Haefliger structures and symplectic/contact structures

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Abstract. For some geometries including symplectic and contact structures on an n-dimensional manifold, we introduce a two-step approach to Gromov's h-principle. From formal geometric data, the first step builds a transversely geometric Haefliger structure of codimension n. This step works on all manifolds, even closed. The second step, which works only on open manifolds and for all geometries, regularizes the intermediate Haefliger structure and produces a genuine geometric structure. Both steps admit relative parametric versions. The proofs borrow ideas from W. Thurston, like jiggling and inflation. Actually, we are using a more primitive jiggling due to R. Thom.

1. Introduction

We consider geometric structures on manifolds, such as the following: symplectic structure, contact structure, foliation of prescribed codimension, immersion or submersion to another manifold. We recall that, in order to provide a given manifold M with such a structure, Gromov's h-principle consists of starting from a *formal* version of the structure on M (this means a *non-holonomic* – that is, non-integrable – section of some jet space) and deforming it until it becomes *genuine* (holonomic) [10]. In the present paper, we introduce a two-step approach to the h-principle for such structures.

From the formal data, the first step builds a Haefliger structure of codimension zero on M, transversely geometric; this concept will be explained below. For each of the geometries above-mentioned, the first step works for every manifold M, even closed.

The second step, which is works for open manifolds only, regularizes the intermediate Haefliger structure, providing a genuine geometric structure. Both steps admit relative parametric versions.

An essential tool in both steps consists of *jiggling.* We recall that Thurston's work on foliations used his famous Jiggling Lemma [19]. As A. Haefliger told us [9], Thurston himself was aware that this lemma applies for getting some h-principles in the sense of Gromov. In a not very popular paper by R. Thom [18], we discovered a more primitive jiggling lemma that is remarkably suitable for the needs of our approach.

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1.1. Groupoids and geometries. According to O. Veblen and J.H.C. Whitehead [20], a geometry in dimension n is defined by an n-dimensional model manifold X (often $\mathbb{R}^n$) and by an open subgroupoid Γ in the groupoid $\Gamma(X)$ of the germs of local C^∞-diffeomorphisms of X; here the topology on $\Gamma(X)$ is meant to be the *sheaf* topology. In what follows, we use the classical notation $\Gamma_n := \Gamma(\mathbb{R}^n)$. Here are examples of such open subgroupoids.

(1) When n is even, $\Gamma_n^{\mathrm{symp}} \subset \Gamma_n$ denotes the subgroupoid of germs preserving the standard symplectic form of $\mathbb{R}^n$.
(2) When n is odd, $\Gamma_n^{\mathrm{cont}} \subset \Gamma_n$ denotes the subgroupoid of germs preserving the standard (positive) contact structure of $\mathbb{R}^n$.
(3) For $n = p + q$, one has the subgroupoid $\Gamma_{n,q}^{\mathrm{fol}} \subset \Gamma_n$ preserving the standard foliation of codimension q (whose leaves are the p-planes parallel to $\mathbb{R}^p$).
(4) When Y is any q-dimensional manifold and $X = \mathbb{R}^p \times Y$, one has the subgroupoid $\Gamma_n^Y \subset \Gamma(X)$ of the germs of the form $(x, y) \mapsto (f(x, y), y)$.

1.2. Γ-foliations. The next concept goes back to A. Haefliger [8]. For an open subgroupoid Γ of $\Gamma(X)$, a Γ-*foliation* on a manifold E is meant to be a codimension-n foliation on E equipped with a *transverse geometry* associated with Γ and invariant by holonomy. More precisely, this foliation is defined by a maximal atlas of submersions $(f_i : U_i \to X)$ from open subsets of E into X such that, for every i, j and every $x \in U_i \cap U_j$, there is a germ $\gamma_{ij} \in \Gamma$ at point $f_j(x)$ verifying

$$[f_i]_x = \gamma_{ij}\,[f_j]_x$$

where $[\text{-}]_x$ stands for the germ at x. When $n = \dim E$, one also speaks of a Γ-*geometry* on E.

Here are examples related to the previous list of groupoids. The first two have been already considered by D. McDuff in [13].

(1) A Γ_n^{symp}-foliation on E amounts to a closed differential 2-form Ω on E, whose kernel is of codimension n at every point. The closedness of Ω is equivalent to the conjunction of the next facts:
    - the codimension-n plane field $(x \in E \mapsto \ker \Omega_x)$ is integrable,
    - Ω is basic[1] with respect to that foliation,
    - Ω is closed on a total transversal.
(2) A Γ_n^{cont}-foliation on E amounts to a codimension-one plane field P on E defined by an equation $A = 0$ where A is a differential form of degree 1, unique up to multiplying by a positive[2] function, which satisfies the next conditions:
    - the n-form $A \wedge (dA)^{(n-1)/2}$ is closed and has a codimension-n kernel K_x at every point $x \in E$; in particular, the field $(x \mapsto K_x)$ is integrable, tangent to a codimension-n foliation denoted by $\mathcal{K}$;
    - K_x is a vector sub-space of P_x for every x;
    - P is invariant by the holonomy of $\mathcal{K}$.
(3) A $\Gamma_{n,q}^{\mathrm{fol}}$-foliation on E consists of a flag $\mathcal{F} \subset \mathcal{G}$ of two nested foliations of respective codimensions n and q in E with $n > q$.

[1]We recall that a form α is said to be basic with respect to a foliation $\mathcal{F}$ if the Lie derivative $L_X\alpha$ vanishes for every vector field X tangent to $\mathcal{F}$; this is the infinitesimal version of the *invariance by holonomy.*

[2]Here, we limit ourselves to co-orientable contact structures.

(4) A Γ_n^Y-foliation on E consists of a codimension-n foliation and a submersion $w : E \to Y$ which is constant on every leaf ([8] p. 145).

1.3. Haefliger's Γ-structures*.* A. Haefliger defined a Γ-*structure* as some class of cocycles valued in Γ (see [8] p. 137). This definition, that makes sense on every topological space and for every topological groupoid Γ, allowed him to build a classifying space $B\Gamma$ for these structures ([8] p. 140). A second description ([7] p. 188) is more suitable for our purpose when the topological space is a manifold M and when the groupoid is an open subgroupoid Γ in the groupoid of germs $\Gamma(X)$ of a n-manifold X. Here it is.

A Γ*-structure on* M *consists of a pair* $\xi = (\nu, \mathcal{F})$ *where*

- ν *is a real vector bundle over* M *of rank* n*, called the* normal bundle*; its total space is denoted by* $E(\nu)$*; and* $Z : M \to E(\nu)$ *denotes the zero section;*
- $\mathcal{F}$ *is a germ along* $Z(M)$ *of* Γ*-foliation in* $E(\nu)$ *transverse to every fibre of* ν*.*

An important feature of Γ-structures is that pulling back by smooth maps (in our restricted setting) is allowed without assuming any transversality: if $f : N \to M$ is a smooth map and ξ is a Γ-structure on M, one defines $f^*\xi$ as the Γ-structure on N whose normal bundle is $f^*\nu$ equipped with the Γ-foliation $F^{-1}(\mathcal{F})$, where F is a bundle morphism over f which is a fibre-to-fibre linear isomorphism.

Let $H^1(M;\Gamma)$ (resp. $H^1_\nu(M;\Gamma)$) denote the space of the Γ-structures on M (resp. those whose normal bundle is ν). It is a topological space since Γ is a topological groupoid; their elements are denoted by $\xi = (\nu, \mathcal{F})$. In what follows, we are mainly interested in the case where $\dim M = n$ and ν is isomorphic to the tangent space τM; in that case, the elements are just denoted $\mathcal{F}$.

In what follows, a Γ-structure on M whose normal bundle is the tangent bundle τM will be called a *tangential* Γ-structure. For short, when it is not ambiguous, we write $\mathcal{F}$ for $(\tau M, \mathcal{F})$. We introduced the general definition of Γ-structure – at least in the smooth case – since we are going to refer to in several places.

1.4. Underlying formal geometries. In the cases of the geometries (1), (2) and (4) given above, that is, for $\Gamma = \Gamma_n^{\mathrm{symp}}$, Γ_n^{cont} or Γ_n^Y, every Γ-structure has an underlying formal Γ-geometry in the sense of Gromov. But, we do not intend to enter Gromov's generality. We just describe what they are. In the case of the geometry (3), every $\Gamma_{n,q}^{\mathrm{fol}}$-structure has an underlying object, somewhat formal, but more complicated than in the cases (1), (2), (4).

(1) Assume n is even. Given $\mathcal{F} \in H^1_{\tau M}(M;\Gamma_n^{\mathrm{symp}})$, one has an associated basic closed 2-form Ω on a neighborhood of $Z(M)$ in the total space TM. Its kernel is everywhere transverse to the fibres. Therefore, Ω defines a non-singular 2-form ω on M by the formula $\omega_x := \Omega_{Z(x)}|T_xM$ for every $x \in M$. This is the underlying *formal symplectic structure.*

(2) Assume n is odd. Given $\mathcal{F} \in H^1_{\tau M}(M;\Gamma_n^{\mathrm{cont}})$, one has a $(n-1)$-plane field Q defined near $Z(M)$ with the following properties:
   - at each point z near $Z(M)$, the plane Q_z is vertical, meaning that it is contained in the fibre of TM passing through z;
   - Q_z carries a symplectic bilinear form, well defined up to a positive factor;

- as a (conformally) symplectic bundle over a neighborhood of $Z(M)$, the plane field Q is invariant by the holonomy of $\mathcal{F}$.

Then, there is a symplectic sub-bundle whose fibre at $x \in M$ is $S_x := Q_{Z(x)}$ (the indeterminancy by a positive factor is irrelevant here). This is the underlying *formal contact structure*. Another way to say the same thing consists of the following: S is the kernel of a 1-form α and there is a 2-form β on M such that β makes S be a symplectic bundle; equivalently, it may be said that $\alpha \wedge \beta^{\frac{n-1}{2}}$ is a volume form.

(3) Assume $n = p + q$. Given $\mathcal{F} \in H^1_{\tau M}(M; \Gamma^{\mathrm{fol}}_{n,q})$, one has an associated foliation $\mathcal{G}$ of codimension q on a neighborhood of $Z(M)$ in the total space TM. The foliation $\mathcal{G}$ induces on M a Γ_q-structure $\gamma := Z^*(\mathcal{G})$, whose normal bundle is $\nu_\gamma := Z^*(\nu_{\mathcal{G}})$; and a monomorphism of vector bundles $\epsilon : \nu_\gamma \hookrightarrow \tau M$. Indeed, for every point $x \in M$, the foliation $\mathcal{G}$ being transverse to $T_x M$ at $Z(x)$, the normal to $\mathcal{G}$ at $Z(x)$ embeds into $T_x M$. The pair (γ, ϵ), an *augmented* Γ_q*-structure* (according to the vocabulary from [4]), plays the role of a formal geometry associated to $\mathcal{F}$.

(4) Assume $n \geq \dim Y$. Given $\mathcal{F} \in H^1_{\tau M}(M, \Gamma^Y_n)$, one has an associated submersion w from a neighborhood of $Z(M)$ in the total space TM to Y. This submersion w induces a *formal submersion* (f, F) from M to Y, that is, a bundle epimorphism from TM to TY whose value at every $x \in M$ is: $f(x) = w(Z(x))$, $F_x := Dw_{Z(x)} | T_x M$.

Observe that all these spaces of formal geometries have natural topologies. Our first theorem yields a converse: for the above geometries, formal Γ-geometries lead to Γ-structures.

Theorem 1.5. *Let M be an n-dimensional manifold, possibly closed. Let Γ be a groupoid in the set of n-dimensional geometries $\{\Gamma^{\mathrm{symp}}_n, \Gamma^{\mathrm{cont}}_n, \Gamma^{\mathrm{fol}}_{n,q}, \Gamma^Y_n\}$. Then, the forgetful map from $H^1_{\tau M}(M; \Gamma)$ to the corresponding space of formal Γ-geometries is a homotopy equivalence.*

Remarks 1.6. 1) Actually, according to R. Palais ([16] Theorem 15), the considered spaces have the property that a weak homotopy equivalence is a genuine homotopy equivalence. Thus, it is sufficient to prove that the mentioned forgetful map is a weak homotopy equivalence, meaning that it induces an isomorphism of homotopy groups in each degree.

