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Stable Kneser Graphs are almost all not weakly Hom-Idempotent

Pablo Torres † Mario Valencia-Pabon‡

Abstract

A graph $G$ is said to be hom-idempotent if there is an homomorphism from $G^2$ to $G$, and weakly hom-idempotent if for some $n \geq 1$ there is a homomorphism from $G^{n+1}$ to $G^n$. Larose et al. [Eur. J. Comb. 19:867-881, 1998] proved that Kneser graphs $KG(n,k)$ are not weakly hom-idempotent for $n \geq 2k + 1$, $k \geq 2$. We show that 2-stable Kneser graphs $KG(n,k)_{2-\text{stab}}$ are not weakly hom-idempotent, for $n \geq 2k + 2$, $k \geq 2$. Moreover, for $s,k \geq 2$, we prove that $s$-stable Kneser graphs $KG(ks+1,k)_{s-\text{stab}}$ are circulant graphs and so hom-idempotent graphs.

Keywords: Cartesian product of graphs, Stable Kneser graphs, Cayley graphs, Hom-idempotent graphs.

1 Introduction

Let $[n]$ denote the set $\{1, \ldots, n\}$. For positive integers $n \geq 2k$, the Kneser graph $KG(n,k)$ has as vertices the $k$-subsets of $[n]$ and two vertices are connected by an edge if they have empty intersection. In a famous paper, Lovász [8] showed that its chromatic number $\chi(K(n,k))$ is equal to $n - 2k + 2$. After this result, Schrijver [10] proved that the chromatic number remains the same when we consider the subgraph $KG(n,k)_{2-\text{stab}}$ of $KG(n,k)$ obtained by restricting the vertex set to the $k$-subsets that are 2-stable, that is, that do not contain two consecutive elements of $[n]$ (where 1 and $n$ are considered also to be consecutive). Schrijver [10] also proved that the 2-stable Kneser graphs are vertex critical (or $\chi$-critical), i.e. the chromatic number of any proper subgraph of $KG(n,k)_{2-\text{stab}}$ is strictly less than $n - 2k + 2$; for this reason, the 2-stable Kneser graphs are also known as the Schrijver graphs. After these general advances, a lot of work has been done concerning properties of Kneser graphs and stable Kneser graphs (see [2, 3, 7, 1, 9] and references therein). For example, it is well known that for $n \geq 2k + 1$ the automorphism group of the Kneser graph $KG(n,k)$ is the symmetric group induced by the permutation action on $[n]$; see [4] for a textbook account. More recently, Braun [1] showed that the automorphism group of the 2-stable Kneser graphs $KG(n,k)_{2-\text{stab}}$ is the dihedral group of order $2n$.

The cartesian product $G \square H$ of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism.

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An homomorphism from a graph $G$ into a graph $H$, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. If $H$ is a subgraph of $G$ and $\phi : G \to H$ has the property that $\phi(u) = u$ for every vertex $u$ of $H$, then $\phi$ is called a retraction and $H$ is called a retract of $G$. If $\phi : G \to H$ is a bijection and $\phi^{-1}$ is also a homomorphism from $H$ to $G$, then $\phi$ is an isomorphism and we write $G \cong H$. In particular, if $G = H$ then $\phi$ is an automorphism if and only if it is bijective. Two graphs $G$ and $H$ are homomorphically equivalent, denoted by $G \leftrightarrow H$, if $G \to H$ and $H \to G$. A graph $G$ is called a core if it has no proper retracts, i.e., any homomorphism $\phi : G \to G$ is an automorphism of $G$. It is well known that any finite graph $G$ is homomorphically equivalent to at least one core $G^*$, as can be seen by selecting $G^*$ as a retract of $G$ with a minimum number of vertices. In this way, $G^*$ is uniquely determined up to isomorphism, and it makes sense to think of it as the core of $G$. It is widely known that Kneser graphs are core. Moreover, it is not difficult to deduce that any $\chi$-critical graph is a core. Therefore, any 2-stable Kneser graph is also a core, because they are $\chi$-critical graphs [10].

