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Asymptotic modelling and design of some microlayered functionally graded heat conductors

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The paper presents a new method of modelling for certain microlayered heat conductors. So far the main attention related to the modelling of composite materials with deterministic structure was mainly restricted to the periodic composites (Bensoussan, Lions, and Papanicolaou [1]). The main feature of the presented approach is that it can be also applied to the class with a non periodic structure. These structures can be referred to as functionally graded in the direction normal to the interfaces between layers. At the same time the non-homogeneous material structure of the composite is slowly varying in this direction. That is why the obtained model equations have functionally but slowly varying coefficients. Moreover, this equations can be applied for designing functionally graded heat conductors with given *a priori* class of temperature fields.

1 Introduction

Composites with a functional gradation of materials (FGM) have their beginning in Japan. As an example we can mention here Japan swords which are made of different kinds of steels. An overview of papers on this subject can be found in Suresh and Mortensen [14].

The structure of functionally graded materials can be either stochastic or deterministic. Some approaches to the modelling of FGM with a deterministic structure in thermomechanics were summarized in monographs Woźniak et al. [15, 16].

Among functionally graded composites with a deterministic structure we can mention those made of large number of thin layers. These composites can be referred to as microlayered. So far, the main attention in thermomechanics of microlayered composites was given to the laminated media which have a periodic structure. We can mention here a series of papers by authors: Jędrzyński and Woźniak [4], Jędrzyński [6], Kaczyński and Mastysiak [7, 8], Łaciński [9], Łaciński and Woźniak [10], Mastysiak [11], Mastysiak, Pauk, and Yevtushenko [12], Michałak and Ostrowski [13].

The scope of this paper is restricted to the modelling and analysis of the heat conduction problems. Obviously the knowledge of the temperature field is necessary to carry out a very important problem of thermal stresses which depend on the temperature distribution provided that the problem can be treated as uncoupled. However the problem of modelling of thermal stresses (equations in linear elastic multilayered media) is much more complicated than that of the heat conduction and will be studied separately.

In this paper the object of analysis are two component microlayered functionally graded with a deterministic structure heat conductors.

The considerations are carried out in the framework of the linearized heat conduction theory on the basis of Fourier equations.

For the basic physical concepts of the heat conduction theory the reader is referred to Carslaw and Jaeger [3].

Two cross sections of functionally graded composites are shown in Figs. 1a,b.

The material gradation shown in Fig. 1a is referred to as transversal while that in Fig. 1b is termed longitudinal.

Let $\Omega = (0, L) \times \Xi$, where Ξ is a region on $O\xi^1\xi^2$ plane, stand for a region occupied by the conductor under consideration. Subsequently $x \in [0, L]$, $\xi = (\xi^1, \xi^2) \in \Xi$. Time coordinate is denoted by t , $t \in [0, T]$.

Define by: $\partial \equiv \frac{\partial}{\partial x}$, $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$, $\alpha = 1, 2$, $\partial_t \equiv \frac{\partial}{\partial t}$, $\bar{\nabla} = \left(\frac{\partial}{\partial \xi^1}, \frac{\partial}{\partial \xi^2} \right)$.

Let n be a number of layers in Ω , $\frac{1}{n} \ll 1$. Every layer is assumed to have a constant thickness λ , $\lambda \equiv \frac{L}{n}$.

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Fig. 1 A structure of FGM a) with the transversal gradation of layers, b) with the longitudinal gradation of layers.

The midplane of the j -th, $j = 1, \dots, n$, layer is situated on the coordinate plane $x = x_j \equiv \frac{\lambda}{2} + (j - 1)\lambda$, $j = 1, \dots, n$, in $Ox\xi^1\xi^2$ space. A midplane of an arbitrary layer is assumed to be its material symmetry plane.

Let $\nu = \nu(x, \xi^1, \xi^2)$ stand for a differentiable function defined in $\bar{\Omega}$, $0 < \nu(x, \xi^1, \xi^2) < 1$. The values of this function on symmetry midplanes will be interpreted as the fraction of the material in which this midplane is situated. Obviously, the fraction of the second component is equal to $1 - \nu(\cdot)$.