2) In the case of symplectic/contact geometry, D. McDuff proved theorems of the same flavor using the *convex integration* technique of Gromov ([13], see also [5] p. 104, 138).

3) Let $\mathcal{F} \in H^1_{\tau M}(M; \Gamma)$. By taking a section s of τM valued in the domain foliated by $\mathcal{F}$ and generic with respect to $\mathcal{F}$, there is an induced Γ-geometry with singularities on $M \cong s(M)$. This seems to be a very natural notion of singular symplectic/contact structure. It follows from Theorem 1.9 that the singular locus may be localized in a ball of M.

1.7. Homotopy and regularization. Our second theorem will allow us to regularize every parametric family of Γ-structures on every manifold M which is *open*, that is, which has no closed connected component; this terminology will be permanently used in what follows.

A *homotopy* (also called a *concordance*[3]) between two Γ-structures $(\nu_i, \mathcal{F}_i)$ $(i = 0, 1)$ on M is a Γ-structure on $M \times [0, 1]$ whose restriction to $M \times 0$ (resp. $M \times 1$) equals $(\nu_0, \mathcal{F}_0)$ (resp. $(\nu_1, \mathcal{F}_1)$). Of course, ν_0 and ν_1 must be isomorphic.

A Γ-structure $(\nu, \mathcal{F})$ is said to be *regular* if the foliation $\mathcal{F}$ is transverse not only to the fibres of ν but also to $Z(M)$ in $E(\nu)$. This bi-transversality of $\mathcal{F}$ induces an isomorphism

[3]This second word emphasizes the difference with a one-parameter family of Γ-structures.

$\nu \cong \tau\big(Z(M)\big)$. In that case, the pull-back $Z^*(\mathcal{F})$ is a Γ-geometry on M, namely the foliation by points equipped with a transverse Γ-geometry.

1.8. The exponential Γ_n-structure. Given a complete Riemannian metric on the n-manifold M, there is a well defined map

$$exp : TM \to M.$$

When restricting exp to a small neighborhood U of $Z(M)$ in TM, we get a submersion to M. The foliation defined by the level sets of $exp|U$ represents a regular Γ_n-structure on M, denoted by $\mathcal{F}_{exp} \in H^1_{\tau M}(M; \Gamma_n)$. Up to isomorphism (vertical isotopy in TM), $\mathcal{F}_{exp}$ does not depend on the Riemannian metric as it is shown by the next construction.

Consider the product $M \times M$ and its diagonal $\Delta \cong M$. We have two projections $p_v, p_h : M \times M \to \Delta$, respectively the vertical and the horizontal projection. A small tube U about Δ equipped with p_v is isomorphic to τM as micro-bundle. Then, the same tube equipped with p_h defines the Γ_n-structure $\mathcal{F}_{exp}$.

We recall the fundamental property of the differential of exp (independent of any Riemannian metric):

$$d(exp|T_xM)_{Z(x)} = Id : T_xM \to T_xM.$$

As a consequence, if $f : M \to Y$ is a smooth map and $v \in T_xM$, one has

$$f \circ exp_x(v) - f(x) = df_x(v) + o(\|v\|). \tag{1.1}$$

Theorem 1.9. *Let X be an n-manifold, let $\Gamma \subset \Gamma(X)$ be an open subgroupoid and let M be a (connected) n-manifold. Assume that M is open (that is, no connected component is closed). Let*

$$s \mapsto \xi_s = (\tau M, \mathcal{F}_s) : \mathbb{D}^k \to H^1_{\tau M}(M; \Gamma)$$

be a continuous family of tangential Γ-structures, parametrized by the compact k-disk ($k \geq 0$), such that for every $s \in \partial\mathbb{D}^k$, the Γ-structure ξ_s is regular and $\mathcal{F}_s$ is tangent to $\mathcal{F}_{exp}$ along $Z(M)$.

Then, there exists a continuous family of concordances

$$s \mapsto \bar{\xi}_s = (\tau M \times [0,1], \bar{\mathcal{F}}_s) : \mathbb{D}^k \to H^1_{\tau M}(M \times [0,1]; \Gamma)$$

such that

- *$\bar{\mathcal{F}}_s = pr_1^*(\mathcal{F}_s)$ for every $s \in \partial\mathbb{D}^k$, where $pr_1 : M \times [0,1] \to M$ is the projection;*
- *$\bar{\mathcal{F}}_s|(M \times 0) = \mathcal{F}_s$ for every $s \in \mathbb{D}^k$;*
- *for every $s \in \mathbb{D}^k$, the Γ-structure $\bar{\xi}_s|(M \times 1)$ is regular and $\bar{\mathcal{F}}_s$ is tangent to $\mathcal{F}_{exp}$ along $Z(M \times 1)$.*

Remark 1.10. The Smale-Hirsch classification of immersions $S \to Y$ (see [17, 12]), where S is a closed manifold of dimension less than $\dim Y$, is covered by Theorem 1.9; in particular, the famous *sphere eversion* amounts to the case where S is the 2-sphere, $Y = \mathbb{R}^3$ and $k = 1$. Let us show it.

Let $(f, F) : TS \to TY$ be a formal immersion. Then thanks to F we have a monomorphism $F_* : \tau S \to f^*\tau Y$ over Id_S. Let ν be a complementary sub-bundle to the image of F_*; when f is an immersion, ν is its normal bundle. Let $\hat{S}$ be a disk bundle in ν; it is a compact manifold with non-empty boundary and $\dim \hat{S} = \dim Y$. Thus, instead of immersing S to Y one tries to immerse $\hat{S}$ to Y; if it is done, the restriction to the 0-section yields an immersion of S to Y with normal bundle ν. The formal immersion $(f, F) : TS \to TY$ easily extends to a formal immersion $(\hat{f}, \hat{F}) : \hat{S} \to Y$ in codimension 0. Since $\hat{F} : T_x\hat{S} \to T_{\hat{f}(x)}Y$ is a linear isomorphism for every $x \in \hat{S}$, the level sets of $exp_Y \circ F$ is a Γ_n^Y-foliation $\mathcal{F}$ near $Z(\hat{S})$, that is, a Γ_n^Y-structure on $\hat{S}$. Moreover, thanks to Equation (1.1), if $\hat{f}$ is an immersion $\mathcal{F}$ is tangent to $\mathcal{F}_{exp}$; here exp stands for $exp_{\hat{S}}$. Then, Theorem 1.9 applies and yields the desired immersion (or family of immersions).

Corollary 1.11. *Let Γ be a groupoid as in Theorem 1.9 and $\xi = (\tau M, \mathcal{F})$ be a tangential Γ-structure on a closed manifold M. Then, after a suitable concordance, all singularities (that is, the points where $\mathcal{F}$ is not transverse to $Z(M) \cong M$) are confined in a ball.*

Proof. Let $B \subset M$ be a closed n-ball. Apply Theorem 1.9 to $\xi|(M \smallsetminus int\, B)$. We are given a regularization concordance C of this restricted Γ-structure. Since this concordance is given on a manifold with boundary, it extends to the whole manifold. Indeed, $B \times [0,1]$ collapses to $\big(B \times \{0\}\big) \cup \big(\partial B \times [0,1]\big)$. $\square$

Remark 1.12. Y. Eliashberg & E. Murphy ([6] Corollary 1.6) gave a similar result for symplectic structures on closed almost symplectic manifolds of dimension greater than 4. Moreover, in the confining ball B their singular symplectic structure is the *negative* cone of an *overtwisted* contact structure on ∂B. Their proof is based on the new techniques in contact geometry initiated by E. Murphy [15] and developped in [1].

1.13. The classical h-principle for Γ-geometries. For a groupoid Γ as listed in 1.1 the h-principle states the following:

If M is an open n-manifold, the space of Γ-geometries on M has the same (weak) homotopy type as the space of formal Γ-geometries on M.

This statement follows from Theorems 1.5 and 1.9.

Proof. We start with a k-parameter family of formal Γ-geometries on M, $k \geq 0$, which are genuine Γ-geometries when the parameter s lies in $\partial\mathbb{D}^k$. Then, for every $s \in \mathbb{D}^k$, the foliation $\mathcal{F}_{exp}$ is a Γ-foliation near $Z(M)$. Thus, Theorem 1.5 applies and yields a k-parameter family of Γ-structures on M which remains unchanged when $s \in \partial\mathbb{D}^k$. Now, since M is open, Theorem 1.9 applies and all the relative homotopy groups of the pair (*formal* Γ-*geometries*, Γ-*geometries*) vanish. $\square$

The article is organized as follows. In Section 2, we detail the tool that goes back to R. Thom [18] and we prove Theorem 1.5 for submersion structures and for foliation structures. The next sections are devoted to the proof of Theorem 1.5 in the case of transversely symplectic structures. The existence part is treated in Section 3. The family of such structures are considered in Section 4; the proof of Theorem 1.5 is completed there when the groupoid is $\Gamma_n^{\rm symp}$. In Section 5, we adapt the proof to the groupoid $\Gamma_n^{\rm cont}$. Finally, in Section 6, we solve the problem of regularizing the Γ-structures on every open manifold.

2. Thom's subdivision and jiggling

Reference [18] is the report of a lecture where R. Thom announced a sort of homological h-principle (ten years before Gromov's thesis). A statement and a sketch of proof are given there;

the details never appeared. From this text, we extracted an unusual subdivision process of the standard simplex and we derived two jiggling formulas[4]. Our jiggling will be vertical while Thom's jiggling is transverse to the fibres in some jet bundle. Nevertheless, we whall speak of Thom's jiggling for it mainly relies on Thom's subdivision. Actually, neither statement nor proof nor formula were written in [18], only words describing the object, a beautiful object indeed.

Here is a good occasion for mentioning that the famous *Holonomic Approximation Theorem* by Y. Eliashberg and N. Mishachev ([5] Chapter 3) is also based on a jiggling process, even if that word is not used there. The difference between their jiggling and ours is that the first one takes place in the manifold itself while the second one is somehow vertical in the total space of a fibre bundle.

Proposition 2.1. *Let Δ^n denote the standard n-simplex. For every positive integer n, there exist a non-trivial subdivision K_n of Δ^n and a simplicial map $\sigma_n : K_n \to \Delta^n$ such that:*
1) (non-degeneracy) *the restriction of σ_n to any n-simplex of K_n is surjective;*
2) (heredity) *for any $(n-1)$-face F of Δ^n, the intersection $K_n \cap F$ is simplicially isomorphic to K_{n-1} and $\sigma_n|F \cong \sigma_{n-1}$.*

Proof. Condition 2) implies $\sigma_n(v) = v$ for any vertex of Δ_n. For K_1, we may take $\Delta^1 = [0, 1]$ subdivised by two interior vertices: $0 < v_1 < v_0 < 1$ and we define σ_1 by $\sigma_1(v_1) = 1$ and $\sigma_1(v_0) = 0$.

For $n = 2$, let A, B, C denote the vertices of Δ^2. The polyhedron K_2 will be built in the following way: subdivide each edge of Δ_2 as K_1 subdivides Δ_1; add an interior triangle with vertices a, b, c so that the line supporting $[b, c]$ is parallel to $[B, C]$ and separates A from a, etc.; join a to the four vertices of $[B, C]$, etc. The simplicial map σ_2 is defined by $a \mapsto A, b \mapsto B, c \mapsto C$ and by imposing to coincide with σ_1 on each edge of Δ^2. Condition 1) is easily checked.

