 *Let $V^{\mathrm{init}} \in H^s(B(0,R))$ satisfy $\nabla_x \cdot V^{\mathrm{init}} = 0$ and the no flux condition* (6). *Denote by V the solution, on $[0,T]$, to* (IE) *with the no flux condition* (6) *given in Theorem 1.1. Consider $\mathscr{V}$ a smooth extension of V satisfying the conditions* (7). *Let $f_\varepsilon^{\mathrm{init}} : \mathbb{R}^N\times\mathbb{R}^N \to [0,\infty)$ be a sequence of integrable functions that satisfy the following requirements*

$$\begin{cases} \displaystyle\iint f_\varepsilon^{\mathrm{init}}\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m}, \\ \displaystyle\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\iint |v-\mathscr{V}^{\mathrm{init}}|^2\ f_\varepsilon^{\mathrm{init}}\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int |\nabla_x \Psi_\varepsilon^{\mathrm{init}}|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\iint \Phi_{\mathrm{e}}\ f_\varepsilon^{\mathrm{init}}\,\mathrm{d}v\,\mathrm{d}x\right\} = 0. \end{cases} \tag{11}$$

Then, the associated solution f_ε of the Vlasov–Poisson equation (V)–(P) *satisfies, as $\varepsilon \to 0$,*

i) ρ_ε converges to n_{e} in $C^0([0,T];\mathscr{M}^1(\mathbb{R}^N) - weak - \star)$;

ii) $\mathscr{H}_{\mathscr{V},\varepsilon}$ converges to 0 uniformly on $[0,T]$;

iii) J_ε converges to J in $\mathscr{M}^1([0,T]\times\mathbb{R}^N)$ weakly-$\star$, the limit J lies in $L^\infty(0,T;L^2(\mathbb{R}^N))$ and satisfies $J\big|_{[0,T]\times B(0,R)} = V$, $\nabla_x\cdot J = 0$ and $J\cdot\nu(x)\big|_{\partial B(0,R)} = 0$.

Remark 1.3 *(i) Here, we were not very precise about the type of solutions to the Vlasov–Poisson system* (V)–(P) *we are considering. We refer to [35, 28] for the construction of global regular solutions to the system and some extra conditions to ensure the propagation of regularity. There are also weaker notions of solutions (weak solutions or renormalized solutions) to which our theorem can apply. We refer the reader to the introduction of [28] for a discussion about these solutions.*

(ii) The second part of the hypothesis (11) *imposes that the initial modulated energy is small; this is a strong hypothesis on the initial data. When the problem is set on the torus, or on the whole space with infinite charge, it can be relaxed, see [29]. In the present framework, going beyond* (11) *would certainly require a fine description of boundary layers on $\{|x| = R\}$. Assuming* (11)*, point* ii) *of the theorem then ensures that the modulated energy remains small at later times. As typical initial data satisfying* (11)*, we can take*

$$f_\varepsilon^{\mathrm{init}}(x,v) = \frac{n_{\mathrm{e}}(x) - \delta_\varepsilon\Delta\chi(x)}{\sigma_\varepsilon^N}\, G\left(\frac{v-\mathscr{V}^{\mathrm{init}}(x)}{\sigma_\varepsilon}\right),$$

where $\chi \in C_c^\infty(B(0,R))$ and where G is a nonnegative function that belongs to the Schwartz space and satisfies $\int G\,\mathrm{d}v = 1$ (for instance, G is a normalized Gaussian $G(v) = (2\pi)^{-N/2}\exp(-|v|^2/2)$). Then, we choose $\sigma_\varepsilon \to 0$ as $\varepsilon\to 0$ and $\delta_\varepsilon = o(\sqrt{\varepsilon})$ (so that $n_{\mathrm{e}} - \delta_\varepsilon\Delta\chi \geq 0$ for ε small enough). Indeed, we easily obtain $\iint \Phi_{\mathrm{e}}\ f_\varepsilon^{\mathrm{init}}\,\mathrm{d}v\,\mathrm{d}x = 0$, $\iint |v - \mathscr{V}^{\mathrm{init}}|^2\ f_\varepsilon^{\mathrm{init}}\,\mathrm{d}v\,\mathrm{d}x = \sigma_\varepsilon^2(\int |v|^2 G(v)\,\mathrm{d}v)(\int n_{\mathrm{e}}\,\mathrm{d}x) \to 0$ and $\Psi_\varepsilon = \delta_\varepsilon/\sqrt{\varepsilon}\chi$, hence $\int |\nabla_x\Psi_\varepsilon^{\mathrm{init}}|^2\,\mathrm{d}x = \delta_\varepsilon^2/\varepsilon\int|\nabla_x\chi|^2\,\mathrm{d}x \to 0$.

We wish to extend this analysis by dealing with more general external potentials. We distinguish two situations depending on the expression of the external potential:

- The quadratic potential

$$\Phi_{\mathrm{ext}}(x) = \frac{1}{2}\sum_{j=1}^N \frac{x_j^2}{\lambda_j^2}, \tag{12}$$

[1]Throughout the paper, we denote by $\mathscr{M}^1(X)$ the space of bounded measures on $X \subset \mathbb{R}^D$. It identifies with the dual space of the separable space $C_0^0(X)$ of the continuous functions that vanish at infinity.

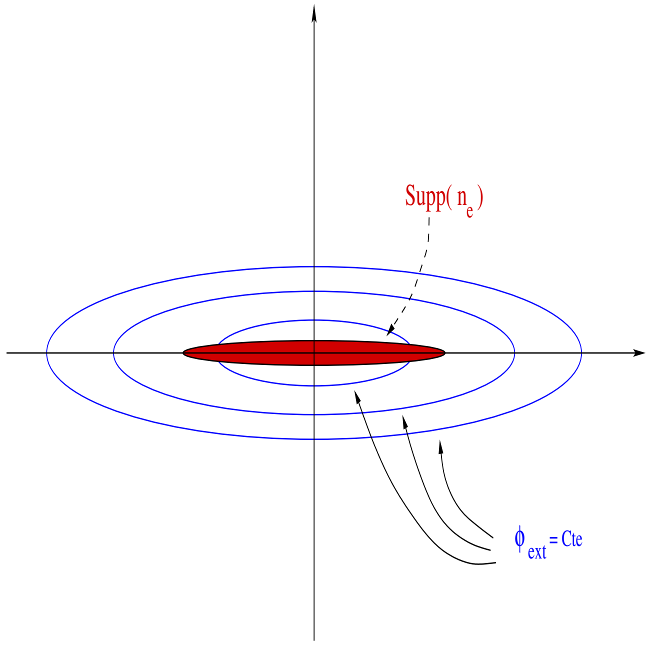

Figure 2: Some level sets of Φ_{ext} and the support of n_{e}.

with $\lambda_j > 0$, $1 \le j \le N$, in dimension $N \ge 2$ is typical to model non neutral plasmas [12] or magneto-optical traps experiments. In this case $\Delta\Phi_{\text{ext}}$ is still a constant, that therefore determines the value of the (uniform) particle density n_{e} on its support. But the problem has lost its symmetries and the shape of the support becomes non trivial. We shall see that ρ_ε tends to a uniform distribution n_{e}, supported in an ellipsoid. However, we point out that the support of n_{e} does not coincide with a level set of Φ_{ext}. An example with $N = 2$ is given in Figure 2. The potential Φ_{e} can be computed rather explicitly, and Theorem 1.2 generalizes directly. See Section 2.1 for a precise statement.

- In the case of a non quadratic potential, under suitable hypotheses on Φ_{ext}, the limiting density n_{e} still has a compact support $\mathcal{K}$ and is still given on $\mathcal{K}$ by $n_{\text{e}} = \Delta\Phi_{\text{ext}}$. However, n_{e} is clearly no longer constant on $\mathcal{K}$. The identification of $\mathcal{K}$ and n_{e} relies on variational techniques, with connection to the obstacle problem. It is still possible to prove the analog of Theorem 1.2, but, since n_{e} becomes non homogeneous, instead of (IE) the limiting equations are now the so-called Lake Equations, see e. g. [26]:

$$\begin{cases} \partial_t V + V \cdot \nabla_x V + \nabla_x p = 0, \\ \nabla_x \cdot (n_{\text{e}} V) = 0. \end{cases} \tag{LE}$$

Such model — also referred to as the Anelastic Equations — arise in the modelling of atmospheric flows [32]; we refer the reader to [30] for the justification of a derivation from the compressible Navier-Stokes system. As a matter of fact, we can observe that the first equation in (LE) may be writen in the following conservative form $\partial_t(n_{\text{e}}V) + \nabla_x \cdot (n_{\text{e}} V \otimes V) + n_{\text{e}} \nabla_x p = 0$. The construction of Φ_{e} and $\mathcal{K}$, and a precise statement of the corresponding convergence theorem can be found in Section 2.2.

Motivated by actual experiments, we will also generalize the results to the case where a Fokker-Planck operator acting on velocities is added to Eq. (V). Our starting point then becomes:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - \nabla_x \Phi_\varepsilon \cdot \nabla_v f_\varepsilon = L f_\varepsilon, \tag{VFP}$$

with

$$Lf = \nabla_v \cdot (vf + \theta \nabla_v f) = \theta \nabla_v \cdot \left(M_{0,\theta} \nabla_v \left(\frac{f}{M_{0,\theta}} \right) \right), \qquad M_{0,\theta}(v) = \frac{1}{(2\pi\theta)^{N/2}} e^{-|v|^2/(2\theta)},$$

for some $\theta > 0$. Equation (VFP) is still coupled to the Poisson equation (P). Using a modified modulated energy, we are able to show in this case that solutions f_ε of (VFP) and (P) with well-prepared data converge when ε and θ tends to 0 in the sense of Theorem 1.2 to $n_\mathrm{e} V$, where V is now the solution of the Lake Equation with friction

$$\begin{cases} \partial_t V + V \cdot \nabla_x V + \nabla_x p + V = 0, \\ \nabla_x \cdot (n_\mathrm{e} V) = 0. \end{cases} \tag{13}$$

On the boundary, we still have the no-flux condition (6). For the sake of completeness, the necessary analog of Theorem 1.1 for the systems (LE) and (13) is sketched in appendix A. See Section 4 for a precise statement on the asymptotic behavior of (VFP) and its proof.

2 The limit density n_e and total potential Φ_e

As said above, we have a clear intuition and explicit formulae for the equilibrium distribution n_e and the potential Φ_e in the specific case of the isotropic external potential (2). Let us discuss in further details how Φ_ext determines n_e and its support, and how the auxiliary potential Φ_e, which plays a crucial role in the analysis through the decomposition (10), can be defined.

We remind the reader the definition of the fundamental solution, hereafter denoted Γ, of $(-\Delta)$ (mind the sign) in the whole space $\mathbb{R}^N$:

$$\Gamma(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{N(N-2)|B_{\mathbb{R}^N}(0,1)| \cdot |x|^{N-2}} & \text{if } N > 2, \\ -\dfrac{\ln|x|}{2\pi} & \text{if } N = 2, \\ -\dfrac{|x|}{2} & \text{if } N = 1. \end{cases} \tag{14}$$

2.1 The case of a general quadratic potential

Let us consider in this section the case of a quadratic potential (12). We have $\Delta \Phi_\mathrm{ext} = \sum_{j=1}^N \frac{1}{\lambda_j^2} > 0$ which is constant in space. It gives the value of the equilibrium density on its support since we still expect $\rho_\varepsilon \to \mathbf{1}_\mathcal{K} \Delta \Phi_\mathrm{ext}$. But it remains to determine this support $\mathrm{Supp}(n_\mathrm{e}) = \mathcal{K} \subset \mathbb{R}^N$ on which we have the volume constraint

$$\mathfrak{m} = \int n_\mathrm{e} \, \mathrm{d}x = |\mathcal{K}| \sum_{j=1}^N \frac{1}{\lambda_j^2}$$

coming from (1). Note that a quick computation reveals that $\mathcal{K}$ can be neither radially symmetric, nor a level set of Φ_ext.

In order to extend Theorem 1.2 for a potential as in (12), we need to construct a domain $\mathcal{K} \subset \mathbb{R}^N$ and a function $\Phi_\mathrm{e} : \mathbb{R}^N \to \mathbb{R}$ such that

$$\Delta \Phi_\mathrm{e}(x) = \left(\sum_{j=1}^N \frac{1}{\lambda_j^2} \right) \mathbf{1}_{\mathbb{R}^N \setminus \mathcal{K}}, \qquad \Phi_\mathrm{e} = 0 \quad \text{in } \mathcal{K}. \tag{15}$$

The starting point is the observation that given $a = (a_1, ..., a_N) \in (\mathbb{R}_+^*)^N$, then the characteristic function of the ellipsoid

$$\mathcal{K}_a = \{x \in \mathbb{R}^N;\ \sum_{j=1}^N x_j^2 / a_j^2 \leq 1\}$$

generates an electric potential which is *quadratic* inside the ellipsoid. This can be found for instance in [22, Chapter VII, § 6]; the computation there is for $N = 3$, but the extension to the case $N \geq 3$ is straightforward, and the two-dimensional case is treated by using arguments from complex analysis in [17].

For $x \in \mathbb{R}^N$, we denote by $\sigma_a(x)$ the largest solution of the equation

$$\sum_{j=1}^{N} \frac{x_j^2}{a_j^2 + \varsigma} = 1$$

(with $\varsigma \in \mathbb{R}$ as unknown). Consequently, $x \in \mathcal{K}_a$ holds if and only if $\sigma_a(x) \leq 0$. By convention, $\sigma_a(0) = -\infty$. This quantity can be seen as an equivalent of the radial coordinate in the ellipsoidal coordinate system. It allows us to construct a solution to (15) where $\mathcal{K}$ is an ellipsoid, the coefficients of which depend on the mass $\mathfrak{m}$ and the λ_j's.

Proposition 2.1 *Let $a = (a_1, ..., a_N) \in (\mathbb{R}_+^*)^N$.*
(i) [22] If $N \geq 3$, then

$$\Gamma \star \mathbf{1}_{\mathcal{K}_a}(x) = \frac{1}{4}\left(\prod_{j=1}^{N} a_j\right) \times \begin{cases} \displaystyle\int_{\sigma_a(x)}^{+\infty} \left(1 - \sum_{j=1}^{N} \frac{x_j^2}{a_j^2 + s}\right)\left(\prod_{j=1}^{N}(a_j^2 + s)\right)^{-1/2} \mathrm{d}s & \text{if } \sigma_a(x) \geq 0, \\ \displaystyle\int_{0}^{+\infty} \left(1 - \sum_{j=1}^{N} \frac{x_j^2}{a_j^2 + s}\right)\left(\prod_{j=1}^{N}(a_j^2 + s)\right)^{-1/2} \mathrm{d}s & \text{if } \sigma_a(x) \leq 0. \end{cases}$$

(ii) If $N = 2$, then

$$\Gamma \star \mathbf{1}_{\mathcal{K}_a}(x) = \frac{1}{4}(a_1 a_2) \times \begin{cases} \displaystyle -\ln\left(\sigma_a(x) + \frac{a_1^2 + a_2^2}{2} + \sqrt{(a_1^2 + \sigma_a(x))(a_2^2 + \sigma_a(x))}\right) \\ \displaystyle \qquad - \int_{\sigma_a(x)}^{+\infty} \sum_{j=1}^{2} \frac{x_j^2}{a_j^2 + s} \frac{\mathrm{d}s}{\sqrt{(a_1^2 + s)(a_2^2 + s)}} & \text{if } \sigma_a(x) \geq 0, \\ \displaystyle -\ln\left(\frac{1}{2}(a_1 + a_2)^2\right) - \int_{0}^{+\infty} \sum_{j=1}^{2} \frac{x_j^2}{a_j^2 + s} \frac{\mathrm{d}s}{\sqrt{(a_1^2 + s)(a_2^2 + s)}} & \text{if } \sigma_a(x) \leq 0. \end{cases}$$

Remark 2.2 *An alternative point of view for the two dimensional case is to work with the electric field instead of the potential. We refer to [17] for expressions of the electric field generated by ellipses in $N = 2$. In the case of a uniform charge distribution, the electric field is linear inside the ellipse, with the same coefficients for the quadratic terms as those coming from the expression in (ii).*

We define the mapping $\mathcal{Z} : (\mathbb{R}_+^*)^N \to (\mathbb{R}_+^*)^N$ by

$$\mathcal{Z}_j(\alpha) = \int_0^{+\infty} \frac{1}{\alpha_j + s} \left(\prod_{j=1}^{N}(\alpha_j + s)\right)^{-1/2} \mathrm{d}s > 0. \tag{16}$$

From Proposition 2.1, we know that the potential generated by $\mathbf{1}_{\mathcal{K}_a}$ is quadratic inside $\mathcal{K}_a$, up to an additive constant. The coefficients of the quadratic terms are the $-(\prod_{j=1}^{N} a_j)\mathcal{Z}_k(\alpha)/4$, $1 \leq k \leq N$. The idea to make the connexion with the external potential Φ_{ext} is now to adapt the a_j's so that the quadratic terms in $\Gamma \star \mathbf{1}_{\mathcal{K}_a}$ (inside $\mathcal{K}_a$) cancel out the quadratic terms of Φ_{ext}, so that $(\Delta\Phi_{\text{ext}})\Gamma \star \mathbf{1}_{\mathcal{K}_a} + \Phi_{\text{ext}}$ is constant in $\mathcal{K}_a$. We observe that $\prod_{j=1}^{N} a_j$ is related to the total charge of the ellipsoid $\mathcal{K}_a$ since

$$\mathfrak{m} = \int n_{\mathrm{e}} = |\mathcal{K}_a| \sum_{j=1}^{N} \lambda_j^{-2} = |B_{\mathbb{R}^N}(0,1)| \left(\prod_{j=1}^{N} a_j\right) \sum_{j=1}^{N} \lambda_j^{-2}.$$

We shall thus need to solve equations in a of the form $\mathcal{Z}(a_1^2, ..., a_N^2) = z$, where $z \in (\mathbb{R}_+^*)^N$ is given. Therefore, we are interested in showing that $\mathcal{Z} : (\mathbb{R}_+^*)^N \to (\mathbb{R}_+^*)^N$ is a smooth diffeomorphism. When $N = 2$, explicit computations may be carried out.

Proposition 2.3 *Assume $N = 2$. Then, for any $\alpha \in (\mathbb{R}_+^*)^2$*

$$\mathcal{Z}(\alpha) = (\mathcal{Z}_1(\alpha), \mathcal{Z}_2(\alpha)) = \left(\frac{2}{\alpha_1 + \sqrt{\alpha_1 \alpha_2}}, \frac{2}{\alpha_2 + \sqrt{\alpha_1 \alpha_2}} \right).$$

Moreover, $\mathcal{Z} : (\mathbb{R}_+^*)^2 \to (\mathbb{R}_+^*)^2$ is a smooth diffeomorphism and its inverse is given by

$$\mathcal{Z}^{-1}(z) = ((\mathcal{Z}^{-1})_1(z), (\mathcal{Z}^{-1})_2(z)) = \left(\frac{2z_2}{z_1(z_1 + z_2)}, \frac{2z_1}{z_2(z_1 + z_2)} \right).$$

Proof. The explicit formula for $\mathcal{Z}(\alpha)$ comes by computing the Abelian integral

$$\int_0^{+\infty} \frac{\mathrm{d}s}{(\alpha_1 + s)^{3/2}(\alpha_2 + s)^{1/2}} = \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{2}{\alpha_2 - \alpha_1} \sqrt{\frac{\alpha_1 + s}{\alpha_2 + s}} \right) \mathrm{d}s = \frac{2}{\alpha_1 + \sqrt{\alpha_1 \alpha_2}},$$

for $\alpha_1 \neq \alpha_2$, and the formula holds true when $\alpha_1 = \alpha_2$ as well. The formula for the inverse then follows by direct substitution. $\square$

For $N \geq 3$, we no longer have simple expressions for $\mathcal{Z}$. However, we shall prove that $\mathcal{Z} : (\mathbb{R}_+^*)^N \to (\mathbb{R}_+^*)^N$ is a smooth diffeomorphism by using the fact that $\mathcal{Z} : (\mathbb{R}_+^*)^N \to (\mathbb{R}_+^*)^N$ is a gradient vector field associated with a strictly concave function.

Proposition 2.4 *Assume $N \geq 2$ and let us define the function $\zeta : (\mathbb{R}_+^*)^N \to \mathbb{R}$ by:*

$$\zeta(\alpha) = \begin{cases} -\displaystyle\int_0^{+\infty} \left(\prod_{k=1}^{N} (\alpha_k + s) \right)^{-1/2} \mathrm{d}s & \text{if } N \geq 3 \\ 4 \ln(\sqrt{\alpha_1} + \sqrt{\alpha_2}) & \text{if } N = 2. \end{cases}$$

Then, $\zeta : (\mathbb{R}_+^*)^N \to \mathbb{R}$ is smooth, strictly concave and it satisfies $\nabla \zeta = \mathcal{Z}$. Furthermore, $\nabla \zeta = \mathcal{Z} : (\mathbb{R}_+^*)^N \to (\mathbb{R}_+^*)^N$ is a smooth diffeomorphism and for any $z \in (\mathbb{R}_+^*)^N$, $\mathcal{Z}^{-1}(z)$ is the unique minimizer for

$$\inf_{\alpha \in (\mathbb{R}_+^*)^N} (z \cdot \alpha - \zeta(\alpha)). \tag{17}$$

In (17) we recognize the minimization problem that defines the Legendre transform of ζ. This gives a way to compute numerically $\mathcal{Z}^{-1}(z)$ through the minimization of a convex function.

