DEVOLUTION OF A PROBLEM AND CONSTRUCTION OF A CONJECTURE,

THE CASE OF THE SUM OF THE ANGLES OF A TRIANGLE

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This study is part of the research project I conducted during the 80s on junior high school students' conceptions of proof in mathematics before the teaching of mathematical proof [in French: *démonstration*]. The first part of this project resulted in the identification of different types of proofs students may rely on. The second part investigated the principle of design of situations which *could* support an evolution of students' conceptions of proofs likely to serve as a basis for teaching mathematical proof; this paper reports on two case studies carried out within this project. It details the principles of design, the implementation and the analysis of a sequence of situations aimed at generating debate on proofs and refutations. It takes up the challenge of rejecting empirical proofs to open the way to intellectual proofs on which teaching could ground the introduction of mathematical proof.

This translation of the report includes comments, notes (Note2020) and new references in order to facilitate the reading and understanding of the contemporary reader.

1 INTRODUCTION

Situation of validation is one of the structuring concepts of the Theory of Didactical Situations. Here are its key characteristics: "The didactical scheme of validation motivates the students to discuss a situation and favors the formulation of their implicit validations, but their reasoning is often insufficient, incorrect, clumsy. They adopt false theories, accept insufficient or false proofs. The didactical situation must lead them to evolve, to revise their opinions, to replace their false theory with a true one. This evolution has a dialectic character as well; a hypothesis must be sufficiently accepted—at least provisionally—even to show that it is false." (Brousseau 1997 p.17)

1.1 DESIGNING A SITUATION OF VALIDATION AT THE SEVENTH-GRADE

The research presented here aims at analyzing the constraints weighting on the design and the implementation of a situation in which students, who have not yet learned the concept of mathematical proof[1], have to make a conjecture and to deal with the problem of proving it or refuting it. The general framework is provided (1) by the *theory of didactical situations* (Brousseau, 1986)[2], of which I have shown elsewhere the relevance for research on the learning of mathematical proof at these school grades (Balacheff, 1987)[3], and (2) the epistemological theses of Imre Lakatos (1976) on the dialectic of proofs and refutations.

[1] Note2020: this expression is adopted as a translation of the French "démonstration" which denote at an elementary level a proof conforming to the Euclidean schema of proof.
[2] Note2020: the *Theory of didactical situation* is now available to the English reader in a book published by Kluwer/Springer (Brousseau, 1997)
[3] Note2020: there is no proper English translation of this article, the anglophone reader may refer to (Balacheff, 1988)

"The **didactical contract** is the rule of the game and the strategy of the didactical situation. It is the justification that the teacher has for presenting the situation. But the evolution of the situation modifies the contract, which then allows new situations to occur. In the same way, knowledge is what is expressed by the rules of the adidactical situation and by the strategies. The evolution of these strategies requires productions of knowledge which in their turn allow the design of new adidactical situations. The didactical contract is not a general pedagogical contract. It depends closely on the specific knowledge in play." (Brousseau 1997 p.31)

As for other mathematical notions, the learning of mathematical proof can be approached with a constructivist perspective but it requires not missing the social dimension which plays an important role in its case; especially with regard to the consequences of the *didactical contract*. It is more particularly on this point that this report on the design and the observation of a particular didactical sequence in the seventh-grade (12-13 years old students) focuses.

A key requirement of the *situation of validation* we want to design is that the students be responsible for the formulation of the conjecture. Indeed, if it were stated by the teacher, this formulation would immediately lose its true character of conjecture for the students. They would only be able to consider it as a true statement and not just as a highly plausible one. Moreover, the problem could become one of finding an agreed proof and not of establishing the truth of a statement. Let us recall, that the status of conjecture for a mathematical statement is a very strong one. It is not mere speculation, but a statement on the truth of which a community commits itself. Therefore, it must have been formulated by it and a consensus must be formed on that formulation. This does not mean, however, that the class as a whole must have a single position on its validity; the research can focus on proofs *as well as* on counter-examples. The complexity of achieving such a situation lies in what it expects from socialization: it is not only a matter of students taking ownership of the problem individually, but also ensuring the sharing of its meaning.