This construction extends to any dimension. If K_{n-1} and σ_{n-1} are known, each facet of Δ^n will be sudivided as K_{n-1}. Then, one puts a small n-simplex δ^n in the interior of Δ^n applying the same rules of parallelism and separation as for $n = 2$. Each vertex v of δ^n will be joined to the vertices of the facet $F(v)$ of Δ^n in front of v, this facet being sudivided by K_{n-1}. The map σ_n maps v to the vertex V which is opposite to $F(v)$ in Δ^n. □

Remarks 2.2. 1) This subdivision may be iterated r times producing a subdivision K_n^r which is arbitrarily fine and a simplicial map $\sigma_n^r : K_n^r \to \Delta^n$ fulfilling the two conditions of Proposition 2.1. More precisely, thinking of σ_n as a map from Δ^n to itself, σ_n^r will denote its r-th iterate and K_n^r is defined by the next formula:

$$K_n^r = \left(\sigma_n^{r-1}\right)^{-1}(K_n)\,. \tag{2.1}$$

We will call σ_n^r an *r-folding map.*

2) Thanks to heredity (condition that the barycentric subdivision does not fulfil), this subdivision of the standard simplex and the r-folding map apply to any polyhedron.

3) It is worth noticing that Thom's subdivision is not *crystalline* in the sense of H. Whitney ([21] Appencice II). Thus, it does not fit Thurston's techniques of *jiggling* (compare [19]).

[4] Thom speaks of "dents de scie" (saw teeth); we keep the word *jiggling* that W. Thurston introduced in [19].

Actually, the above construction has an unfolding property which is stated in the next proposition.

Proposition 2.3. *With the above notations, for every n-simplex τ of K_n^{r-1}, the restriction $\sigma_n^r|\tau$ is homotopic to $\sigma_n^{r-1}|\tau$ among piecewise linear maps $\tau \to \Delta^n$ which are compatible with the face operators.*

Proof. According to formula (2.1) it is sufficient to prove the proposition for $\sigma_n \equiv \sigma_n^1$. In that case, $\sigma_n^{r-1} = Id$. The homotopy is obvious for $n = 1$; it consists of shrinking the middle interval δ^1 to the barycenter of Δ^1 and shrinking its image at the same time. Recursively, the homotopy of σ_n is known on the faces of Δ^n. Then, it is sufficient to define the homotopy on the interior small n-simplex δ^n. As when $n = 1$, the homotopy consists of shrinking δ^n and its image simultaneously to the barycenter of Δ^n. $\square$

2.4. First jiggling formula. Let M be an n-manifold and $\tau M = (TM \xrightarrow{p} M)$ be its tangent bundle. Choose an auxiliary Riemannian metric on τM and an arbitrarily small open disk sub-bundle U so that, for every $x \in M$, the exponential map $exp_x : U_x \to M$ is an embedding. Take a combinatorial triangulation T of M so fine that every n-simplex τ of T is covered by $exp_x(U_x)$ for every $x \in \tau$. Let T^r be the r-th Thom subdivision of T and $\sigma^r : T^r \to T$ be the corresponding simplicial folding map. The r-th *jiggling map* $j^r : M \to TM$ is defined in the following way. For each $x \in M$, the point $j^r(x)$ is the unique point in U_x such that

$$exp_x(j^r(x)) = \sigma^r(x). \tag{2.2}$$

This formula defines j^r as a piecewise smooth section $M \to TM$. We have the following properties.

Proposition 2.5. 1) *Let τ be an n-simplex of T, let x be a point in τ and let δ be an n-simplex of T^r passing through x. Then, $j^r(\delta)$ goes to $exp_x^{-1}(\tau)$ as $r \to \infty$. The convergence is uniform for $x \in \tau$.*
2) *The map j^r is homotopic to the 0-section Z among PL maps which are transverse to the exponential foliation $\mathcal{F}_{exp}$ on each n-simplex of their domain.*

Proof. 1) The diameter of the simplices of T^r goes to 0 as r goes to ∞. Then, for $y \in \delta$, the point $j^r(y) := exp_y^{-1}(\sigma^r(y))$ is close to $exp_x^{-1}(\sigma^r(y))$. Since σ^r is a surjective simplicial map onto τ, we have the C^0 closeness of $j^r(\delta)$ and $exp_x^{-1}(\tau)$. A similar argument holds for the derivatives.
2) On the one hand, the leaves of $\mathcal{F}_{exp}$ in U are n-disKs. We define $exp^u : U \to U$, $u \in [0,1]$, to be the map which is the homothety by u in each fibre of exp. It is a homotopy from Id_U to $exp|U$ which restricts to a homotopy from j^r to σ^r. On the other hand, according to Proposition 2.3, σ^r is homotopic to Id_M through PL maps which are non-degenerate on each n-simplex of their domain, hence transverse to $\mathcal{F}_{exp}$. $\square$

Remarks 2.6.
1) Any piecewise smooth map defined on an n-manifold M and smooth on each n-simplex of a triangulation T may be approximated by a smooth map with the same polyhedral image. It is sufficient to precompose with a smooth homeomorphism such that, for every simplex τ in the

$(n-1)$-skeleton of T and every $x \in \tau$, all partial derivatives in directions transverse τ vanish at x. Then, even if the concept of Γ-structure is restricted to the smooth category, there is no trouble to pull-back a Γ-structure by j^r; it will be well defined up to homotopy.

2) In general a jiggling, for instance based on the iterated barycentric triangulation, does not share the properties stated in Proposition 2.5 (non-degeneracy and PL-homotopy).

2.7. Second jiggling formula. Here, we consider a trivial bundle ε^n of rank n whose base is an n-manifold M equipped with a *colored* triangulation[5]. Let $\Delta^n \subset \mathbb{R}^n$ be a non-degenerate n-simplex whose vertices are colored. The coloring defines a simplicial map $c : T \to \Delta^n$. We have a first jiggling $j^1 : M \to M \times \mathbb{R}^n$, $x \mapsto \big(x, c(x)\big)$. Then, the Thom process defines a r-th jiggling

$$j^r(x) = \big(x, c \circ \sigma^r(x)\big). \tag{2.3}$$

The first item of Proposition 2.5 holds true for this formula: $j^r(\delta)$ tends to $\{x\} \times \Delta^n$ when n goes to ∞.

2.8. Proof of theorem 1.5 in the easy cases. For two of the four geometries considered, namely the submersion geometry Γ_n^Y and the foliation geometry $\Gamma_{n,q}^{\text{fol}}$, the jiggling method yields directly a simple proof of theorem 1.5.

For the *submersion geometry*, we begin by proving that the forgetful map is π_0-surjective. One is given an n-manifold M, a q-manifold Y and a *formal submersion* (in the sense of Subsection 1.4 (4)), that is, a pair (f, F), where $f : M \to Y$ is a smooth map and where $F : TM \to TY$ is a bundle epimorphism above f. One seeks for a one-parameter family (f_u, F_u) of formal submersions, $u \in [0,1]$, such that $(f_0, F_0) = (f, F)$, and such that (f_1, F_1) underlies some Γ_n^Y-structure $\xi = (\tau M, \mathcal{F})$. According the definitions given in Subsections 1.2 and 1.4, we have to find a pair $(w, \mathcal{F})$ formed with a submersion valued in Y and a codimension-n foliation, both defined near $Z(M)$ in TM, such that:

- w is constant on each leaf of $\mathcal{F}$;
- $f_1(x) = w(Z(x))$ and $F_1 = Dw_{Z(x)}|T_xM$.

This will work by taking $w = exp \circ F$ which is clearly a submersion on some neighborhood U of $Z(M)$ in TM; here, Y is endowed with some auxiliary Riemannian metric and $exp : TY \to Y$ is the associated exponential map. The only somehow delicate point is to find $\mathcal{F}$ as a subfoliation of the foliation $\mathcal{W}$ whose leaves are $w^{-1}(y)$, $y \in Y$. Let P be an n-dimensional plane field on U transverse to every fibre T_xM and contained in the kernel of the differential of w.

Let T be a triangulation of M. Consider the iterated Thom subdivisions T^r. By Proposition 2.5 (1), for r large enough, the r-th Thom jiggling j^r maps every n-simplex of T^r into U and transversely to P. Fix such an r. Then, on some small open neighborhood V of $j^r(T^r)$ in TM, there is a C^0-small perturbation of P, among the n-plane fields tangent to $\mathcal{W}$, yielding an integrable plane field on V. In the present situation where the dimension of the simplicial complex $j^r(T^r)$ is not larger than the codimension of P, the wanted integrating perturbation can be easily constructed by induction on the dimension of the simplices (see the very beginning of Section 6 in [19]).

[5] A triangulation T of dimension n is colored when each vertex has a color in $\{0, 1, \dots, n\}$ such that two vertices of the same simplex have different colors. The first barycentric subdivision of any triangulation is colored.

Let $S : M \to TM$ be a smooth section so close to j^r, that $S(M) \subset V$. For every $u \in [0,1]$, one has the section $S_u := uS$ valued in U. Set $f_u := w \circ S_u$, and $F_u(v_x) := Dw_{uS(x)} v_x$. The structure $S^*(w, \mathcal{F})$ is really a Γ_n^Y-structure whose underlying formal structure is (f_1, F_1). The π_0-surjectivity is proved.

More generally, one is given a parametric family of formal submersions (f_s, F_s), $s \in \mathbb{D}^k$, which are underlying some Γ_n^Y-structures $\mathcal{F}_s \in H^1_{\tau M}(M; \Gamma_n^Y)$ for every $s \in \partial\mathbb{D}^k$. First, one constructs, on some open neighborhood U of the zero section in TM and for every $s \in \mathbb{D}^k$, an n-plane field P_s as above: P_s is contained in $\ker D(exp \circ F_s)$, and transverse to every fibre $T_x M$. One arranges that P_s depends smoothly on s, and coincides with the tangent space to $\mathcal{F}_s$ for every $s \in \partial\mathbb{D}^k$. The construction of such a family is easy, by convexity[6] of the space of the n-plane fields on U contained in $\ker(D(exp \circ F_s))$ and transverse to the fibres.

Let T be a triangulation of M, and let T^r be a Thom subdivision whose order r is large enough so that the same jiggling $j^r(T^r)$ is transverse to P_s for every $s \in \mathbb{D}^k$; since the considered family is compact, such an r certainly exists. Then, the integrating perturbation can be chosen smoothly with respect to s and coinciding with the identity for every $s \in \partial\mathbb{D}^k$. A single neighborhood V and a single section S fit all parameters $s \in \mathbb{D}^k$. We get on V a parametric family $(\mathcal{P}_s)$ of Γ_n^Y-foliations transverse to every fibre $T_x M$; and $\mathcal{P}_s = \mathcal{F}_s | V$ for every $s \in \partial\mathbb{D}^k$.

Define S and S_u as above. Set $\mathcal{F}_s := S^*(\mathcal{P}_s) \in H^1_{\tau M}(M, \Gamma_n^Y)$. Set $f_{s,u} := exp \circ F_s \circ S_u$ and $F_{s,u}(v_x) := D(exp \circ F_s)_{uS(x)} v_x$. This is a one-parameter family of $\mathbb{D}^k$-parametrized families of formal submersions, between $(f_{s,0}, F_{s,0}) = (f_s, F_s)$ and $(f_{s,1}, F_{s,1})$, which is the formal submersion underlying $\mathcal{F}_s$.

Finally, the families $(\mathcal{F}_s)$ and $(\mathcal{F}_s)|\partial\mathbb{D}^k$ are homotopic as mappings $\partial\mathbb{D}^k \to H^1_{\tau M}(M, \Gamma_n^Y)$. The homotopy consists of pulling $\mathcal{F}_s$ back through S_u. The proof of Theorem 1.5 is complete for the groupoid Γ_n^Y. □

In the case of the *foliation geometry* on a manifold M of dimension $n = p + q$, we are given a parametric family of augmented Γ_q-structures (ξ_s, ϵ_s), $s \in \mathbb{D}^k$. Moreover, for every $s \in \partial\mathbb{D}^k$, the augmented Γ_q-structure (ξ_s, ϵ_s) is underlying some $\Gamma_{n,q}^{\mathrm{fol}}$-structure $\mathcal{F}_s \in H^1_{\tau M}(M; \Gamma_{n,q}^{\mathrm{fol}})$.

Denote $\xi_s := (\nu, \mathcal{X}_s)$ this family of Γ_q-structures. Of course, the normal vector bundle ν over M does not depend on $s \in \mathbb{D}^k$. Recall that $\epsilon_s : \nu \hookrightarrow TM$ is a monomorphism of vector bundles.