Proof. The smoothness of ζ is clear and $\nabla \zeta = \mathcal{Z}$ follows from direct computations. If $N = 2$, the strict concavity of ζ is straightforward and the fact that $\nabla \zeta = \mathcal{Z} : (\mathbb{R}_+^*)^2 \to (\mathbb{R}_+^*)^2$ is a smooth diffeomorphism comes from Proposition 2.3: for any $z \in (\mathbb{R}_+^*)^2$, $\mathcal{Z}^{-1}(z)$ is a critical point of the strictly convex (since ζ is strictly concave) function $\alpha \mapsto z \cdot \alpha - \zeta(\alpha)$, hence is the unique minimizer of that function. We assume now $N \geq 3$. Then, for each $s \in \mathbb{R}_+$, the function

$$\varpi_s : \alpha \in (\mathbb{R}_+^*)^N \mapsto \left(\prod_{k=1}^{N} (\alpha_k + s) \right)^{-1/2}$$

is logarithmically strictly convex since $\ln \circ \varpi_s(\alpha) = (-1/2) \sum_{k=1}^{N} \ln(\alpha_k + s)$ and $\mathrm{Hess}(\ln \circ \varpi_s, \alpha) = (1/2)\mathrm{Diag}((\alpha_1 + s)^{-2}, ..., (\alpha_N + s)^{-2})$. Consequently, $-\zeta(\alpha) = \int_0^{+\infty} \varpi_s(\alpha) \, \mathrm{d}s$ is a strictly convex function of α. Let us show that the Jacobian determinant of $\mathcal{Z}$ never vanishes, that is $\mathrm{Hess}(\zeta, \alpha)$

is everywhere negative definite. For that purpose, for $v \in \mathbb{R}^N$, we write $-v^T \mathrm{Hess}(\zeta, \alpha) v = \int_0^{+\infty} v^T \mathrm{Hess}(\varpi_s, \alpha) v \, \mathrm{d}s$, and thus it suffices to show that $\mathrm{Hess}(\varpi_s, \alpha)$ is positive definite for any $s \geq 0$. Now, we write $\varpi_s(\alpha) = \exp(\ln \circ \varpi_s(\alpha))$, thus $\partial^2_{j,k} \varpi_s(\alpha) = \exp(\ln \circ \varpi_s(\alpha))[\partial^2_{j,k}(\ln \circ \varpi_s)(\alpha) + \partial_j(\ln \circ \varpi_s)(\alpha) \partial_k(\ln \circ \varpi_s)(\alpha)]$. Therefore, if $v \neq 0$, we obtain

$$\begin{aligned} v^T \mathrm{Hess}(\varpi_s, \alpha) v = \varpi_s(\alpha) &\left[v^T \mathrm{Hess}(\ln \circ \varpi_s, \alpha) v + \left(\sum_{j=1}^N v_j \partial_j (\ln \circ \varpi_s)(\alpha) \right)^2 \right] \\ &\geq \varpi_s(\alpha) v^T \mathrm{Hess}(\ln \circ \varpi_s, \alpha) v = \varpi_s(\alpha) \sum_{j=1}^N \frac{v_j^2}{2(\alpha_j + s)^2} > 0, \end{aligned}$$

as wished.

Let us now fix $z \in (\mathbb{R}_+^*)^N$ and consider the minimization problem (17). In view of the negativity of ζ, this infimum μ belongs to $[0, +\infty)$. Since ζ is strictly concave, this problem has at most one minimizer. Let us show that it has at least one by considering a minimizing sequence $(\alpha^n)_{n \geq 0} \in (\mathbb{R}_+^*)^N$. We claim that the sequence $(\alpha^n)_{n \geq 0}$ is bounded. Indeed, we have $z \cdot \alpha^n - \zeta(\alpha^n) \to \mu \in \mathbb{R}_+$, and since $\zeta \leq 0$, this implies $z \cdot \alpha^n = \zeta(\alpha^n) + \mu + o(1) \leq \mu + o(1)$. Using that all components of z are positive, the claim follows. As a consequence, we may assume, up to a subsequence, that there exists $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{R}_+^N$ such that $\alpha^n \to \alpha$ as $n \to +\infty$. In particular, $\zeta(\alpha^n) = z \cdot \alpha^n - \mu + o(1)$ converges. We now prove that at most two components of α vanish. For otherwise, Fatou's lemma would yield

$$\begin{aligned} +\infty = \int_0^{+\infty} \left(\prod_{k=1}^N (\alpha_k + s) \right)^{-1/2} \mathrm{d}s &= \int_0^{+\infty} \liminf_{n \to +\infty} \left(\prod_{k=1}^N (\alpha_k^n + s) \right)^{-1/2} \mathrm{d}s \\ \leq \liminf_{n \to +\infty} \int_0^{+\infty} \left(\prod_{k=1}^N (\alpha_k^n + s) \right)^{-1/2} \mathrm{d}s &= \liminf_{n \to +\infty} (-\zeta(\alpha^n)), \end{aligned}$$

contradicting the convergence of $(\zeta(\alpha^n))_{n \in \mathbb{N}}$. It remains to show that α has no zero component to ensure that $\mu + o(1) = z \cdot \alpha^n - \zeta(\alpha^n) \to z \cdot \alpha - \zeta(\alpha)$ so that $\alpha \in (\mathbb{R}_+^*)^N$ is actually a minimizer for (17). We then assume that $\alpha_1 = 0$, for instance, and show that for sufficiently small $\delta > 0$, $z \cdot (\delta, \alpha_2, ..., \alpha_N) - \zeta(\delta, \alpha_2, ..., \alpha_N) < z \cdot (0, \alpha_2, ..., \alpha_N) - \zeta(0, \alpha_2, ..., \alpha_N)$. This reaches a contradiction for n large enough. We thus compute

$$\begin{aligned} D(\delta) &= \Big(z \cdot (\delta, \alpha_2, ..., \alpha_N) - \zeta(\delta, \alpha_2, ..., \alpha_N) \Big) - \Big(z \cdot (0, \alpha_2, ..., \alpha_N) - \zeta(0, \alpha_2, ..., \alpha_N) \Big) \\ &= z_1 \delta + \int_0^{+\infty} (\delta + s)^{-1/2} \left(\prod_{k=2}^N (\alpha_k + s) \right)^{-1/2} \mathrm{d}s - \int_0^{+\infty} s^{-1/2} \left(\prod_{k=2}^N (\alpha_k + s) \right)^{-1/2} \mathrm{d}s \\ &= \delta \Big(z_1 - \int_0^{+\infty} \frac{1}{s^{1/2} (\delta + s)^{1/2} [s^{1/2} + (\delta + s)^{1/2}]} \left(\prod_{k=2}^N (\alpha_k + s) \right)^{-1/2} \mathrm{d}s \Big). \end{aligned}$$

As $\delta \to 0$, we have, by monotone convergence,

$$\int_0^{+\infty} \frac{\left(\prod_{k=2}^N (\alpha_k + s) \right)^{-1/2}}{s^{1/2} (\delta + s)^{1/2} [s^{1/2} + (\delta + s)^{1/2}]} \, \mathrm{d}s \to \int_0^{+\infty} \frac{1}{2 s^{3/2}} \left(\prod_{k=2}^N (\alpha_k + s) \right)^{-1/2} \mathrm{d}s = +\infty,$$

hence for δ sufficiently small, $D(\delta) < 0$, as claimed. Therefore, $\alpha \in (\mathbb{R}_+^*)^N$ and α is a minimizer for (17). It then follows that $\nabla \zeta(\alpha) = z$ as wished. $\square$

We may now construct a solution to (15).

Corollary 2.5 (Construction of the function $\Phi_{\rm e}$ for quadratic potentials). *Let $N \geq 2$ and assume that*

$$\Phi_{\rm ext}(x) = \frac{1}{2}\sum_{j=1}^{N}\frac{x_j^2}{\lambda_j^2},$$

with $\lambda_j > 0$, $1 \leq j \leq N$. Let us also fix $\mathfrak{m} > 0$. Then, there exists a unique $a \in (\mathbb{R}_+^*)^N$ such that

$$|\mathcal{K}_a|\left(\sum_{j=1}^{N}\frac{1}{\lambda_j^2}\right) = \mathfrak{m} \qquad \text{and} \qquad \frac{\mathfrak{m}}{2|B_{\mathbb{R}^N}(0,1)|\sum_{k=1}^{N}\lambda_k^{-2}}\ \mathcal{Z}(a_1^2,...,a_N^2) = \left(\frac{1}{\lambda_j^2}\right)_{1\leq j\leq N}. \quad (18)$$

Therefore, there exists a constant κ, depending only on the λ_j's, N and $\mathfrak{m}$ such that the function

$$\Phi_{\rm e} = \Phi_{\rm ext} + \left(\sum_{j=1}^{N}\frac{1}{\lambda_j^2}\right)\Gamma\star\mathbf{1}_{\mathcal{K}_a} + \kappa$$

is convex and satisfies

$$-\Delta\Phi_{\rm e} = \left(\sum_{j=1}^{N}\frac{1}{\lambda_j^2}\right)\mathbf{1}_{\mathbb{R}^N\setminus\mathcal{K}_a} \text{ with, furthermore, } \Phi_{\rm e} = 0 \text{ in } \mathcal{K}_a \text{ and } \Phi_{\rm e} > 0 \text{ in } \mathbb{R}^N\setminus\mathcal{K}_a. \quad (19)$$

Proof. We define $\lambda > 0$ such that $\lambda^{-2} = \sum_{j=1}^{N}\lambda_j^{-2}$ and the constant κ by the formulas $4\kappa = -\lambda^{-2}\left(\prod_{j=1}^{N}a_j\right)\int_0^{+\infty}\left(\prod_{j=1}^{N}(a_j^2+s)\right)^{-1/2}\mathrm{d}s$ if $N \geq 3$ and $4\kappa = -\lambda^{-2}(a_1a_2)\ln((a_1+a_2)^2/2)$ if $N = 2$. The existence (and uniqueness) of a satisfying the conditions (18) then ensures that $\Phi_{\rm e} = 0$ in $\mathcal{K}_a$ and $-\Delta\Phi_{\rm e} = \lambda^{-2}\mathbf{1}_{\mathbb{R}^N\setminus\mathcal{K}_a}$. Then, from the formulas in Proposition 2.1, we get, in $\{\sigma_a > 0\}$,

$$\begin{aligned}
\frac{4\lambda^2}{\prod_{j=1}^{N}a_j}\Phi_{\rm e}(x) &= \Gamma\star\mathbf{1}_{\mathcal{K}_a}(x) + \frac{4\lambda^2}{\prod_{j=1}^{N}a_j}\Phi_{\rm ext}(x) + \frac{4\lambda^2}{\prod_{j=1}^{N}a_j}\kappa \\
&= \int_{\sigma_a(x)}^{+\infty}\left(1-\sum_{j=1}^{N}\frac{x_j^2}{a_j^2+s}\right)\left(\prod_{j=1}^{N}(a_j^2+s)\right)^{-1/2}\mathrm{d}s \\
&\quad + \sum_{k=1}^{N}x_k^2\int_0^{+\infty}\left(\prod_{j=1}^{N}(a_j^2+s)\right)^{-1/2}\frac{\mathrm{d}s}{a_j^2+s} - \int_0^{+\infty}\left(\prod_{j=1}^{N}(a_j^2+s)\right)^{-1/2}\mathrm{d}s \\
&= \int_0^{\sigma_a(x)}\left(\sum_{j=1}^{N}\frac{x_j^2}{a_j^2+s}-1\right)\left(\prod_{j=1}^{N}(a_j^2+s)\right)^{-1/2}\mathrm{d}s. \qquad (20)
\end{aligned}$$

The last integral is positive if $\sigma_a(x) > 0$ since, when $0 \leq s < \sigma_a(x)$, $\sum_{j=1}^{N}\frac{x_j^2}{a_j^2+s}-1 > \sum_{j=1}^{N}\frac{x_j^2}{a_j^2+\sigma_a(x)}-1 = 0$. In order to see that $\Phi_{\rm e}$ is convex, we notice that $\Phi_{\rm e} \equiv 0$ in $\{\sigma_a \leq 0\} = \mathcal{K}_a$ and that, from (20), we have, when $\sigma_a(x) > 0$, and for any direction $\omega \in \mathbb{S}^{N-1}$,

$$\begin{aligned}
\partial_\omega^2\Phi_{\rm e}(x) &= \frac{\prod_{j=1}^{N}a_j}{2\lambda^2}\int_0^{\sigma_a(x)}\left(\sum_{j=1}^{N}\frac{\omega_j^2}{a_j^2+s}\right)\left(\prod_{j=1}^{N}(a_j^2+s)\right)^{-1/2}\mathrm{d}s \\
&\quad + \frac{\prod_{j=1}^{N}a_j}{\lambda^2}\left(\sum_{j=1}^{N}\frac{x_j\omega_j}{a_j^2+\sigma_a(x)}\right)^2\left(\sum_{j=1}^{N}\frac{x_j^2}{(a_j^2+\sigma_a(x))^2}\right)\left(\prod_{j=1}^{N}(a_j^2+\sigma_a(x))\right)^{-1/2},
\end{aligned}$$

since $\sum_{j=1}^{n} x_j^2/(a_j^2+\sigma_a(x)) = 1$, which is indeed > 0. $\square$

Clearly, the ellipsoid $\mathcal{K}_a$ is not a level set of the external potential Φ_{ext} (except when all the λ_j's are all equal). It is interesting to study the limiting case of a very asymmetric external potential. For instance in $N=2$, we consider a trapping potential (12) with a large aspect ratio $A = \lambda_1/\lambda_2 \gg 1$. Direct computations (using Proposition 2.3) lead to

$$a_1 = \sqrt{\frac{\mathfrak{m}}{\pi}} \frac{\lambda_1}{\sqrt{1+\frac{\lambda_2^2}{\lambda_1^2}}} \ ; \ a_2 = \sqrt{\frac{\mathfrak{m}}{\pi}} \frac{\lambda_2}{\sqrt{1+\frac{\lambda_1^2}{\lambda_2^2}}}.$$

Hence

$$\frac{a_1}{a_2} = \frac{\lambda_1^2}{\lambda_2^2} = A^2$$

Thus, the aspect ratio of the particles' cloud is much larger than the aspect ratio of the external potential: this is an effect of the strong repulsion, see Figure 2 for a typical picture. A similar phenomenon occurs in higher dimensions. For $N=3$ with cylindrical symmetry, explicit formulae corresponding to our $\mathcal{Z}$ function are given for instance in [41]. It is easy to check that for a strongly oblate external potential ("pancake shape"), the aspect ratio of the cloud is again of the order of the square of the aspect ratio of the external potential. We can now state the analog of Theorem 1.2 for a general quadratic Φ_{ext}.

Theorem 2.6 *Let Φ_{ext} be any quadratic potential* (12) *to which we associate, by virtue of Corollary 2.5, the ellipsoid $\mathcal{K}_a$ and the potential Φ_e. Let $V^{\text{init}} \in H^s(\mathring{\mathcal{K}}_a)$ satisfy $\nabla_x \cdot V^{\text{init}} = 0$ in $\mathring{\mathcal{K}}_a$ and the no flux condition* (6) *on $\partial\mathcal{K}_a$. Denote by V the solution on $[0,T]$ to* (IE) *with the no flux condition* (6) *given in Theorem 1.1 and consider $\mathscr{V}^{\text{init}}$ a smooth extension of V in $\mathbb{R}^N$ satisfying the following conditions, where $R>0$ is such that $\mathcal{K}_a \subset B(0,R)$,*

$$\mathscr{V}\Big|_{\mathcal{K}_a} = V, \qquad \mathscr{V}\Big|_{\mathbb{R}^N\setminus B(0,2R)} = 0, \qquad \mathscr{V}(t,x)\cdot\nu(x)\Big|_{\partial\mathcal{K}_a} = 0.$$

Let $f_\varepsilon^{\text{init}} : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ be a sequence of integrable functions that satisfy (11)*. Then, the associated solution f_ε of the Vlasov–Poisson equation* (V)–(P) *satisfies, as $\varepsilon \to 0$,*

i) ρ_ε converges to $n_e = \left(\sum_{j=1}^N \lambda_j^{-2}\right) \mathbf{1}_{\mathcal{K}_a}$ in $C^0(0,T;\mathscr{M}^1(\mathbb{R}^N) - weak - \star)$;

ii) $\mathscr{H}_{\mathscr{V},\varepsilon}$ converges to 0 uniformly on $[0,T]$;

iii) J_ε converges to J in $\mathscr{M}^1([0,T]\times\mathbb{R}^N)$ weakly-$\star$, the limit J lies in $L^\infty(0,T;L^2(\mathbb{R}^N))$ and satisfies $J\big|_{[0,T]\times\mathcal{K}_a} = V$, $\nabla_x\cdot J = 0$ and $J\cdot\nu(x)\big|_{\partial\mathcal{K}_a} = 0$.

The existence of a smooth extension $\mathscr{V}$ of V on $\mathbb{R}^N$ satisfying the above mentioned constraints follows from [27, Chapter I: Theorem 2.1 p. 17 & Theorem 8.1 p. 42]. See Lemma B.1 in the appendix for a divergence-free extension.

2.2 The case of a general potential

We wish now to extend the above results to a general confining potential. When Φ_{ext} is not quadratic, the equilibrium density n_e cannot be expected to be constant on its support. In turn, the limiting equation will be more complicated than the Incompressible Euler system. Besides, the determination of the domain $\mathcal{K} = \{\Phi_e = 0\}$ is a non trivial issue, and its geometry might be quite involved [37]. In the following we write $\Omega = \mathring{\mathcal{K}}$ for the interior of $\mathcal{K}$.

The pair (n_e,Ω) should be thought of through energetic consideration. As it will be detailed below, the total energy of the system (V)-(P) is

$$\iint \frac{|v|^2}{2} f_\varepsilon \, dv \, dx + \frac{1}{2\varepsilon}\int \Phi_\varepsilon\rho_\varepsilon \, dx + \frac{1}{\varepsilon}\int \Phi_{\text{ext}}\rho_\varepsilon \, dx.$$

It is natural to investigate solutions whose energy does not diverge when ε tends to 0. Hence we are interested in configurations close to the ground state $n_{\rm e}$ defined by the variational problem where only the electrostatic part of the energy is involved: namely, we wish to minimize

$$\mathcal{E}[\rho] \stackrel{\rm def}{=} \int \Phi_{\rm ext}(x)\,{\rm d}\rho(x) + \frac{1}{2}\iint \Gamma(x-y)\,{\rm d}\rho(y)\,{\rm d}\rho(x),$$

for a fixed $\mathfrak{m} > 0$ over the convex subset $\mathscr{M}^+_{\rm ext}(\mathfrak{m})$ made of nonnegative Borel measures ρ of total mass $\mathfrak{m} > 0$ such that $\int \Phi_{\rm ext}\,{\rm d}\rho$ is finite. This problem, which is often referred to as the generalized Gauss variational problem, is quite classical and the basis of the theory dates back to [16]. We refer the reader to [36, Chapter 1] for the case $N = 2$, and to [7, Theorem 1.2] when $N \geq 3$ for the existence of a minimizer under suitable assumptions on $\Phi_{\rm ext}$. In what follows, we shall assume that $\Phi_{\rm ext}$ fulfils the following requirements:

h1) $\Phi_{\rm ext} : \mathbb{R}^N \to \mathbb{R}_+$ is continuous, nonnegative and satisfies $\Phi_{\rm ext}(x) \to +\infty$ as $|x| \to +\infty$,

h2) If $N = 2$ or $N = 1$, we have $\lim_{|x|\to+\infty} (\Phi_{\rm ext} + \mathfrak{m}\Gamma)(x) = +\infty$.

The following statement collects from [7, 36, 37] the results we shall need.

Theorem 2.7 *We assume that the potential $\Phi_{\rm ext}$ satisfies the hypotheses h1) and h2).*
(i) The functional $\mathcal{E}$ is strictly convex on $\mathscr{M}^+_{\rm ext}(\mathfrak{m})$.
(ii) The problem

$$\inf\left\{\mathcal{E}[\rho]\ ;\ \rho \in \mathscr{M}^+_{\rm ext}(\mathfrak{m})\right\} \tag{21}$$

has a unique minimizer $n_{\rm e}$ which has a compact support of positive capacity. Moreover, there exists a constant C_* such that

$$\begin{cases} \Gamma \star n_{\rm e} + \Phi_{\rm ext} \geq C_* & \textit{quasi everywhere}, \\ \Gamma \star n_{\rm e} + \Phi_{\rm ext} = C_* & \textit{quasi everywhere on } {\rm Supp}(n_{\rm e}). \end{cases} \tag{22}$$

(iii) Conversely, assume that $\rho_0 \in \mathscr{M}^+_{\rm ext}(\mathfrak{m})$ and C_0 are such that

$$\begin{cases} \Gamma \star \rho_0 + \Phi_{\rm ext} \geq C_0 & \textit{quasi everywhere}, \\ \Gamma \star \rho_0 + \Phi_{\rm ext} = C_0 & \textit{quasi everywhere on } {\rm Supp}(\rho_0). \end{cases}$$

Then, ρ_0 is the minimizer for (21): $\rho_0 = n_{\rm e}$.

We then define the potential $\Phi_{\rm e} \stackrel{\rm def}{=} \Gamma \star n_{\rm e} + \Phi_{\rm ext} - C_*$. The constant C_* in (22) is called the modified Robin constant and *quasi everywhere* (q. e.) means up a set of zero capacity (which is a bit stronger than to be Lebesgue-negligible); see [37, Definition 2.11]. If $N = 1$, (22) holds pointwise.