The classical position in the 80s was to consider errors as indicators of *misconceptions*. This term used to come with expressions like "naive theory", "private concepts", "beliefs" or "mathematics of the child". Such views downplayed learners as rational knowers and prevented recognizing that their way of knowing was situated with limits that errors evidence, but true knowledge anyway (Confrey 1990). Hence the choice to use the term **conception** to denote all possible ways of knowing and understanding a mathematical concept, be it in a right or a wrong way, an elementary or an advanced one. In the middle of the 90s, I designed the model cK¢ as a proposition to characterize mathematical conceptions (Balacheff 1995, 2013)

The concept of **theorem-in-act** designates the properties of the relations grasped and used by the subject in a problem-solving situation, it being understood that this does not mean that he is able to explain or justify them (Vergnaud 1981 p.220).

The problem, if it cannot be formulated by the teacher, can only come from the confrontation of the students with a situation specific enough with respect to the targeted conjecture. We consider with Ernst Mach (1908) that the source of a problem is "the disagreement between thoughts and facts, or the disagreement of thoughts with each other" (ibid. p.254). Then the challenge of ensuring that such a disagreement is noticed by students, and is clearly recognized so that the problem can be formulated. A favorable situation for this is one in which they have erroneous or incomplete but *stable conceptions* backed by *theorems-in-act* (Vergnaud, 1981, 1990)[4] which confrontation with facts or other conceptions lead effectually to questions. These conceptions must be such that they allow explicit expectations that can be discussed in order to be confirmed or refuted.

The formulation of a conjecture can be very complex because of the conceptual constructions, the recognition of the objects of knowledge and the relations it requires. At the level that interests us, the seventh grade, we wish to minimize the impact of these problems of formulation, be it the use of natural language or of symbolic representations.

In line with these reflections, the theorem on the invariance of the sum of the measurements of the angles of a triangle has been selected. On the one hand, the idea of the possibility of such an invariant should not fail to challenge the widespread conception at this school level according to which the value of this sum should depend on the "size" of the triangle; the design of the situation must allow this confrontation. On the other hand,

[44] Note2020: see (Vergnaud, 2009) in English

formulating the statement: “the sum (of the measures) of the angles of a triangle is 180°”, is of a low linguistic complexity; it should be within the reach of seventh-grade students.

Finally, this property is prone to several types of proofs in terms of content and levels, a sample of which are evoked below. This richness let consider the whole process, from the construction of the conjecture to its establishment as a theorem, plausible in whatever classes.

1.2 Preliminary comment on a classical approach

Another origin of the choice of the case of the sum of the angles of a triangle is an observation of its teaching reported by two master students[5] attending my didactics of mathematics lectures in 1983. They had observed a seventh-grade lesson concerning this theorem following a classical scheme, which consists in supporting the statement of the conjecture based on a purely practical approach, then asking the students to admit the truth of it because they cannot establish the proof. Here is what reported these master students:

> *"The teacher has each student draw a triangle and measure the angles with a protractor. Several different difficulties appear:*
>
> > *- the children do not know how to place the zero of their protractors;*
> > *- they don't understand why the protractor has to be aligned on one side of the triangle;*
> > *- if the triangle is smaller than the protractor, they do not understand why the sides of the triangle have to be extended to read the angle.*
>
> [...] *the teacher has the angles added up, and everyone writes their result on the blackboard. The students notice by themselves that the results are often around 179°, 180°, 181°, 182°, with a predominance of 182°. The teacher asks those who have found a sum farther away than 180° (165°, 228°, ...) repeat the measurements.*
> *The next lesson starts with a repetition of the previous lesson: the students start the same work again. This time they all find a sum of the angles between 178° and 184°, and the average result is 182°* [...] *(we heard several students tell their neighbors while they were measuring the angles, that they had to find 182°).*
> [...] *Then the teacher announces that the sum of the angles of a triangle is 180°, which is the reason for the strong reactions of the students. They are skeptical and invoke the following arguments: "we don’t find 180° by measuring", "if the triangle is big it should be more", "next year we'll be told it's 183°". The first argument is the one that appeared the most frequently."*

Such a teaching approach is rather common: after a few measurements, the fact that the result should be 180° is *revealed*, with or without the presentation of a proof. The basis of this approach is a simplistic understanding of the relationship between action and thought that expects the natural emergence of the conjecture from observing a few measurements. But, as the teaching sequence reported above shows, things are not that simple. Students' conceptions, which are not sufficiently considered here, strongly resist their transformation. The idea that any “quantity” associated with a triangle is an increasing function of its "size", as it is commonly observed for the area and the perimeter, constitutes a true knowledge that here stands as an *obstacle*. Moreover, the *didactical contract*, and with it the students’ idea of the existence of conventional or *ad hoc* responses, creates an obstacle of another kind by leading them to regulate their conduct on the basis of clues that are not intrinsic to the knowledge at stake. Then arises the problem of the meaning of the theorem for students when it is asserted as such, in particular the question of how it will take its place among their prior knowledge, and ultimately the question of the status of the proof which may be provided by the teacher.