For every $s \in \partial\mathbb{D}^k$, denote by $\mathcal{G}_s$ the foliation of codimension q tangent to $\mathcal{F}_s$ on a neighborhood U of $Z(M)$ in TM ; that is, if $\mathcal{F}_s$ is viewed as a codimension-n foliation, we have $\mathcal{F}_s \subset \mathcal{G}_s$. For every $x \in M$, define $\tau_s(x) := (T\mathcal{G}_s)_{Z(x)} \cap T_x M$ to be the p-plane tangent to the foliation $\mathcal{G}_s \cap T_x M$ at $Z(x)$. Thus, $\tau M = \tau_s \oplus \epsilon_s(\nu)$. The family (τ_s) extends to a $\mathbb{D}^k$-parametrized family (τ_s) of p-plane fields on M complementary to $\epsilon_s(\nu)$.

For every $s \in \partial\mathbb{D}^k$, after pushing $\mathcal{G}_s$ by a vertical isotopy in TM, whose 1-jet at every point of the zero section is the identity, and after restricting to some smaller neighborhood, one can moreover assume that the trace $\mathcal{G}_s \cap (T_x M \cap U)$ is the restriction to $T_x M \cap U$ of the linear p-dimensional foliation parallel to $\tau_s(x)$. Then, the family $(\mathcal{G}_s)$ extends to a $\mathbb{D}^k$-parametrized family $(\mathcal{G}_s)$ of foliations of codimension q, transverse to the fibres of $TM \to M$ and defined on some neighborhood U of $Z(M)$ independent of s. Indeed, for every $s \in \mathbb{D}^k \setminus \partial\mathbb{D}^k$, we define $\mathcal{G}_s$ as the pullback of $\mathcal{X}_s$ through the linear projection of $TM = \tau_s \oplus \epsilon_s(\nu)$ onto $\epsilon_s(\nu)$ parallel to τ_s.

[6]Compare footnote 14.

Just as in the case of the submersion geometry, one constructs a $\mathbb{D}^k$-parametrized smooth family (P_s) of n-plane fields on U, transverse to the fibres T_xM, and contained in $T\mathcal{G}_s$. For $s \in \partial\mathbb{D}^k$, one has $P_s = T\mathcal{F}_s$. For a large enough integer r, a C^0-small perturbation of P_s yields a foliation $\mathcal{P}_s$ of codimension n, contained in $\mathcal{G}_s$, on some open neighborhood V of the jiggled zero section $j^r(T^r)$. For $s \in \partial\mathbb{D}^k$, one has $\mathcal{P}_s = \mathcal{F}_s$.

Define sections S, S_u as above. For every $s \in \mathbb{D}^k$, set $\mathcal{F}'_s := S^*(\mathcal{P}_s) \in H^1_{\tau M}(M, \Gamma^{\mathrm{fol}}_{n,q})$. The underlying augmented Γ_q-structure is homotopic to the given one (ξ_s, ϵ_s). The homotopy consists of the 1-parameter family of $\mathbb{D}^k$-parametrized families of augmented Γ_q-structures $(\nu, S^*_u(\mathcal{G}_s), \epsilon_s)$ (note that $\mathcal{G}_s$ is defined on the whole of the open set U). Finally, the $\partial\mathbb{D}^k$-parametrized families $\mathcal{F}_s$ and $\mathcal{F}'_s$ of $\Gamma^{\mathrm{fol}}_{n,q}$-structures are homotopic: the homotopy consists of $\mathcal{F}_{s,u} := S^*_u(\mathcal{F}_s)$. The proof of Theorem 1.5 is complete for the groupoid $\Gamma^{\mathrm{fol}}_{n,q}$. □

3. Existence of transversely symplectic Γ_n-structures

In this section we prove a slightly more general statement than the existence part of Theorem 1.5 for the groupoid Γ^{symp}_n: we consider any symplectic bundle of rank n. We are going to use a more informative notation: a Γ^{symp}_n-structure on M will be denoted by $\xi = (\nu, \mathcal{F}, \Omega)$ where Ω is a closed 2-form whose kernel is $\mathcal{F}$.

Theorem 3.1. *Let $\nu = (E \to M)$ be a symplectic bundle of even rank n over a manifold M of dimension $\leq n+1$. Then there exists a Γ^{symp}_n-structure ξ on M whose normal bundle $\nu(\xi)$ is isomorphic to ν as a symplectic bundle.*

Moreover, if a real cohomology class $\bar{a} \in H^2(M, \mathbb{R})$ is given, ξ can be chosen so that the cohomology class $[Z^*\Omega]$ equals $\bar{a}$, where Ω is the closed 2-form underlying ξ.

We think of this problem as a lifting problem that we attack by obstruction theory. Let us explain how it works. As for any groupoid of germs, there are a classifying space[7] $B\Gamma^{\mathrm{symp}}_n$ and a canonical isomorphism

$$\Gamma^{\mathrm{symp}}_n(M) \cong [M, B\Gamma^{\mathrm{symp}}_n]$$

where $[-,-]$ stands for the set of homotopy classes of maps.

This classifying space is the source of two maps. The first one is $\beta : B\Gamma^{\mathrm{symp}}_n \to BSp(n;\mathbb{R})$: if $f : M \to B\Gamma^{\mathrm{symp}}_n$ classifies a Γ^{symp}_n-structure $\xi = (\nu, \mathcal{F}, \Omega)$ up to concordance, $\beta \circ f$ classifies its normal bundle ν. The second one is $\kappa : B\Gamma^{\mathrm{symp}}_n \to K(\mathbb{R}, 2)$, where the target is the Eilenberg-MacLane space classifying the functor $H^2(-, \mathbb{R})$: the composed map $\kappa \circ f$ classifies the cohomology class of the closed 2-form $Z^*\Omega$. Finally, the pair (β, κ) defines a map

$$\pi^{\mathrm{symp}} : B\Gamma^{\mathrm{symp}}_n \to BSp(n;\mathbb{R}) \times K(\mathbb{R}, 2)$$

that we see as a homotopy fibration. For Theorem 3.1, we are given a map $M \to BSp(n;\mathbb{R}) \times K(\mathbb{R}, 2)$ and we have to lift this map to $B\Gamma^{\mathrm{symp}}_n$. Since M is $(n+1)$-dimensional, Theorem 3.1 is a direct corollary of the next statement (the $(n-1)$-connectedness would be sufficient for Theorem 1.5). Indeed, thanks to the long exact sequence associated with π^{symp}, this map induces a monomorphism up to the n-th homotopy group (see, for instance Hatcher's book [11], Section 4.3).

[7]The contravariant homotopy functor $\Gamma^{\mathrm{symp}}_n(-)$ satisfies the axiom of gluing (Mayer-Vietoris) and wedge sum; then, the classifying space exists according to E. Brown's Theorem [2].

Theorem 3.2. (Haefliger, McDuff) *The homotopy fibre of π^{symp}, denoted by $F\pi^{\mathrm{symp}}$, is n-connected.*

A. Haefliger ([8], Section 6) showed that the $(n-1)$-connectedness of this homotopy fibre is a consequence of the h-principle. D. McDuff ([13], Theorem 6.1) proved the n-connectedness thanks to the convex integration technique.

3.3. What do we have to prove for Theorem 3.2? We have to prove that the k-th homotopy group $\pi_k(F\pi^{\mathrm{symp}})$ vanishes when $k \leq n$. An element of this group is represented by a Γ_n^{symp}-structure $\xi = (\varepsilon^n, \mathcal{F}, \Omega)$ on the k-sphere with the following properties:

- The normal bundle is trivial as a symplectic bundle; this means that its underlying symplectic bilinear form is the standard form ω_0 of $\mathbb{R}^n$ on each fibre.
- The closed 2-form Ω, which is defined in a neighborhood of the 0-section $Z(S^k)$ in $\mathbb{S}^k \times \mathbb{R}^n$, is assumed to be exact.

Let (p_1, p_2) denote the two projections of $\mathbb{S}^k \times \mathbb{R}^n$ onto its factors. Recall that the kernel of Ω is the tangent space to the codimension-n foliation $\mathcal{F}$ and that $\mathcal{F}$ is transverse to the fibres of p_1.

We have to extend this structure ξ over the $(k+1)$-ball $\mathbb{D}^{k+1}$ or, equivalently, to show that it is homotopic to the trivial structure $\xi_0 := (\varepsilon^n, \mathcal{F}_0, \Omega_0)$ where $\Omega_0 = p_2^*\omega_0$.

According to Moser's Lemma with k parameters [14], there exists a vertical isotopy of $\mathbb{S}^k \times \mathbb{R}^n$, keeping Z fixed, which reduces us to the case where the germ at $Z(x)$ of the form induced by Ω on $p_1^{-1}(x)$ equals ω_0 for every $x \in \mathbb{S}^k$. After this vertical Moser isotopy, take a trivial tube $U = \mathbb{S}^k \times B^n$ in the domain of Ω, where B^n is an n-ball of small radius. Now, Theorem 3.2 directly follows from the next lemma, as we will see just after its statement.

Lemma 3.4. *Given the above-mentioned data, there exist a section $s : \mathbb{S}^k \to U$, a neighborhood W of $s(\mathbb{S}^k)$ in U and an ambient diffeomorphism ψ such that:*

(1) *ψ is the time-1 map of a* vertical *isotopy (ψ_t); set $W_0 := \psi(W)$;*
(2) *ψ sends the pair (W, Ω) to (W_0, Ω_0);*
(3) *the isotopy ψ_t is Hamiltonian with respect to ω_0 in each fibre.*

Here, "vertical" means that the isotopy preserves each fibre of p_1.

Remarks 3.5.
1) The statement holds true for every symplectic vector bundle of rank n, equipped (near the 0-section) with two forms exact forms Ω and Ω_0 which define Γ_n^{symp}-foliations and induce the same symplectic form on each fibre.
2) Moreover, the two first items are valid for a pair of Γ_n-foliations without any transverse geometry.

Proof of Theorem 3.2. Since ψ is vertical, $s_0 := \psi \circ s$ is a section of the trivial bundle ε^n over $\mathbb{S}^n$. Also, recall the 0-section Z. Then, we have a sequence of homotopies of Γ_n^{symp}-structures on $\mathbb{S}^k$:

- a first homotopy from $Z^*\Omega$ to $s^*\Omega$;
- then, a homotopy from $s^*\Omega$ to $s_0^*\Omega_0$ defined by the isotopy (ψ_t);
- a last homotopy from $s_0^*\Omega_0$ to $Z^*\Omega_0$.

The last structure obviously extends to the $(k+1)$-ball. $\square$

Shortly said, the proof of Lemma 3.4 will consist of taking a jiggled section in the sense of formula (2.3) whose simplices are very vertical, then covering it by boxes which *trivialize* the kernel of Ω and pushing these boxes by some vertical Hamiltonian isotopy until $\ker \Omega$ becomes horirontal. This isotopy is done recursively on the boxes. There are two main problems:

- rectifying the $(j+1)$-th box should not destroy what was gained for the j-th box;
- manage the vertical isotopies to be Hamiltonian and not just symplectic; if not, they could not extend.

Proof of Lemma 3.4. We limit ourselves to $k = n$; for $k < n$, it is the same argument by replacing the base $\mathbb{S}^k$ with an n-dimensional base $\mathbb{S}^k \times \mathbb{D}^{n-k}$. Let $B^n := p_2(U)$ and let Δ^n be a non-degenerate and colored n-simplex in the interior of B^n.

Take a decreasing sequence

$$\varepsilon_0 > \cdots > \varepsilon_j > \cdots > \varepsilon_{n-1}.$$

When α is a strict closed j-face of Δ^n, let $N(\alpha)$ denote the closed ε_j-neighborhood of α in $\mathbb{R}^n$. Set

$$N(\Delta^n) := \Delta^n \bigcup_{\alpha} N(\alpha)$$

where the union is taken over all faces of Δ^n. For a suitable choice of the sequence (ε_j) we may arrange that :

(1) $N(\alpha) \cap N(\beta) = \emptyset$ if α and β are two disjoint faces;
(2) if $\alpha \cap \beta \neq \emptyset$ and if α and β are not nested, then $N(\alpha) \cap N(\beta)$ is interior to $N(\alpha \cap \beta)$;
(3) $N(\Delta^n) \subset B^n$.