Let $A$ be a group and $S$ a subset of $A$ that is closed under inverses and does not contain the identity. The Cayley graph $Cay(A,S)$ is the graph whose vertex set is $A$, two vertices $u,v$ being joined by an edge if $u^{-1} v \in S$. Cayley graphs of cyclic groups are often called circulants.

A graph $G$ is said vertex-transitive if its automorphism group $\text{AUT}(G)$ acts transitively on its vertex-set. It’s well known that Cayley graphs and Kneser graphs are vertex-transitive. However, 2-stable Kneser graphs are not vertex-transitive in general. For example, no automorphism in $KG(6,2)_{2-\text{stab}}$ sends $\{1,3\}$ to $\{1,4\}$, since $\text{AUT}(KG(6,2)_{2-\text{stab}})$ is the dihedral group of order 12 acting on the set $\{1,2,\ldots,6\}$.

We write $G^n$ for the $n$-fold cartesian product of a graph $G$. A graph $G$ is said hom-idempotent if there is a homomorphism from $G^2$ to $G$, and weakly hom-idempotent if for some $n \geq 1$ there is a homomorphism from $G^{n+1}$ to $G^n$. Larose et al. [7] showed that the Kneser graphs are not weakly hom-idempotent. However, the technique used by Larose et al. [7] cannot be extended directly to the 2-stable Kneser graphs.

A subset $S \subseteq [n]$ is s-stable if any two of its elements are at least "at distance $s$ apart" on the $n$-cycle, that is, if $s \leq |i - j| \leq n - s$ for distinct $i, j \in S$. For $s, k \geq 2$ and $n \geq ks$, the $s$-stable Kneser graph $KG(n,k)_{s-\text{stab}}$ is the subgraph of $KG(n,k)$ obtained by restricting the vertex set of $KG(n,k)$ to the $s$-stable $k$-subsets of $[n]$.

In this paper, we show that almost all 2-stable Kneser graphs are not weakly hom-idempotent. Moreover, for $s, k \geq 2$, we show that $s$-stable Kneser graphs $KG(ks + 1,k)_{s-\text{stab}}$ are circulant graphs and so hom-idempotent graphs. In the sequel, we will use the term modulo $[n]$ to denote arithmetic operations on the set $[n]$ where $n$ represents the 0.

## 2 2-stable Kneser graphs

As we have mentioned in the previous section, Braun [1] showed that the automorphism group of the 2-stable Kneser graph $KG(n,k)_{2-\text{stab}}$ is the dihedral group $D_{2n}$ of order $2n$. We denote the elements of $D_{2n}$ as follows (arithmetic operations are taken modulo $[n]$):

- **Rotations**: Let $\sigma^0$ be the identity permutation on $[n]$ and, for $1 \leq i \leq n - 1$, let $\sigma^i = \sigma^{i-1} \circ \sigma^1$, where $\sigma^1$ is the circular permutation $(1,2,\ldots,n-1,n)$.

- **Reflexions:**
denotes the set of all shifts of \( G \). 

**Proposition 1.** For \( 1 \leq i \leq n \), let \( \rho_i \) be the permutation formed by the product of the transpositions \((i+1, i)\) for \( i \) is a fix point.

**Lemma 1.** Let \( n \geq 2k+2 \). Then, the only two shifts of the 2-stable Kneser graph \( KG(n, k)_{2\text{-stab}} \) are the rotations \( \sigma^1 \) and \( \sigma^{n-1} \).

**Proof.** It is very easy to deduce that the circular permutations \( \sigma^1 = (1, 2, 3, \ldots, n-1, n) \) and \( \sigma^{n-1} = (\sigma^1)^{-1} = (n, n-1, n-2, \ldots, 2, 1) \) are both shifts of the graph \( KG(n, k)_{2\text{-stab}} \). In order to prove that they are the only two shifts of \( KG(n, k)_{2\text{-stab}} \), we will proceed by cases. The arithmetic operations are taken modulo \( [n] \).