[5] F. Barrachin and P. Bouchet

An objective of the research project presented here is to show how a didactical analysis sheds light on this complexity and brings elements of answer as to the conditions of the construction of a conjecture in the classroom and its establishment as a theorem.

2 ANGLE CONCEPTIONS AND PROOFS OF THE THEOREM

2.1 MATHEMATICAL DEFINITIONS AND ANGLE CONCEPTIONS

2.1.1 THE MAIN DEFINITIONS

A historical study of the concept of angle would be beyond the scope of this research project. Considering the school level targeted, the purpose is limited to identifying the main features of this concept by pointing out some historical milestones.

I deliberately leave aside algebraic definitions and considerations of angle measurements, as well as the concepts of oriented angle or vector angle; the survey is limited to elementary plane geometry. Likewise, I have left aside the discussion of the confusion sometimes noticed between the notions of angle and angle measurement.

Then, four main types of definitions can be distinguished: angle as the inclination to one another of two lines, angle as a figure formed by two half-lines, angle as a region of the plane and angle assimilated to rotation.

- **The angle, inclination to one another of two lines**

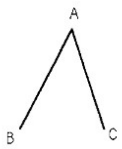
  This is the classical definition of the Euclidean Elements. Several different formulations can be found in the many translations of the *Elements*, we quote here a classical one:

  > *"A plane angle is the inclination to one another of two lines in a plan and do not line in a straight line."* (Heath, 1956, p. 153)

  This definition excludes flat angles and it encourages to consider only angles of measurement less than 180°. The definition of the *Elements* has been subjected to numerous criticisms and has undergone various reformulations (e.g. Heath, 1956, p. 176 sqq).

- **The angle, figure formed by two half-lines**

  This definition goes back to Aristotle (Smith, 1925, p. 277). Here is, for example, a formulation from a late French edition of Legendre's elements:

  The figure formed by two intersecting straight lines AB and AC is called an angle. Point A is the apex of the angle; lines AB and AC are the sides of the angle. (Legendre 1875, p.2)

  Hilbert proposed a more precise formulation:

  > "*Let α be any arbitrary plane and h, k any two distinct half-rays lying in α and emanating from the point O so as to form a part of two different straight lines. We call the system formed by these two half-rays h, k an angle and represent it by the symbol $\angle$(h, k)or $\angle$(k, h).*" *(Hilbert, 1899, p. 9)*

  Hilbert notes that "this definition excludes flat and concave angles" (ibid.), then he introduces the notions of inside and outside of an angle.

  This definition dominated mathematics education in France before the First World War (Berdonneau, 1981, p. 215 sqq). It reappeared under the definition of the angle adopted by the 1970 French curriculum reform for ninth-grade classes, the geometrical angle—as opposed to the common-sense angle—being defined as a class of isometric pairs of semi-straight lines of the same origin (Berdonneau op. cit.).

- **The angle, region of the plane**

Two different types of definition related to the conception of the angle as a region of the plane can be distinguished. One type that we qualify dynamic, like Henrici's definition:

> *"The part of a pencil of half-rays, described by a half-ray turning about its end point C from one position* a *to another* b, *is called an angle. The centre C of the pencil is called the VERTEX, and the first and last positions* a *and* b *of the describing ray are the LIMITS of the angle"* (Henrici, 1879, p. 47)

and a static type like the definition of Louis Bertrand[6]:

> *"Two lines AB, CD that intersect in a plane, make four parts ASD, DSB, BSC, CSA, which are angles: an angle is therefore a part of the plane such that it has as its limit two lines that meet, and end at their meeting point. The straight lines AS, SD which limit an angle ASD are its legs; its vertex S, is their point of intersection; its size is the very size of the portion of the plane which is limited by its legs."* (Bertrand, 1812, pp. 4–5)[7]

This conception of the angle has reached its peak with the introduction of the notion of angular sector in the teaching of "Modern Mathematics"; the angle was then defined as a class of superimposable sectors.

- **angle, rotation**

Although Sir Thomas Heath (1956, p. 177) suggests Carpus of Antioch as one of his precursors, this definition will not really impose itself in teaching. At the time of the modernist debate on the renewal of the teaching of mathematics, it was defended by some mathematicians. Notably, Gustave Choquet, after considering the different conceptions of the angle, proposed...