Now, take a colored triangulation T of the base $\mathbb{S}^n$, its Thom subdivision T^r and the associated jiggling j^r given by formula (2.3). We are going to construct bi-foliated *boxes* associated with each simplex of T^r whose plaques are respectively contained in the leaves of $\mathcal{F}$ and in the fibres of p_1; the boundary of a box has a part tangent to $\mathcal{F}$ and another part tangent to the fibres. Let τ be a k-simplex of T^r; with τ, the coloring of T associates some face $\tau^\perp$ of $\Delta^n \subset B^n$. The box $B(\tau)$ is defined in the following way. Its base $p_1(B(\tau))$ equals $star(\tau)$, the star of τ in T^r. In the fibre over the barycenter $b(\tau)$, we take the domain $N(\tau^\perp)$. Finally, $B(\tau)$ is the union of all plaques of $\mathcal{F}$ passing through $N(\tau^\perp)$ and contained in $p_1^{-1}(star(\tau))$. If the diameter of the base is small enough, that is, if the order r of the subdivision is large enough, the holonomy of $\mathcal{F}$ over the base is C^0 close to *Identity*. Therefore, each plaque in $B(\tau)$ cannot get out of U; thus, it covers $star(\tau)$.

Look at two faces τ and τ' of the same simplex σ of T^r. Assume first that τ and τ' are disjoint. Apply the above condition (1) to $\tau^\perp$ and $\tau'^\perp$; by the holonomy argument, if r is large enough, the boxes $B(\tau)$ and $B(\tau')$ are disjoint. Assume now that τ and τ' are not disjoint but not nested. Then, by (2), we have

$$B(\tau) \cap B(\tau') \subset B(\tau \cap \tau').$$

Nevertheless, if τ and τ' do not belong to the same n-simplex and if $\partial star(\tau) \cap \partial star(\tau') \neq \emptyset$, then $B(\tau)$ and $B(\tau')$ could intersect badly. This is corrected in the following way.

Again, for r large enough, the leaves of $\mathcal{F}$ meeting $j^r(\tau)$ intersect the fibre over $b(\tau)$ in $N(\tau^\perp)$. This guarantees that $j^r(T^r)$ is covered by the interior of the boxes. From now on, r is fixed. For $1 > \eta > 0$, the *η-reduced box* associated with τ is defined by

$$B_\eta(\tau) := B(\tau) \cap p_1^{-1}\big((1-\eta)star(\tau)\big)$$

where the homothety is applied from the barycenter $b(\tau)$. Fix $\eta > 0$ small enough so that the η-reduced open boxes still cover the jiggling. Now, we are sure that $B_\eta(\tau)$ and $B_\eta(\tau')$ are disjoint once τ and τ' are disjoint.

The desired open set W is the union $V_0 \cup \cdots \cup V_k \cup \cdots \cup V_{n-1}$, where V_k denotes the interior of the η-reduced boxes associated with each k-simplex; the section s is any smooth approximation of j^r valued in W. We are ready to perform the isotopy. It is done step by step, in the boxes associated with the vertices of T^r first, then with the edges etc. For $x \in star(\tau)$, lifting the segment $[x, b(\tau)]$ to $\mathcal{F}$ yields a holonomy diffeomorphism between fibres of box

$$(hol\,\mathcal{F})_x^{b(\tau)} : B(\tau)_x \to B(\tau)_{b(\tau)} \tag{3.1}$$

which is an ω_0-symplectomorphism since Ω is closed. Similarly, we have the holonomy of $\mathcal{F}_0$ which also give an ω_0-symplectomorphism. The steps are numbered from 0 to n.

If v is a vertex in T^r, we define ψ^0 in $B_\eta(v)$ by the next formula. For $z \in B_\eta(v)$ and $x = p_1(z)$,

$$\psi^0(z) = (hol\,\mathcal{F}_0)_v^x \circ (hol\,\mathcal{F})_x^v(z). \tag{3.2}$$

Since the reduced boxes are disjoint, this formula simultaneously applies to the reduced boxes associated with all vertices. By shrinking the segment $[x, v]$ to $[x, x+t(v-x)]$ and by replacing v with $x + t(v-x)$ in formula (3.2), we define an interpolation between $\psi^0(z)$ and z. As a consequence ψ^0 is the time-1 map of a vertical isotopy of embeddings (ψ_t^0) which is easily checked to be symplectic. Since the components of the domain of (ψ_t^0) are contractible, this is actually a Hamiltonian isotopy[8] which therefore extends to a global Hamiltonian isotopy supported in U, still denoted by (ψ_t^0). Let $\mathcal{F}_1$ (resp. Ω_1) be the direct image of $\mathcal{F}$ (resp. Ω) by ψ_1^0; all reduced boxes are transported in this way, becoming $B_\eta^1(\tau)$ for each $\tau \in T^r$. Observe that $\mathcal{F}_1$ is horizontal in the reduced new boxes associated with vertices.

The next step (numbered 1) deals with the edges. Let e be an edge in T^r with end points v_0, v_1. For $z \in B_\eta^1(e)$ and $x = p_1(z)$, define $\psi^1(z)$ by:

$$\psi^1(z) = (hol\,\mathcal{F}_0)_{b(e)}^x \circ (hol\,\mathcal{F}_1)_x^{b(e)}(z). \tag{3.3}$$

Observe that $\psi^1(z) = z$ when $z \in B_\eta^1(v_i)$, $i = 0, 1$; indeed, this box covers the barycenter $b(e)$ and $\mathcal{F}_1$ is horizontal there. Moreover, ψ^1 is the time-1 map of a symplectic isotopy (ψ_t^1) relative to the reduced boxes of the vertices; this isotopy, called the *step-1 isotopy*, follows from an interpolation formula analogous to the one defining (ψ_t^0).

If e and e' are two edges, after condition (2), the domain where their η-reduced boxes could intersect is contained in a domain where $\mathcal{F}_1$ is horizontal and, hence, $\psi_t^1 = Id$ on this domain. Therefore, ψ_t^1 is well defined on the union V_1 of closed η-reduced boxes associated with the vertices and edges. Unfortunately, it is not a Hamiltonian isotopy of embeddings; some vertical loops in V_1 may sweep out some non-zero ω_0-area. Thus, it could not extend to an ambient vertical symplectic isotopy. The needed correction is offered by the next claim, following

[8]The infinitesimal generator X_t of an ω_0-symplectic isotopy satisfies that $\iota(X_t)\omega_0$ is a closed 1-form; it is said to be Hamiltonian if this form is the differential of a function.

well-known ideas (compare V. Colin [3], Lemme 4.4).

Claim. 1) *There is a real combinatorial cocycle* $\mu = \mu_{\Omega_1}$ *of the triangulation* T^r *such that, for each triangle* τ*, the real number* $< \mu, \tau >$ *measures the* ω_0*-area swept out by the loop* $\{x\} \times (\partial\tau^\perp)$ *through the isotopy* (ψ_t^1) *for every* $x \in (1-\eta)star(\tau)$*; in particular, this area does not depend on* x.

2) *When* Ω_1 *is exact,* μ *is a coboundary.*

3) *There is an ambient vertical* ω_0*-symplectic isotopy* $(g_t)_{t\in[0,1]}$*, supported in* U*, which is stationary on* V_0 *and such that* $\mu_{g_1^*\Omega_1} = 0$.

The third item, together with the first item, means that the step-1 isotopy (ψ_t^1) becomes Hamiltonian when $\mathcal{F}_1$ stands for the foliation tangent to $\ker g_1^*\Omega_1$ instead of $\ker \Omega_1$.

The proof of the claim is postponed to the end of the section. We first finish the proof of Lemma 3.4 by applying the claim in the next way.

After the step-0 isotopy, the cocycle μ_{Ω_1} is calculated and the Hamiltonian isotopy (g_t) is derived. Let $\tilde{\mathcal{F}}_1$ denote the foliation tangent to $\ker g_1^*\Omega_1$; let $\tilde{B}_\eta^1(\tau) := (g_t)^{-1}\left(B_\eta^1(\tau)\right)$. Now, the straightening formula 3.3 of the box $\tilde{B}_\eta^1(e)$ is applied $\tilde{\mathcal{F}}_1$ instead of $\mathcal{F}_1$. The associated isotopy (ψ_t^1) becomes ω_0-Hamiltonian. Hence, it extends to a vertical isotopy supported in U, denoted likewise, which is ω_0-Hamiltonian on each fibre U_x. This finishes step 1 of the isotopy.

The next steps of the induction are similar, except that the question of being a Hamiltonian isotopy is not raised again since, up to homotopy, every loop in W is already contained in $V_0 \cup V_1$. In the end of this induction, we have a proof of Lemma 3.4 by taking $\psi = \psi_1^n$. $\square$

Proof of the claim.

1) Let e be an edge of T^r; its end points are denoted v_0 and v_1. Let x be a point in the base of $B_\eta(e)$. Set $\gamma := \{x\} \times e^\perp$. We first compute the ω_0-area swept out by the vertical arc $(\psi_1^1)^{-1}(\gamma)$ through the step-1 isotopy. Denote this area by $\mathcal{A}(x,e)$; any other arc with the same end points would give the same area.

There are two natural "squares", C and C_0, appearing for this computation. The square C (resp. C_0) is generated by the holonomy of $\mathcal{F}_1$ (resp. $\mathcal{F}_0$) over $[b(e), x]$ with initial vertical arc $e^\perp$ in the fibre $U_{b(e)}$. They have common horizontal edges: $\beta_i := e \times v_i^\perp$ for $i = 0, 1$. Orient $e^\perp$ from $v_0^\perp$ to $v_1^\perp$; thus, γ and $(\psi_1^1)^{-1}(\gamma)$ are oriented by carrying the orientation of $e^\perp$ by the respective holonomies; and also C_0 and C are oriented by requiring $\{b(e)\} \times e^\perp$ to define the boundary orientation. Then, we have

$$\mathcal{A}(x,e) = \int_C \Omega_0 - \int_{C_0} \Omega_0 \,. \tag{3.4}$$

The second summand is 0 by construction. Similarly, we have $\int_C \Omega_1 = 0$. Then, if Λ is any primitive of $\Omega_1 - \Omega_0$, we derive

$$\mathcal{A}(x,e) = -\int_C d\Lambda \,. \tag{3.5}$$

We now use a specific choice of primitive. Recall the zero-section $Z : \mathbb{S}^n \to U$. For $t \in [0,1]$, let c_t denote the contraction $(x,v) \mapsto (x,tv)$ and let $c : U \times [0,1] \to U$ be the corresponding homotopy from $Z \circ p_1$ to Id_U. This yields the next formula:

$$\Omega_1 - \Omega_0 = d\left[p_1^*\theta + \int_0^1 \iota(\partial_t)\, c^*(\Omega_1 - \Omega_0)\right] \tag{3.6}$$

where θ is a primitive of the exact form $Z^*\Omega_1$ (observe that $Z^*\Omega_0 = 0$); the integral is just the mean value of a one-parameter family of 1-forms. This primitive of $\Omega_1 - \Omega_0$ also reads

$$\Lambda_0 := p_1^*\theta + \int_0^1 c_t^*\iota(v\partial_v)(\Omega_1 - \Omega_0)\,, \tag{3.7}$$

which vanishes on every vertical vector since Ω_1 and Ω_0 coincide on the fibres. Orient β_0 as the horizontal lift of $[b(e), x]$ and β_1 as the opposite of the oriented horizontal lift. We have

$$\int_C d\Lambda_0 = \int_{\beta_1} \Lambda_0 + \int_{\beta_0} \Lambda_0\,. \tag{3.8}$$

Now, we consider a triangle τ in T^r and we look at the ω_0-area $\mathcal{A}(x, \partial\tau)$ swept out by $\{x\} \times (\partial\tau)^\perp$ when x belongs to $(1-\delta)star(\tau)$. The vertices of τ are denoted by v_i, $i = 0, 1, 2$, cyclically ordered; the oriented edges are $e_j := [v_{j-1}, v_j]$ where $j-1$ is taken modulo 3. There are two particular horizontal lifts of $[b(e_j), x]$, denoted by $\beta_{j,k}$ with $k = j$ or $j-1$ depending on whether its origin is $(b(e_j), v_j^\perp)$ or $(b(e_j), v_{j-1}^\perp)$. If $k = j-1$, it is oriented as $[b(e_j), x]$; if $k = j$, it has the opposite orientation. By summing up the area swept out by each edge of $\{x\} \times (\partial\tau)^\perp$, we have

$$\mathcal{A}(x, \partial\tau) =< \Lambda_0, \beta_{1,1} + \beta_{2,1} + \beta_{2,2} + \beta_{0,2} + \beta_{0,0} + \beta_{1,0} > \tag{3.9}$$

where the bracketing stands for the integration over chain.