- **Rotations.** Clearly, the identity permutation \( \sigma^0 \) is not a shift. Now, we claim that for each \( i \in \{2, 3, \ldots, n-2\} \), there exists a vertex \( v_i \) in \( KG(n, k)_{2\text{-stab}} \) such that \( \{1, i+1\} \subseteq v_i \). In fact, vertex \( v_i \) can be computed as follows:
  - If \( 2 \leq i \leq 2k-1 \) then, set \( v_i = \{1, 1+2, 1+2, \ldots, 1+(k-1)\} \) if \( i \) is even, otherwise set \( v_i = \{1, 2+2, 2+2, \ldots, 2+2(k-1)\} \).
  - If \( 2k \leq i \leq n-2 \) then, set \( v_i = \{1, 1+2, 1+2, \ldots, 1+(k-2), i+1\} \).

Now, for each \( 2 \leq i \leq n-2 \), we know that \( \sigma^i(1) = 1+i \) and therefore, \( 1+i \in \sigma^i(v_i) \) which implies that \( \{v_i, \sigma^i(v_i)\} \) is not an edge of \( KG(n, k)_{2\text{-stab}} \). Thus, for \( 2 \leq i \leq n-2 \), \( \sigma^i \) is not a shift of \( KG(n, k)_{2\text{-stab}} \).

- **Reflexions.** We consider two cases:
  - Case \( n \) odd. For each \( 1 \leq i \leq n \), let \( v_i \) be a vertex in \( KG(n, k)_{2\text{-stab}} \) such that \( i \in v_i \). Trivially, such vertex \( v_i \) always exists. Now, we know that \( i \) is a fix point under the permutation \( \rho_i \) and thus, \( i \in \rho_i(v_i) \) which implies that \( \{v_i, \rho_i(v_i)\} \) is not an edge of \( KG(n, k)_{2\text{-stab}} \). Thus, for \( 1 \leq i \leq n \), \( \rho_i \) is not a shift of \( KG(n, k)_{2\text{-stab}} \).
  - Case \( n \) even. Analogous to the previous case, we can show that \( \rho_i \) is not a shift of \( KG(n, k)_{2\text{-stab}} \), for \( 1 \leq i \leq \frac{n}{2} \). Now, for each \( 1 \leq i \leq \frac{n}{2} \), let \( v_i = \{i+1, i+2, i+2, \ldots, i+1+2(k-2)+i-2\} \). Clearly, \( v_i \) is a 2-stable set, since \( i+1+2(k-2)+i-2 \) and \( i-2 \) are not consecutive integers modulo \( [n] \). So, \( v_i \) is a vertex of \( KG(n, k)_{2\text{-stab}} \) such that \( \{i+1, i-2\} \subseteq v_i \). However, \( \{i+1, i-2\} \subseteq \delta_i(v_i) \) which implies that \( \{v_i, \delta_i(v_i)\} \) is not an edge of \( KG(n, k)_{2\text{-stab}} \). Thus, for \( 1 \leq i \leq \frac{n}{2} \), \( \delta_i \) is not a shift of \( KG(n, k)_{2\text{-stab}} \).

\( \square \)

Larose et al. [7] showed the following useful results:

**Proposition 1 (Proposition 2.3 in [7]).** A graph \( G \) is hom-idempotent if and only if \( G \leftrightarrow Cay(Aut(G^*), S_{G^*}) \), where \( Aut(G^*) \) denotes the automorphism group of the core \( G^* \) of \( G \) and \( S_{G^*} \) denotes the set of all shifts of \( G^* \).
Theorem 1 (Theorem 5.1 in [7]). Let $G$ be a $\chi$-critical graph. Then $G$ is weakly hom-idempotent if and only if it is hom-idempotent.

Proposition 2. Let $n \geq 2k + 2$ and let $G$ denotes the graph $KG(n,k)_{2-stab}$. Then, $G \not\rightarrow \text{Cay}({\text{Aut}}(G), S_G)$, where $S_G$ are the shifts of $G$.