> to *"identify angles to rotations around a point O; then show that the choice of O doesn't matter."* (Choquet, 1964, p. 97)

Actually, from the beginning of the 20th century it could be noticed "that nearly all of the text-books which give definitions different from those in group 2 [angle as rotation] add to them something pointing to a connection between an angle and rotation: a striking indication that the essential nature of an angle is closely connected with rotation." (Heath, 1956, p. 179).

What Georges Papy (1967, p. 289) expressed in a striking shortcut by:

"ANGLE=ROTATION
which has lost its CENTRE!"

2.1.2 *A DIFFICULT NOTION*[8]

"The notion of angle is undoubtedly the one that raises the most discussions and difficulties in the teaching of geometry", wrote Gustave Choquet (1964, p. 96). While there is unanimous recognition of these difficulties, there is no consensus on the solutions that can be brought about to overcome them, particularly with regard to

[6] Note2020: Bertrand de Genève, quoted by (Fourrey, 1938, p. 44)
[7] Note2020: My free translation from the French
[8] Note2020: Understanding the learning difficulty of angle has a long history, it is still a challenge for research in mathematics education (e.g. Devichi & Munier, 2013; Tanguay & Venant, 2016)

the definition. This is evidenced, for example, by the abundance of formulations that can be found in school textbooks in use along the 20th century, which are all attempts to solve this difficult teaching problem (c.f. esp. Berdonneau, 1981, pp. 215–224)[9].

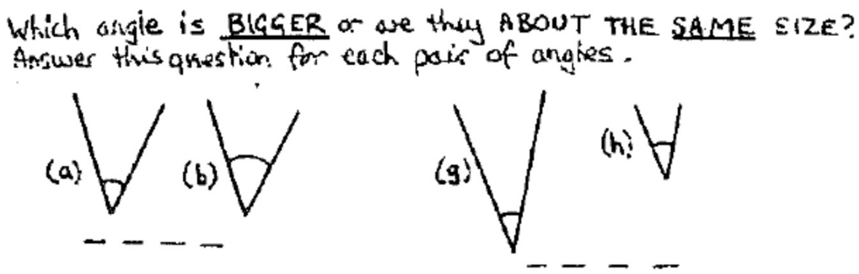
The main difficulty is due to the student development of a conception of angle which consists of its identification with the drawing representing it on a paper or on the blackboard. The angle is then conceived as two segments with a common vertex and distinct supports. With such a conception, two drawings that differ only in the length of the segments that make them up appear as representing two different objects. This conception is widespread and very resistant. It could be thought to derive from the very definitions used, such as the Legendre definition quoted above. For this reason, some textbooks try preventing the possible error by stating that:

> *"the size of an angle depends only on its opening and not on the length of its sides, which, theoretically, should be considered unlimited"* (in Précis de géométrie - Thuret, 1934).

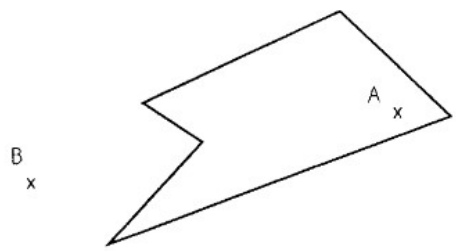
This kind of remark has all but disappeared from recent textbooks[10] in which angle is defined as an equivalence class of angular sectors or pairs of half-lines of common ends.

Yet this conception remains present at the end of primary education and the beginning of secondary education, a fact that is well known to teachers. However, it seems to be little studied.

Geoff Giles (1981, p. 11) attests the presence of this conception in half of the 4881 subjects examined on a task of classifying a set of angles according to their size. For her own research, Gillian Susan Close (1982, pp. 120, 122) proposed a questionnaire on angles to 87 English students aged 11 to 12 years old, in mainstream education, including the items below. She obtained 53 false answers for item 1, and 49 false answers for item 2.



In common French language, the word *angle* (trans. angle) is synonymous with *coin* (trans. corner) which refers to a relatively well-defined region; one can without too much ambiguity fix an appointment "*au coin de la rue*" (trans. at the corner of a street), it will then not come to mind placing oneself elsewhere than in the vicinity of the corresponding crossroads. Similarly, in mathematics, working with shapes, if a student is asked to show an angle of a polygon, he or she will be expected to point to a fairly precise region; usual practice will lead to the assumption that point A in the figure below is in an angle of the polygon shown, but not point B.