Proof. The statements for $N \geq 3$ can be found in [7, Theorem 1.2]. When $N = 2$, we refer the reader to [36, Theorem 1.3 for (ii) and Theorem 3.3 for (iii)]. If $N = 2$, the strict convexity (i) is not explicited in [36]. Thus we give proofs of (i) for $N = 2$, and (i)-(iii) for $N = 1$.

The argument for (i) is that if ρ_0, $\rho_1 \in \mathscr{M}^+_{\rm ext}(\mathfrak{m})$ and $\theta \in (0, 1)$, then

$$\begin{aligned}
&\mathcal{E}[(1-\theta)\rho_0 + \theta\rho_1] - (1-\theta)\mathcal{E}[\rho_0] - \theta\mathcal{E}[\rho_1] \\
&\quad = \frac{1}{2}\iint \Gamma(x-y)\,{\rm d}[(1-\theta)\rho_0 + \theta\rho_1](y)\,{\rm d}[(1-\theta)\rho_0 + \theta\rho_1](x) \\
&\qquad - \frac{1}{2}(1-\theta)\iint \Gamma(x-y)\,{\rm d}\rho_0(y)\,{\rm d}\rho_0(x) - \frac{1}{2}\theta\iint \Gamma(x-y)\,{\rm d}\rho_1(y)\,{\rm d}\rho_1(x) \\
&\quad = -\frac{1}{2}\theta(1-\theta)\iint \Gamma(x-y)\,{\rm d}[\rho_0-\rho_1](y)\,{\rm d}[\rho_0-\rho_1](x).
\end{aligned}$$

Unless $\rho_0 = \rho_1$, the last integral is shown to be positive if $N \geq 3$ in [7, Lemma 3.1]. The case $N = 2$ is dealt with in [36, Lemma 1.8], under the restriction that $\rho_0 - \rho_1$ has compact support. Actually, the method used in [7], which consists in writing $\Gamma(x)$ as an integral of Gaussians $\mathsf{e}^{-|x|^2/2t}$, can be extended to the case $N = 2$ as we check now. The starting point is the equality (see [2, equation (12)])

$$\ln \frac{1}{r} = \int_0^{+\infty} \frac{1}{2t} \left(\mathsf{e}^{-r^2/2t} - \mathsf{e}^{-1/2t} \right) \, \mathrm{d}t.$$

Therefore, denoting $\rho \stackrel{\text{def}}{=} \rho_0 - \rho_1$ and $r \stackrel{\text{def}}{=} |x - y|$ and using the dominated convergence theorem (on each of the sets $\{|x - y| < 1\}$ and $\{|x - y| \geq 1\}$), we obtain

$$\begin{aligned}
\iint \ln \frac{1}{|x-y|} \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) &= \lim_{T \to +\infty} \int_{1/T}^{T} \frac{1}{2t} \iint \left(\mathsf{e}^{-r^2/2t} - \mathsf{e}^{-1/2t} \right) \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \, \mathrm{d}t \\
&= \lim_{T \to +\infty} \int_{1/T}^{T} \frac{1}{2t} \iint \mathsf{e}^{-r^2/2t} \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \, \mathrm{d}t \\
&= \lim_{T \to +\infty} \int_{1/T}^{T} \frac{1}{4\pi} \iint \int \mathsf{e}^{-t|\xi|^2/2 - i\xi\cdot(x-y)} \, \mathrm{d}\xi \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \, \mathrm{d}t \\
&= \lim_{T \to +\infty} \int_{1/T}^{T} \int \frac{1}{4\pi} \mathsf{e}^{-t|\xi|^2/2} |\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi \, \mathrm{d}t \\
&= \int \frac{1}{2\pi|\xi|^2} |\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi,
\end{aligned}$$

where, for the second equality, we use $\int \mathrm{d}\rho = 0$, and for the third one, we write $\mathsf{e}^{-r^2/2t}$ as the Fourier transform of a two dimensional Gaussian. This clearly shows that $\iint \Gamma(x - y) \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x)$ is positive unless $\rho = 0$, ensuring the strict convexity of $\mathcal{E}$ on $\mathscr{M}^+_{\mathrm{ext}}(\mathfrak{m})$. When $N = 1$, we argue in a similar way by observing that

$$-r = \int_0^{+\infty} \frac{1}{\sqrt{2\pi t}} \left(\mathsf{e}^{-r^2/2t} - 1 \right) \, \mathrm{d}t.$$

Indeed, $\mathsf{e}^{-r^2/2t} - 1 = \int_0^r \partial_u(\mathsf{e}^{-u^2/2t}) \, \mathrm{d}u = -\int_0^r (u/t)\mathsf{e}^{-u^2/2t} \, \mathrm{d}u$, thus

$$\begin{aligned}
-\int_0^{+\infty} \frac{1}{\sqrt{t}} \left(\mathsf{e}^{-r^2/2t} - 1 \right) \, \mathrm{d}t &= \int_0^{+\infty} \frac{1}{\sqrt{t}} \int_0^r (u/t)\mathsf{e}^{-u^2/2t} \, \mathrm{d}u \, \mathrm{d}t = \int_0^r \int_0^{+\infty} \frac{u}{t^{3/2}} \mathsf{e}^{-u^2/2t} \, \mathrm{d}t \, \mathrm{d}u \\
&= \int_0^r \int_0^{+\infty} 2\sqrt{2}\mathsf{e}^{-\tau^2} \, \mathrm{d}\tau \, \mathrm{d}u = \int_0^r \sqrt{2\pi} \, \mathrm{d}u = r\sqrt{2\pi},
\end{aligned}$$

where we have used the change of variable $\tau = u/\sqrt{2t}$. Owing to this relation, we can follow the same lines as above:

$$\begin{aligned}
-\frac{1}{2} \iint |x - y| \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) &= \lim_{T \to +\infty} \int_{1/T}^{T} \frac{1}{2\sqrt{2\pi t}} \iint \left(\mathsf{e}^{-r^2/2t} - 1 \right) \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \, \mathrm{d}t \\
&= \lim_{T \to +\infty} \int_{1/T}^{T} \frac{1}{2\sqrt{2\pi t}} \iint \mathsf{e}^{-r^2/2t} \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \, \mathrm{d}t \\
&= \lim_{T \to +\infty} \int_{1/T}^{T} \frac{1}{4\pi} \iint \int \mathsf{e}^{-t\xi^2/2 - i\xi(x-y)} \, \mathrm{d}\xi \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \, \mathrm{d}t \\
&= \lim_{T \to +\infty} \int_{1/T}^{T} \int \frac{1}{4\pi} \mathsf{e}^{-t\xi^2/2} |\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi \, \mathrm{d}t \\
&= \int \frac{1}{2\pi|\xi|^2} |\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi.
\end{aligned}$$

It only remains to prove (ii)-(iii) for $N = 1$. This is tackled in [37], but with $\Gamma(x) = -\ln|x|$. The very same arguments apply to the case $\Gamma(x) = -|x|/2$. $\square$

Remark 2.8 *To motivate the above computation, one can remark that for $N \geq 1$ and under the condition $\int \mathrm{d}\rho = 0$, we have, at least formally,*

$$\iint \Gamma(x-y)\,\mathrm{d}\rho(y)\,\mathrm{d}\rho(x) = \int |\nabla\Delta^{-1}\rho|^2 dx = (2\pi)^{-N}\int \frac{1}{|\xi|^2}|\hat{\rho}(\xi)|^2\,\mathrm{d}\xi.$$

The minimization of the functional $\mathcal{E}$ is connected to an *obstacle problem*. This connection is explained in details in [37, Section 2.5].

Proposition 2.9 *If n_e is the minimizer of Theorem 2.7, then $h = \Gamma \star n_\mathrm{e}$ is the unique solution to the obstacle problem*

To find $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that
$$\int \nabla\phi\cdot\nabla(g-\phi)\,\mathrm{d}x \geq 0,$$
holds for any $g \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$, with $g - \phi$ compactly supported and $\phi \geq \psi$ q. e.

where $\psi(x) \stackrel{\mathrm{def}}{=} C_* - \Phi_{\mathrm{ext}}(x)$.

We then define the coincidence set

$$\mathcal{K} \stackrel{\mathrm{def}}{=} \{\Phi_\mathrm{e} = 0\} = \{\Gamma\star n_\mathrm{e} = C_* - \Phi_{\mathrm{ext}}\}$$

and claim that $\mathcal{K}$ is compact. Indeed, as $|x| \to +\infty$, we have $-\Gamma\star n_\mathrm{e}(x) \sim -\mathfrak{m}\Gamma(x)$ and $\Phi_{\mathrm{ext}}(x) + \mathfrak{m}\Gamma(x) \gg 1$ whatever is the dimension N by h1)-h2), thus $\mathcal{K} = \{\Gamma\star n_\mathrm{e} = C_* - \Phi_{\mathrm{ext}}\}$ is bounded. Moreover, by (22), the set $\mathrm{Supp}(n_\mathrm{e}) \setminus \mathcal{K}$ has zero capacity. We give some examples in section 2.3 below where $\mathrm{Supp}(n_\mathrm{e}) \subsetneq \mathcal{K}$, due to the presence of points or regions where $\Delta\Phi_{\mathrm{ext}}$ vanishes. These points are precisely defined in [20, Section 3.6] and called 'shallow points' and it is shown in this paper (see Proposition 3.12 there) that it is possible to pass from $\mathrm{Supp}(n_\mathrm{e})$ to $\mathcal{K}$ by simply adding these 'shallow points'. This fact is illustrated in section 2.3 below.

For a general potential Φ_{ext}, the variational viewpoint and the theory of the obstacle problem provide a definition for the equilibrium distribution n_e, the domain $\mathcal{K}$ (which is not always the support of n_e) and the potential Φ_e. The regularity of Φ_{ext} is not "transferred" to the solution Φ_e or $\Gamma\star n_\mathrm{e}$ beyond $C^{1,1}$ regularity (see [13], [6]) since the Laplacian of these functions is discontinuous. In addition, the topology of $\mathcal{K}$ is difficult to analyse in general: $\mathcal{K}$ may have empty interior or may exhibit cusps. Hence, these regularity issues for both $\mathcal{K}$ and n_e need to be discussed individually. Let us then list the properties, which very likely are far from optimal, that we need to deal with the asymptotic regime: there exists $s > 1 + N/2$ such that

H1) $\mathcal{K}$ has a non empty interior Ω and $\partial\Omega$ is of class C^1.

H2) $\Phi_{\mathrm{ext}} \in C^{s+3}(\mathbb{R}^N)$, $\Delta\Phi_{\mathrm{ext}}$ is bounded away from zero on $\mathcal{K}$.

The C^1 regularity assumption H1) on $\partial\Omega$ excludes the presence of cusps in $\mathcal{K}$. Let us point out two regularity results derived from the obstacle problem theory.

Proposition 2.10 *Let Φ_{ext} be a potential satisfying* h1) *and* h2) *and consider n_e the minimizer of* (21) *given by Theorem 2.7 and let $\mathcal{K} \stackrel{\mathrm{def}}{=} \{\Phi_\mathrm{e} = 0\}$.*

i) [23] Assume that H1) and H2) are satisfied. Then $\bar{\Omega} = \mathcal{K}$ and the boundary $\partial\Omega$ is C^{s+1}.

ii) [13], [6], [20] Assume that $\Phi_{\mathrm{ext}} \in C^{1,1}(\mathbb{R}^N)$. Then, $\Gamma\star n_\mathrm{e} \in C^{1,1}$ and $n_\mathrm{e} = \mathbf{1}_\Omega(\Delta\Phi_{\mathrm{ext}})$ as measures.

Remark 2.11 *A consequence of ii) is that, in* (22)*, we may replace 'quasi everywhere' by 'everywhere' since all the functions involved are continuous. Note that $H2$) then implies that n_e is C^{s+1} and bounded from below on $\mathcal{K}$.*

Remark 2.12 *Under the low regularity assumption $\Phi_\mathrm{ext} \in C^1(\mathbb{R}^N)$ and when $N \geq 2$, it follows from [6, Theorem 2] that $\Gamma \star n_\mathrm{e} \in C^1$. This prevents the singular part of the measure n_e from being a Dirac mass or a finite sum of Dirac masses, since the fundamental solution Γ is unbounded (if $N \geq 2$) near the origin. This however may happen in dimension 1 (see the examples in section 2.3 below) and in these cases, the relation $n_\mathrm{e} = \mathbf{1}_\Omega(\Delta\Phi_\mathrm{ext})$ (as measures) might not be true.*

Proof. For the first point, notice that $\bar{\Omega} = \mathcal{K}$ by H1). In addition, we also have from [23, Theorem 1] or [14, Chapter 2, Theorem 1.1], since we assume that $\Delta\Phi_\mathrm{ext} \in C^{s+1}_\mathrm{loc}(\mathbb{R}^N)$ and does not vanish on $\mathcal{K}$, that the boundary $\partial\Omega$ is automatically of class C^{s+1} (and even $C^{s+1+\beta}$ for any $\beta \in (0, 1)$). If Φ_ext is analytic, then $\partial\Omega$ is also analytic.

For the second statement, we first invoke the regularity result of [13] (see also [6], [14]) saying that since $\Phi_\mathrm{ext} \in C^{s+3}(\mathbb{R}^N) \subset C^{1,1}(\mathbb{R}^N)$, then $\Gamma \star n_\mathrm{e}$ belongs to $C^{1,1}(\mathbb{R}^N)$. Consequently, in the distributional sense, the compactly supported measure $n_\mathrm{e} = -\Delta(\Gamma \star n_\mathrm{e})$ belongs to $L^\infty(\mathbb{R}^N)$. Since $\Gamma \star n_\mathrm{e} = C_* - \Phi_\mathrm{ext}$ in Ω (q. e., hence everywhere by continuity of the functions), we infer that $n_\mathrm{e} = \Delta\Phi_\mathrm{ext}$ in Ω. If we make the assumption H1), $\partial\Omega$ is of class C^1 and we then deduce that $n_\mathrm{e} = (\Delta\Phi_\mathrm{ext})\mathbf{1}_\Omega$ as a measure. If assumption H1) is not satisfied, then, as noticed in [20, Theorem 3.10], it follows from [24, Chapter 2, Lemma A.4] that $n_\mathrm{e} = \Delta\Phi_\mathrm{ext}$ holds almost everywhere in $\mathcal{K}$, which concludes. $\square$

With these assumptions H1) and H2), we can establish the following statement, where we point out that the limit problem is the Lake Equation (LE) instead of the mere incompressible Euler system, since now the equilibrium distribution n_e is inhomogeneous. We obviously need a smooth enough solution to the Lake Equation (LE): we may refer to the works [26], [33] (when the domain Ω is simply connected) and [25] (without simple connectedness assumption on the domain Ω), which rely on a vorticity formulation *à la* Yudovitch and are then restricted to the dimension $N = 2$. We provide in the appendix (see Theorem A.1) a well-posedness result analogous to Theorem 1.1 valid in any dimension and without simple connectedness assumption on Ω.

Theorem 2.13 *Let Φ_ext be a potential satisfying $h1$) and $h2$) and consider n_e the minimizer of* (21) *given by Theorem 2.7 and let $\mathcal{K} \stackrel{\text{def}}{=} \{\Phi_\mathrm{e} = 0\}$. Assume in addition that $H1$) and $H2$) are satisfied. Let $V^\mathrm{init} \in H^s(\Omega)$ satisfy $\nabla_x \cdot (n_\mathrm{e} V^\mathrm{init}) = 0$ in Ω and the no flux condition* (6)*. Denote by V the solution on $[0, T]$ to the Lake Equation* (LE)*, with the no flux condition* (6) *and initial condition V^init, given in Theorem A.1 and consider $\mathscr{V}^\mathrm{init}$ a smooth extension of V^init satisfying the following conditions, where $R > 0$ is such that $\Omega \subset B(0, R)$,*

$$\mathscr{V}\Big|_\Omega = V, \qquad \mathscr{V}\Big|_{\mathbb{R}^N \setminus B(0,2R)} = 0, \qquad \mathscr{V}(t, x) \cdot \nu(x)\Big|_{\partial\Omega} = 0.$$

Let $f^\mathrm{init}_\varepsilon : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ be a sequence of integrable functions that satisfy (11)*. Then, the associated solution f_ε of the Vlasov–Poisson equation* (V)–(P) *satisfies, as $\varepsilon \to 0$,*

i) *ρ_ε converges to n_e in $C^0(0, T; \mathscr{M}^1(\mathbb{R}^N) - weak - \star)$;*

ii) *$\mathscr{H}_{\mathscr{V},\varepsilon}$ converges to 0 uniformly on $[0, T]$;*

iii) *J_ε converges to J in $\mathscr{M}^1([0, T] \times \mathbb{R}^N)$ weakly-$\star$, the limit J lies in $L^\infty(0, T; L^2(\mathbb{R}^N))$ and satisfies $J\big|_{[0,T]\times\Omega} = V$, $\nabla_x \cdot J = 0$ and $J \cdot \nu(x)\big|_{\partial\Omega} = 0$.*

The existence of a smooth extension $\mathscr{V}$ of V follows from [27, Chapter I: Theorem 2.1 p. 17 & Theorem 8.1 p. 42].

For convex potentials Φ_ext, the only situation where we have been able to check the hypotheses H1) and H2) (except the quadratic potentials for which $\Delta\Phi_\mathrm{ext}$ is constant) is the case of the space dimension $N = 1$ (see Proposition 2.14) and the case of a radial potential (see Proposition 2.15 below).

2.3 About hypothesis $H1)$ for convex potentials $\Phi_{\rm ext}$

For the problem we have in mind, it is natural to assume that the confining potential $\Phi_{\rm ext}$ is smooth and convex. In this case, one may think that the coincidence set $\mathcal{K}$ or $\mathrm{Supp}(n_{\rm e})$ is convex. We have not been able to find such a result in the literature for a general convex, coercive and smooth enough confining potential $\Phi_{\rm ext}$. Actually, the obstacle problem is, in most cases, set on a bounded convex domain G with suitable boundary conditions instead of the whole space $\mathbb{R}^N$.

For the obstacle problem in bounded convex domains G, we can find a convexity result for the coincidence set $\mathcal{K}$ in [15, Theorem 6.1] in the specific assumptions that $\Delta\Phi_{\rm ext}$ is constant and with the boundary condition $\Gamma \star n_{\rm e} = 1 + \psi = 1 + C_* - \Phi_{\rm ext}$ on ∂G. Just after [15, Theorem 6.1], an example is given (in a bounded convex domain G) showing that the assumption $\Phi_{\rm ext}$ smooth and strictly convex (and $\Gamma \star n_{\rm e} > \psi$ on ∂G) is not sufficient to guarantee that $\mathcal{K}$ is convex. Roughly speaking, $\Delta\Phi_{\rm ext}$ is constant for quadratic potentials.

Turning back to the obstacle problem in the whole space $\mathbb{R}^N$, the only convexity result we are aware of is [6, Corollary 7], which corresponds to the case where $\Delta\Phi_{\rm ext}$ is constant. Extending this result to space depending functions $\Delta\Phi_{\rm ext}$ is a delicate issue (see however [14, Chapter 2, section 3], which is not sufficient for our situation).

In the one dimensional case and for a convex potential $\Phi_{\rm ext}$, there is a simple characterization of $\mathcal{K}$, as explicited in the following Proposition.

Proposition 2.14 (The one dimensional case with a convex potential) *Assume that $N = 1$ and that $\Phi_{\rm ext} : \mathbb{R} \to \mathbb{R}$ is of class C^1, piecewise C^2, nonnegative, convex (i.e. $\Phi'_{\rm ext}$ is nondecreasing) and that $\Phi_{\rm ext}(x) - \mathfrak{m}|x|/2 \gg 1$ for $|x| \gg 1$ (so that h1) and h2) are satisfied). We denote by $\partial\Phi'_{\rm ext}$ the piecewise continuous function associated with the second order derivative of $\Phi_{\rm ext}$. Then, the minimizer $n_{\rm e}$ for (21) is given by*

$$n_{\rm e} = (\partial\Phi'_{\rm ext})\Big|_{]a_-,a_+[} \tag{23}$$

where a_+ and a_- are defined by the equations

$$\frac{\mathfrak{m}}{2} = \Phi'_{\rm ext}(a_+) \qquad \textit{and} \qquad -\frac{\mathfrak{m}}{2} = \Phi'_{\rm ext}(a_-). \tag{24}$$

Furthermore, $\mathrm{Supp}(n_{\rm e}) = \mathrm{Supp}(\partial\Phi'_{\rm ext}) \cap [a_-, a_+]$ *and* $\{\Phi_{\rm e} = 0\} = [a_-, a_+]$. *In addition, the potential $\Phi_{\rm e}$ is convex.*

Proof. As a first observation, notice that (24) has at least one (possibly non unique) solution since $\Phi'_{\rm ext}$ is continuous, nondecreasing and tends to $\geq \mathfrak{m}/2$ (resp. $\leq -\mathfrak{m}/2$) in view of our hypothesis. If the limit at $+\infty$ is $\mathfrak{m}/2$, it follows from the convexity of $\Phi_{\rm ext}$ that $\Phi_{\rm ext}(x) - \Phi_{\rm ext}(y) \leq \mathfrak{m}/2$.

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(resp. $-\infty$) (resp. $-\mathfrak{m}/2$) Since $\Phi'_{\rm ext}$ is nondecreasing, and if $b_+ > a_+$ also solves $\mathfrak{m}/2 = \Phi'_{\rm ext}(b_+)$, this implies that, on $[a_+, b_+]$, $\Phi'_{\rm ext} \equiv \mathfrak{m}/2$, thus $\partial\Phi'_{\rm ext} \equiv 0$ and this does not change $n_{\rm e}$.