Function $\nu(\cdot)$ is assumed to satisfy the following restrictions:

1. $\lambda |\partial \nu(x, \xi)| < 1$, $(x, \xi) \in \bar{\Omega}$,
2. $\lambda |\partial_\alpha \nu(x, \xi)| < 1$, $(x, \xi) \in \bar{\Omega}$, $\alpha = 1, 2$.

For transversally graded materials $\nu \equiv \nu(x)$, $x \in [0, L]$ and for longitudinally graded materials $\nu \equiv \nu(\xi^1, \xi^2)$, $(\xi^1, \xi^2) \in \Xi$. If $\nu = \text{const}$ then we deal with the λ -periodic conductor.

The parts of region Ω occupied by pertinent material components will be denoted by Ω' and Ω'' .

Subsequently we denote $\Delta \equiv (-\frac{\lambda}{2}, \frac{\lambda}{2})$, $\Delta(x) \equiv (x - \frac{\lambda}{2}, x + \frac{\lambda}{2})$, $x \in (0, L)$.

Moreover $\Delta'(x_j, \xi) \equiv (x_j - \frac{\lambda}{2}\nu(x_j, \xi), x_j + \frac{\lambda}{2}\nu(x_j, \xi))$ and $\Delta''(x_j, \xi) \equiv \Delta(x_j, \xi) \setminus \bar{\Delta}'(x_j, \xi)$. It follows that $\Omega' = \sum_{j=1}^n \bigcup_{\xi \in \Xi} \Delta'(x_j, \xi)$, $\Omega'' = \sum_{j=1}^n \bigcup_{\xi \in \Xi} \Delta''(x_j, \xi)$ and that the j -th layer occupies region $\Delta(x_j) \times \Xi$.

The set of interfaces between Ω' and Ω'' is denoted by I , $I \equiv \bar{\Omega}' \cap \bar{\Omega}''$.

Let material components of heat conductor occupying parts Ω' and Ω'' of Ω assumed to be isotropic and homogeneous. The property of these components are determined by heat conductor coefficients k' , k'' and specific heats c' , c'' .

Define:

$$k(x, \xi) = \begin{cases} k' & \text{if } (x, \xi) \in \Omega' \\ k'' & \text{if } (x, \xi) \in \Omega'' \end{cases},$$

$$c(x, \xi) = \begin{cases} c' & \text{if } (x, \xi) \in \Omega' \\ c'' & \text{if } (x, \xi) \in \Omega'' \end{cases}.$$

Denote by $\theta = \theta(x, \xi^1, \xi^2, t)$ temperature field in $\bar{\Omega} \times [0, T)$ and by $q(\cdot)$, $q^1(\cdot)$, $q^2(\cdot)$ the components of heat flux vector. Let the heat conductor be subjected to the known heat sources field $f(\cdot)$ defined in Ω and for almost every $t \in [0, T)$. The heat conduction equations will be assumed in the well known form:

$$c(x, \xi) \partial_t \theta(x, \xi, t) + \partial q(x, \xi, t) + \partial_1 q^1(x, \xi, t) + \partial_2 q^2(x, \xi, t) + f(x, \xi, t) = 0,$$

$$q(x, \xi, t) = -k(x, \xi) \partial \theta(x, \xi, t), \quad q^\alpha(x, \xi, t) = -k(x, \xi) \partial_\alpha \theta(x, \xi, t), \quad \alpha = 1, 2$$
(1)

which hold a.e. in Ω and for every $t \in [0, T)$.

Heat flux component $q(\cdot)$ is continuous on interfaces I for every instant $t \in [0, T)$.