[9] Note2020: it's worth noting Gustave Choquet comment: "[...] note that we were able to conveniently construct much of the geometry without ever talking about angles.... The abuse of angles in our teaching is due to historical reasons: on the one hand, angles are among the primitive terms of Euclid's axiomatic, and for a long time the postulate of parallels was stated in angular form; on the other hand, Euclid's continuators believed it necessary, once they had understood the notions of oriented angle and angle of straight lines well enough, to make unjustified universal use of them." (ibid. p.97—DeepL free translation)

[10] Note2020: the reader should not forget that these words were written in the beginning of the 80s.

The everyday use and the mathematical meaning of "angle" are somewhat in contradiction. In the absence of a study similar to that of Gillian Susan Close, this observation leads us to hypothesize that, like young English students, French seventh-grade students will have a conception that favors the identification of the angle to its graphical representation.

2.2 Proofs of the theorem on the sum of the angles of a triangle

This project is part of my study of proofs that students may propose before mathematical proof is taught to them. I distinguished two main categories: pragmatic vs intellectual proof. Pragmatic proofs use efficient action based on manipulating concrete objects, while intellectual proofs are based on the formulations of properties and of their relations. **Pragmatic proofs** are fundamentally about concrete action, **intellectual proofs** are fundamentally about discourse.

In this section we draw on a few historical examples but, as in the previous section, the aim is simply to set the context and witness the complexity of the notion of angle by providing examples of proofs and debates that have effectively appeared in mathematics or in mathematics education.

After describing *pragmatic proofs* that we have been able to identify, we will present *intellectual proofs* by grouping them into two categories according to the types of conceptions of angle that support them: the *inclination to one another of two lines* or *rotation*.

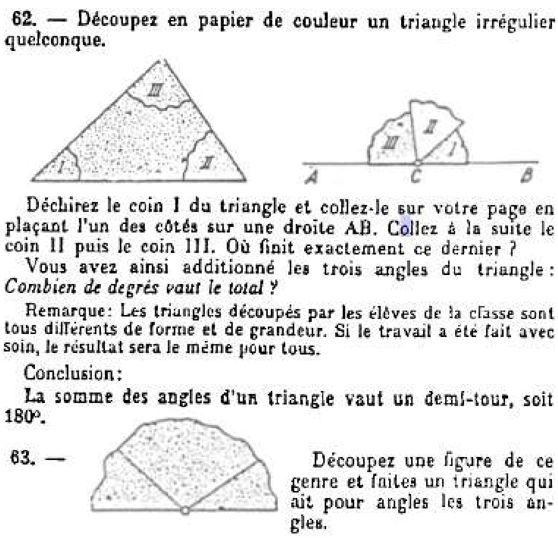
2.2.1 Pragmatic proofs

2.2.1.1 ***Measurement***

A first type of pragmatic proof, falling within the category *naïve empiricism*, consists in carrying out measurements and calculations for a few triangles and concluding that the property observed on these few cases will always be verified. The use of such a proof should be easily disqualified by highlighting the inevitable uncertainty of measurement.

2.2.1.2 Cutting up

A classic pragmatic proof consists of drawing a triangle on paper, tearing it apart to separate the three corners, and then joining them adjacently as shown in the figure below.


62. — **Découpez en papier de couleur un triangle irrégulier quelconque.**

Déchirez le coin I du triangle et collez-le sur votre page en plaçant l'un des côtés sur une droite AB. Collez à la suite le coin II puis le coin III. Où finit exactement ce dernier ?

Vous avez ainsi additionné les trois angles du triangle : *Combien de degrés vaut le total ?*

Remarque : Les triangles découpés par les élèves de la classe sont tous différents de forme et de grandeur. Si le travail a été fait avec soin, le résultat sera le même pour tous.

Conclusion :

La somme des angles d'un triangle vaut un demi-tour, soit 180°.

63. — Découpez une figure de ce genre et faites un triangle qui ait pour angles les trois angles.

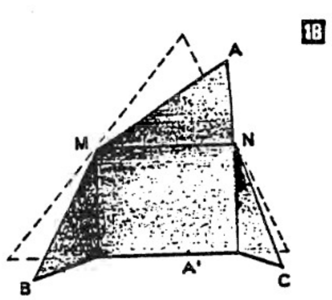
Excert from "Enseignement de la Géométrie" (Grosgurin, 1926)
Book for the fifth grade

Naive empiricism qualifies a pragmatic type of proof which consists in drawing from the observation of a small number of cases the certainty of the truth of an assertion.