Let us use the characterization (iii) in Theorem 2.7 and look for the measure $n_{\rm e}$ under the form $n_{\rm e} = (\partial\Phi'_{\rm ext})\Big|_{]a_-,a_+[}$, which is piecewise continuous. This function $n_{\rm e}$ satisfies the mass constraint if and only if

$$\mathfrak{m} = \int_{a_-}^{a_+} \partial\Phi'_{\rm ext}\ \mathrm{d}x = \Phi'_{\rm ext}(a_+) - \Phi'_{\rm ext}(a_-). \tag{25}$$

Now, let us compute $\Gamma \star n_{\rm e} + \Phi_{\rm ext}$ in $[a_-, a_+]$ and investigate under which condition this function

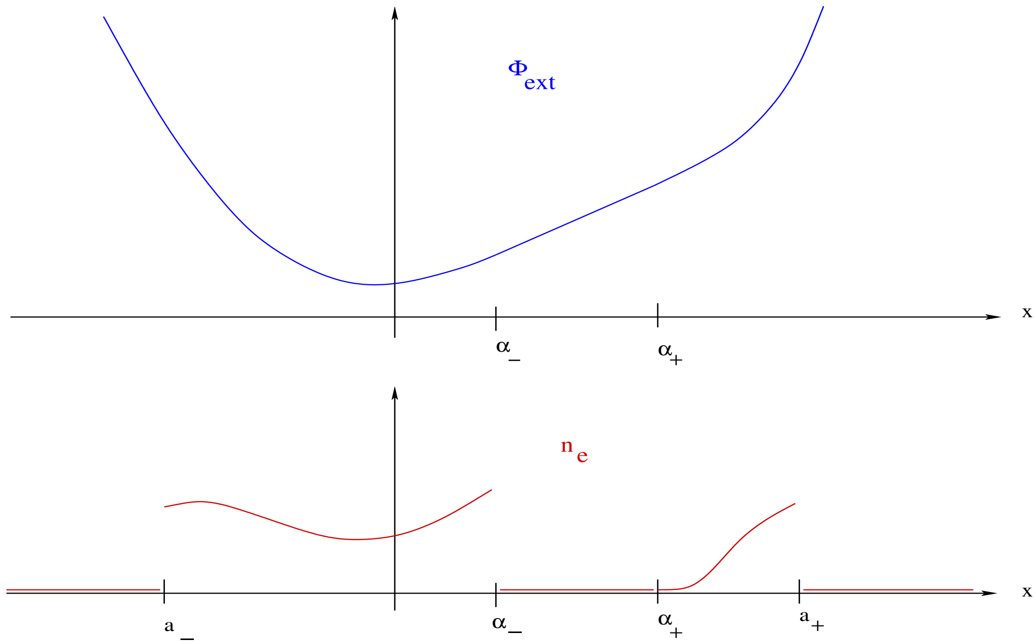

Figure 3: The potential $\Phi_{\rm ext}$ and the corresponding measure $n_{\rm e}$ for example 2

is constant (in $[a_-, a_+]$). Elementary computations give, for $a_- \le x \le a_+$:

$$\begin{aligned}\Gamma \star n_{\rm e}(x) &= -\frac{1}{2}\int_{a_-}^{a_+} |y-x|(\partial\Phi'_{\rm ext})(y)\, {\rm d}y \\ &= -\frac{1}{2}\Phi'_{\rm ext}(a_+)(a_+ - x) + \frac{1}{2}\Phi'_{\rm ext}(a_-)(x-a_-) + \frac{1}{2}\int_{a_-}^{a_+} {\rm sgn}(y-x)\Phi'_{\rm ext}(y)\, {\rm d}y \\ &= -\frac{1}{2}\Phi'_{\rm ext}(a_+)(a_+ - x) + \frac{1}{2}\Phi'_{\rm ext}(a_-)(x-a_-) + \frac{1}{2}\Phi_{\rm ext}(a_+) + \frac{1}{2}\Phi_{\rm ext}(a_-) - \Phi_{\rm ext}(x).\end{aligned}$$

As a consequence, $\Gamma \star n_{\rm e} + \Phi_{\rm ext}$ is constant in $[a_-, a_+]$ if and only if $\Phi'_{\rm ext}(a_+) + \Phi'_{\rm ext}(a_-) = 0$. Combining this with the mass constraint $\Phi'_{\rm ext}(a_+) - \Phi'_{\rm ext}(a_-) = \mathfrak{m}$ yields the relation (24). It then follows that, on $[a_-, a_+]$,

$$\begin{aligned}\Gamma \star n_{\rm e} + \Phi_{\rm ext} = C_* &\stackrel{\rm def}{=} \frac{1}{2}\left(\Phi_{\rm ext}(a_+) + \Phi_{\rm ext}(a_-) - a_+\Phi'_{\rm ext}(a_+) - a_-\Phi'_{\rm ext}(a_-)\right) \\ &= \frac{1}{2}\left(\Phi_{\rm ext}(a_+) + \Phi_{\rm ext}(a_-)\right) - \frac{\mathfrak{m}}{4}(a_+ - a_-).\end{aligned}$$

It remains to check that $\Gamma \star n_{\rm e} + \Phi_{\rm ext} \ge C_*$ in $\mathbb{R}$. To see this, note that $\Phi_{\rm e} \stackrel{\rm def}{=} \Gamma \star n_{\rm e} + \Phi_{\rm ext} - C_*$ is convex since its (distributional) second order derivative is equal to the piecewise continuous function $\partial_x \Phi'_{\rm ext}\mathbf{1}_{\mathbb{R}\setminus[a_-,a_+]}$, and $\Phi_{\rm e} \equiv 0$ on $[a_-, a_+]$, hence is ≥ 0 everywhere. This finishes the proof. $\square$

Let us give some examples illustrating Proposition 2.14.

Example 1 (1D): If $\Phi_{\rm ext}$ is of class C^2 and $\Phi''_{\rm ext}$ is positive on $\mathbb{R}$, then $n_{\rm e}(x) = \Phi''_{\rm ext}(x)\mathbf{1}_{[a_-,a_+]}(x)$ and is absolutely continuous with respect to the Lebesgue measure. We then have ${\rm Supp}(n_{\rm e}) = [a_-, a_+]$.

Example 2 (1D): the potential $\Phi_{\rm ext}$ is C^1, piecewise C^2, but is affine on the interval $[\alpha_-,\alpha_+]$ (hence it is not strictly convex), where its slope belongs to $]-\mathfrak{m}/2,+\mathfrak{m}/2[$ (see figure 3). In addition, the second order derivative $\Phi''_{\rm ext}$ is discontinuous at α_- and continuous at α_+ and $\partial\Phi'_{\rm ext}$ is positive except on $[\alpha_-,\alpha_+]$. In this case, we may still define $a_\pm$ as the unique solutions to $\Phi'_{\rm ext}(a_\pm)=\pm\mathfrak{m}/2$, and we have $\mathrm{Supp}(n_{\rm e})=[a_-,\alpha_-]\cup[\alpha_+,a_+]\subsetneq[a_-,a_+]=\{\Phi_{\rm e}=0\}$ and this is then a disconnected set. If the slope in the region $[\alpha_-,\alpha_+]$ where $\Phi_{\rm ext}$ is affine does not belong to $]-\mathfrak{m}/2,+\mathfrak{m}/2[$, then the support of $n_{\rm e}$ is an interval as in Example 1.

Example 1 fits the hypotheses of Theorem 2.13, but not Example 2 since $n_{\rm e}$ is not bounded away from zero (near α_+). In particular, for Example 2, we have to face new difficulties in solving the Cauchy problem (see Theorem A.1) for the Lake Equation (LE). If in the one dimensional situation one can easily check that the support of $n_{\rm e}$ (instead of $\mathcal{K}$) is smooth, in a similar higher dimensional case, the regularity of Supp $(n_{\rm e})$ is certainly not easy to analyse since we can not rely on the results in [23, Theorem 1] or [14, Chapter 2, Theorem 1.1]. All these issues motivate hypothesis H2).

Let us give now examples which do not fit the regularity hypotheses required in Proposition 2.14. These expressions are justified through the characterization (iii) in Theorem 2.7 and simple computation of $\Gamma\star n_{\rm e}$.

Example 3 (1D): Take the potential $\Phi_{\rm ext}(x)=|x|$. Then, hypothesis h2) exactly means $\mathfrak{m}<1$. In that case, we have $n_{\rm e}=\mathfrak{m}\delta_0$ and $\mathrm{Supp}(n_{\rm e})=\{0\}=\{\Phi_{\rm e}=0\}$.

Example 4 (1D): Take two reals $a<b$ and a convex potential $\Phi_{\rm ext}$ which is affine on $]-\infty,a]$, on $[a,b]$ and on $[b,+\infty[$. Assume also that h2) is satisfied, that is $\mathfrak{m}<\min(\Phi'_{\rm ext}(+\infty),-\Phi'_{\rm ext}(-\infty))$.
Then, $n_{\rm e}=\min\left(\frac{1}{2}\left(\mathfrak{m}+\frac{\Phi_{\rm ext}(b)-\Phi_{\rm ext}(a)}{b-a}\right)_+,\mathfrak{m}\right)\delta_a+\min\left(\frac{1}{2}\left(\mathfrak{m}-\frac{\Phi_{\rm ext}(b)-\Phi_{\rm ext}(a)}{b-a}\right)_+,\mathfrak{m}\right)\delta_b$.
As a consequence:
- if $-\mathfrak{m}<\frac{\Phi_{\rm ext}(b)-\Phi_{\rm ext}(a)}{b-a}<\mathfrak{m}$, then $\mathrm{Supp}(n_{\rm e})=\{a,b\}$ and $\{\Phi_{\rm e}=0\}=[a,b]$;
- if $\frac{\Phi_{\rm ext}(b)-\Phi_{\rm ext}(a)}{b-a}\le-\mathfrak{m}$, then $n_{\rm e}=\mathfrak{m}\delta_b$ and $\mathrm{Supp}(n_{\rm e})=\{b\}=\{\Phi_{\rm e}=0\}$;
- if $\frac{\Phi_{\rm ext}(b)-\Phi_{\rm ext}(a)}{b-a}\ge\mathfrak{m}$, then $n_{\rm e}=\mathfrak{m}\delta_a$ and $\mathrm{Supp}(n_{\rm e})=\{a\}=\{\Phi_{\rm e}=0\}$.

Example 5 (1D): Consider the potential $\Phi_{\rm ext}(x)=|x|+x^2/2+\max(x-1,0)$:
- if $\mathfrak{m}\le 2$, then $n_{\rm e}=\mathfrak{m}\delta_0$, $\Gamma\star n_{\rm e}(x)=-\mathfrak{m}|x|/2$ and $\mathrm{Supp}(n_{\rm e})=\{0\}=\{\Phi_{\rm e}=0\}$;
- if $2\le\mathfrak{m}\le 4$, then $n_{\rm e}=2\delta_0+\mathbf{1}_{[-\mathfrak{m}/2+1,\mathfrak{m}/2-1]}$ and $\mathrm{Supp}(n_{\rm e})=[-\mathfrak{m}/2+1,\mathfrak{m}/2-1]=\{\Phi_{\rm e}=0\}$;
- if $4\le\mathfrak{m}\le 6$, then $n_{\rm e}=2\delta_0+(\mathfrak{m}/2-2)\delta_1+\mathbf{1}_{[-\mathfrak{m}/2+1,1]}$ and $\mathrm{Supp}(n_{\rm e})=[-\mathfrak{m}/2+1,1]=\{\Phi_{\rm e}=0\}$;
- if $\mathfrak{m}\ge 6$, then $n_{\rm e}=2\delta_0+\delta_1+\mathbf{1}_{[-\mathfrak{m}/2+1,\mathfrak{m}/2-2]}$ and $\mathrm{Supp}(n_{\rm e})=[-\mathfrak{m}/2+1,\mathfrak{m}/2-2]=\{\Phi_{\rm e}=0\}$.

Examples 3, 4 and 5 show that the single convexity hypothesis on $\Phi_{\rm ext}$ does not guarantee that $n_{\rm e}$ is a restriction of the nonnegative measure $\partial_x^2\Phi_{\rm ext}$ (in the distributional sense). It appears in these examples that $n_{\rm e}$ is nondecreasing with respect to the mass $\mathfrak{m}$, and thus that we always have $n_{\rm e}\le\partial_x^2\Phi_{\rm ext}$ in the distributional sense. It is an open problem to determine whether this holds true in higher dimensions. Here again, these issues motivate the regularity assumptions on $\Phi_{\rm ext}$ in H2).

The other situation where we may verify hypothesis H1) is the radial case (see [7, Corollary 1.4] for a related result in dimension $N\ge 3$ for C^2 potentials $\Phi_{\rm ext}$). Let $\varphi_{\rm ext}:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function of class C^1 and piecewise C^2. Consider now the potential $\Phi_{\rm ext}:\mathbb{R}^N\to\mathbb{R}$ given by $\Phi_{\rm ext}(x)=\varphi_{\rm ext}(|x|)$. It is then clear that $\varphi_{\rm ext}$ is convex if and only if $\Phi_{\rm ext}$ is convex.

Proposition 2.15 (The radial case with a convex potential) *Assume that $N\ge 2$ and that*

$\Phi_{\rm ext} : \mathbb{R}^N \to \mathbb{R}$ *is as above. Then, the minimizer* $n_{\rm e}$ *for* (21) *is given by*

$$n_{\rm e}(x) = \mathbf{1}_{B(0,R)}(x)\Delta\Phi_{\rm ext}(x), \tag{26}$$

where R *is defined by the equation*

$$\mathfrak{m} = \int_{B(0,R)} \Delta\Phi_{\rm ext}(x)\,{\rm d}x \qquad \textit{or, equivalently,} \qquad N|B(0,1)|R^{N-1}\varphi'_{\rm ext}(R) = \mathfrak{m}. \tag{27}$$

Furthermore, $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,R) \setminus B(0,R_{\min})$*, where* $R_{\min} \overset{\rm def}{=} \max\{\varphi'_{\rm ext} = 0\} \le R$*. In addition, the potential* $\Phi_{\rm e}$ *is convex.*

Proof. The existence of R is clear. We may have non uniqueness only in the case where $\Phi_{\rm ext}$ is constant on a ball $B(0,R_0)$ (of positive radius), since $\varphi'_{\rm ext}$ is nondecreasing. The potential $\Phi_{\rm e}$ may be searched for under the form of a radial function, and we find the expressions

$$\Phi_{\rm e}(x) = (\varphi_{\rm ext}(R) - \varphi_{\rm ext}(|x|) + \varphi'_{\rm ext}(R)\Gamma(R))\mathbf{1}_{B(0,R)} + \varphi'_{\rm ext}(R)\Gamma(x)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)},$$

where $\Gamma(R)$ stands for $\Gamma(y)$ for any $y \in \partial B(0,R)$. $\square$

Let us give some examples illustrating Proposition 2.15.

Example 1 (radial): If $\varphi_{\rm ext}$ is of class C^2 and $\varphi''_{\rm ext}$ is positive on $\mathbb{R}_+$, then $n_{\rm e}(x) = \mathbf{1}_{B(0,R)}(x)\Delta\Phi_{\rm ext}(x)$ and is absolutely continuous with respect to the Lebesgue measure. We then have $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,R)$.

Example 2 (radial): The potential $\varphi_{\rm ext}$ is C^1, piecewise C^2, but is constant on the interval $[0,R_0]$ (hence it is not strictly convex). It does not matter whether the second order derivative of $\varphi_{\rm ext}$ is continuous or not at R_0. We define $R \ge R_0 > 0$ by the relation $\mathfrak{m} = \int_{B(0,R)} \Delta\Phi_{\rm ext}\,{\rm d}x$, or, equivalently, $N|B(0,1)|R^{N-1}\varphi'_{\rm ext}(R) = \mathfrak{m}$. Then, $n_{\rm e} = \mathbf{1}_{B(0,R)\setminus B(0,R_0)}\Delta\Phi_{\rm ext}$, $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,R) \setminus B(0,R_0) \subsetneq \bar{B}(0,R) = \{\Phi_{\rm e} = 0\}$ and this set is then neither starshaped nor simply connected. Here again, if $\varphi_{\rm ext} \in C^2$, this potential does not fit hypothesis H2) since $\Delta\Phi_{\rm ext}$ is not bounded away from 0 near R_0.

Let us give now examples which do not fit the regularity hypotheses required in Proposition 2.14. These expressions are justified through the characterization (iii) in Theorem 2.7 and simple computation of $\Gamma \star n_{\rm e}$.

Example 3 (radial): Take the potential $\varphi_{\rm ext}(r) = r$, that is $\Phi_{\rm ext}(x) = |x|$. Then, $\Delta\Phi_{\rm ext} = (N-1)/r > 0$, $n_{\rm e} = (N-1)|x|^{-1}\mathbf{1}_{B(0,R)}$, with $N|B(0,1)|R^{N-1} = \mathfrak{m}$, and $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,R) = \{\Phi_{\rm e} = 0\}$.

Example 5 (radial): Consider the potential $\varphi_{\rm ext}(r) = r + \max(r-1,0)$:
- if $\mathfrak{m} \le N|B(0,1)|$, then $n_{\rm e} = (N-1)|x|^{-1}\mathbf{1}_{B(0,R)}$, with $R = (\mathfrak{m}/N|B(0,1)|)^{1/N-1}$ and $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,R) = \{\Phi_{\rm e} = 0\}$;
- if $N|B(0,1)| \le \mathfrak{m} \le 2N|B(0,1)|$, then $n_{\rm e} = (N-1)|x|^{-1}\mathbf{1}_{B(0,1)} + (\mathfrak{m} - N|B(0,1)|)\delta_{\partial B(0,1)}$ and $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,1) = \{\Phi_{\rm e} = 0\}$;
- if $\mathfrak{m} \ge 2N|B(0,1)|$, then $n_{\rm e} = (N-1)|x|^{-1}\mathbf{1}_{B(0,1)} + N|B(0,1)|\delta_{\partial B(0,1)} + 2(N-1)|x|^{-1}\mathbf{1}_{B(0,R)\setminus B(0,1)}$, where $R \ge 1$ is such that $2N|B(0,1)|(R^{N-1}-1) + N|B(0,1)| = \mathfrak{m}$, and $\mathrm{Supp}(n_{\rm e}) = \bar{B}(0,R) = \{\Phi_{\rm e} = 0\}$.

Since we assume $\varphi_{\rm ext}$ convex and with 0 as a minimum point, it follows that $\varphi_{\rm ext}$ has a right-derivative at 0, hence the singularity in $1/|x|$ at the origin for $n_{\rm e}$ is the worst we can have. The radial Example 5 also shows that we may have Dirac masses on a sphere (of positive radius).

Our next results guarantees that $\mathcal{K}$ has non empty interior when the confining potential Φ_{ext} is C^1 and convex.

Proposition 2.16 *We assume that* 0 *is a minimum point of* Φ_{ext} *and that the potential* Φ_{ext} *is of class* C^1 *and convex. Then, there exists* $r_0 > 0$ *such that* $B_{r_0}(0) \subset \mathcal{K}$*. In particular,* $\mathcal{K}$ *has non empty interior.*

Proof. We follow the argument of [24, Chapter 5, Theorem 6.2], where we work on $h \stackrel{\text{def}}{=} \Gamma \star n_{\text{e}}$ and shall use that it is a solution to the obstacle problem given in Proposition 2.9 with the obstacle $\psi = C_* - \Phi_{\text{ext}}$.

We first consider the case $N \geq 3$ and notice that $\text{Supp}(n_{\text{e}})$ has a positive capacity: we fix some $a \in \text{Supp}(n_{\text{e}})$ such that $C_* = h(a) + \Phi_{\text{ext}}(a)$ (see (22)). Now, since $N \geq 3$, we observe that (with $c_N > 0$) $h(a) = \Gamma \star n_{\text{e}}(a) = c_N |\cdot|^{2-N} \star n_{\text{e}} > 0$ and that 0 is actually a global minimum point of Φ_{ext}, thus $C_* > \Phi_{\text{ext}}(a) \geq \Phi_{\text{ext}}(0)$ and it follows that $\psi(0) = C_* - \Phi_{\text{ext}}(0) > 0$. On the other hand, $h(x) \sim \mathfrak{m}\Gamma(x)$ tends to $0 < \psi(0)$ at infinity, thus there exists an $R_0 > 0$ such that $h(x) \leq \psi(0)/2$ when $|x| \geq R_0$. For x_0 that will be close to 0, we let $v(x) \stackrel{\text{def}}{=} \psi(x_0) + (x - x_0) \cdot \nabla\psi(x_0)$ be the affine tangent to ψ at x_0. Since ψ is concave (Φ_{ext} is convex), we have $\psi \leq v$ in $\mathbb{R}^N$. Furthermore, if x_0 is sufficiently close to 0 (depending on R_0), then $\nabla\psi(x_0)$ is small (since ψ is C^1 and achieves a minimum at 0) and thus $v > \psi(0)/2 > 0$ on $\partial B(0, R_0)$. Since $\Delta v \equiv 0$, we may now apply [24, Chapter 4, Theorem 8.3] to infer $h \leq v$ in $B(0, R_0)$ (this is a maximum type principle proved using the comparison function $g \stackrel{\text{def}}{=} \min(h, v)\mathbf{1}_{B(0,R_0)} + h\mathbf{1}_{\mathbb{R}^N \setminus B(0,R_0)}$ in the formulation of the obstacle problem given in Proposition 2.9). In particular, $\psi(x_0) \leq h(x_0) \leq v(x_0) = \psi(x_0)$, which means that, as wished, $x_0 \in \mathcal{K}$.