On the other hand we shall also design the various gradation of material components in a multilayered medium in order to realize certain temperature fields. This is an inverse problem with the respect to the boundary value problem which can be called “the design problem”.

$$g(x, \xi) = \begin{cases} g' & \text{if } (x, \xi) \in \Omega' \\ g'' & \text{if } (x, \xi) \in \Omega''. \end{cases}$$

We also introduce the locally periodic function $\bar{h}(\cdot, y, \xi) \in L^2(\Delta)$ for every $y \in [0, L]$, $\xi = (\xi^1, \xi^2) \in \bar{\Xi}$, the fragment of which in interval $(y - \frac{\lambda}{2}, y + \frac{\lambda}{2})$ is shown in Fig. 2.

$$\langle \bar{g} \rangle(y, \xi) \equiv \frac{1}{\lambda} \int_{y-\frac{\lambda}{2}}^{y+\frac{\lambda}{2}} \bar{g}(x, y, \xi) dx.$$
$$\theta^j(x, \xi, t) = \vartheta(x, \xi, t) + \bar{h}(x, x_j, \xi) \psi(x, \xi, t) \quad (2)$$

for every $x \in \bar{\Delta}(x_j)$, $\xi \in \Xi$, $t \in [0, T]$, $j = 1, \dots, n$, where $\vartheta(\cdot)$, $\psi(\cdot)$ are certain differentiable functions defined on $\bar{\Omega} \times [0, T]$ and $\bar{h}(\cdot, y, \xi) \in L^2(\Delta)$ is the locally periodic function introduced in Sect. 2.

The above assumption plays a similar role to the fairly close approximation in the well known method of asymptotic expansions in the framework of the homogenization procedure; for example c.f. Jikov et al. [5] p. 25.

From the purely formal point of view we also define

$$\theta_\varepsilon^y(x, \xi, t) = \vartheta(x, \xi, t) + \varepsilon \bar{h}_\varepsilon(x, y, \xi) \psi(x, \xi, t) \quad (3)$$

for $x \in \bar{\Delta}_\varepsilon(y)$, $\xi \in \Xi$, $t \in [0, T]$, and arbitrary $y \in [0, L]$.

The heat fluxes related to the temperature field $\theta_\varepsilon^y(\cdot)$ are

$$\begin{aligned} q_\varepsilon^y(x, \xi, t) &= -\bar{k}(x, y, \xi) \partial \theta_\varepsilon^y(x, \xi, t), \\ \tilde{q}_\varepsilon^{y\alpha}(x, \xi, t) &= -\bar{k}(x, y, \xi) \partial_\alpha \theta_\varepsilon^y(x, \xi, t). \end{aligned} \quad (4)$$

If $\varepsilon \rightarrow 0$ then

$$\begin{aligned} q_\varepsilon^y(x, \xi, t) &\rightharpoonup -\langle \bar{k} \rangle(y, \xi) \partial \vartheta(x, \xi, t) - \langle \bar{k} \partial \bar{h} \rangle(y, \xi) \psi(x, \xi, t), \\ q_\varepsilon^{y\alpha}(x, \xi, t) &\rightharpoonup -\langle \bar{k} \rangle(y, \xi) \partial_\alpha \vartheta(x, \xi, t), \\ \bar{c}(x, \xi, t) \partial_t \theta_\varepsilon^y(x, \xi, t) &\rightharpoonup \langle \bar{c} \rangle(y, \xi) \partial_t \vartheta(x, \xi, t) \end{aligned} \quad (5)$$

in $L_{loc}^2(R)$.

Notice that the weak limit passage has to be used as a tool of a modelling since the convergence in the strong sense does not hold.

Let us substitute the right hand side of decomposition (3) into (4) for heat flux. Bearing in mind the heat flux continuity across interfaces, after limit passage $\varepsilon \rightarrow 0$ we arrive at

$$\psi(x, \xi, t) \rightarrow \frac{\nu(x, \xi)(1 - \nu(x, \xi))(k' - k'')}{\nu(x, \xi)k'' + (1 - \nu(x, \xi))k'} \partial \vartheta(x, \xi, t). \quad (6)$$

The above limit passages applied to fields $\theta_\varepsilon^y(\cdot)$ lead to the model equations or every $(x, \xi) \in (0, L) \times \Xi \times [0, T]$ summarized in the next section.