Since $\Omega_1 - \Omega_0$ vanishes on $(1-\eta)star(\tau) \times \{v_i^\perp\}$, we have

$$< \Lambda_0, \beta_{i,i} + \beta_{i+1,i} >=< \Lambda_0, [b(e_i), b(e_{i+1})] \times v_i^\perp > .$$

By summation, we have

$$\mathcal{A}(x, \partial\tau) = \sum_i < \Lambda_0, [b(e_i), b(e_{i+1})] \times v_i^\perp > \tag{3.10}$$

which implies that $\mathcal{A}(x, \partial\tau)$ does not depend on x. The combinatorial cochain μ is now defined by the next formula:

$$< \mu, \tau >= \sum_i < \Lambda_0, [b(e_i), b(e_{i+1})] \times v_i^\perp > . \tag{3.11}$$

If an arbitrary primitive Λ of $\Omega - \Omega_0$ is used, the above formula becomes

$$< \mu, \tau >= \sum_i < \Lambda, \{b(e_i)\} \times [v_{i-1}^\perp, v_i^\perp] > + < \Lambda, [b(e_i), b(e_{i+1})] \times v_i^\perp > . \tag{3.12}$$

Indeed, a change of primitive consists of adding a closed 1-form; and the integral of this on the polygon P considered in formula (3.12) is zero since P bounds a 2-cell.[9]

[9]The cochain μ is a cocycle. Regarding the second item, this fact is not important and left to the reader. Note that the previous calculation uses a local primitive of Ω only.

2) Since T^r is a finite simplicial set, we only have to prove that $<\mu, \Sigma>= 0$ for every 2-cycle Σ of T^r. Here, the exactness of Ω_1 is used. Summing formula (3.12) over all triangles of Σ yields a sum of integrals of Λ over horizontal polygons in regions where $d\Lambda = 0$ (one polygon for each vertex of Σ). Then, these integrals are null. Therefore, there exists a combinatorial 1-cochain α of T^r such that $\mu = \partial^*\alpha$ where ∂^* stands for the combinatorial co-differential.

3) We are going to use this 1-cochain α in order to correct Ω_1 by a certain vertical isotopy. Let e be an oriented edge in T^r with origin v_- and extremity v_+. The value $\alpha(e)$ is used in the following way. In the fibre over $b(e)$, we find an ω_0-Hamiltonian isotopy $(g^e_t)_{t\in[0,1]}$, compactly supported in $U_{b(e)}$ and fixing $(V_0)_{b(e)}$, such that the area swept out by the arc $\gamma_e := b(e)\times[v_-^\perp, v_+^\perp]$ is $-\alpha(e)$.[10] Observe that the Hamiltonian function is not required to vanish in the fixed domain, but only to be constant on each connected component of the fiber $(V_0)_{b(e)}$ over $b(e)$.

Then, the infinitesimal generator X_t of the desired isotopy (g_t) is chosen in finitely many fibres. By a suitable partition of unity there is an extention which is Hamiltonian in each fibre, compactly supported and vanishing in V_0. Note that the Hamiltonian has to be constant in each connected component of the fibre $(V_0)_x$, but these constants may vary with x.

The 2-form $g_1^*\Omega_1 - \Omega_1$ has a primitive associated with the isotopy, named the *Poincaré* primitive,

$$A = \int_0^1 g_t^*\left(\iota(X_t)\Omega_1\right) dt. \tag{3.13}$$

Since X_t vanishes on V_0, the 1-form A vanishes over here and we have:

$$< A, \gamma_e >= -\alpha(e). \tag{3.14}$$

Now, $\Lambda + A$ is a primitive of $g_1^*\Omega_1 - \Omega_0$. According to formula (3.12), the combinatorial cochain $\mu_{g_1^*\Omega_1}$ associated with the 2-form $g_1^*\Omega_1$ vanishes and the claim is proved. □

4. Parametric family of transversely symplectic Γ_n-structures

In this section, we prove the parametric version of Theorem 1.5 for the groupoid $\Gamma = \Gamma_n^{\mathrm{symp}}$. We emphasize that the required k-connectedness of the homotopy fibre $F\pi^{\mathrm{symp}}$ depends only on the dimension of M and not on the number of parameters in the family. Indeed, there is no integrability condition with respect to the parameter. Moreover, we insist that a common jiggling will be used in the proof; its order is bounded by compactness of the parameter space.

We consider the same setting as in Theorem 3.1: $\nu = (E \to M)$ is a bundle of even rank n over a manifold M of dimension $\leq n+1$ equipped with a k-parameter family $(\omega_u)_{u\in\mathbb{D}^k}$ of symplectic bilinear forms ω_u on E. It is understood that k is positive.

Theorem 4.1. *Assume there is a family $(\xi_u)_{u\in\partial\mathbb{D}^k}$ of Γ^{symp}-structures, namely a family $(\Omega_u)_{u\in\partial\mathbb{D}^k}$ of closed 2-forms defined near the zero section Z of E, such that Ω_u induces ω_u on the fibres of ν for every $u \in \partial\mathbb{D}^k$*[11].

[10] In dimension $n = 2$, this is possible only if $|\lambda(e)|$ is less than the ω_0-area of $U_{b(e)}$. This last condition is satisfied when r is large enough.

[11] In other words, the symplectic normal bundles equal $(\nu, \omega_u)_{u\in\partial\mathbb{D}^k}$.

Then, this family extends over the whole $\mathbb{D}^k$ such that Ω_u induces ω_u on the fibres of ν for every $u \in \mathbb{D}^k$. Moreover, the family of cohomology classes $[Z^*\Omega_u]_{u\in\mathbb{D}^k}$ may be arbitrarily chosen among those which extend the boundary data.

Proof. We start with a cell decomposition $\mathcal{C}$ of M fine enough so that, for every $u \in \partial\mathbb{D}^k$ and every cell $C \in \mathcal{C}$, there is a fibered isotopy of $E_{|C}$ (depending smoothly on u) whose time-1 map ψ_u satisfies: $(\psi_u)_*\Omega_u = \theta_u^*\Omega_0$, where θ_u is a linear symplectic trivialization of $(\nu_{|C}, \omega_u)$, depending smoothly on $u \in \mathbb{D}^k$, and where Ω_0 stands for the pull-back of ω_0 by the projection $C \times \mathbb{R}^n \to \mathbb{R}^n$.

The theorem will be proved by induction on an order of the simplices of $\mathcal{C}$ for which their dimension is a non-decreasing function. Skipping the intermediate dimensions we jump to the $(n+1)$-cells. Thus, we are reduced to consider the n-trivial bundle over $\mathbb{S}^n$ and a family (Ω_u) of exact 2-forms on a small disK bundle U about the zero section Z, which induce the standard form ω_0 on each fibre U_x, $x \in \mathbb{S}^n$, (here a parametric version of Moser's lemma is applied again). This family fulfills the condition that $\Omega_u = \Omega_0$ for every $u \in \partial\mathbb{D}^k$. Let T be a triangulation of $\mathbb{S}^n$ and let T^r be a Thom subdivision whose order r is large enough so that the same jiggling $j^r(T^r)$ fits the proof of Lemma 3.4 for every $u \in \mathbb{D}^k$; since the considered family is compact, such an r certainly exists.

Each step of that proof may be performed with parameters using this fixed jiggling. Here it is worth noticing that the vertical isotopy given by Lemma 3.4 is stationary when $\Omega_u = \Omega_0$, in particular when $u \in \partial\mathbb{D}^k$. $\square$

The proof of Theorem 1.5 is now completed for the groupoid Γ_n^{symp}. $\square$

5. Transversely contact Γ_n-structures

Here, we prove a theorem which implies Theorem 1.5 for Γ_n^{cont}-structures. Our setting is not the one of tangential Γ-structures. It is the following. Given an odd natural integer n, a manifold M and a vector bundle $\nu = (E \to M)$ of rank n, we recall that a Γ_n^{cont}-structure on M with normal bundle $\nu = (E \to M)$ is given by $\xi = (A, \mathcal{K})$, where A is a 1-form and $\mathcal{K}$ is a codimension-n foliation, both defined near the 0-section Z in E, such that:

- $A \wedge dA^{\frac{n-1}{2}}$ induces a germ of volume form on E_x for every $x \in M$;
- $\ker(A \wedge dA^{\frac{n-1}{2}}) = T\mathcal{K}$;
- $\ker A$ contains $T\mathcal{K}$ and is invariant by the holonomy of $\mathcal{K}$.

As in the symplectic case, the next Theorem was known to A. Haefliger [8] when $\dim M < n+1$ and to D. McDuff [13] when $\dim M = n+1$.

Theorem 5.1. *Assume M is a manifold of dimension not greater than the rank of the vector bundle ν. Let (α, β) be* formal contact data*, that is, a section α of ν^* and a section β of $\wedge^2\nu^*$ such that $\alpha \wedge \beta^{\frac{n-1}{2}}$ is a non-vanishing section of $\wedge^n\nu^*$. Then, there exists a Γ_n^{cont}-structure $\xi = (A, \mathcal{K})$ on M with normal bundle ν such that, for all $x \in M$, the next two conditions are fulfilled:*

$$\left\{ \begin{array}{l} \ker A_{Z(x)} \cap \nu_x = \ker \alpha(x) \\ (dA)_{Z(x)} = \beta(x). \end{array} \right. \tag{5.1}$$

Moreover, this statement holds true in a relative parametric version.

Proof. For simplicity, we do not formulate any homotopy statement at the level of classifying spaces. Nevertheless, the strategy of proof is similar to the one we used for $\Gamma_n^{\rm symp}$-structures. It is even simpler since every contact isotopy is Hamiltonian.

Let us first consider the non-parametric version. The construction of ξ is performed step by step over each cell of a cell decomposition of M. We are looking on the last n-cell e^n only. Let $C := \partial e^n \times [0,1] \cong \mathbb{S}^{n-1} \times [0,1]$ be a collar neighborhood of the boundary, on which we are given a $\Gamma_n^{\rm cont}$-structure $\xi = (A, \mathcal{K})$ which fufills (5.1).