It should be noted that, like the previous one, this evidence falls within the category *naïve empiricism* and is as prone to practical error as is the use of measurement; students' cut-outs are most often uncertain and their assemblages approximate. One could put forward that on the one hand, students' notions of "angle" (i.e. corner) and "angles of a triangle" may not be identical, and that, on the other hand students may not be convinced that cutting ensures the conservation of properties.

2.2.1.3 FOLDING

A triangle having been drawn on a sheet of paper, it is cut out and then folded as shown in the following picture: bring the vertex A to its opposite side by folding around the midpoints (which can be determined by folding) of the sides adjacent to it, then fold the *corners* of the resulting trapezium inwards.



2e PROCÉDÉ : Sur le second triangle ABC, déterminer les milieux M, de AB, N de AC, et plier le triangle autour de MN (FIG. 18). A vient en A′ sur BC; plier le triangle BMA′ de façon que B vienne en A′, et le triangle CNA′ de façon que C vienne en A′. Vérifier que les trois angles de ABC sont ainsi construits en A′.
Conclure : **« La somme des angles d'un triangle est égale à... » (Cette propriété sera démontrée plus tard.)**

Excerpt from (*Mathématiques 6°*, 1969)
E. Rich series, Hatier

2.2.2 ANGLE, THE INCLINATION TO ONE ANOTHER OF TWO LINES

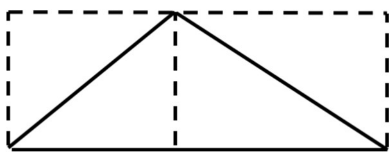
2.2.2.1 THE ORIGINS

The theorem on the sum of the angles of a triangle is believed to be Pythagorean, not only the general proof is attributed to them but also its discovery. According to Geminus, quoted by Heath:

> *"The ancients investigated the theorem of the two right angles in each individual species of triangle, first in the equilateral, again in the isosceles, and afterwards in the scalene triangle, and later geometers demonstrated the general theorem to the effect that in any triangle the three interior angles are equal to two right angles."* (Heath, 1921, p. 135)

We are interested here in reporting the attempt to reconstruct this approach proposed by Sir Thomas Heath in his translation of the *Euclid Elements* (Heath, 1956, pp. 316–322), following Henkel and Cantor to whom he refers. Having recalled that equilateral triangles or regular hexagons were used for paving, he continues:

> *"it would then be clear that six angles equal to an angle of an equilateral triangle are equal to four right angles, and therefore that the three angles of an equilateral triangle are equal to two right angles.* […] *Next it would be inferred, as the result of drawing the diagonal of* any *rectangle and observing the equality of the triangles forming the two halves, that the sum of the angles of* any *right-angle triangle is equal to two right angles, and hence (the two congruent right-angled triangles being then placed so as to form one isosceles triangle) that the same is true of* any isosceles *triangle.*



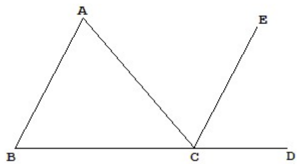
Only the last step remained, namely that of observing that any *triangle could be regarded as the half of a rectangle (drawn as indicated in the next figure) or simply that any triangle could be divided into two right-angled triangles, whence it would be inferred that in general the sum of the angles of any triangle is equal to two right angles."* (ibid. pp. 318-319)

It should be added that Heath himself considers this reconstruction to be highly speculative (ibid. p.319).

2.2.2.2 Euclid's proof

The most classical proof is that of the Euclidean Elements (Book 1, proposition 32). Here is the Heath' translation (1956, pp. 316–317):

> *"In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three angles of the triangle are equal to two right angles."*
>
> *Let ABC be a triangle, and let one side of it BC be produced to D;*
> *I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, CAB are equal to two right angles.*
> *For let CE be drawn through the point C parallel to the straight-line AB.*
> *Then, since AB is parallel to CE, and AC has fallen upon them, the alternate angles BAC, ACE are equal to one another.*
> *Again, since AB is parallel to CA, and the straight-line BD has fallen upon them, the exterior angle ECD is equal to the interior and opposite angle ABC.*



> *But the angle ACE was also proved equal to the angle BAC; therefore, the whole angle ACD is equal to the two interior and opposite angles BAC, ABC.*
> *Let the angle ACB be added to each; therefore, the angles ACD, ACB are equal to the three angles ABC, BCA, CAB.*
> *But the angles ACD, ACB are equal to two right angles; therefore, the angles ABC, BCA, CAB are also equal to two right angles.*
> *Therefore etc.*
>
> *Q. E. D.*

A **thought experiment** is an intellectual type of proof which invokes action by internalizing it and detaching it from its realization on a particular representative. It remains marked by the anecdotal temporality, but the operations and relationships underlying the proof are made explicit otherwise than by the result of their implementation.