Proof. We know that the automorphism group of the graph $KG(n,k)_{2-stab}$ is the dihedral group $D_{2n}$ on $[n]$. Moreover, by Lemma 1, we known that the only two shifts of $KG(n,k)_{2-stab}$ are the circular permutations $\sigma^1 = (1,2,\ldots,n-1,n)$ and its inverse permutation $\sigma^{n-1} = (\sigma^1)^{-1} = (n,n-1,\ldots,2,1)$. Therefore, the Cayley graph $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ is a 2-regular graph, that is, each vertex (i.e. each automorphism in $D_{2n}$) has exactly two neighbors. In fact, notice that the vertex identity $\sigma^0$ has as neighbors the vertices $\sigma^1$ and $\sigma^{n-1}$, and as it’s well known, any Cayley graph is vertex-transitive, and thus, any vertex in $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ has exactly two neighbors. So, $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ is a disjoint union of cycles, which implies that $2 \leq \chi(\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})) \leq 3$. Moreover, we known that $\chi(\text{Cay}(n,k)_{2-stab}) = n - 2k + 2 \geq 4$, because $n \geq 2k + 2$. Therefore $KG(n,k)_{2-stab} \not\rightarrow \text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$. \hfill $\Box$

As mentioned in the previous section, we know that any 2-stable Kneser graph is a core. Therefore, by Propositions 1 and 2, and by Theorem 1, we have the following result.

Theorem 2. For any $n \geq 2k + 2$, the 2-stable Kneser graphs $KG(n,k)_{2-stab}$ are not weakly hom-idempotent.

3 $s$-stable Kneser graphs $KG(ks + 1, k)_{s-stab}$

Let $G$ denote the complement graph of the graph $G$, i.e. $G$ has the same vertex set of $G$ and two vertices are adjacent in $G$ if and only if they are not adjacent in $G$. Let $p$ be a positive integer. The $p$th power of a graph $G$, that we denoted by $G^{(p)}$, is the graph having the same vertex set as $G$ and where two vertices are adjacent in $G^{(p)}$ if the distance between them in $G$ is at most equal to $p$, where the distance of two vertices in a graph $G$ is the number of edges on the shortest path connecting them.

Let $n \geq 2k$ be positive integers. The Cayley graphs $\text{Cay}(Z_n, \{k, k + 1, \ldots, n - k\})$, that we denoted by $G(n,k)$, are known as circular graphs [11, 6], where $Z_n$ denote the cyclic group of order $n$. It is well known that the Kneser graph $KG(n,k)$ contains an induced subgraph isomorphic to $G(n,k)$. In fact, let $C(n,k)$ be the subgraph of $KG(n,k)$ obtained by restricting the vertex set of $KG(n,k)$ to the shifts modulo $[n]$ of the $k$-subset $\{1,2,\ldots,k\}$, that is, $\{1,2,\ldots,k\}, \{2,3,\ldots,k+1\}, \ldots, \{n,1,2,\ldots,k-1\}$. Define $\phi : G(n,k) \rightarrow C(n,k)$ by putting $\phi(u) = \{u+1, u+2, \ldots, u+k\}$ where the arithmetic operations are taken modulo $[n]$. Clearly, $\phi$ is a graph isomorphism. Notice also that the graph $G(n,k)$ is isomorphic to the graph $\overline{C_n^{(k-1)}}$, i.e. the complement graph of the $(k-1)$th power of a cycle $C_n$. Vince [11] has shown that $\chi(G(n,k)) = \lceil \frac{n}{k} \rceil$.

In the remaining of this section, we will always assume w.l.o.g. that any vertex $v = \{v_1, v_2, \ldots, v_k\}$ of the $s$-stable Kneser graph $KG(ks + 1, k)_{s-stab}$ is such that $v_1 < v_2 < \ldots < v_k$, where $s, k \geq 2$. For $i \in [k-1]$, let $l_i(v) = v_{i+1} - v_i$ and $l_k(v) = v_1 + (ks+1) - v_k$. If $C$ is the cycle on $ks + 1$ points labeled by integers $1, 2, \ldots, ks + 1$ in the clockwise direction and $v = \{v_1, v_2, \ldots, v_k\}$ is a vertex of the $s$-stable Kneser graph $KG(ks + 1, k)_{s-stab}$, then $l_i(v)$ gives the distance in the clockwise direction between $v_i$ and $v_{i+1}$ in $C$.
Lemma 2. Let $s, k \geq 2$ and let $v = \{v_1, v_2, \ldots, v_k\}$ be a vertex of $KG(k+1, k)_{s-stab}$. Then, $l_i(v) \in \{s, s+1\}$ for all $i \in [k]$. Moreover, there exists exactly one $i' \in [k]$ such that $l_i(v) = s+1).