Since the formal data (α, β) extends over e^n, there is a trivialization of ν over e^n, $(p_1, p_2) : E|e^n \to e^n \times \mathbb{R}^n$, in which $(\alpha(x), \beta(x))$ is independent of $x \in e^n$. On $\mathbb{R}^n$ equipped with (α, β), we may think of β as a closed differential form with constant coefficients; by taking a primitive, we have a unique contact form α_0 such that :

$$(5.2) \qquad \begin{cases} \alpha_0(0) = \alpha \\ d\alpha_0 = \beta. \end{cases}$$

We derive a *trivial* $\Gamma_n^{\rm cont}$-structure $\xi_0 = (A_0, \mathcal{K}_0)$ on e_n such that

$$(5.3) \qquad \begin{cases} A_0 = p_2^*(\alpha_0) \\ dA_0 = p_2^*(d\alpha_0). \end{cases}$$

Hence, $\mathcal{K}_0$ is the horizontal foliation. Now, there is a Moser type lemma[12] which we are going to present below. This allows us to perform some vertical isotopy which reduces to the case where, in a small tube U about the zero section and for every point $x \in C$, we have

$$(5.4) \qquad A_{|U_x} = A_{0|U_x}$$

Lemma 5.2. *Let $(\alpha_t)_{t\in[0,1]}$ be a path of contact forms in a manifold V^n. Let L be a hypersurface in V. It is assumed that the Reeb vector field R_t of α_t is never tangent to L. Then, we have the following:*

1) *The next equation whose unknown is X_t can be solved near L:*

$$(5.5) \qquad L_{X_t}\alpha_t + \dot\alpha_t = 0.$$

2) *Let $(\alpha_t)_{t\in[0,1]}$ be a path of germs in $(\mathbb{R}^{2p+1}, 0)$ of contact forms. If $\ker \alpha_t(0)$ is independent of t, then these germs are isotopic.*
3) *The previous statements hold true with parameters and in a relative version.*

Proof. 1) The vector X_t decomposes as $X_t = Y_t + Z_t$ with $Y_t \in \ker \alpha_t$ and $Z_t = \alpha_t(X_t)\, R_t$. Let us recall that R_t generates in each point the kernel of $d\alpha_t$. Then, Equation (5.5) becomes the following system:

$$(5.6) \qquad \begin{cases} R_t \cdot (\alpha_t(X_t)) + \dot\alpha_t(R_t) = 0 \\ \iota(Y_t)\left(d\alpha_{t|\ker\alpha_t}\right) + d\left(\alpha_t(X_t)\right)_{|\ker\alpha_t} + \dot\alpha_{t|\ker\alpha_t} = 0 \end{cases}$$

Fix $t \in [0,1]$. The first equation of this system is a differential equation along the orbits of R_t whose unknown function is $\alpha_t(X_t)$. It has a unique solution if $\alpha_t(X_t)$ is required to equal

[12] The statement comes from Eliashberg-Mishachev's book [5] where the proof is left to the reader. We only add the relative and parametric version.

0 along L (here the transversality assumption is used). Then, the component Z_t of X_t is determined. Once $\alpha_t(X_t)$ is known, the second equation of (5.6) has a unique solution since the form induced by $d\alpha_t$ on $\ker\alpha_t$ is symplectic.

2) Here L is the hyperplane which is the common kernel of the contact forms in the considered path. Replace the germs with genuine representatives. Following the solution of 1), we have $X_t(0) = 0$ for every $t \in [0,1]$. Therefore, the flow φ_t of X_t keeps the origin fixed. It is well defined on some neighborhood of the origin up to $t = 1$ and it is the identity on L. We deduce from Equation (5.5) that the following is satisfied near the origin for every $t \in [0,1]$:

$$\varphi_t^*(L_{X_t}\alpha_t + \dot\alpha_t) = 0, \tag{5.7}$$

and the latter is obtained by taking the time derivative of the next equation

$$\varphi_t^*\alpha_t = \alpha_0\,. \tag{5.8}$$

So, the desired isotopy is obtained by integrating X_t.

3) Considering the equations which are solved, this claim is clear. □

We continue the proof of Theorem 5.1. In order to derive (5.4) from Lemma 5.2, we use $x \in C$ as a parameter and, in each fibre E_x, we consider $\alpha_t = tA_{|E_x} + (1-t)A_{0|E_x}$. Due to the formal data, α_t is a contact form near the origin of E_x for every $t \in [0,1]$.

After this Moser type reduction, we have to state and prove a lemma similar to Lemma 3.4. Actually, it is not useful to write it down explicitly since it is the same: the holonomy maps are contactomorphisms; thus, in each fibre of a box the vertical isotopy preseves the contact distribution $\ker A_0 \cap E_x$. Then, it is Hamiltonian with respect to $A_0|E_x$[13]. Therefore, it extends globally since extending such an isotopy amounts to extend its Hamiltonian function; thus, no obstruction is encountered. This finishes the proof of the non-parametric version.

For the relative parametric version of Theorem 5.1, we have to consider a family $(\alpha_u, \beta_u)_{u\in\mathbb{D}^k}$ which underlies a family of Γ_n^{cont}-structures when $u \in \partial\mathbb{D}^k$. Thanks to the relative parametric version of Lemma 5.2, we may follow word for word the proof we gave for Γ_n^{symp}.

□

6. Open manifolds

This section is devoted to the proof of Theorem 1.9. Let us recall the setting: Γ is an open subgroupoid of the structural groupoid $\Gamma(X)$ of a model n-manifold X (see Section 1); M is an open manifold of dimension n and $\xi \in H^1_{\tau M}(M;\Gamma)$ is a Γ-structure on M whose normal bundle is τM, the tangent space to M. Let TM denote its total space and $Z : M \to TM$ denote the 0-section. The associated Γ-foliation defined near $Z(M)$ in TM is denoted by $\mathcal{F} = \mathcal{F}_\xi$. We recall a topological fact about open manifolds whose proof is available in [5] Section 4.3.

Proposition 6.1. *Given an open n-manifold M, there exists an $(n-1)$-polyhedron $K \subset M$, called a* spine *of M, so that the inclusion is a homotopy equivalence. More precisely, for any*

[13] The Hamiltonian function of a vertical isotopy of contactomorphisms whose infinitesimal generator is X_t is (in our setting) the time dependent function $z \in U \mapsto A_0(X_t)(z)$.

regular neighborhood V of K, there exists a compressing *isotopy of embeddings $f_t : M \to M$, $t \in [0,1]$, from Id_M to an embedding $f_1 : M \to V$, which is stationary on a neighborhood of K and such that $t' > t$ implies $f_{t'}(M) \subset f_t(M)$.*

We are going to prove the next statement from which Theorem 1.9 will be easily derived.

Theorem 6.2. *Let $K \subset M$ be an $(n-1)$-dimensional polyhedron in an n-manifold M (open or closed). Let ξ_s, $s \in \mathbb{D}^k$, be a k-parameter family of Γ-structures on M with normal bundle τM. When $s \in \partial\mathbb{D}^k$, it is assumed that the associated foliation $\mathcal{F}_s$ is tangent to $\mathcal{F}_{exp}$ along $Z(M)$. Then, there exist an open neighborhood V of K in M and a k-parameter family $\bar\xi_s$ of Γ-structures on $M \times [0,1]$ – that is, concordances of ξ_s – such that:*

- $\bar\xi_s | V \times \{0\} = \xi_s | V$;
- $\bar\xi_s | V \times \{1\}$ *is regular and its associated foliation is tangent to $\mathcal{F}_{exp}$;*
- $\bar\xi_s = p_1^*(\xi_s | V)$ *for every $s \in \partial\mathbb{D}^k$, where p_1 denotes the projection $M \times [0,1] \to M$.*

In other words, $(\bar\xi_s)_{s\in\mathbb{D}^k}$ is a family of regularization concordances on a neighborhood of a K, relative to the boundary of the parameter space.

6.3. Proof of Theorem 1.9 from Theorem 6.2. Here, M is an open manifold. A spine K of M may be chosen (Proposition 6.1) and Theorem 6.2 applies to these data: $K \subset M$, $(\xi_s)_{s\in\mathbb{D}^k}$. So, we have a family $\bar\xi_s$ of regularization concordances on some neighborhood V of K in M, relative to $\partial\mathbb{D}^k$. We have to extend this family to a family of regularization concordances over the whole of M, still relative to $\partial\mathbb{D}^k$. We may assume there exists ρ close to 1 so that ξ_s is regular on M when $\|s\| \in [\rho, 1]$.

We first insert the family of concordances described as follows, where $t \in [0,1]$ is the parameter of the concordance:

- for $\|s\| \le \rho$, we put the concordance $t \mapsto f_t^*\xi_s$, where $(f_t)_{t\in[0,1]}$ is the isotopy of embeddings given by Proposition 6.1;
- for $\rho \le \|s\| \le 1$, we put the concordance $t \mapsto f^*_{\left(\frac{1-\|s\|}{1-\rho}t\right)}\xi_s$.

When $t = 1$ (that is the *end* of these concordances depending on $s \in \mathbb{D}^k$) and when $\|s\| \ge \rho$, the structures are regular on M. Then, denoting by $S : [1,2] \to [0,1]$ the shift $t \mapsto t-1$, we continue, for $t \in [1,2]$ with the concordances $(f_1 \times S)^*\bar\xi_s$ when $\|s\| \le \rho$; these ones are stationary when $\|s\| = \rho$. Thus, we are allowed to extend them by the stationary concordances when $\rho \le \|s\| \le 1$. Of course, the previous piecewise description can be made smooth if desired. □

6.4. Proof of Theorem 6.2 without parameters ($k = 0$).

We start with a Γ-structure ξ on M. Let $\mathcal{F}$ be its associated Γ-foliation defined in some small neighbohood U of $Z(M)$. Let ξ^u (resp. $\mathcal{F}^u$) be the underlying Γ_n-structure (resp. Γ_n-foliation) of ξ (resp. $\mathcal{F}$) where the transverse geometry is forgotten.

The proof will consist of two steps: in the first step, we will make a specific regularization of ξ^u by some Γ_n-concordance over $M \times [0,3]$; in the second step, the geometric Γ-structure of the concordance will be defined only over a small neighborhood of $K \times [0,3]$. Finally, we get the Γ-regularization of ξ near K.

1st step. Fix a small $\varepsilon > 0$. As in Thurston [19], we consider a one-parameter family P_t, $t \in [0,3]$, of n-plane fields on U with the following properties:

- P_t is transverse to the fibres for every t.
- P_t is tangent to $\mathcal{F}$ when $t \in [0, 1+\varepsilon]$.
- P_t is tangent to $\mathcal{F}_{exp}$ when $t \in [2,3]$.

Such a plane field exists by barycentric combination in the convex set[14] of the plane fields transverse to the fibres of τM.

Let T be a triangulation of M containing a subdivision of K (also called K) as a sub-complex and fine enough with respect to the open covering $\{exp_x(U_x) \mid x \in M\}$ in order that formula (2.2) makes sense. Here, we recall that formula which holds for x in any simplex of T:

$$\exp_x(j^r(x)) = \sigma^r(x).$$

We now consider the Thom jiggling given by formula (2.2); its order r is chosen large enough so that the n-simplices of $j^r(T^r)$ are transverse to P_t for every $t \in [0,3]$.

The first piece of the concordance, when $t \in [0, 1+\varepsilon]$, actually a Γ-concordance, consists of moving the zero section from Z to j^r by traversing any homotopy valued in U and stationary when $t \in [1, 1+\varepsilon]$. The concordance of Γ-structure is given by pulling ξ back by this homotopy of maps $M \to U$ (look at Remark 2.6 1) about smoothness).

We now describe the second piece of the concordance, when $t \in [1,2]$. We consider the codimension n-plane field $\tilde{P}$ in $U \times [0,3]$ defined by

$$\tilde{P}(x,t) := P_t(x) \oplus \mathbb{R}\partial_t. \tag{6.1}$$

It is tangent to $\mathcal{F} \times [0, 1+\varepsilon]$ and to $\mathcal{F}_{exp} \times [2,3]$. The trace of $\tilde{P}$ on each $(n+1)$-cell of $j^r(M) \times [1,2]$ is one-dimensional. Then, this trace is integrable. Thus, there is a C^0-small smooth approximation of $\tilde{P}$, relative to $t \in [0,1] \cup [2,3]$ and still denoted by $\tilde{P}$, which is integrable near $j^r(M) \times [1,2]$. Now, the pair $(j^r(M) \times [1,2], \tilde{P})$ defines a concordance of Γ_n-structure s. This finishes the second piece.

The third piece of the concordance when $t \in [2,3]$ consists of keeping the foliation $\mathcal{F}_{exp}$ and applying the homotopy from j^r to the 0-section Z as provided by Proposition 2.5 2). On the whole, we built a specific regularization concordance of the underlying Γ_n-structure ξ^u, which is nearly sufficient for our purpose.

We need more of *good position.* Let K^r denote the $(n-1)$-dimensional complex which is the r-th Thom subdivision of K. Let $\tilde{K}^r$ be the image of $Z(K^r) \times [0,3]$ along the concordance built above. This is an n-complex whose n-cells are not transverse to $\tilde{P}$. When $t \in [1,2]$, the only reason for non-transversality is that $\tilde{K}^r$ and $\tilde{P}$ share the ∂_t-direction. Let $\tilde{K}^r_{[t,t']}$ (resp. $\tilde{K}^r_t$) denote the restriction of $\tilde{K}^r$ over $M \times [t,t']$ (resp. $M \times \{t\}$).