Alexis Claude Clairaut, a famous mathematician of the 18th century, looked for linking Euclidean proofs to intuition, as he wrote in the preface of his *Elemens de Géométrie*: "I *carefully avoids giving any proposition in the form of a theorem; that is to say, those propositions in which one demonstrates that such and such a truth is without showing how one managed to discover it*." (1741, p. vij of the preface)[11]. He then offered an "intuitive" explanation of the origins of this proof. It takes the form of a *thought experiment* that implements the conception of angle as the inclination to one another of two lines[12]:

> *" As it is important in practice, as we have already said, that the angles are measured exactly, we must not be satisfied with just taking them, even with the most perfect instruments, we still have to find a way to check their measurements, to make the correction, if necessary. Yet this means is simple and easy.*

[11] Note2020: DeepL free translation, here is the French original : "j'évite avec soin de donner aucune proposition sous la forme de théorème ; c'est à dire de ces propositions où l'on démontre que telle ou telle vérité est sans faire voir comment on est parvenu à la découvrir."

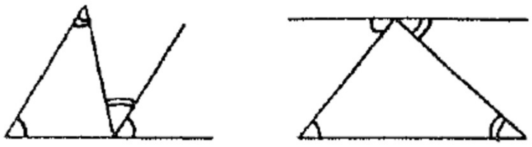
[12]Note2020: I wrongly indicated in the original report that Heath (1956, p.321) attributed such an idea to Proclus. Actually, Heath mentioned the Proclus idea of exploiting the notion of angle as the inclination to one another of two lines to provide a proof independently of parallels which was not the aim of Clairaut.

Let's go back to the ABC triangle. We feel that the size of the angle C must result from the size of the angles A & B; because as we increase or decrease these angles, the position of the lines CA, BC change, & consequently, the angle C, which these lines make between them. Now if this angle depends on the size of the angles A & B, it must be assumed that the number of degrees contained in the angles A & B must determine the number of degrees contained in angle C, & that it can thus be used as a check on the operations carried out to determine the angles A & B, since it will be certain that the angles A & B have been properly measured, if, by subsequently measuring angle C, it is found to contain the right number of degrees in relation to the size of the angles A & B.

To find out how much of the magnitude of angles A & B can be concluded from the magnitude of angle C, let's examine what happens at this angle, if the lines AC, BC, come closer or further apart. Suppose, for example, that BC turning around point B, moves away from AB, to approach BE, it is clear that while BC was turning, angle B is continuously opening; & that, on the contrary, angle C is getting closer and closer ; which first of all leads us to presume that, in this case, the decrease of angle C equals the increase of angle B, & that the sum of the three angles A, B, C, is always the same, whatever the inclination of the lines AC, BC, on line AE." [13] (ibid. pp. 63-64).

Clairaut went on stating that "*this alleged induction carries with it its mathematical proof*" (ibid.), and then he provided the classical proof of the Elements.

This proof of Euclid is the one most classically found in French school books until the *New Math* reform. The diagrams below sketch the main variants of the proof found in textbooks (the drawing on the right illustrates the one attributed to the Pythagorean school):



New Math French textbooks introduced a notion of *geometric angle*—equivalence class of a pair of superimposable rays of the same origin—and its measurement, the "*écart*" (magnitude of the smallest rotation that maps one of the rays into the other). However, the corresponding proofs using geometric transformations (see below) are framed by the classical Euclid heuristic: the transformations allow to "transport" angles adjacently around the same vertex. This is an example of a transition from *pragmatic proofs* to *intellectual proofs*, here at a rather high level of abstraction but still in the filiation of conceptions and proofs mentioned above.