4 Model equations

Substituting the right hand side of formula (6) into formula (5) and denoting these values by $q_0(\cdot)$, $q_0^\alpha(\cdot)$, $\alpha = 1, 2$ we obtain that:

$$\begin{aligned} q_0(x, \xi, t) &= -k_0(x, \xi) \partial \vartheta(x, \xi, t), \\ q_0^\alpha(x, \xi, t) &= -k_0^\alpha(x, \xi) \partial_\alpha \vartheta(x, \xi, t), \end{aligned}$$

where

$$\begin{aligned} k_0(x, \xi) &\equiv \frac{k'k''}{(1 - \nu(x, \xi))k' + \nu(x, \xi)k''}, \\ k_0^\alpha(x, \xi) &= k' \nu(x, \xi) + k''(1 - \nu(x, \xi)). \end{aligned} \quad (7)$$

The heat conduction equation takes the form

$$\partial q_0(x, \xi, t) + \partial_1 q_0^1(x, \xi, t) + \partial_2 q_0^2(x, \xi, t) + c_0(x, \xi) \partial_t \vartheta(x, \xi, t) + f(x, \xi, t) = 0, \quad (8)$$

where

$$c_0(x, \xi) = c' \nu(x, \xi) + c''(1 - \nu(x, \xi))$$

for $x \in (0, L)$, $\xi \in \Xi$, $t \in [0, T]$.

After finding solution for $\vartheta(\cdot)$ from the initial boundary value problem under consideration we calculate the approximate form of temperature field. To this end define $h_j(x, \xi) = \bar{h}(x, x_j, \xi)$

$$\tilde{\vartheta}^j(x, \xi, t) = \vartheta(x, \xi, t) + h_j(x, \xi) \frac{\nu(x, \xi)(1 - \nu(x, \xi))(k' - k'')}{\nu(x, \xi)k'' + (1 - \nu(x, \xi))k'} \partial\vartheta(x, \xi, t) \quad (9)$$

for every $x \in \bar{\Delta}(x_j)$, $\xi \in \Xi$, $t \in [0, T]$, $j = 1, \dots, n$.

Evidently for heat conductors which are periodic in the direction $0x$ axis (for an arbitrary $\xi \in \Xi$) we have $\nu = \nu(\xi)$. In this case the above equations represent the model of longitudinal heat conductors. If $\nu(\cdot) \in (0, 1)$ is constant then above model equations have constant coefficients. In this case the obtained results coincide with the well known homogenized model for periodically layered composites.

5 Example of application

The subsequent considerations are restricted to multilayered functionally graded heat conductors with a transversal gradation. Moreover we consider stationary problems and we neglect heat sources. From the formal point of view the functionally graded conductor under consideration can be defined as the fourtuple

$$FGM \equiv ((0, L) \times \Xi, k', k'', \nu(\cdot)), \quad \nu = \nu'(x) \quad \text{or} \quad \nu = \nu''(x), \quad \nu'(x) + \nu''(x) = 1, \quad x \in (0, L).$$

Let us assume that $(0, L) \times \Xi$, k', k'' are known.

The problem we are going to solve is: how for the given *a priori* temperature field $\vartheta(\cdot)$ $\vartheta \in C^1((0, L) \times \Xi)$ and $\vartheta(\cdot) \geq 0$ find the pertinent distribution of material component (defined by function $\nu(\cdot)$).

To simplify solution of the problem under consideration let us also assume that:

$$\bar{\nabla} \cdot \bar{\nabla} \vartheta(x, \cdot) \equiv 0 \forall x \in (0, L).$$

It follows that Eq. (8) reduces to the form:

$$\partial(k_0 \partial\vartheta) = 0 \forall \xi \in \Xi.$$

Hence the solution for $\partial\vartheta(\cdot)$ of above equation takes the form:

$$\partial\vartheta(x, \xi) = \frac{[(k'' - k')\nu(x) + k']}{k'} g(\xi), \quad \xi \in \Xi,$$

where $g(\cdot)$ is an arbitrary differentiable function.

