Asymptotic modelling of some functionally graded materials

Czeslaw Wozniak, Monika Wagrowska

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Czesław Woźniak[1,*] and **Monika Wągrowska**[2,**]

[1,] Technical University in Łódź, Poland
[2,] University of Life Sciences SGGW-Warsaw, Poland

1 Object of analysis

The object of analysis is a multilayered functionally graded laminated heat conductor. Region occupied by this heat conductor is denoted by $\Omega = (0, L) \times \Xi$, where Ξ is a region on the $0\xi_1\xi_2$ plane and $x \in (0, L)$. Region Ω is divided into n layers of the same thicknesses $\lambda = L/n$, where λ is sufficiently small where compared with L. It is assumed that there is known function $\nu(\cdot) \in C^1([0, L])$, which takes the values from interval $[0, 1]$.

Denote $x_j = x_j^\lambda \equiv \lambda/2 + (j-1)\lambda, j = 1, \dots, n = L/\lambda$. Let $\Omega_R^\lambda \equiv \bigcup_{j=1}^{n} (x_j^\lambda - \frac{1}{2}\lambda\nu(x_j^\lambda), x_j^\lambda + \frac{1}{2}\lambda\nu(x_j^\lambda)) \times \Xi$ be a part of Ω occupied by the reinforcement material and $\Omega_M^\lambda \equiv \Omega - \overline{\Omega}_R^\lambda$ is a part of the Ω occupied by the matrix material. Hence $I^\lambda \equiv \Omega - (\Omega_R^\lambda \cup \Omega_M^\lambda)$ is a set of interfaces. By A_R, A_M we denote symmetric positive definite, second order tensors, representing heat conduction in both materials. Hence the distribution of material properties of composite is given by

$$A_\lambda(x, \xi) = \begin{cases} A_M & \text{if } (x, \xi) \in \Omega_M^\lambda \\ A_R & \text{if } (x, \xi) \in \Omega_R^\lambda \end{cases} \tag{1}$$

Let w_λ, q_λ be a scalar temperature and vector flux heat field, respectively. Thus the heat conduction problem under consideration is described by constitutive equations

$$q_\lambda = -A_\lambda \nabla w_\lambda, \tag{2}$$

and by the balance equations

$$\nabla \cdot q_\lambda = f, \tag{3}$$

where f is the known *a priori* heat source field, as well as the prescribed boundary condition.

Due to the discontinuity of the field A_λ the direct solution of the heat conduction problem is rather difficult from the computational point of view.

2 Aim of the contribution

If $\nu = const$, then we deal with a periodic multilayered composite structure and the well known homogenization procedure can be applied to Eqs. (1), (2), [1].

The aim of this contribution is to obtain the asymptotic model of the problem under consideration for an arbitrary but fixed $\nu(\cdot) \in C^1([0, L])$. To this end we apply the concept of G - convergence of the heat conduction tensor $A_\lambda(\cdot)$, provided that $\lambda \to 0$. In this case $A_\lambda \xrightarrow{G} A_0$, where A_0 is said to be effective heat conduction tensorfield and Equations (1), (2) lead to the asymptotic model equations:

$$q_0 = -A_0 \nabla u, \quad \nabla \cdot q_0 = f, \tag{4}$$

At the same time we are going to obtain the approximation of the temperature field w_λ given by

$$\widetilde{w}_\lambda = u + N_\lambda \cdot \nabla u + O(\lambda^2), \tag{5}$$

where $N_\lambda(\cdot)$ is a vector field which has to be determined together with $A_0(\cdot)$. Hence our aim is to obtain $A_0(\cdot)$ and $N_\lambda(\cdot)$ for the given *a priori* $\nu(\cdot)$.

* e-mail: czeslaw.wozniak@p.lodz.pl, Phone: +48 42 631 35 64, Fax: +48 42 631 35 64
** Corresponding author: e-mail: monika_wagrowska@sggw.pl

3 Modelling procedure

Let $h_\lambda \in C^0([0,L])$ be a scalar function satisfying conditions $h_\lambda(0) = h_\lambda(L), h_\lambda(x_j^\lambda \pm \frac{\lambda}{2}\nu(x_j^\lambda)) = \pm\frac{\lambda}{2}, j = 1,\ldots,n,$ which is sequentially linear in every $(x_j^\lambda - \frac{\lambda}{2}\nu(x_j^\lambda), x_j^\lambda + \frac{\lambda}{2}\nu(x_j^\lambda))$. The proposed modelling procedure will be based on two assumptions that the temperature field $w_\lambda(\cdot)$ can be approximated by:

$$\widetilde{w}_\lambda(x,\xi) = u(x,\xi) + h_\lambda(x)\nu(x,\xi) + O(\lambda), \tag{6}$$

where $u(\cdot), \nu(\cdot) \in C^1(\overline{\Omega})$ and are independent of λ.

The second assumption is that the heat flux vector field is continuous across all interfaces $I^\lambda = x_j^\lambda \pm \frac{\lambda}{2}\nu(x_j^\lambda)$. Subsequently we shall use tensor notation in which $x = x_1, \xi = (x_2, x_3)$ and superscrips k, l run over $1,2,3$. Moreover $\nu_R \equiv \nu$ and $\nu_M \equiv 1 - \nu$.

Fundamental assertion. Setting

$$N_\lambda^k \equiv -h_\lambda \frac{\nu^R \nu^M [A^{1k}]}{\nu^R A_M^{11} + \nu^M A_R^{11}}, \quad [A^{kl}] \equiv A_R^{kl} - A_M^{kl} \tag{7}$$

we obtain:

$$\widetilde{w}_\lambda = u + N_\lambda^k \partial_k u + O(\lambda^2). \tag{8}$$

Moreover the effective heat conduction tensor field has the form:

$$A_0^{kl} \equiv \nu^R A_R^{kl} + \nu_M A_M^{kl} - \frac{\nu^R \nu^M [A^{1k}][A^{1l}]}{\nu^R A_M^{11} + \nu^M A_R^{11}}. \tag{9}$$

By the property of the G-limit, tensor $A_0^{kl}(\cdot)$ introduced above is uniquely defined. The proof of this assertion will be given separately.

References

[1] V.V Jikov, C.M. Kozlov, O.A Oleinik.: Homogenization of differential operators and integral functionals, Springer Verlag, Berlin-Heidelberg 1994.