Proof. By definition, $l_i(v) \geq s$ for any $i \in [k]$. The result follows from the fact that $\sum_{i=1}^{k} l_i(v) = ks + 1$.

Lemma 3. Let $s, k \geq 2$. The number of vertices of the graph $KG(k+1, k)_{s-stab}$ is equal to $ks + 1$.

Proof. Again, let $C$ be the cycle on $ks + 1$ points labeled by integers $1, 2, \ldots, ks + 1$ in the clockwise direction. From Lemma 2, we have that each vertex of $KG(k+1, k)_{s-stab}$ is uniquely determined by a clockwise circular interval of length $s + 1$ in $C$. Trivially there exist $ks + 1$ distinct clockwise circular intervals of length $s + 1$ in $C$ and the lemma holds.

Proposition 3. Let $s, k \geq 2$. Then, $G(k+1, k) \simeq KG(k+1, k)_{s-stab}$.

Proof. Let $C$ be a cycle on $ks + 1$ points. We assume that the vertices of $G(k+1, k)$ are disposed over $C$ in clockwise increasing order from $0$ to $ks$. By Lemmas 2 and 3, we are able to define the application $\phi : G(k+1, k) \rightarrow KG(k+1, k)_{s-stab}$ as follows: let $u$ be a vertex of $G(k+1, k)$ such that $u = jk + i$, where $0 \leq j \leq s - 1$ and $0 \leq i \leq k - 1$. Then, $\phi(u) = \{u_1, \ldots, u_k\}$ where,

$$u_r = \begin{cases} j + 1 + (r - 1)s, & \text{if } 1 \leq r \leq k - i \\ j + 2 + (r - 1)s, & \text{if } k - i + 1 \leq r \leq k. \end{cases}$$

Finally, define $\phi(k) = \{s + 1, 2s + 1, \ldots, ks + 1\}$. It is not difficult to prove that $\phi$ is a bijective function. It remains to show that $\phi$ is indeed a graph isomorphism. Let $u, v$ be two vertices in $C(k+1, k)$. In the sequel, we assume that $v > u$. In fact, if $u > v$ we can always swap $u$ and $v$. Let $u = jk + i$, where $0 \leq j \leq s - 1$ and $0 \leq i \leq k - 1$. Let $t = v - u$, where $1 \leq t \leq k - 1$. By construction, $\phi(v) \setminus \{v_p : k - i - t + 1 \leq p \leq k - i\} \subset \phi(u)$, where the arithmetic operations are taken modulo $[k]$. As $t \leq k - 1$ then, we must have that $\phi(u) \cap \phi(v) \neq \emptyset$. Analogously, let $k(s + 1) + 2 \leq v - u \leq ks$ and let $t = ks + 1 - (v - u)$. Then, $\phi(u) \setminus \{u_p : k - i - t + 1 \leq p \leq k - i\} \subset \phi(v)$, where the arithmetic operations are taken modulo $[k]$. Again, as $t \leq k - 1$ then, we must have that $\phi(u) \cap \phi(v) \neq \emptyset$. Notice that, by the previous argument, if $v - u = k$ or $v - u = (s - 1) + 1$ then, $\phi(u) \cap \phi(v) = \emptyset$. Therefore, let $t = v - u$, where $k + 1 \leq t \leq k(s - 1)$ and assume that there exists $r \in [k]$ such that $u_r \in \phi(u) \cap \phi(v)$. We consider the following cases:

- **Case $1 \leq r \leq k - i$.** As $u = jk + i$ then, $v = jk + i + xk + y$, where $0 \leq j \leq s - 1$, $0 \leq i \leq k - 1$, $1 \leq x \leq s - 1$, and $0 \leq i + y \leq k - 1$. If $u_r = v_r$, with $1 \leq r' \leq k - (i + y)$ then, $u_r = j + 1 + (r - 1)s = j + x + 1 + (r' - 1)s = v_r$ and thus $x \geq s$ which is a contradiction. If $k - (i + y) + 1 \leq r' \leq k$ then, $j + 1 + (r - 1)s = j + x + 2 + (r' - 1)s$ which implies that $x = (r - r')s - 1$. However, $r \leq k - i$ and $r' \geq k - i - y + 1$ and so, $x \leq (y - 1)s - 1$. Moreover, as $x \geq 1$ then, $y \geq 2$ and thus $x = s - 1$, but it implies that $t \geq k(s - 1) + 2$ which is a contradiction.

- **Case $k - i + 1 \leq r \leq k$.** As the previous case, $v = (j + x)k + (i + y)$, with $1 \leq x \leq s - 1$ and $0 \leq i + y \leq k - 1$. If $u_r = j + 2 + (r - 1)s = j + x + 1 + (r - 1)s = v_r$, with $1 \leq r' \leq k - (i + y)$ then, $x = (r - r')s + 1$ which implies that $x \geq (y + 1)s + 1 \geq s$, that is a contradiction. If $k - (i + y) + 1 \leq r' \leq k$ then, $j + 2 + (r - 1)s = j + x + 2 + (r' - 1)s$. Then, $x = (r - r')s$ and as $x \geq 1$ then $r - r' > 0$, but it implies that $x \geq s$ which is again a contradiction.
Therefore, vertices $u, v$ in $C(ks + 1, k)$ are adjacent if and only if vertices $\phi(u), \phi(v)$ in $KG(ks + 1, k)_{s-\text{stab}}$ are adjacent.

Notice that if $n \geq 2r + 3$ the automorphism group of the $r^{th}$ power of a $n$-cycle, $C_n^{(r)}$, is the dihedral group $D_{2n}$ (see Remark 1) and thus, $\text{AUT}(C_n^{(r)})$ is also $D_{2n}$.

**Remark 1.** For $r \geq 1$ and $n \geq 2r + 3$ the automorphism group of $C_n^{(r)}$ is the dihedral group $D_{2n}$.

In fact, it is not hard to see that $D_{2n}$ is a subgroup of $\text{AUT}(C_n^{(r)})$. Let $\alpha \in \text{AUT}(C_n^{(r)})$ and let $N[i]$ denote the closed neighborhood of a vertex $i$ in $C_n^{(r)}$. Assume that $\alpha$ sends two consecutive vertices $i, j$ in the cycle to non consecutive vertices $i', j'$ in the cycle, respectively. Observe that $|N[i] \cap N[j]| = 2r$, but $|N[i'] \cap N[j']| \leq 2r - 1$ since there exists at least two vertices in $N[i'] \setminus N[j']$. Therefore, any automorphism of $C_n^{(r)}$ sends consecutive vertices to consecutive vertices. From this fact it is easy to see that any automorphism of $C_n^{(r)}$ belongs to $D_{2n}$.

Therefore, by Proposition 3, the automorphism group of the graph $KG(ks + 1, k)_{s-\text{stab}}$ is the dihedral group $D_{2(ks+1)}$. However, the problem to compute the automorphism group of the graph $KG(n, k)_{s-\text{stab}}$ for $k \geq 2$, $s > 2$, and $n > ks + 1$ is still open.

Another direct consequence of Proposition 3 is that $\chi(KG(ks + 1, k)_{s-\text{stab}}) = s + 1$. This fact has been found also by Meunier (see Proposition 1 in [9]).

Let $\text{Cay}(A, S)$ be a Cayley graph. If $a^{-1}Sa = S$ for all $a \in A$, then $\text{Cay}(A, S)$ is called a normal Cayley graph.

**Lemma 4 ([5]).** Any normal Cayley graph is hom-idempotent.

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the circulant graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). Therefore, by Proposition 3 and Lemma 4, we have the following result.

**Theorem 3.** Let $s, k \geq 2$. Then, $KG(ks + 1, k)_{s-\text{stab}}$ is hom-idempotent.

**References**