Let us now turn to the dimensions $N = 2$ and $N = 1$. Then, it may happen that $\psi(0) \leq 0$, but since $h(x) \sim \mathfrak{m}\Gamma(x)$ tends to $-\infty < \psi(0)$ at infinity, the previous argument still applies. $\square$

If one is able to prove that $\mathcal{K}$ is convex and assuming that Φ_{ext} satisfies H2), then H1) is automatically true. Indeed, any point of $\partial\mathcal{K}$ has then a positive density and we may then apply the regularity result of L. Caffarelli (see *e.g.*, [14, Chapter 2, Theorem 3.10]) which ensures that $\partial\Omega$ is of class C^1 (hence C^{s+1} by H2)).

We conclude with a result from [20, Theorem 3.24] on the topology of $\mathcal{K}$ valid only in space dimension two (the proof uses complex analysis).

Proposition 2.17 ([20]) *We assume* $N = 2$*. Suppose that* Φ_{ext} *is of class* C^2 *and that its Hessian is everywhere positive definite. Then,* $\text{supp}(n_{\text{e}})$ *is simply connected, and equal to the closure of its interior. Moreover, if* Φ_{ext} *is* $C^{2,\alpha}$ *for some* $\alpha \in]0, 1[$*, then* $\partial\mathcal{K}$ *is a* $C^{1,\beta}$ *Jordan curve, for some* $\beta \in]0, 1[$.

The above result does not prevent cusps in $\partial\mathcal{K}$, but just says that the boundary $\partial\mathcal{K}$ possesses a $C^{1,\beta}$ parametrization.

3 Asymptotic analysis

This section is devoted to the analysis of the asymptotic regime $\varepsilon \to 0$. We shall point out the difficulties and necessary adaptations between the case of quadratic potentials, Theorem 1.2 and Theorem 2.6, and the general case, Theorem 2.13. For the existence theory of the Vlasov–Poisson equation, we refer the reader to [1] for weak solutions and more recently to [28, 35] where strong solutions and regularity issues are discussed. Further details and references can be found in the survey [18].

3.1 A useful estimate on $\Phi_{\rm e}$

Before we turn to the analysis of the asymptotic regime $\varepsilon \to 0$, it is convenient to set up an estimate that describes the behavior of $\Phi_{\rm e}$ close to the neighborhood of $\partial\mathcal{K}$. In the isotropic case, $\Phi_{\rm ext}$ being given by (2), the potential $\Phi_{\rm e}$ is defined by (9), and we observe that there exists $C > 0$ such that

$$0 \leq (|x| - R)\ |\nabla_x \Phi_{\rm e}(x)| \leq C\,\Phi_{\rm e}(x) \tag{28}$$

holds for any x with $|x| \geq R$. More generally, for a quadratic potential (12), we can establish the following property, based on the formulas in Section 2.1.

Lemma 3.1 *Let $\Phi_{\rm e}$ be the quadratic potential defined as in Corollary 2.5. Let $\mathscr{V} : \mathbb{R}^N \to \mathbb{R}^N$ be smooth, compactly supported and such that $\mathscr{V} \cdot \nu\big|_{\partial\mathcal{K}_a} = 0$. Then, there exists a positive constant C, depending only on N, $\Phi_{\rm ext}$ and $\mathscr{V}$ such that we have, for any $x \in \mathbb{R}^N$,*

$$|\mathscr{V} \cdot \nabla\Phi_{\rm e}(x)| \leq C\Phi_{\rm e}(x). \tag{29}$$

Proof. Since $\mathscr{V}$ is compactly supported and $\Phi_{\rm e}$ is positive in $\{\sigma_a > 0\}$, we just need to prove the inequality for x close to $\partial\mathcal{K}_a$, that is for $\sigma_a(x)$ small. We still define $\lambda > 0$ so that $\lambda^{-2} = \sum_{j=1}^N \lambda_j^{-2}$. From (20), and by Taylor expansion of the integral, we infer that for $0 < \sigma_a(x) \ll 1$ and $1 \leq k \leq N$,

$$\lambda^2 \partial_k \Phi_{\rm e}(x) = \frac{x_k}{2}\left(\prod_{j=1}^N a_j\right)\left(\sigma_a(x)\frac{1}{a_k^2}\left(\prod_{j=1}^N a_j^2\right)^{-1/2} + \mathcal{O}(\sigma_a^2(x))\right) = \frac{x_k\sigma_a(x)}{2a_k^2} + \mathcal{O}(\sigma_a^2(x)).$$

Let $\mathscr{X}(x)$ stands for the vector with components x_k/a_k^2. In particular, for $0 < \sigma_a(x) \ll 1$, we get

$$|\nabla\Phi_{\rm e}(x)| = \mathcal{O}(\sigma_a(x)) \qquad \text{and} \qquad \frac{\nabla\Phi_{\rm e}(x)}{|\nabla\Phi_{\rm e}(x)|} = \frac{\mathscr{X}(x)}{|\mathscr{X}(x)|} + \mathcal{O}(\sigma_a(x)), \tag{30}$$

where the unit vector field $x \mapsto \frac{\mathscr{X}(x)}{|\mathscr{X}(x)|}$ is smooth near $\partial\mathcal{K}_a$ and is the (outward) normal on $\partial\mathcal{K}_a$. Now, observe that

$$\begin{aligned} 0 = \partial_k\left(\sum_{j=1}^N \frac{x_j^2}{a_j^2 + \sigma_a(x)}\right) &= 2\frac{x_k}{a_k^2 + \sigma_a(x)} - \left(\sum_{j=1}^N \frac{x_j^2}{(a_j^2 + \sigma_a(x))^2}\right)\partial_k\sigma_a(x) \\ &= 2\frac{x_k}{a_k^2} - \partial_k\sigma_a(x)\left(\sum_{j=1}^N \frac{x_j^2}{a_j^4}\right) + \mathcal{O}(\sigma_a(x)). \end{aligned}$$

Therefore, for $0 < \sigma_a(x) \ll 1$ and $1 \leq k \leq N$, we have

$$\lambda^2\partial_k\Phi_{\rm e}(x) = \frac{1}{4}\sigma_a(x)\partial_k\sigma_a(x)\left(\sum_{j=1}^N \frac{x_j^2}{a_j^4}\right) + \mathcal{O}(\sigma_a^2(x)) = \frac{1}{8}\partial_k\left(\sigma_a^2(x)\left(\sum_{j=1}^N \frac{x_j^2}{a_j^4}\right)\right) + \mathcal{O}(\sigma_a^2(x)).$$

As a consequence,

$$\lambda^2\Phi_{\rm e}(x) = \frac{1}{8}\sigma_a^2(x)\left(\sum_{j=1}^N \frac{x_j^2}{a_j^4}\right) + \mathcal{O}(\sigma_a^3(x)) \geq \frac{\sigma_a^2(x)}{C}, \tag{31}$$

holds for some $C > 0$. Going back to (30), we arrive at

$$\begin{aligned} \mathscr{V}(x)\cdot\nabla\Phi_{\rm e}(x) = \mathscr{V}(x)\cdot\left(\frac{\nabla\Phi_{\rm e}(x)}{|\nabla\Phi_{\rm e}(x)|}\right) \times |\nabla\Phi_{\rm e}(x)| &= \mathscr{V}(x)\cdot\left(\frac{\mathscr{X}(x)}{|\mathscr{X}(x)|} + \mathcal{O}(\sigma_a(x))\right) \times \mathcal{O}(\sigma_a(x)) \\ &= (\mathcal{O}(\sigma_a(x)) + \mathcal{O}(\sigma_a(x))) \times \mathcal{O}(\sigma_a(x)) = \mathcal{O}(\sigma_a^2(x)) = \mathcal{O}(\Phi_{\rm e}(x)), \end{aligned}$$

by (31) and since $\mathscr{V} \cdot \frac{\mathscr{X}}{|\mathscr{X}|}$ vanishes when $\sigma_a = 0$ in view of the no flux condition satisfied by $\mathscr{V}$. This finishes the proof. $\square$

In the more general setting considered in Theorem 2.13, the result is the following and simply relies on the use of a local chart.

Lemma 3.2 *We assume that $\partial\Omega$ is of class C^1 and that H2) is satisfied. Then, there exists a constant C such that, for any $x \in \mathbb{R}^N$,*

$$|\mathscr{V} \cdot \nabla\Phi_{\mathrm{e}}(x)| \leq C\Phi_{\mathrm{e}}(x). \tag{32}$$

Proof. We have already seen that $\partial\Omega$ is actually of class C^{s+1}. Since Φ_{e} is positive in $\mathbb{R}^N \setminus \mathcal{K}$ and $\mathscr{V}$ has compact support, by a compactness argument, it suffices to show that (32) holds near any point $a \in \partial\Omega$. Possibly translating and rotating the axis, we assume $a = 0$ and that the inward normal to Ω at $a = 0$ is $e_1 = (1, 0, ..., 0)$. We let $x_1 = \Theta(x_\perp)$, where $x_\perp = (x_2, ..., x_N)$, be a C^2 parametrization of $\partial\Omega$ near 0, with $\nabla\Theta(0) = 0$, hence $\Theta(x_\perp) = \mathcal{O}(|x_\perp|^2)$.

We now consider the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ defined by $\varphi(y) \overset{\text{def}}{=} \Phi_{\mathrm{e}}(y_1 + \Theta(y_\perp), y_\perp)$, where $y_\perp = (y_2, ..., y_N) \in \mathbb{R}^{N-1}$. Then, $\varphi(y) = 0$ when $y_1 \geq 0$, hence, for $2 \leq j \leq N$ and $1 \leq k \leq N$ and if $y_1 = 0$, $\partial_k\varphi(y) = \partial^2_{j,k}\varphi(y) = 0$; moreover, $\partial^2_{1,1}\varphi(0, y_\perp) = \Delta\Phi_{\mathrm{ext}}(0, y_\perp)$ in view of the equality $\Delta\Phi_{\mathrm{ext}}(x) = \Delta_x\Phi_{\mathrm{e}}(x) = (\Delta_y\varphi - (\Delta_\perp\Theta)\partial_1\varphi + \sum_{j=2}^N(\partial_j\Theta)^2\partial^2_{1,j}\varphi)(x_1 - \Theta(x_\perp), x_\perp)$ in $\{x_1 \leq \Theta(x_\perp)\}$.

It follows from these relations that, by the Taylor formula and by using $\Delta\Phi_{\mathrm{ext}}(0) > 0$ and $y_1 = x_1 - \Theta(x_\perp) \leq 0$,

$$\varphi(y) = \varphi(y) - \varphi(0, y_\perp) - y_1\partial_1\varphi(0, y_\perp) = y_1^2\int_0^1 (1-t)\partial_1^2\varphi(ty_1, y_\perp)\ \mathrm{d}t \geq \frac{y_1^2}{C},$$

and we deduce

$$\Phi_{\mathrm{e}}(x) \geq \frac{(x_1 - \Theta(x_\perp))^2}{C}. \tag{33}$$

Still by the Taylor formula, we have, for $2 \leq j \leq N$,

$$\partial_j\varphi(y) = y_1^2\int_0^1 (1-t)\partial^3_{1,1,j}\varphi(ty_1, y_\perp)\ \mathrm{d}t = \mathcal{O}(y_1^2)$$

and

$$\partial_1\varphi(y) = y_1\partial_1^2\varphi(0, y_\perp) + y_1^2\int_0^1 (1-t)\partial_1^3\varphi(ty_1, y_\perp)\ \mathrm{d}t = y_1\Delta\Phi_{\mathrm{ext}}(0, y_\perp) + \mathcal{O}(y_1^2).$$

Now, we write $\partial_1\Phi_{\mathrm{e}}(x) = \partial_1\varphi(y)$ (with $y = (x_1 - \Theta(x_\perp), x_\perp)$) and $\nabla_\perp\Phi_{\mathrm{e}}(x) = \nabla_\perp\varphi(y) - \partial_1\varphi(y)\nabla_\perp\Theta(y_\perp)$, thus

$$\mathscr{V}(x)\cdot\nabla\Phi_{\mathrm{e}}(x) = \mathscr{V}_1(x)\partial_1\Phi_{\mathrm{e}}(x) + \mathscr{V}_\perp(x)\cdot\nabla_\perp\Phi_{\mathrm{e}}(x) = \mathscr{V}_1(x)\partial_1\varphi(y) + \mathscr{V}_\perp(x)\cdot\nabla_\perp\varphi(y) - \partial_1\varphi(y)\mathscr{V}_\perp(x)\cdot\nabla_\perp\Theta(y_\perp).$$

Note that $\nabla_\perp\varphi(y) = \mathcal{O}(y_1^2)$. Furthermore, since $\mathscr{V} \cdot \nu = 0$ on $\partial\Omega = \{x_1 = \Theta(x_\perp)\}$ and $\nu(x) = (1, -\nabla_\perp\Theta(x_\perp))/|(1, -\nabla_\perp\Theta(x_\perp))|$, we deduce

$$\begin{aligned}\mathscr{V}(x) \cdot \nabla\Phi_{\mathrm{e}}(x) &= \mathcal{O}(y_1^2) + \partial_1\varphi(y)\Big([\mathscr{V}_1(x) - \mathscr{V}_\perp(x) \cdot \nabla_\perp\Theta(x_\perp)] - [\mathscr{V}_1(\Theta(x_\perp), x_\perp) - \mathscr{V}_\perp(\Theta(x_\perp), x_\perp) \cdot \nabla_\perp\Theta(x_\perp)]\Big)\\ &= \mathcal{O}(y_1^2) + \mathcal{O}(|y_1|) \times \mathcal{O}(|x_1 - \Theta(x_\perp)|) = \mathcal{O}(y_1^2).\end{aligned}$$

We conclude by using (33). $\square$

3.2 Basic a priori estimates

Now that we have in hand the limiting density profile $n_{\rm e}$ and the associated potential field $\Phi_{\rm e}$, we derive some basic a priori estimates from (V)–(P).

Using the splitting of Poisson equation as in (10), (V) recasts as

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon - \frac{1}{\varepsilon}\nabla_x\Phi_{\rm e}\cdot\nabla_v f_\varepsilon - \frac{1}{\sqrt{\varepsilon}}\nabla_x\Psi_\varepsilon\cdot\nabla_v f_\varepsilon = 0.$$

Let us compute the time variation of the following energies:

- Kinetic energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \frac{|v|^2}{2}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{\varepsilon}\iint v\cdot\nabla_x\Phi_{\rm e}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{\sqrt{\varepsilon}}\iint v\cdot\nabla_x\Psi_\varepsilon\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x,$$

- Leading order potential energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \Phi_{\rm e}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = \iint v\cdot\nabla_x\Phi_{\rm e}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x,$$

- Fluctuations potential energy

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int |\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x &= \int \nabla_x\Psi_\varepsilon\cdot\partial_t\nabla_x\Psi_\varepsilon\,\mathrm{d}x = -\int \Psi_\varepsilon\partial_t\Big(\frac{n_{\rm e}-\rho_\varepsilon}{\sqrt{\varepsilon}}\Big)\,\mathrm{d}x\\ &= -\int \Psi_\varepsilon\frac{1}{\sqrt{\varepsilon}}\nabla_x\cdot\Big(\int vf_\varepsilon\,\mathrm{d}v\Big)\,\mathrm{d}x = \frac{1}{\sqrt{\varepsilon}}\iint v\cdot\nabla_x\Psi_\varepsilon\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

By summing these relations, we conclude with the following claim (which applies for all three cases for $\Phi_{\rm ext}$).

Proposition 3.3 *The solution $(f_\varepsilon, \Phi_\varepsilon = \frac{1}{\varepsilon}\Phi_{\rm e} + \frac{1}{\sqrt{\varepsilon}}\Psi_\varepsilon)$ of* (V)–(P) *satisfies the following energy conservation equality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\iint \frac{|v|^2}{2}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\iint \Phi_{\rm e}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int |\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x\Big\} = 0.$$

Furthermore, the total charge is conserved

$$\iint f_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint f_\varepsilon(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m}.$$

3.3 Convergence of the density and the current

We assume a uniform bound on the energy at the initial time, namely

$$\sup_{0<\varepsilon<1}\iint \frac{1}{2}|v|^2\ f_\varepsilon^{\rm init}\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\iint \Phi_{\rm e}\ f_\varepsilon^{\rm init}\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int |\nabla_x\Psi_\varepsilon^{\rm init}|^2\,\mathrm{d}x < \infty, \tag{34}$$

where $\Psi_\varepsilon^{\rm init}$ solves the Poisson equation (10). Then, Proposition 3.3 ensures that the energy remains uniformly bounded for positive times. Thus, possibly at the price of extracting subsequences, we can suppose that

$$f_\varepsilon \rightharpoonup f \text{ weakly-}\star \text{ in } \mathscr{M}^1([0,T]\times\mathbb{R}^N\times\mathbb{R}^N),\quad \rho_\varepsilon = \int f_\varepsilon\,\mathrm{d}v \rightharpoonup \rho \text{ weakly-}\star \text{ in } \mathscr{M}^1([0,T]\times\mathbb{R}^N).$$

Going back to the Poisson equation, we observe that

$$n_{\rm e} - \rho_\varepsilon = \sqrt{\varepsilon}\nabla_x\cdot(\nabla_x\Psi_\varepsilon)$$

where, by Proposition 3.3, $\nabla_x\Psi_\varepsilon$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))$. Consequently, we establish the following claim.

Lemma 3.4 *The sequence ρ_ε converges to $n_{\rm e} = \rho$ strongly in $L^\infty(0,T;H^{-1}(\mathbb{R}^N))$ and weakly-$\star$ in $\mathscr{M}^1([0,T]\times\mathbb{R}^N)$. The limit f is supported in $[0,T]\times\bar\Omega\times\mathbb{R}^N$. The sequence $J^\varepsilon = \int v f_\varepsilon \,\mathrm{d}v$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$; it admits a subsequence which converges, say weakly$-\star$ in $\mathscr{M}^1([0,T]\times\mathbb{R}^N)$; the limit J is divergence free, supported in $[0,T]\times\bar\Omega$ and may be written $\int v f \,\mathrm{d}v = n_{\rm e} W$ for some $W \in \mathscr{M}^1([0,T]\times\mathbb{R}^N)$.*

Proof. Proposition 3.3 tells us that $|v|^2 f_\varepsilon$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))$. Hence, by using Cauchy-Schwarz' inequality, we get

$$\int |J_\varepsilon| \,\mathrm{d}x \le \iint |v| \sqrt{f_\varepsilon}\,\sqrt{f_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x \le \left(\iint |v|^2 \, f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\right)^{1/2} \left(\iint f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}, \tag{35}$$

which leads to the asserted uniform estimate on the current. We can thus also assume $J_\varepsilon \rightharpoonup J$ weakly$-\star$ in $\mathscr{M}^1([0,T]\times\mathbb{R}^N)$. Furthermore, since the second order moment in v of f_ε is uniformly bounded, we check that

$$\rho = \int f \,\mathrm{d}v, \qquad J = \int v \; f \,\mathrm{d}v.$$

Note that ρ_ε and J_ε satisfy (5). Letting ε go to 0 yields

$$\partial_t\rho + \nabla_x\cdot J = 0 = \partial_t n_{\rm e} + \nabla_x\cdot J = 0 + \nabla_x\cdot J = 0.$$

Thus, J is divergence-free. Finally, since $\lim_{|x|\to\infty}\Phi_{\rm e}(x) = +\infty$ and the second order moment in v of f_ε is uniformly bounded, $\{f_\varepsilon,\, \varepsilon > 0\}$ is tight, and we can write

$$\begin{aligned}
\int_0^T \iint f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \mathfrak{m}T = \int_0^T \int \rho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t& \\
\xrightarrow[\varepsilon\to 0]{} \int_0^T \iint f \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \mathfrak{m}T &= \int_0^T \int \rho \,\mathrm{d}x\,\mathrm{d}t = T\int n_{\rm e}\,\mathrm{d}x \\
&= \int_0^T \iint_{\overline{\Omega}} f \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \iint_{\mathbb{R}^N\setminus\overline{\Omega}} f \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\overline{\Omega}} n_{\rm e}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\overline{\Omega}} \rho\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \iint_{\overline{\Omega}} f \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega} n_{\rm e}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\Omega} \rho\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \iint_{\Omega} f \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

It proves that $\mathrm{supp}(f) \subset [0,T]\times\overline{\Omega}\times\mathbb{R}^N$, and thus $\mathrm{supp}(J) \subset [0,T]\times\overline{\Omega}$. In particular, we note that $f([0,T]\times\partial\Omega\times\mathbb{R}^N) = 0$, and $J([0,T]\times\partial\Omega) = 0$. $\square$

In order to define the normal trace of J over $\partial\Omega$ (that is the sphere $\partial B(0,R)$ in the case (2)), we shall use the theory introduced in [8]. As a consequence of the discussion above, we start by observing that J belongs to the set $\mathscr{DM}^{\rm ext}(\mathbb{R}^N)$ of extended divergence-measure fields over $\mathbb{R}^N$, see [8, Definition 1.1]. Therefore, according to [8, Theorem 3.1], J admits a normal trace $J\cdot\nu\big|_{\partial\Omega}$ defined as a continuous linear functional over $Lip(\gamma,\partial\Omega)$, $\gamma > 1$ (see [8, Equation (2.1)]) with

$$\big\langle J\cdot\nu\big|_{\partial\Omega}, \phi\big\rangle = \int_\Omega \hat\phi\nabla_x\cdot J + \int_\Omega J\cdot\nabla_x\hat\phi,$$

where the function $\hat\phi \in Lip(\gamma,\Omega)$ in the right-hand side is an extension of $\phi \in Lip(\gamma,\partial\Omega)$. However, by $\nabla_x\cdot J = 0$ and the support property on J, we can rewrite

$$\big\langle J\cdot\nu\big|_{\partial\Omega}, \phi\big\rangle = 0 + \int_{\mathbb{R}^N} J\cdot\nabla_x\hat\phi = -\big\langle \nabla_x\cdot J, \hat\phi\big\rangle = 0.$$

Another way to see this is to observe that the normal trace from Ω must be the same as the normal trace from $\mathbb{R}^N \setminus \overline{\Omega}$, which is clearly zero since J has support in $\overline{\Omega}$. Consequently,

$$J \cdot \nu\big|_{\overline{\Omega}} = 0 \qquad \text{in} \quad [C(0,T; Lip(\gamma, \overline{\Omega}))]^*.$$

Remark that this is not a pointwise relation. In particular, it may happen that $J_\varepsilon^{\mathrm{init}} \cdot \nu\big|_{\overline{\Omega}} = \rho_\varepsilon V_\varepsilon^{\mathrm{init}} \cdot \nu\big|_{\overline{\Omega}}$ is nonzero, but this does not prevent the time integral of $J \cdot \nu\big|_{\overline{\Omega}}$ to vanish.