Define $\alpha \equiv \frac{k''}{k'}$ and assume that $\alpha \in (0, 1)$, what does not restrict considerations.

From relations (7₁) and $\partial\vartheta(x, \xi) = \frac{[(k'' - k')\nu(x) + k']}{k'} g(\xi)$, $\xi \in \Xi$ it can be easily shown that:

1⁰ Solution to the problem exists if $\forall \xi \in \Xi$ function $\partial\vartheta(\cdot, \xi)$ is strongly monotone and $\partial\vartheta(0, \xi) \neq 0$, $\partial\vartheta(L, \xi) \neq 0$.

2⁰ If $\vartheta(\cdot, \xi)$, $\forall \xi \in \Xi$, is convex in $[0, L]$ then

$$\alpha \partial\vartheta(L, \xi) = \partial\vartheta(0, \xi),$$

$$\partial\vartheta(L, \xi) \leq g(\xi) \leq \frac{1}{\alpha} \partial\vartheta(0, \xi). \quad (10)$$

3⁰ If $\vartheta(\cdot, \xi)$, $\forall \xi \in \Xi$, is concave in $[0, L]$ then:

$$\alpha \partial\vartheta(0, \xi) = \partial\vartheta(L, \xi),$$

$$\partial\vartheta(0, \xi) \leq g(\xi) \leq \frac{1}{\alpha} \partial\vartheta(L, \xi). \quad (11)$$

From relation $\partial\vartheta(x, \xi) = \frac{[(k'' - k')\nu(x) + k']}{k'} g(\xi)$, $\xi \in \Xi$ we obtain that:

1⁰ if $\vartheta(\cdot, \xi)$ is not linear with respect to $x \in [0, L]$

$$\nu(x) = \frac{1}{1 - \alpha} \left(1 - \frac{\partial\vartheta(x, \xi)}{g(\xi)} \right), \quad \xi \in \Xi, x \in [0, L], \quad (12)$$

where: $\partial\vartheta(x, \cdot)$, $g(\cdot)$ are harmonic in Ξ and $g(\xi)$ is interrelated either by (10) or (11).

2⁰ If $\vartheta(\cdot, \xi)$ is linear with respect to $x \in [0, L]$ then $\nu(\cdot)$ is constant in $[0, L]$ and can take an arbitrary value from $(0, 1)$.

6 Illustrative example

For the sake of simplicity let us restrict the following consideration to the one dimensional problem, setting $\vartheta = \vartheta(x)$, $x \in [0, L]$.

Solution to the design problem will be discussed for cube, quadratic and linear distribution of $\vartheta(\cdot)$ for a transversally graded composites.

I $\vartheta(\cdot)$ is a cubic function

1⁰ if $\vartheta(\cdot)$ – is convex in $[0, L]$, $\vartheta(x) = cx^3 + ax^2 + bx$, $a, b, c > 0$
then from (12) the formula for $\nu(\cdot)$ takes form:

$$\nu(x) = \left[1 - \frac{3cx^2 + 2ax}{3cL^2 + 2aL} \right], x \in [0, L].$$

From this relation we obtain that in this example $\nu(\cdot)$ is concave in $[0, L]$. The obtained result is shown in Fig. 3.

Fig. 3 Diagram of $\nu(\cdot)$ for the given cube distribution of temperature ϑ for a transversally graded composite.

2⁰ if $\vartheta(\cdot)$ – is concave in $[0, L]$, $\vartheta(x) = cx^3 - ax^2 + bx$, $a, b, c > 0$
then from (12) we obtain that:

$$\nu(x) = \frac{3cx^2 - 2ax}{3cL^2 - 2aL}, x \in [0, L].$$

From this relation we obtain that in this example $\nu(\cdot)$ is convex in $[0, L]$. The obtained result is shown in Fig. 4.