When the n-cells of $\tilde{K}^r_{[t,t']}$ are prismatic (that is, simplex $\times$ interval) which is always the case when $[t,t'] \subset [1,2]$, they will receive the *standard subdivision* defined by H. Whitney ([21], Appendix II)[15]; this latter only depends on an order chosen on the set of vertices of $\tilde{K}^r_t$.

[14] Take an n-plane field Q transverse to the fibres. The above-mentioned convex set is affinely isomorphic to $\hom(Q, \tau^v TM)$ where τ^v stands for the sub-bundle of $\tau(TM)$ tangent to the fibres of $TM \to M$.

[15] This subdivision that W. Thurston names *crystalline* is clearly explained inside the proof of his famous *Jiggling* Lemma.

Claim. *There exist a subdivision $t_1 = 1, t'_1, ..., t_i, t'_i, ..., t_N = 3$ and a small piecewise smooth vertical isotopy, its time-one map being denoted by ψ, such that:*

(i) $\psi|\tilde{K}^r_{t_i} = Id$ *for every* $i = 1, ..., N$;

(ii) *for every n-simplex τ of the standard subdivision of $\tilde{K}^r_{[t_i,t'_i]}$ (resp. $\tilde{K}^r_{[t'_i,t_{i+1}]}$), the image $\psi(\tau)$ is smoothly embedded in $U \times [0,1]$ and quasi-transverse to $\tilde{P}$. Here, quasi-transverse means transverse when* $\dim \tau \geq n$ *and no tangency when* $\dim \tau < n$.

Proof of the claim. We search for a *jiggling in time*. We are going to do it for $\tilde{K}^r_{[1,2]}$; the jiggling in time of $\tilde{K}^r_{[2,3]}$ is similar, but a bit more complicated due to the fact that the cell decomposition is not purely prismatic. A numbering of the vertices of $\tilde{K}^r_1$ is fixed: $v_1, v_2, \ldots$; this numbering propagates to the corresponding vertices of $\tilde{K}^r_t$ for every $t \in [1,2]$.

The time subdivision is chosen so that, for every $(n-1)$-simplex $c \subset \tilde{K}^r_t$ and every $x \in c$, the hyperplane $H_t(x,c) := T_x(c) + P_t(x)$ varies very little in TU when t traverses $[t_i, t_{i+1}]$, uniformly when x runs in any star. Let t'_i be the middle of this interval. Each $\tilde{K}^r_{[t_i,t'_i]}$ (resp. $\tilde{K}^r_{[t'_i,t_{i+1}]}$) receives the *standard subdivision* of the prismatic cells. On $\tilde{K}^r_{[t_i,t'_i]}$, the desired embedding ψ and its isotopy from Id are constructed recursively on the numbered stars of vertices $star(v_1), star(v_2), \ldots$. Precisely, there is a locally finite family of isotopies $\chi_1, \chi_2, \ldots$ (that is, only finitely many supports intersect) and ψ will be the composition of their time-one map: $\cdots \circ \chi^1_2 \circ \chi^1_1$. The reversed isotopies are used over the interval $[t'_i, t_{i+1}]$.

Let v_1 be the *first* vertex of $\tilde{K}^r_{t_i}$. Let X_1 be a small vertical vector in $T_{v_1}U$ which is chosen linearly independent form all above-mentioned hyperplanes $H_{t_i}(x,c)$ where c is any n-simplex of $\tilde{K}^r_{t_i}$ passing through v_1. Let v'_1 be the corresponding vertex in $\tilde{K}^r_{t'_i}$. By definition of the standard subdivision of $\tilde{K}^r_{[t_i,t'_i]}$, the vertex v'_1 is joined to $star(v_1, \tilde{K}^r_{t_i})$, the star of v_1 in $\tilde{K}^r_{t_i}$. In affine notation χ^1_1 is obtained by replacing in $\tilde{K}^r_{[t_i,t'_i]}$

$$star(v'_1, \tilde{K}^r_{[t_i,t'_i]}) \quad \text{with} \quad \big(v'_1 + X_1\big) * lk(v'_1, \tilde{K}^r_{[t_i,t'_i]}), \tag{6.2}$$

where $lk(a,-)$ stands for the *link* of a, that is, the union of simplices in $star(a,-)$ which do not contain a. After this step, we get the property that every n-simplex in $B_1 := \chi^1_1\left(\tilde{K}^r_{[t_i,t'_i]}\right)$ passing through $v'_1 + X_1$ is transversal to $\tilde{P}$.

The second step consists of a similar construction related to the star of v'_2 in B_1 by using a small vertical vector X_2 which is linearly independent of the hyperplanes made with the n-simplices of B_1 passing through v'_2. This yields χ^1_2, the time-one map of the second isotopy which allows us to gain the transversality of new n-simplices to $\tilde{P}$. When $star(v'_2, B_1)$ meets $star(v'_1 + X_1, B_1)$, the vector X_2 is chosen so small that the property gained in the first step is preserved. And so on. $\square$

In what follows, we still denote by $\tilde{K}^r$ the outcome of the previous jiggling. As a result, every n-cell of $\tilde{K}^r_{[1,3]}$ is transverse to $\tilde{P}$. This simplicial complex $\tilde{K}^r$ collapses[16] successively to

[16] A simplicial complex L collapses to K if there is a sequence of elementary collapses $L_q \searrow L_{q+1}$ starting with L and ending with K. An elementary collapse means that L_q is the union of L_{q+1} and a simplex σ so that $\sigma \cap L_{q+1}$ is made of the boundary of σ with an open facet removed.

$\tilde{K}^r_{[0,2]}$ and then to $\tilde{K}^r_{[0,1]}$.

2nd step.

We now focus on $\tilde K^r_{[0,1]}$ on which $\tilde K^r$ collapses. Since the cells of $\tilde K^r_{[1,3]}$ of positive dimension are quasi-transverse to the foliation $\tilde P$, the collapse $\tilde K^r \searrow \tilde K^r_{[0,1]}$ extends to a collapse of pair

$$(\tilde K^r, \tilde P) \searrow (\tilde K^r_{[0,1]}, \tilde P)\,. \tag{6.3}$$

Let $N(\tilde K^r)$ denote a small neighborhood of $\tilde K^r$ in $U\times[0,3]$. From the sequence of elementary collapses, one derives step by step an embedding of pair

$$\Phi : \big(N(\tilde K^r), \tilde P\big) \to \big(U\times[0,1+\varepsilon),\, \mathcal F\times[0,1+\varepsilon)\big) \tag{6.4}$$

which induces the inclusion $N(\tilde K^r_{[0,1]}) \hookrightarrow U\times[0,1+\varepsilon)$. Since $\mathcal F\times[0,1+\varepsilon]$ is a Γ-foliation, $\tilde P_{|N(\tilde K^r)}$ is so by pulling back through Φ. Therefore, $\Phi^*\big(\mathcal F\times[0,1+\varepsilon)\big)$ is a regularization concordance of the Γ-structure which is induced near K. $\square$

This last process associated with collapses is named *inflation* in W. Thurston's article [19].

6.5. Relative parametric version of Theorem 6.2. Here, the data consist of a family $(\xi_s)_{s\in\mathbb D^k}$ of Γ-structures whose normal bundle is the tangent bundle τM. It is assumed that, for every $s\in\partial\mathbb D^k$, the associated foliation $\mathcal F_s$ is tangent to $\mathcal F_{exp}$ along the 0-section $Z(M)$, hence ξ_s is regular. Without loss of generality, we may assume ξ_s is regular when $1\geq\|s\|\geq\rho$ for some ρ close to 1. The proof just consists of two remarks.

1) The previous proof (see Subsection 6.4) works directly for our k-parameter family of data if we do not take care of the boundary condition. Indeed, observe that a common order r of Thom jiggling j^r can be chosen for all $s\in\mathbb D^k$ since the family of n-plane fields $P_{s,t}$ we have to look at is compact. Thus, if the jiggling is vertical enough , that is, for r large enough, its n-simplices are transverse to $P_{s,t}$ for every $(s,t)\in\mathbb D^k\times[0,3]$. The 0-parameter process applies for every $s\in\mathbb D^k$ and yields a regularization in a fixed neighborhood V of K in M. Precisely, we have formulas (6.3) and (6.4) depending on the parameter $s\in\mathbb D^k$, yielding regularization concordances $\Phi_s^*\big(\mathcal F_s\times[0,1+\varepsilon)\big)$.

2) We may assume that $P_{s,t}$ is tangent to $\mathcal F_s$ for every $s\in\{1\geq\|s\|\geq\rho\}$ and $t\in[0,3]$. For $\|s\|\in[\rho,1]$, set $\mu(s):=\frac{1-\|s\|}{1-\rho}$. Recall that Φ_s is the time-one map of an isotopy of embeddings $\Phi_s^w : N(\tilde K^r)\to U\times[0,3]$, $w\in[0,1]$, relative to $N(\tilde K^r_{[0,1]})$ and such that $\Phi_s^0 = Id$ and $\Phi_s^1\big(N(\tilde K^r)\big)\subset N(\tilde K^r_{[0,1+\varepsilon]})$.

We finish, for $s\in\{1\geq\|s\|\geq\rho\}$, with the regularization concordance $(\Phi_s^{\mu(s)})^*\big(\mathcal F_s\times[0,3]\big)$. When $\|s\|=1$, this is the trivial concordance $\mathcal F_s\times[0,3]$. Then, the relative version is proved. $\square$

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References

[1] M. S. Borman, Y. Eliashberg & E. Murphy, *Existence and classification of overtwisted contact structures in all dimensions,* arXiv:1404.6157.

[2] E. Brown, *Cohomology theories,* Annals of Math. 75 (1962), 467-484.

[3] V. Colin, *Livres ouverts en géométrie de contact,* in *Séminaire Bourbaki, vol. 2006/2007,* Astérisque 317, Soc. Math. France (2008), 91-117.

[4] Y. Eliashberg & N. Mishachev, *Wrinkling of smooth mappings III: Foliations of codimension greater than one,* Topol. Methods Nonlinear Anal. 11 n°2 (1998), 321-350.

[5] Y. Eliashberg & N. Mishachev, *Introduction to the h-principle*, G.S.M 48, Amer. math. Soc., 2002.

[6] Y. Eliashberg & E. Murphy, *Making cobordisms symplectic,* arXiv:1504.06312.

[7] A. Haefliger, *Feuilletages sur les variétés ouvertes*, Topology 9 (1970), 183-194.

[8] A. Haefliger, *Homotopy and integrability*, 133-175 in: Manifolds-Amsterdam 1970, L.N.M. 197, Springer, 1971.

[9] A. Haefliger, private communication, Jan. 2014.

[10] M. Gromov, *Partial Differential Relations,* Springer, 2002. 1980.

[11] A. Hatcher, *Algebraic topology,* Cambridge University Press, 2002.

[12] M. Hirsch, *Immersions of manifolds,* Trans. Amer. Math. Soc. 93 (1959), 242-276.

[13] D. McDuff, *Application of convex integration to symplectic and contact geometry,* Ann. Inst. Fourier (Grenoble) 37 (1987), 107-133.

[14] J. Moser, *On the volume elements on a manifold,* Trans. Amer. Math. Soc. 120 (1965), 286-294.

[15] E. Murphy, *Loose Legendrian embeddings in high dimensional contact manifolds, arXiv:1201.2245.*

[16] R. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966), 1-16.

[17] S. Smale, *The classification of immersions of spheres in euclidean spaces*, Ann. of Math. 69 (1959), 327-344.

[18] R. Thom, *Remarques sur les problèmes comportant des inéquations différentielles globales,* Bull. Soc. Math. France 87 (1959), 455-461.

[19] W. Thurston, *The theory of foliations of codimension greater than one*, Comment. Math. Helv. 49 (1974), 214-231.

[20] O. Veblen &J. H. C. Whitehead, *A set of axioms for differential geometry,* Proceedings of the National Academy of Sciences 17, 10 (1931), 551-561.

[21] H. Whitney, *Geometric Integration Theory,* Princeton Univ. Press, 1957.

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