3.4 Passing to the limit: modulated energy

We now study the modulated energy

$$\mathscr{H}_{\mathscr{V},\varepsilon} = \frac{1}{2} \iint |v - \mathscr{V}|^2 \; f_\varepsilon \, \mathrm{d}v \, \mathrm{d}x + \frac{1}{2} \int |\nabla_x \Psi_\varepsilon|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \iint \Phi_{\mathrm{e}} \; f_\varepsilon \, \mathrm{d}v \, \mathrm{d}x,$$

where all the terms integrated are nonnegative. Let us compute as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}_{\mathscr{V},\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}t} \iint \Big(\frac{|\mathscr{V}|^2}{2} - \mathscr{V} \cdot v \Big) f_\varepsilon \, \mathrm{d}v \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int \Big(\rho_\varepsilon \frac{|\mathscr{V}|^2}{2} - \mathscr{V} \cdot J_\varepsilon \Big) \, \mathrm{d}x,$$

by using Proposition 3.3. We thus have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}_{\mathscr{V},\varepsilon} = \int (\rho_\varepsilon \mathscr{V} - J_\varepsilon) \cdot \partial_t \mathscr{V} \, \mathrm{d}x + \int \frac{|\mathscr{V}|^2}{2} \partial_t \rho_\varepsilon \, \mathrm{d}x - \int \mathscr{V} \cdot \partial_t J_\varepsilon \, \mathrm{d}x.$$

Here, we are assuming that the solution f_ε of the Vlasov–Poisson system (V)–(P) is regular enough so that we can perform all the calculations that follow. Integrating the Vlasov equation, we obtain

$$\partial_t J_\varepsilon + \nabla_x \cdot \mathbb{P}_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \rho_\varepsilon \nabla_x \Psi_\varepsilon + \frac{1}{\varepsilon} \rho_\varepsilon \nabla_x \Phi_{\mathrm{e}} = 0, \tag{36}$$

where we rewrite

$$\frac{1}{\sqrt{\varepsilon}} \rho_\varepsilon \nabla_x \Psi_\varepsilon = \frac{\rho_\varepsilon - n_{\mathrm{e}}}{\sqrt{\varepsilon}} \nabla_x \Psi_\varepsilon + \frac{n_{\mathrm{e}}}{\sqrt{\varepsilon}} \nabla_x \Psi_\varepsilon = -\Delta_x \Psi_\varepsilon \nabla_x \Psi_\varepsilon + \frac{n_{\mathrm{e}}}{\sqrt{\varepsilon}} \nabla_x \Psi_\varepsilon,$$

and

$$\Delta_x \Psi_\varepsilon \nabla_x \Psi_\varepsilon = \nabla_x \cdot \big(\nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon \big) - \nabla_x \Big(\frac{|\nabla_x \Psi_\varepsilon|^2}{2} \Big).$$

Combining these relations to the charge conservation (5) and integration by parts, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}_{\mathscr{V},\varepsilon} = & \int (\rho_\varepsilon \mathscr{V} - J_\varepsilon) \cdot \partial_t \mathscr{V} \, \mathrm{d}x + \int J_\varepsilon \cdot \nabla_x \left(\frac{|\mathscr{V}|^2}{2} \right) \mathrm{d}x \\
& - \int D_x \mathscr{V} : (\mathbb{P}_\varepsilon - \nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon) \, \mathrm{d}x + \int \nabla_x \cdot \mathscr{V} \frac{|\nabla_x \Psi_\varepsilon|^2}{2} \, \mathrm{d}x \\
& + \frac{1}{\varepsilon} \int \rho_\varepsilon \mathscr{V} \cdot \nabla_x \Phi_{\mathrm{e}} \, \mathrm{d}x + \int \frac{n_{\mathrm{e}}}{\sqrt{\varepsilon}} \mathscr{V} \cdot \nabla_x \Psi_\varepsilon \, \mathrm{d}x,
\end{aligned}$$

where $D_x \mathscr{V}$ stands for the jacobian matrix of the vector field $\mathscr{V}$. For the last integral, since n_{e} is supported in Ω, we write it as

$$\int_\Omega \frac{n_{\mathrm{e}}}{\sqrt{\varepsilon}} \mathscr{V} \cdot \nabla_x \Psi_\varepsilon \, \mathrm{d}x = 0$$

by integration by parts and using that $\nabla_x \cdot (n_{\mathrm{e}} \mathscr{V}) = 0$ in Ω and the no-flux condition (6).

Let us set

$$\mathbb{P}_{\mathscr{V},\varepsilon} \stackrel{\text{def}}{=} \int (v-\mathscr{V})\otimes(v-\mathscr{V})\ f_\varepsilon\,\mathrm{d}v = \mathbb{P}_\varepsilon - \mathscr{V}\otimes J_\varepsilon - J_\varepsilon\otimes\mathscr{V} + \rho_\varepsilon \mathscr{V}\otimes\mathscr{V}.$$

A direct substitution leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\mathscr{V},\varepsilon} &= \int (\rho_\varepsilon\mathscr{V} - J_\varepsilon)\cdot\big(\partial_t\mathscr{V} + (\mathscr{V}\cdot\nabla_x)\mathscr{V}\big)\,\mathrm{d}x \\ &\quad - \int D_x\mathscr{V} : (\mathbb{P}_{\mathscr{V},\varepsilon} - \nabla_x\Psi_\varepsilon\otimes\nabla_x\Psi_\varepsilon)\,\mathrm{d}x \\ &\quad + \int \nabla_x\cdot\mathscr{V}\frac{|\nabla_x\Psi_\varepsilon|^2}{2}\,\mathrm{d}x + \frac{1}{\varepsilon}\int\rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x.\end{aligned}\tag{37}$$

We shall use the shorthand notation $A \lesssim B$ when the inequality $A \le CB$ holds for some constant $C>0$, the value of which might vary from a line to another. As a matter of fact, we can dominate the second and third integrals of the right-hand side by

$$\|D_x\mathscr{V}\|_\infty\left(\iint|v-\mathscr{V}|^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \int|\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x\right) \le \|D_x\mathscr{V}\|_\infty\ \mathscr{H}_{\mathscr{V},\varepsilon}.$$

Let us distinguish the case of the isotropic potential in order to point out the difficulties. When Φ_{ext} is given by (2), we remind the reader that Φ_{e} is supported in $\{|x|\ge R\}$, radially symmetric and increasing in $|x|$, see (9). Combining this with (28) allows us to estimate the last term in (37) as follows:

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int\rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x\right| &= \left|\frac{1}{\varepsilon}\int_{\{|x|>R\}}\rho_\varepsilon\frac{\mathscr{V}\cdot x/|x|}{|x|-R}\ (|x|-R)|\nabla_x\Phi_{\mathrm{e}}|\,\mathrm{d}x\right| \\ &\lesssim \frac{1}{\varepsilon}\int_{\{|x|>R\}}\rho_\varepsilon\Phi_{\mathrm{e}}\,\mathrm{d}x\ \left\|\frac{\mathscr{V}\cdot x/|x|}{|x|-R}\right\|_\infty \lesssim \mathscr{H}_{\mathscr{V},\varepsilon},\end{aligned}$$

where we have used that $\frac{\mathscr{V}\cdot x/|x|}{|x|-R}$ belongs to $L^\infty(\mathbb{R}^N)$ since $\mathscr{V}$ is smooth, compactly supported, and $\mathscr{V}\cdot\nu = 0$ on $\partial B(0,R)$. For a quadratic external potential (12), we can proceed similarly by using Lemma 3.1. When dealing with a general potential, we made hypothesis H2) so that Lemma 3.2 applies and (29) allows us to estimate

$$\left|\frac{1}{\varepsilon}\int\rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x\right| \lesssim \frac{1}{\varepsilon}\int\rho_\varepsilon\Phi_{\mathrm{e}}\,\mathrm{d}x \lesssim \mathscr{H}_{\mathscr{V},\varepsilon}.$$

Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\mathscr{V},\varepsilon} \lesssim \mathscr{H}_{\mathscr{V},\varepsilon} + r_\varepsilon \tag{38}$$

where we have set

$$r_\varepsilon \stackrel{\text{def}}{=} \int(\rho_\varepsilon\mathscr{V} - J_\varepsilon)\cdot\big(\partial_t\mathscr{V} + (\mathscr{V}\cdot\nabla_x)\mathscr{V}\big)\,\mathrm{d}x.$$

The Grönwall lemma yields

$$\mathscr{H}_{\mathscr{V},\varepsilon}(t) \le e^{Ct}\left(\mathscr{H}_{\mathscr{V},\varepsilon}(0) + \int_0^t e^{-C\tau} r_\varepsilon(\tau)\,\mathrm{d}\tau\right),$$

for a certain constant $C>0$. The assumption (11) on the initial data is that $\lim_{\varepsilon\to0}\mathscr{H}_{\mathscr{V},\varepsilon}(0) = 0$. Hence, we are left with the task of proving that $\int_0^t r_\varepsilon(\tau)\,\mathrm{d}\tau$ tends to 0 as $\varepsilon\to0$. We have

$$\begin{aligned}\int_0^t r_\varepsilon(\tau)\,\mathrm{d}\tau \quad \xrightarrow[\varepsilon\to0]{} \quad & \int_0^t\int(n_{\mathrm{e}}\mathscr{V} - J)\cdot\big(\partial_t\mathscr{V} + (\mathscr{V}\cdot\nabla_x)\mathscr{V}\big)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_0^t\int_\Omega(n_{\mathrm{e}}\mathscr{V} - J)\cdot\big(\partial_t\mathscr{V} + (\mathscr{V}\cdot\nabla_x)\mathscr{V}\big)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= -\int_0^t\int_\Omega(n_{\mathrm{e}}\mathscr{V} - J)\cdot\nabla_x p\,\mathrm{d}x\,\mathrm{d}\tau = 0,\end{aligned}$$

since $n_e\mathscr{V}$ and J are divergence free on Ω and their normal trace vanish. $\square$

It is worth pointing out that the regularity assumption of the sequence of solutions f_ε was only made to justify the computations leading to (38). If one consider less regular solutions, we have to assume that these solutions were constructed through a regularization procedure and that the previous calculations were done on these regularizations and hence (38) will still hold.

3.5 Identification of the limit

Let us observe that if the initial datum satisfies (11), then (34) holds true. Let us first justify i): we shall show that $\int \rho_\varepsilon \chi \, \mathrm{d}x \to \int n_e \chi \, \mathrm{d}x$ uniformly on $[0,T]$ as $\varepsilon \to 0$ for any $\chi \in C_0^0(\mathbb{R}^N)$. We start by observing that

$$\left| \int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x \right| \leq \mathfrak{m}\|\chi\|_\infty \tag{39}$$

holds for any $\chi \in C_0^0(\mathbb{R}^N)$. Next, consider $\chi \in C_c^1(\mathbb{R}^N)$. The charge conservation (5) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x = \int \partial_t\rho_\varepsilon \chi\,\mathrm{d}x = -\int \nabla_x \cdot J_\varepsilon\,\chi\,\mathrm{d}x = \int J_\varepsilon \cdot \nabla_x\chi\,\mathrm{d}x,$$

hence the uniform bound (35) on J_ε implies a uniform bound on $\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x$ for $0 \leq t \leq T$. By virtue of the Ascoli-Arzelà theorem, the set $\left\{t \mapsto \int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x,\ \varepsilon > 0\right\}$ is therefore relatively compact in $C([0,T])$ for any fixed $\chi \in C_c^1(\mathbb{R}^N)$. This property extends to any $\chi \in C_0^0(\mathbb{R}^N)$ by virtue of (39). Indeed, for any $\delta > 0$, we can pick $\chi_\delta \in C_c^1(\mathbb{R}^N)$ such that $\|\chi - \chi_\delta\|_\infty \leq \delta/\mathfrak{m}$. It follows that

$$\int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x = \int \rho_\varepsilon(t,x)(\chi - \chi_\delta)(x)\,\mathrm{d}x + \int \rho_\varepsilon(t,x)\chi_\delta(x)\,\mathrm{d}x$$

where, owing to (39), the former integral is uniformly dominated by δ and the latter lies in a compact set of $C([0,T])$. Therefore $\left\{t \mapsto \int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x,\ \varepsilon > 0\right\}$ can be covered by a finite number of balls with radius 2δ in $C([0,T])$. Finally, since $C_0^0(\mathbb{R}^N)$ is separable, we apply a diagonal argument to extract a subsequence such that $\int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x$ converges uniformly in $C([0,T])$ for any element χ of a numerable dense set in $C_0^0(\mathbb{R}^N)$. By uniqueness of the limit, we find

$$\lim_{\varepsilon\to 0}\int \rho_\varepsilon(t,x)\chi(x)\,\mathrm{d}x = \int n_e\chi\,\mathrm{d}x.$$

Going back to (39), we check that the convergence holds for any $\chi \in C_0^0(\mathbb{R}^N)$.

The manipulations detailed in the previous Section prove ii). In order to establish iii), it is convenient to introduce the following functional: given λ a non negative bounded measure on $[0,T]\times\mathbb{R}^N$, and μ a vector valued bounded measure on $[0,T]\times\mathbb{R}^N$, we set

$$\mathscr{K}(\lambda,\mu) \stackrel{\mathrm{def}}{=} \sup_\Theta \left\{ \int \mu\cdot\Theta - \frac{1}{2}\int \lambda|\Theta|^2 \right\}$$

where the supremum is taken over continuous functions $\Theta : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$. According to [4, Prop. 3.4], we have:

Lemma 3.5 ([4]) *If μ is absolutely continuous with respect to λ, denoting by $\mathbb{V}$ the Radon-Nikodym derivative of μ with respect to λ, we have*

$$\mathscr{K}(\lambda,\mu) = \frac{1}{2}\int \lambda|\mathbb{V}|^2 \in [0,\infty],$$

otherwise $\mathscr{K}(\lambda,\mu) = +\infty$.

Clearly $(\lambda,\mu)\mapsto \mathscr{K}(\lambda,\mu)$ is a convex and lower semi–continuous (for the weak-$\star$ convergence) functional. Let $\eta:[0,T]\to[0,\infty)$ be a continuous non negative function. Reasoning as in [5], we show that $J\in L^\infty(0,T;L^2(\mathbb{R}^N))$ since

$$\begin{aligned}\mathscr{K}(\eta\rho_\varepsilon,\eta J_\varepsilon) &= \frac{1}{2}\int_0^T\int_{\mathbb{R}^N}\frac{|J_\varepsilon(t,x)|^2}{\rho_\varepsilon(t,x)}\,\eta(t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^T\int_{\mathbb{R}^N}\frac{1}{\rho_\varepsilon(t,x)}\,\Big|\int_{\mathbb{R}^N} v\sqrt{f_\varepsilon(t,x,v)}\,\sqrt{f_\varepsilon(t,x,v)}\,\mathrm{d}v\Big|^2\,\eta(t)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{1}{2}\int_0^T\iint_{\mathbb{R}^N\times\mathbb{R}^N}|v|^2 f_\varepsilon(t,x,v)\,\eta(t)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_0^T\eta\,\mathrm{d}t\end{aligned}$$

becomes, as ε tends to 0

$$\mathscr{K}(\eta n_{\mathrm{e}},\eta J)\lesssim \|\eta\|_{L^1(0,T)}.$$

Reasoning the same way, we get

$$\begin{aligned}\mathscr{K}(\rho_\varepsilon,J_\varepsilon-\rho_\varepsilon\mathscr{V}) &= \frac{1}{2}\int_0^T\int_{\mathbb{R}^N}\frac{|J_\varepsilon-\rho_\varepsilon\mathscr{V}|^2}{\rho_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{1}{2}\int_0^T\iint_{\mathbb{R}^N\times\mathbb{R}^N}|v-\mathscr{V}|^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\le\int_0^T\mathscr{H}_{\mathscr{V},\varepsilon}\,\mathrm{d}t.\end{aligned}$$

It follows that $\mathscr{K}(n_{\mathrm{e}},J-n_{\mathrm{e}}\mathscr{V})=0$, which identifies the limit J and ends the proof of iii).

Finally, we can check that the initial data for the limit equation is meaningful by establishing some time–compactness on the sequence J_ε. Let

$$\mathscr{W}_R=\big\{\Theta:[0,T]\times\mathbb{R}^N\to\mathbb{R}^N,\ \Theta\text{ of class }C^1,\ \mathrm{supp}(\Theta)\subset[0,T]\times\overline{\Omega},\ \nabla_x\cdot(n_{\mathrm{e}}\Theta)=0\big\},$$

which is a closed subspace of the Banach space C^1 (endowed with the sup norm for the function and its first order derivatives). Multiplying (36) by a function in $\mathscr{W}_R$, we shall get rid of the stiff terms. Indeed, for such a trial function Θ, we deduce from (36)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int J_\varepsilon\cdot\Theta\,\mathrm{d}x=\int J_\varepsilon\cdot\partial_t\Theta\,\mathrm{d}x-\int\Theta\cdot(\nabla_x\cdot\mathbb{P}_\varepsilon)\,\mathrm{d}x-\frac{1}{\sqrt{\varepsilon}}\int\rho_\varepsilon\Theta\cdot\nabla_x\Psi_\varepsilon\,\mathrm{d}x, \tag{40}$$

since $\Theta\cdot\nabla_x\Phi_{\mathrm{e}}=0$ pointwise in view of the supports. By using the estimates deduced from Proposition 3.3, we observe that the first two terms are bounded in $L^\infty(0,T)$. For the last one, we use the Poisson equation (10) and integration by parts to infer

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}\int\rho_\varepsilon\Theta\cdot\nabla_x\Psi_\varepsilon\,\mathrm{d}x &= \frac{1}{\sqrt{\varepsilon}}\int n_{\mathrm{e}}\Theta\cdot\nabla_x\Psi_\varepsilon\,\mathrm{d}x-\int\Delta_x\Psi_\varepsilon\Theta\cdot\nabla_x\Psi_\varepsilon\,\mathrm{d}x\\ &= 0+\int\nabla_x\Psi_\varepsilon\cdot\nabla_x(\Theta\cdot\nabla_x\Psi_\varepsilon)\,\mathrm{d}x\\ &= \frac{1}{2}\int\Theta\cdot\nabla_x(|\nabla_x\Psi_\varepsilon|^2)\,\mathrm{d}x+\sum_{1\le j,k\le N}\int\partial_{x_j}\Psi_\varepsilon\partial_{x_j}\Theta_k\partial_{x_k}\Psi_\varepsilon\,\mathrm{d}x\end{aligned}$$

where we have used that $n_{\mathrm{e}}\Theta(t,\cdot)$ is divergence free. For quadratic external potentials, an integration by parts shows that the first integral is zero (since $n_{\mathrm{e}}\Theta$ is divergence free). In any cases, the right hand side can be dominated by $\|\nabla\Theta\|_\infty\|\nabla\Psi_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}$ and it is thus bounded in $L^\infty(0,T)$. Reporting this into (40) allows us to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int J_\varepsilon\cdot\Theta\,\mathrm{d}x\text{ is bounded in }L^\infty(0,T).$$

Since $\mathscr{W}_R$ is separable, we can boil down a diagonal argument to justify that J_ε is relatively compact in $C^0(0,T;\mathscr{W}_R'-\text{weak}-\star)$: we can assume that the extracted subsequence is such that $\int J_\varepsilon\cdot\Theta\,\mathrm{d}x$ converges uniformly on $[0,T]$ for any $\Theta\in\mathscr{W}_R$. $\square$

4 Asymptotic analysis of the Vlasov–Poisson–Fokker–Planck system

In this Section we state and prove a Theorem analogous to Theorem 1.2 when the basic equation is (VFP), which includes a Fokker–Planck operator, coupled with (P).