Fig. 4 Diagram of $\nu(\cdot)$ for the given cube distribution of temperature ϑ for a transversally graded composite.

II $\vartheta(\cdot)$ is a quadratic function

1⁰ if $\vartheta(\cdot)$ – is convex in $[0, L]$, $\vartheta(x) = ax^2 + bx$, $a, b > 0$
then formula for $\nu(\cdot)$ takes form:

$$\nu(x) = \frac{L-x}{L}, x \in [0, L].$$

Diagram of $\nu(\cdot)$ for the given convex quadratic distributions of temperature ϑ is shown in Fig. 5.

Fig. 5 Diagram of $\nu(\cdot)$ for the given quadratic distributions of temperature ϑ for a transversally graded composite.

2^0 if $\vartheta(\cdot)$ – is concave in $[0, L]$ $\vartheta(x) = -ax^2 + bx$ $a > 0, b > 0$
then formula (12) takes the form:

$$\nu(x) = \frac{x}{L}, x \in [0, L].$$

Diagram of $\nu(\cdot)$ for the given concave quadratic distributions of temperature ϑ is shown in Fig. 6.

Fig. 6 Diagram of $\nu(\cdot)$ for the given quadratic distributions of temperature ϑ for a transversally graded composite.

It can be seen that for different quadratic distribution of temperature $\vartheta(\cdot)$:

if $\vartheta(\cdot)$ is a convex function then the form of function $\nu(\cdot)$ is given by the linear decreases function in form $\frac{L-x}{L}$ – in this case the distribution of $\vartheta(\cdot)$ does not depend of coefficients a, b ,

if $\vartheta(\cdot)$ is a concave function then the form of function $\nu(\cdot)$ is given by the linear increases function in form $\frac{x}{L}$ – in this case the distribution of $\vartheta(\cdot)$ does not depend of coefficients a, b .

III $\vartheta(\cdot)$ has a linear distribution if $\vartheta(x) = bx, b > 0$

then formula (12) takes form:

$$\nu(x) = \frac{g-b}{g(1-\alpha)}.$$

Diagram of $\nu(\cdot)$ for the linear distribution of temperature ϑ is shown in Fig. 7.

Fig. 7 Diagram of $\nu(\cdot)$ for the linear distribution of temperature ϑ for a transversally graded composites – periodic composites.

These examples were only illustrative examples more complicated examples will be presented in separated form.

In this contribution it was shown that the asymptotic modelling procedure which is well known in homogenization of periodically layered heat conductors can be also extended to the functionally graded (not periodic) heat conductors. In the contrast to the modelling of periodic laminates the coefficients in the model equations (8) are not constant but represented by smooth functions. These functions depend on material fractions $\nu'(\cdot)$, $\nu''(\cdot)$, where $\nu'(\cdot) + \nu''(\cdot) = 1$, which were postulated as smooth. In every specific case fractions $\nu'(\cdot)$, $\nu''(\cdot)$ are assumed to be known *a priori*. However, the above fractions can be also interpreted as unknowns if we are going to design the material structure of a layered composite. Moreover, these functions have to be slowly varying. This property is very important mainly for the application numerical procedures for solutions obtaining to specific boundary value problem. This fact makes it possible not only to solve specific boundary value problem for the functionally graded heat conductors but also to design new functionally graded composites by finding a suitable form of fractions $\nu'(\cdot)$, $\nu''(\cdot)$. This problem was studied in Sect. 5 for transversally graded laminates but can be also extended to designing of longitudinally graded composites. The effective properties of these conductors are represented by smooth functions given by formulas (7). It can be seen that these functional coefficients have a very simple analytical form and in the special case of periodically layered heat conductors coincide with the well known formulas obtained by homogenization. Let us also observe that the design problem formulated in Sect. 5 has a physical sense mainly in the case of longitudinal gradation of material properties. In this case we can design multilayered heat conductors which have different conductivities in the direction along the layers. Such situations take place in many engineering problems in which we deal with multilayered walls, the conductivity of which should be different in different parts of these walls.

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