For the well-posedness issues of the system (VFP) coupled to (P), we refer the reader to [3, 10]. The role of the external potential is precisely investigated in [11]. The associated moment system reads

$$\begin{cases} \partial_t \rho_\varepsilon + \nabla_x \cdot J_\varepsilon = 0, \\ \partial_t J_\varepsilon + \nabla_x \cdot \mathbb{P}_\varepsilon + \rho_\varepsilon \nabla_x \Phi_\varepsilon = -J_\varepsilon, \end{cases}$$

where we still use the notation $J_\varepsilon = \int v f_\varepsilon \,\mathrm{d}v$, $\mathbb{P}_\varepsilon = \int v \otimes v f_\varepsilon \,\mathrm{d}v$. As $\varepsilon \to 0$, we expect as before that $\rho_\varepsilon \to n_\mathrm{e} = \mathbf{1}_\Omega \Delta \Phi_\mathrm{ext}$ and that the behavior of the current is driven by the Lake Equation with friction

$$\begin{cases} \partial_t V + V \cdot \nabla_x V + \nabla_x p = -V, \\ \nabla_x \cdot (n_\mathrm{e} V) = 0. \end{cases} \tag{LE$_f$}$$

If Φ_ext is quadratic as in (12) (possibly isotropic), the domain Ω is an ellipsoid (possibly a ball) as in Section 2.1 and (LE$_f$) becomes the Incompressible Euler system with friction

$$\begin{cases} \partial_t V + \nabla_x \cdot (V \otimes V) + \nabla_x p = -V, \\ \nabla_x \cdot V = 0. \end{cases} \tag{41}$$

For a more general confining potential Φ_ext, we make assumptions h1), h2), H1) and H2) as in Section 2.2. Since we work with finite charge data, the limit equation (LE$_f$) holds in Ω, completed with the no flux boundary condition (6), namely

$$V(t,x) \cdot \nu(x)\Big|_{\partial\Omega} = 0.$$

Like in the previous section we associate with V, smooth solution of (LE$_f$), a smooth compactly supported extension $\mathscr{V}$ defined on $[0,T] \times \mathbb{R}^N$ such that $\mathscr{V} \cdot \nu(x)\Big|_{\partial\Omega} = 0$.

We shall investigate this asymptotics in the specific case where the "temperature" $\theta = \theta_\varepsilon$ goes to 0 as $\varepsilon \to 0$. In this context, we can derive an analog of Proposition 3.3 that accounts for the dissipation mechanisms induced by the Fokker–Planck operator.

Proposition 4.1 *The solution* $(f_\varepsilon, \Phi_\varepsilon = \frac{1}{\varepsilon}\Phi_\mathrm{e} + \frac{1}{\sqrt{\varepsilon}}\Psi_\varepsilon)$ *of* (VFP)–(P) *satisfies the following entropy dissipation inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\iint |v|^2\ f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\iint \Phi_\mathrm{e}\ f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x + \theta_\varepsilon \iint f_\varepsilon \ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int |\nabla_x \Psi_\varepsilon|^2 \,\mathrm{d}x\right\} = -\mathscr{D}_\varepsilon$$

where we denote

$$\mathscr{D}_\varepsilon = \iint |v\sqrt{f_\varepsilon} + 2\theta_\varepsilon \nabla_v \sqrt{f_\varepsilon}|^2 \,\mathrm{d}v\,\mathrm{d}x \geq 0.$$

Furthermore, the total charge is conserved

$$\iint f_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint f_\varepsilon(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m}.$$

Uniform estimates are not directly included in this statement since the function $z \mapsto z\ln(z)$ changes sign. Nevertheless, we can establish such uniform estimates.

Corollary 4.2 *We assume that there exists some (large) $\lambda > 1$ such that*

$$\int \exp(-\lambda \Phi_{\mathrm{ext}})\, \mathrm{d}x < \infty. \tag{42}$$

We suppose also that $0 < \varepsilon \le 1/(8\lambda)$ and $0 < \theta_\varepsilon \le 1$. Let $f_\varepsilon^{\mathrm{init}} : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ be a sequence of integrable functions that satisfy the following requirements

$$\begin{aligned} &\iint f_\varepsilon^{\mathrm{init}}\, \mathrm{d}v\, \mathrm{d}x = \mathfrak{m}, \\ &\sup_{0<\varepsilon\le 1/(8\lambda),\ 0<\theta_\varepsilon\le 1} \left\{ \frac{1}{2} \iint |v|^2\ f_\varepsilon^{\mathrm{init}}\, \mathrm{d}v\, \mathrm{d}x + \theta_\varepsilon \iint f_\varepsilon^{\mathrm{init}} |\ln(f_\varepsilon^{\mathrm{init}})|\, \mathrm{d}v\, \mathrm{d}x \right. \\ &\qquad\qquad \left. + \frac{1}{2} \int |\nabla_x \Psi_\varepsilon^{\mathrm{init}}|^2\, \mathrm{d}x + \frac{1}{\varepsilon} \iint \Phi_{\mathrm{e}}\ f_\varepsilon^{\mathrm{init}}\, \mathrm{d}v\, \mathrm{d}x \right\} < \infty, \end{aligned} \tag{43}$$

with

$$\Delta_x \Psi_\varepsilon^{\mathrm{init}} = \frac{1}{\sqrt{\varepsilon}} \Big(n_{\mathrm{e}} - \int f_\varepsilon^{\mathrm{init}}\, \mathrm{d}v \Big).$$

Let $0 < T < \infty$ and let $(f_\varepsilon, \Phi_\varepsilon = \frac{1}{\varepsilon}\Phi_{\mathrm{e}} + \frac{1}{\sqrt{\varepsilon}}\Psi_\varepsilon)$ be the associated solution of (VFP)–(P)*. Then, uniformly for $0 < \varepsilon \le 1/(8\lambda)$ and $0 < \theta_\varepsilon \le 1$:*

i) $f_\varepsilon(1 + |v|^2 + \theta_\varepsilon |\ln(f_\varepsilon)|) + \varepsilon^{-1}\Phi_{\mathrm{e}} f_\varepsilon$ *is bounded in* $L^\infty(0,T; L^1(\mathbb{R}^N \times \mathbb{R}^N))$,

ii) $\nabla_x \Psi_\varepsilon$ *is bounded in* $L^\infty(0,T; L^2(\mathbb{R}^N))$,

iii) $\mathscr{D}_\varepsilon$ *is bounded in* $L^1(0,T)$.

Remark 4.3 *In any dimension $N \ge 1$,* (42) *is always true for quadratic potentials. If $N = 1$ or $N = 2$, hypothesis* (42) *is satisfied if h2) is, since $\Phi_{\mathrm{ext}}(x) + \mathfrak{m}\Gamma(x) \to +\infty$ when $|x| \to +\infty$ and that $\Gamma(x) = -|x|/2$ or $-\ln|x|/(2\pi)$. Therefore, hypothesis* (42) *needs to be verified only for $N \ge 3$.*

Proof. We first observe that hypothesis (42) implies

$$\int \exp(-\lambda \Phi_{\mathrm{e}})\, \mathrm{d}x < \infty.$$

Indeed, we have $\Phi_{\mathrm{e}} = \Gamma \star n_{\mathrm{e}} - C_* + \Phi_{\mathrm{ext}} \ge \Phi_{\mathrm{ext}} - C_*$. We write, for $h \ge 0$,

$$\begin{aligned} f_\varepsilon \ln(f_\varepsilon) \le f_\varepsilon |\ln(f_\varepsilon)| &= f_\varepsilon \ln(f_\varepsilon) - 2 f_\varepsilon \ln(f_\varepsilon) \big(\mathbf{1}_{e^{-h} \le f_\varepsilon \le 1} + \mathbf{1}_{0 \le f_\varepsilon < e^{-h}} \big) \\ &\le f_\varepsilon \ln(f_\varepsilon) + 2h f_\varepsilon + \frac{4}{e} e^{-h/2}, \end{aligned} \tag{44}$$

and denote

$$E_\varepsilon(f_\varepsilon) = \frac{1}{2} \iint |v|^2\ f_\varepsilon\, \mathrm{d}v\, \mathrm{d}x + \frac{1}{\varepsilon} \iint \Phi_{\mathrm{e}}\ f_\varepsilon\, \mathrm{d}v\, \mathrm{d}x + \frac{1}{2} \int |\nabla_x \Psi_\varepsilon|^2\, \mathrm{d}x.$$

We now use (44) with $h(x,v) = |v|^2/(8\theta_\varepsilon) + \Phi_{\mathrm{e}}(x)/(4\varepsilon\theta_\varepsilon)$ to infer

$$\begin{aligned} \theta_\varepsilon \iint f_\varepsilon \ln(f_\varepsilon)\, \mathrm{d}v\, \mathrm{d}x \le \theta_\varepsilon \iint f_\varepsilon |\ln(f_\varepsilon)|\, \mathrm{d}v\, \mathrm{d}x &\le \theta_\varepsilon \iint f_\varepsilon \ln(f_\varepsilon)\, \mathrm{d}v\, \mathrm{d}x + \frac{\theta_\varepsilon}{2} E_\varepsilon(f_\varepsilon) \\ &\quad + \theta_\varepsilon \frac{4}{e} \iint \exp(-|v|^2/(16\theta_\varepsilon) - \Phi_{\mathrm{e}}(x)/(8\varepsilon\theta_\varepsilon))\, \mathrm{d}v\, \mathrm{d}x. \end{aligned} \tag{45}$$

The last term is equal to

$$\theta_\varepsilon \frac{4}{e} \int \exp(-|v|^2/(16\theta_\varepsilon)\, \mathrm{d}v \int \exp(-\Phi_{\mathrm{e}}(x)/(8\varepsilon\theta_\varepsilon))\, \mathrm{d}x \le \theta_\varepsilon \frac{4}{e} \int \exp(-|v|^2/16)\, \mathrm{d}v \int \exp(-\lambda\Phi_{\mathrm{e}}(x))\, \mathrm{d}x,$$

thus tends to zero as $\theta_\varepsilon \to 0$ (uniformly for $0 < \varepsilon < 1/(8\lambda)$). Using the dissipation of the entropy given in Proposition 4.1, we then infer

$$\begin{aligned}
E_\varepsilon(f_\varepsilon^{\mathrm{init}}) + \theta_\varepsilon \iint f_\varepsilon^{\mathrm{init}} |\ln(f_\varepsilon^{\mathrm{init}})| \,\mathrm{d}v\,\mathrm{d}x &\geq E_\varepsilon(f_\varepsilon^{\mathrm{init}}) + \theta_\varepsilon \iint f_\varepsilon^{\mathrm{init}} \ln(f_\varepsilon^{\mathrm{init}}) \,\mathrm{d}v\,\mathrm{d}x \\
&\geq E_\varepsilon(f_\varepsilon) + \theta_\varepsilon \iint f_\varepsilon \ln(f_\varepsilon) \,\mathrm{d}v\,\mathrm{d}x \\
&\geq E_\varepsilon(f_\varepsilon) + \theta_\varepsilon \iint f_\varepsilon |\ln(f_\varepsilon)| \,\mathrm{d}v\,\mathrm{d}x - \frac{\theta_\varepsilon}{2} E_\varepsilon(f_\varepsilon) + o_{\theta_\varepsilon \to 0}(1)
\end{aligned}$$

and the conclusion follows since $\theta_\varepsilon \leq 1$. $\square$

Untill the end of the Section, we shall make hypothesis (42). Since we are dealing with the regime

$$0 < \varepsilon \ll 1, \qquad 0 < \theta_\varepsilon \ll 1,$$

the estimates in Proposition 4.2 do not provide L^1-weak compactness on the particle distribution function and its moments; we still need to work with convergences in spaces of finite measures. The first step in the investigation of the asymptotic behavior is summarized in the following claim.

Lemma 4.4 *We make assumptions* (42) *and* (43). *Up to a subsequence, we can assume that f_ε converges to f weakly–$\star$ in $\mathscr{M}^1((0,T) \times \mathbb{R}^N \times \mathbb{R}^N)$. Then, ρ_ε converges to $n_{\mathrm{e}} = \int f \,\mathrm{d}v$ in $L^\infty(0,T; H^{-1}(\mathbb{R}^N))$ and in $C^0(0,T; \mathscr{M}^1(\mathbb{R}^N) - \text{weak–}\star)$. Moreover, we can assume that $J_\varepsilon \rightharpoonup J = \int v f \,\mathrm{d}v$ in $\mathscr{M}^1([0,T] \times \mathbb{R}^N)$, the limit J is divergence–free and supported in $[0,T] \times \bar{\Omega}$.*

Proof. We follow the arguments of the previous Section. We identify the limit of ρ_ε by coming back to the Poisson equation $\sqrt{\varepsilon}\nabla_x \cdot \nabla_x \Psi_\varepsilon = n_{\mathrm{e}} - \rho_\varepsilon$. The time compactness then appears as a consequence of the charge conservation, together with the estimates on the current. We obtain the $L^\infty(0,T; L^1(\mathbb{R}^N))$ estimate on J_ε as in (35). Letting ε go to 0 in the charge conservation equation, we obtain $\partial_t n_{\mathrm{e}} + \nabla_x \cdot J = 0 = \nabla_x \cdot J$. Still reproducing the arguments of the previous section, based on the conservation of the total charge, we arrive at the following conclusion:

$$\mathrm{supp}(f) \subset [0,T] \times \bar{\Omega} \times \mathbb{R}^N, \qquad \mathrm{supp}(J) \subset [0,T] \times \bar{\Omega}.$$

Furthermore, J belongs to the set $\mathscr{DM}^{\mathrm{ext}}(\mathbb{R}^N)$, it admits a normal trace $J \cdot \nu\big|_{\partial\Omega}$, which actually vanishes. $\square$

It remains to identify the limit J. As in the case of the pure Vlasov–Poisson equation, the idea consists in introducing a suitable functional intended to compare f_ε to the expected limit. Let $\mathscr{N}_\varepsilon : \mathbb{R}^N \to (0, \infty)$ be a given function such that

$$\int \mathscr{N}_\varepsilon \,\mathrm{d}x = \mathfrak{m} = \int n_{\mathrm{e}} \,\mathrm{d}x = \iint f(0,x,v) \,\mathrm{d}v\,\mathrm{d}x$$

and let us set

$$M_{\mathscr{V},\theta_\varepsilon}(t,x,v) = \frac{1}{(2\pi\theta_\varepsilon)^{N/2}} \exp\left(- \frac{|v - \mathscr{V}(t,x)|^2}{2\theta_\varepsilon} \right).$$

A natural candidate to replace the functional $\mathscr{H}_{\mathscr{V},\varepsilon}$ would be the relative entropy of f_ε with respect to $n_{\mathrm{e}}(x) M_{\mathscr{V},\theta_\varepsilon}(t,x,v)$ associated with the non-negative convex function $z \mapsto z\ln(z) - z + 1$, namely

$$\iint \left(f_\varepsilon \ln\left(\frac{f_\varepsilon}{n_{\mathrm{e}} M_{\mathscr{V},\theta_\varepsilon}} \right) - f_\varepsilon + n_{\mathrm{e}} M_{\mathscr{V},\theta_\varepsilon} \right) \mathrm{d}v\,\mathrm{d}x,$$

but the first term is clearly meaningless since n_{e} has compact support. Therefore, we introduce

$$\mathscr{N}_\varepsilon(x) = \frac{\mathfrak{m}}{Z_\varepsilon} \exp\Big(- \frac{\Phi_{\mathrm{e}}(x)}{\varepsilon\theta_\varepsilon} \Big), \qquad \text{where} \quad Z_\varepsilon = \int \exp\Big(- \frac{\Phi_{\mathrm{e}}(y)}{\varepsilon\theta_\varepsilon} \Big) \,\mathrm{d}y, \tag{46}$$

and the following modulated functional

$$\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} = \theta_\varepsilon \iint \left(f_\varepsilon \ln\left(\frac{f_\varepsilon}{\mathscr{N}_\varepsilon M_{\mathscr{V},\theta_\varepsilon}}\right) - f_\varepsilon + \mathscr{N}_\varepsilon M_{\mathscr{V},\theta_\varepsilon}\right) \mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int |\nabla_x \Psi_\varepsilon|^2\,\mathrm{d}x. \tag{47}$$

In fact, $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}$ is up to the term $\frac{1}{2}\int |\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x$, nothing but the relative entropy of f_ε with respect to $\mathscr{N}_\varepsilon M_{\mathscr{V},\theta_\varepsilon}$ associated with the non-negative convex function $G : (0,+\infty) \ni z \mapsto z\ln(z) - z + 1$. This implies in particular that the integrand in the first integral of (47) is simply $G(f_\varepsilon) - G(N_\varepsilon M_{\mathscr{V},\theta_\varepsilon}) - G'(f_\varepsilon)(f_\varepsilon - N_\varepsilon M_{\mathscr{V},\theta_\varepsilon})$, thus pointwise nonnegative, and vanishes only when $f_\varepsilon = N_\varepsilon M_{\mathscr{V},\theta_\varepsilon}$. By definition of N_ε and $M_{\mathscr{V},\theta_\varepsilon}$ and using the fact $\iint f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m} = \iint \mathscr{N}_\varepsilon M_{\mathscr{V},\theta_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x$ in view of our normalizations, we infer

$$\begin{aligned}
\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} &= \theta_\varepsilon \iint f_\varepsilon \ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\iint |v-\mathscr{V}|^2\; f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\int \Phi_{\mathrm{e}} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int |\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2}N\mathfrak{m}\theta_\varepsilon\ln(2\pi\theta_\varepsilon) - \theta_\varepsilon\mathfrak{m}\ln\left(\frac{\mathfrak{m}}{Z_\varepsilon}\right) \\
&= \mathscr{H}_{\mathscr{V},\varepsilon} + \theta_\varepsilon\iint f_\varepsilon\ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}N\mathfrak{m}\theta_\varepsilon\ln(2\pi\theta_\varepsilon) - \theta_\varepsilon\mathfrak{m}\ln\left(\frac{\mathfrak{m}}{Z_\varepsilon}\right).
\end{aligned} \tag{48}$$

This second expression of $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}$ justifies the choice we have made for $\mathscr{N}_\varepsilon$. Actually, for our purpose, the exact normalization $\iint \mathscr{N}_\varepsilon M_{\mathscr{V},\theta_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m}$ is not necessary, though natural in a modulated entropy argument, only the fact that $\ln(\mathscr{N}_\varepsilon M_{\mathscr{V},\theta_\varepsilon}) \approx -\Phi_{\mathrm{e}}(x)/(\varepsilon\theta_\varepsilon) - |v-\mathscr{V}|^2/(2\theta_\varepsilon)$ is used. This is related to the fact that the temperature θ_ε is small in the regime we are considering.

Let us now compare $\mathscr{H}_{\mathscr{V},\varepsilon}$ and $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}$ more precisely. As a first step, note that, on the one hand,

$$\theta_\varepsilon\ln(2\pi\theta_\varepsilon) \to 0$$

when $\theta_\varepsilon \to 0$; and on the other hand, that

$$|\Omega| = \int_\Omega \exp\Big(-\frac{\Phi_{\mathrm{e}}(y)}{\varepsilon\theta_\varepsilon}\Big)\,\mathrm{d}y \le Z_\varepsilon = \int \exp\Big(-\frac{\Phi_{\mathrm{e}}(y)}{\varepsilon\theta_\varepsilon}\Big)\,\mathrm{d}y \le \int \exp\Big(-\lambda\Phi_{\mathrm{e}}(y)\Big)\,\mathrm{d}y < +\infty$$

if $\theta_\varepsilon \le 1$ and $\varepsilon \le 1/(8\lambda)$, thus, as $\theta_\varepsilon \to 0$,

$$\theta_\varepsilon\mathfrak{m}\ln\left(\frac{\mathfrak{m}}{Z_\varepsilon}\right) \to 0.$$

The inequality (45) implies

$$\begin{aligned}
\mathscr{H}_{\mathscr{V},\varepsilon} &= \mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} - \theta_\varepsilon\iint f_\varepsilon\ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}N\mathfrak{m}\theta_\varepsilon\ln(2\pi\theta_\varepsilon) + \theta_\varepsilon\mathfrak{m}\ln\left(\frac{\mathfrak{m}}{Z_\varepsilon}\right) \\
&\le \mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} - \theta_\varepsilon\iint f_\varepsilon\ln(f_\varepsilon)\mathbf{1}_{f_\varepsilon\le 1}\,\mathrm{d}v\,\mathrm{d}x + o_{\varepsilon\to 0}(1) \\
&\le 2\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} + o_{\varepsilon\to 0}(1).
\end{aligned} \tag{49}$$

Then, let us compute the time derivative of the modulated entropy $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}$. We get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\theta_\varepsilon\iint f_\varepsilon\ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\iint|v-\mathscr{V}|^2\; f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\int\Phi_{\mathrm{e}} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + \frac{1}{2}\int|\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x\Big\} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\theta_\varepsilon\iint f_\varepsilon\ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\iint|v|^2\; f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\int\Phi_{\mathrm{e}} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int|\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x\Big\} \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}\Big\{-\iint v\cdot\mathscr{V} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\iint|\mathscr{V}|^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\Big\}.
\end{aligned}$$

Bearing in mind the computation for proving Proposition 4.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} = -\mathscr{D}_\varepsilon + \frac{\mathrm{d}}{\mathrm{d}t}\left\{-\int J_\varepsilon\cdot\mathscr{V}\,\mathrm{d}x + \frac{1}{2}\int \rho_\varepsilon|\mathscr{V}|^2\,\mathrm{d}x\right\}.$$

Reasoning as in the previous section, and by using the moment equations, we are led to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} &= -\mathscr{D}_\varepsilon + \int \mathscr{V}\cdot J_\varepsilon\,\mathrm{d}x \\ &\quad + \int (\rho_\varepsilon\mathscr{V} - J_\varepsilon)\cdot\big(\partial_t\mathscr{V} + (\mathscr{V}\cdot\nabla_x)\mathscr{V}\big)\,\mathrm{d}x \\ &\quad - \int D\mathscr{V} : (\mathbb{P}_{\mathscr{V},\varepsilon} - \nabla_x\Psi_\varepsilon\otimes\nabla_x\Psi_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int \rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x,\end{aligned}$$

using once again that $\nabla_x\cdot(n_{\mathrm{e}}\mathscr{V}) = 0$ and the no-flux condition (6). Let us set

$$\mathscr{D}_{\mathscr{V},\varepsilon} = \iint \left|(v-\mathscr{V})\sqrt{f_\varepsilon} + 2\theta_\varepsilon\nabla_v\sqrt{f_\varepsilon}\right|^2\mathrm{d}v\,\mathrm{d}x \geq 0.$$

We rewrite

$$\mathscr{D}_\varepsilon = \mathscr{D}_{\mathscr{V},\varepsilon} - \int \rho_\varepsilon|\mathscr{V}|^2\,\mathrm{d}x + 2\int \mathscr{V}\cdot J_\varepsilon\,\mathrm{d}x.$$

Accordingly, we can reorganize terms as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} &= -\mathscr{D}_{\mathscr{V},\varepsilon} + \int (\rho_\varepsilon\mathscr{V} - J_\varepsilon)\cdot\big(\partial_t\mathscr{V} + (\mathscr{V}\cdot\nabla_x)\mathscr{V} + \mathscr{V}\big)\,\mathrm{d}x \\ &\quad - \int D\mathscr{V} : (\mathbb{P}_{\mathscr{V},\varepsilon} - \nabla_x\Psi_\varepsilon\otimes\nabla_x\Psi_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int \rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x.\end{aligned}$$

We can summarize the previous manipulations within the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} + \mathscr{D}_{\mathscr{V},\varepsilon} \leq \frac{1}{\varepsilon}\int \rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x + r_\varepsilon + \int D\mathscr{V} : (\mathbb{P}_{\mathscr{V},\varepsilon} - \nabla_x\Psi_\varepsilon\otimes\nabla_x\Psi_\varepsilon)\,\mathrm{d}x, \tag{50}$$

where, for any $0 < t \leq T$,

$$\int_0^t r_\varepsilon\,\mathrm{d}\tau = \int_0^t\int (\rho_\varepsilon\mathscr{V} - J_\varepsilon)(\partial_\tau\mathscr{V} - \mathscr{V}\cdot\nabla_x\mathscr{V} + \mathscr{V})\,\mathrm{d}x\,\mathrm{d}\tau$$

tends to 0 as $\varepsilon \to 0$. We wish to strengthen this result as follows.

Lemma 4.5 *We make assumptions* (42) *and* (43) *and suppose that $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} + \mathscr{D}_{\mathscr{V},\varepsilon} \lesssim \mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} + r_\varepsilon$$

where, for any $0 < t \leq T$, $\lim_{\varepsilon\to 0}\int_0^t r_\varepsilon\,\mathrm{d}\tau = 0$.

Proof. We can also reproduce the arguments in the previous section used to estimate

$$\frac{1}{\varepsilon}\int \rho_\varepsilon\mathscr{V}\cdot\nabla_x\Phi_{\mathrm{e}}\,\mathrm{d}x \lesssim \frac{1}{\varepsilon}\int \rho_\varepsilon\Phi_{\mathrm{e}}\,\mathrm{d}x.$$

For the last term in (50), we have

$$\int D\mathscr{V} : (\mathbb{P}_{\mathscr{V},\varepsilon} - \nabla_x\Psi_\varepsilon\otimes\nabla_x\Psi_\varepsilon)\,\mathrm{d}x \leq \|D\mathscr{V}\|_\infty\left(\iint |v-\mathscr{V}|^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \int |\nabla_x\Psi_\varepsilon|^2\,\mathrm{d}x\right),$$

so that, using (49),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}+\mathscr{D}_{\mathscr{V},\varepsilon} &\lesssim \frac{1}{\varepsilon}\iint f_\varepsilon \Phi_{\mathrm{e}}\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{2}\iint |v-\mathscr{V}|^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{2}\int |\nabla_x \Psi_\varepsilon|^2\,\mathrm{d}x+r_\varepsilon\\ &\lesssim \mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}+r_\varepsilon+o_{\varepsilon\to 0}(1).\end{aligned}$$

It allows us to conclude by coming back to (50). □

Let us now state our main result concerning the Vlasov-Poisson-Fokker-Planck system. We recall that we may work either with a quadratic potential Φ_{ext} (and then the domain Ω) is an ellipsoid), or with a general potential where h1), h2), H1) and H2) are satisfied.

Theorem 4.6 *If $N \geq 3$, we make assumption (42), that is we assume that there exists some (large) $\lambda > 1$ such that*

$$\int \exp(-\lambda\Phi_{\mathrm{ext}})\,\mathrm{d}x < \infty.$$

Denote by V the solution, on $[0,T]$, to the Lake Equation with friction (LE_f) *with the no-flux condition* (6) *given by Theorem A.1 and consider a smooth extension $\mathscr{V}$ to V. Let $f^{\mathrm{init}}_\varepsilon : \mathbb{R}^N\times\mathbb{R}^N \to [0,\infty)$ be a sequence of integrable functions satisfying*

$$\iint f^{\mathrm{init}}_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m} \qquad \text{and} \qquad \mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}(f^{\mathrm{init}}_\varepsilon) \to 0,$$

where $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}$ is defined in (47). *Consider then the associated solutions f_ε of the Vlasov–Poisson–Fokker–Planck equation* (VFP)–(P). *Then, we have, as $\varepsilon \to 0$ and $\theta_\varepsilon \to 0$,*

- *i) ρ_ε converges to n_{e} in $C^0(0,T;\mathscr{M}^1(\mathbb{R}^N)-\text{weak}-\star)$;*
- *ii) $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon} \to 0$ uniformly on $[0,T]$;*
- *iii) J_ε converges to J in $\mathscr{M}^1([0,T]\times\mathbb{R}^N)$, where $J\big|_{[0,T]\times\Omega} = V$, $\nabla_x\cdot J = 0$ and $J\cdot\nu(x)\big|_{\partial\Omega} = 0$.*

Remark 4.7 *We have seen (see Remark 4.3) that the integrability assumption $\int \exp(-\lambda\Phi_{\mathrm{ext}})\,\mathrm{d}x < \infty$ is automatically satisfied if $N = 1$, 2 by h2) or for quadratic potentials. When $N \geq 3$, it is also true if Φ_{ext} is convex and tends to $+\infty$ at infinity.*

Remark 4.8 *One may construct an admissible family of initial conditions following the lines of Remark 1.3. In particular, taking G a normalized Gaussian, it is enough to choose θ_ε and σ_ε such that $\theta_\varepsilon \int f_\varepsilon \ln f_\varepsilon \to 0$, which imposes $\theta_\varepsilon \ln\sigma_\varepsilon \to 0$.*

Proof. It is clear that if $\mathscr{H}^{\mathrm{FP}}_{\mathscr{V},\varepsilon}(f^{\mathrm{init}}_\varepsilon) \to 0$, then (43) is satisfied. Item i) has already been discussed. Applying the Grönwall lemma, we deduce readily that ii) holds from Lemma 4.5. Coming back to (49), we infer that $\iint |v-\mathscr{V}|^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x$ tends to 0. Then, we appeal to Lemma 3.5 to conclude that J belongs to $L^\infty(0,T;L^2(\mathbb{R}^N))$ and that $J = n_{\mathrm{e}}V$. We can also justify some time–compactness as in the pure Vlasov–Poisson case.

A Smooth solutions of the Lake Equations

Theorem A.1 *Let Ω be a smooth ($\partial\Omega$ of class C^{s+1} is enough) bounded open set in $\mathbb{R}^N$, γ be a real constant, $s \in \mathbb{N}$ such that $s > 1 + N/2$ and $n_{\mathrm{e}} : \Omega \to \mathbb{R}$ in H^{s+1} such that $\inf_\Omega n_{\mathrm{e}} > 0$. Let $V^{\mathrm{init}} : \Omega \to \mathbb{R}^N$ be a divergence free vector field in H^s satisfying the no flux condition $V^{\mathrm{init}}\cdot\nu = 0$ on $\partial\Omega$. There exists $T > 0$ and a unique solution $V \in L^\infty(0,T;H^s(B(0,R)))$ of*

$$\begin{cases} \partial_t V + V\cdot\nabla_x V + \nabla_x p = -\gamma V,\\ \nabla_x\cdot(n_{\mathrm{e}}V) = 0,\end{cases}$$

with the no flux condition (6)*. Moreover, we have*

$$\sup_{0\leq t\leq T}\Big(\|V(t)\|_{H^s}+\|\partial_t V(t)\|_{H^{s-1}}+\|\nabla_x p(t)\|_{H^s}+\|\partial_t\nabla_x p(t)\|_{H^{s-1}}\Big)\leq C(T)$$

for some positive constant $C(T)$ depending on γ, T, $n_{\rm e}$ and the initial datum.

Proof. The scheme of proof is exactly the same as in [38]. We shall denote $\hat V \stackrel{\rm def}{=} n_{\rm e}V$, which is divergence free. Applying $\nabla\cdot(n_{\rm e}\,.)$ to the equation, we see that the pressure p satisfies, for any t, the elliptic equation

$$\begin{aligned}-\nabla_x\cdot(n_{\rm e}\nabla_x p)&=\nabla_x\cdot(n_{\rm e}V\cdot\nabla_x V)=\nabla_x\cdot\left(\hat V\cdot\nabla_x\left(\frac{1}{n_{\rm e}}\hat V\right)\right)\\&=\hat V\cdot\nabla_x\left(\nabla_x\cdot\left(\frac{1}{n_{\rm e}}\hat V\right)\right)+\sum_{1\leq j,k\leq N}\partial_j\hat V_k\partial_k\left(\frac{\hat V_j}{n_{\rm e}}\right)\\&=\hat V\cdot\nabla_x\left(\hat V\cdot\nabla_x\left(\frac{1}{n_{\rm e}}\right)\right)+\sum_{1\leq j,k\leq N}\partial_j\hat V_k\partial_k\left(\frac{\hat V_j}{n_{\rm e}}\right),\end{aligned}\tag{51}$$

where we have used that $\hat V$ is divergence free as well as the identity $\nabla_x\cdot(\mathcal V\cdot\nabla_x\mathcal U)-\mathcal V\cdot\nabla_x(\nabla_x\cdot\mathcal U)=\sum_{1\leq j,k\leq N}\partial_j\mathcal V_k\partial_k\mathcal U_j$. We may further impose a suitable Neumann boundary condition for p on $\partial\Omega$. We recall that for $\sigma>N/2$, H^σ is an algebra. Notice that if $V\in H^s$, with $s>1+N/2$, then the right-hand side of (51) is in H^{s-1} and

$$\left\|\hat V\cdot\nabla_x\left(\hat V\cdot\nabla_x\left(\frac{1}{n_{\rm e}}\right)\right)+\sum_{1\leq j,k\leq N}\partial_j\hat V_k\partial_k\left(\frac{\hat V_j}{n_{\rm e}}\right)\right\|_{H^{s-1}}\leq C\|V\|^2_{H^s},$$

where C depends on $\inf_\Omega n_{\rm e}$ (which is assumed positive) and the H^{s+1} norm of $n_{\rm e}$.

Since $n_{\rm e}$ is in H^{s+1} and bounded away from zero and since the boundary is assumed of class C^{s+1}, it follows from classical elliptic estimates that (51) endowed with the Neumann condition on $\partial\Omega$ has a unique solution $p\in H^{s+1}(\Omega)\Big|_{\mathbb R}$, enjoying the estimate

$$\|p\|_{H^{s+1}}\leq C\|V\|^2_{H^s},\tag{52}$$

where C depends on $\inf_\Omega n_{\rm e}$ and the H^{s+1} norm of $n_{\rm e}$. Assume now that V is a smooth solution of (A.1) and let us perform an H^s estimate. For any $\alpha\in\big(\mathbb N\cup\{0\}\big)^d$ with $|\alpha|\leq s$, we have

$$\frac{\rm d}{{\rm d}t}\int_\Omega|\partial^\alpha V|^2\,{\rm d}x=-2\int_\Omega\partial^\alpha V\cdot\partial^\alpha\left((V\cdot\nabla_x)V\right)\,{\rm d}x-2\int_\Omega\partial^\alpha V\cdot\partial^\alpha\nabla_x p\,{\rm d}x-2\gamma\int_\Omega|\partial^\alpha V|^2\,{\rm d}x$$

Using classical commutator estimates, the Sobolev imbedding $H^s\subset W^{1,\infty}$ and the H^{s+1} estimate (52) on p, we then deduce

$$\frac{\rm d}{{\rm d}t}\int_\Omega|\partial^\alpha V|^2\,{\rm d}x\leq-2\int_\Omega\partial^\alpha V\cdot\left((V\cdot\nabla_x)\partial^\alpha V\right)\,{\rm d}x+C(\|V\|_{H^s}+\|V\|^2_{H^s}+\|V\|^3_{H^s}).$$

We use integration by parts for the first integral (recall that $V\cdot\nu=0$ on the boundary), which then becomes $\int_\Omega(\nabla_x\cdot V)|\partial^\alpha V|^2\,{\rm d}x\leq C\|V\|^3_{H^s}$. This yields

$$\frac{\rm d}{{\rm d}t}\int_\Omega|\partial^\alpha V|^2\,{\rm d}x\leq C(\|V\|_{H^s}+\|V\|^2_{H^s}+\|V\|^3_{H^s}),$$

and it follows that, for some $T_0>0$ depending only on γ, $n_{\rm e}$ and $V^{\rm init}$, we have $\|V\|_{L^\infty(0,T_0;H^s)}\leq 2\|V^{\rm init}\|_{H^s}$. The conclusion of the theorem follows from a suitable viscous approximation where a careful treatment of boundary terms is needed, see [39]. □

B Construction of an extended divergence–free velocity

Lemma B.1 *Let $V \in L^\infty(0,T;H^s(B(0,R),\mathbb{R}^N))$ be a divergence free vector field in H^s, with $s > 1 + N/2$, satisfying the no flux condition $V \cdot \nu = 0$ on $\partial B(0,R)$. There exists a solenoidal extension $\mathscr{V}$ of the vector field V defined on the whole space and compactly supported. Namely, $\mathscr{V} \in L^\infty(0,T;H^s(\mathbb{R}^N,\mathbb{R}^N))$ and it satisfies:*
i) (7),
ii) $\nabla_x \cdot \mathscr{V} = 0$ *in* $\mathbb{R}^N$.

Proof. Let us assume $N = 2$ or $N = 3$. Since $\nabla_x \cdot V = 0$ in the ball $B(0,R)$, which is convex, there exists $h \in L^\infty(0,T;H^{s+1}(B(0,R),\mathbb{R}^N))$ such that $V = \nabla \times h$. Then, by standard extension results (see, e. g., [27, Chapter I: Theorem 2.1 p. 17 & Theorem 8.1 p. 42]), there exists an extension $\tilde{h} \in L^\infty(0,T;H^{s+1}(\mathbb{R}^N,\mathbb{R}^N))$ to h. Considering a cut-off function $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $\chi(x) = 1$ for $x \in B(0,3R/2)$ and denoting $\mathscr{V} = \nabla \times (\chi\tilde{h})$, we see that $\mathscr{V}$ enjoys the desired properties. For $N \geq 4$, the construction is similar but involves differential forms. The arguments generalize to the case where Ω is an ellipsoid. □

Remark B.2 *In the case of a general potential, n_e is not uniform, and it is possible to construct an extension $\mathscr{V} \in L^\infty(0,T;H^s(\mathbb{R}^N,\mathbb{R}^N))$ which satisfies:*
i) (7),
ii) $\nabla_x \cdot (n_\mathrm{e}\mathscr{V}) = 0$ *in* $\mathbb{R}^N$.

However, it requires further topological hypotheses on Ω. Assuming H1, and assuming also that $\mathcal{K}$ is connected, and $\partial\Omega$ has a finite number of connected components, we may apply [21, Corollary 3.2]: $n_\mathrm{e}V$ is divergence free in the smooth domain Ω, hence we can construct a divergence free extension $\mathscr{J} : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ to $n_\mathrm{e}V$. Using a cut-off function, we may take $\mathscr{J}$ compactly supported in an arbitrary neighborhood of $\mathcal{K}$, the latter can be chosen so that $\Delta\Phi_{\mathrm{ext}}$ remains > 0. Finally, we set $\mathscr{V} = \mathscr{J}/(\Delta\Phi_{\mathrm{ext}})$, which is well-defined even when $\Delta\Phi_{\mathrm{ext}}$ vanishes.

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References

[1] A. A. Arsenev. Global existence of a weak solution of Vlasov's system of equations. *U. S. S. R. Comput. Math. Math. Phys.*, 15:131–143, 1975.

[2] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields*, 108(4):517–542, 1997.

[3] F. Bouchut. Existence and uniqueness of a global smooth solution for the VPFP system in three dimensions. *J. Func. Anal.*, 111:239–258, 1993.

[4] Y. Brenier. A homogenized model for vortex sheets. *Arch. Rational Mech. Anal.*, 138(4):319–353, 1997.

[5] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4):737–754, 2000.

[6] L. Caffarelli. The obstacle problem revisited. *The J. of Fourier Anal. and Appl.*, 4(3&4):383–402, 1998.

[7] D. Chafaï, N. Gozlan, and P-A. Zitt. First order asymptotics for confined particles with singular pair repulsions. *Ann. Appl. Probab.*, 24(6):2371–2413, 2014.

[8] G.-Q. Chen and H. Frid. Extended divergence-measure fields and the Euler equations for gas dynamics. *Commun. Math. Phys.*, 236:251–280, 2003.

[9] J. Dalibard and C. Cohen-Tannoudji. Atomic motion in laser light: connection between semiclassical and quantum descriptions. *Journal of Physics B: Atomic and Molecular Physics*, 18(8):1661, 1985.

[10] P. Degond. Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimension. *Ann. Scient. Ecole Normale Sup.*, 19:519–542, 1986.

[11] J. Dolbeault. Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states). *J. Math. Pures et Appl.*, 78(2):121–157, 1999.

[12] D.H.E. Dubin and T.M. O'Neil. Trapped nonneutral plasmas, liquids, and crystals (the thermal equilibrium states). *Reviews of Modern Physics*, 71(1):87–172, 1999.

[13] J. Frehse. On the regularity of the solution of a second order variational inequality. *Boll. Un. Mat. Ital. (4)*, 6:312–315, 1972.

[14] A. Friedman. *Variational principles and free-boundary problems.* Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1982. A Wiley-Interscience Publication.

[15] A. Friedman and D. Phillips. The free boundary of a semilinear elliptic equation. *Trans. Amer. Math. Soc.*, 282(1):153–182, 1984.

[16] O. Frostman. Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Meddelanden Mat. Sem. Univ. Lund*, 3, 1935.

[17] M. A. Furman. Compact complex expressions for the electric field of two-dimensional elliptical charge distributions. *Am. J. Phys.*, 62 (12):1134–1140, 1994.

[18] R. T. Glassey. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), 1996.

[19] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Comm. Partial Differential Equations*, 36(8):1385–1425, 2011.

[20] H. Hedenmalm and N. Makarov. Coulomb gas ensembles and Laplacian growth. *Proc. Lond. Math. Soc. (3)*, 106(4):859–907, 2013.

[21] T. Kato, M. Mitrea, G. Ponce, and M. Taylor. Extension and representation of divergence-free vector fields. *Math. Research Letters*, 7:643–650, 2000.

[22] O. D. Kellogg. *Foundations of potential theory.* Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31. Springer-Verlag, Berlin-New York, 1967.

[23] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977.

[24] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[25] C. D. Levermore, M. Oliver, and E. S. Titi. Global well-posedness for models of shallow water in a basin with a varying bottom. *Indiana Univ. Math. J.*, 45(2):479–510, 1996.

[26] C. D. Levermore, M. Oliver, and E. S. Titi. Global well-posedness for the lake equations. *Phys. D*, 98(2-4):492–509, 1996.

[27] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications (Volume 1).* Travaux et recherches mathématiques. Dunod, 1968.

[28] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system. *Invent. Math.*, 105:415–430, 1991.

[29] N. Masmoudi. From Vlasov-Poisson system to the incompressible Euler system. *Comm. Partial Differential Equations*, 26(9-10):1913–1928, 2001.

[30] N. Masmoudi. Rigorous derivation of the anelastic approximation. *J. Math. Pures Appl.*, 88(3):230–240, 2007.

[31] J.T. Mendonça, R. Kaiser, H. Terças, and J. Loureiro. Collective oscillations in ultra-cold atomic gas. *Physical Review A*, 78:013408, 2008.

[32] Y. Ogura and N. Phillips. Scale analysis for deep and shallow convection in the atmosphere. *J. Atmos. Sci.*, 19:173–179, 1962.

[33] M. Oliver. Classical solutions for a generalized Euler equation in two dimensions. *J. Math. Anal. Appl.*, 215(2):471–484, 1997.

[34] A. Olivetti. *Effets collectifs et particules en interaction : des systèmes à longue portée aux atomes froids.* PhD thesis, Univ. Nice Sophia Antipolis, 2011.

[35] K. Pfaffelmoser. Global existence of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differ. Equ.*, 95:281–303, 1992.

[36] E.B. Saff and V. Totik. *Logarithmic potentials with external fields.* Grundlehren der mathematischen Wisenschatftens. Springer, 1997.

[37] S. Serfaty. Coulomb gases and Ginzburg–Landau vortices, 2014. Lecture Notes of the "nachdiplom vorleisung" course, ETH Zürich, Forschungsinstitut für Mathematik.

[38] R. Temam. On the Euler equations of incompressible perfect fluids. *J. Functional Analysis*, 20(1):32–43, 1975.

[39] R. Temam. Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids. In *Turbulence and Navier-Stokes equations (Proc. Conf., Univ. Paris-Sud, Orsay, 1975)*, pages 184–194. Lecture Notes in Math., Vol. 565. Springer, Berlin, 1976.

[40] T. Walker, D. Sesko, and C. Wieman. Collective behavior of optically trapped neutral atoms. *Physical Review Letters*, 64:408–412, 1990.

[41] D.J. Wineland, J.J. Bollinger, W.M. Itano, and J.D. Prestage. Angular momentum of trapped atomic particles. *J. Opt. Soc. Am. B*, 2(11):1721, 1985.