[13] Note2020: DeepL free translation, here is the French original: " Comme il importe dans la pratique, ainsi que nous l'avons déjà dit, que les angles soient exactement mesurés, il ne faut pas se contenter de les prendre, même avec les instruments les plus parfaits, il faut encore trouver le moyen de vérifier leurs mesures, pour en faire la correction, s'il étoit nécessaire. Or ce moyen est simple et facile. Reprenons le triangle ABC. On sent que la grandeur de l'angle C doit résulter de celle des angles A & B ; car qu'on augmentât, ou qu'on diminuât ces angles, la position des lignes CA, BC changeroit, & par conséquent, l'angle C, que ces lignes font entr'elles. Or si cet angle dépend de la grandeur des angles A & B, on doit présumer que ce que les angles A & B renferment de degrés doit déterminer le nombre de degré que doit renfermer l'angle C, & qu'ainsi il pourra servir de vérification aux opérations qu'on aura faites pour déterminer les angles A & B, puisqu'on sera sûr qu'on aura bien mesuré les angles A & B, si, en mesurant ensuite l'angle C, on lui trouve le nombre de degrés qui lui conviendra relativement à la grandeur des angles A & B.

Pour trouver comment de la grandeur des angles A & B, on peut conclure celle de l'angle C, examinons ce qui arriveroit à cet angle, si les lignes AC, BC, venoient à se rapprocher, ou à s'écarter l'une de l'autre. Supposons, par exemple, que BC tournant autour du point B, s'écarte de AB, pour s'approcher de BE, il est clair que pendant que BC tourneroit, l'angle B s'ouvriroit continuellement ; & qu'au contraire, l'angle C se resserre de plus en plus ; ce qui d'abord pourroit faire présumer que, dans ce cas, la diminution de l'angle C égaleroit l'augmentation de l'angle B, & qu'ainsi la somme des trois angles A, B, C, seroit toujours la même, quelle que fut l'inclinaison des lignes AC, BC, sur la ligne AE. "

SOMME DES ÉCARTS DES ANGLES GÉOMÉTRIQUE D'UN TRIANGLE

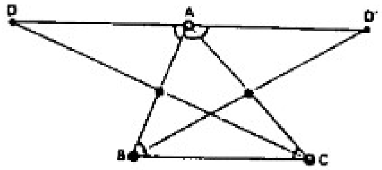


fig. 14

Soit un triangle (A, B, C), désignons par S la symétrie centrale par rapport au milieu de (A, B) et S' la symétrie par rapport au milieu de (A, C) soient :

$$D = S(C) \qquad \text{et} \qquad D' = S'(B)$$

C et D sont donc dans deux demi-plans différents de bord commun (AB) les secteurs angulaires saillants fermés de bords [A, D) et [A, B) pour l'un et [A, B) et [A, C) pour l'autre sont alors adjacents donc

$$E(\widehat{DAB}) + E(\widehat{BAC}) = E(\widehat{DAC}) \qquad \text{avec} \qquad \widehat{DAB} = \widehat{ABC}$$

car ils ont des représentants isométriques par S.

Mais de la même manière la symétrie centrale S' permet d'obtenir

$$\widehat{CAD'} = \widehat{ACB} \qquad \text{et} \qquad \overrightarrow{AD'} = -\overrightarrow{CB} = -\overrightarrow{AD}$$

donc D' et D sont dans deux demi-plans distincts de bord commun (AC) les deux secteurs angulaires saillants fermés de côtés [A, D) et [A, C) pour l'un, [A, C) et [A, D') pour l'autre sont donc adjacents et on a :

$$E(\widehat{DAC}) + E(\widehat{CAD'}) = E(\widehat{DAD'}) = k$$

En utilisant la relatic . E $(\widehat{DAC}) = E(\widehat{BAC}) + E(\widehat{ABC})$ on obtient finalement :

$$E(\widehat{BAC}) + E(\widehat{ABC}) + E(\widehat{ACB}) = k$$

THÉORÈME

La somme des écarts angulaires des angles géométriques d'un triangle est égale à l'écart angulaire k d'un angle plat.

Excerpt from (*Mathématiques 3°*, 1972)
Queysanne-Revuz Series – Fernand Nathan

2.2.2.3 LEGENDRE'S CRITIQUE

We cannot close this section on Euclid's proof without mentioning the criticisms of several mathematicians throughout history, which questioned the essential basis of this proof, namely the Euclid's fifth postulate:

> *"5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two rights angles" (Trans. Heath, 1956, p. 155)*

The French mathematician Adrien Marie Legendre, in particular, spent close to thirty years trying to establish this theorem without using Euclid's postulate, rewriting its proof in each of the twelve editions of its Elements published from 1794 to 1823 (Le Rest, 1982, pp. 144–148).

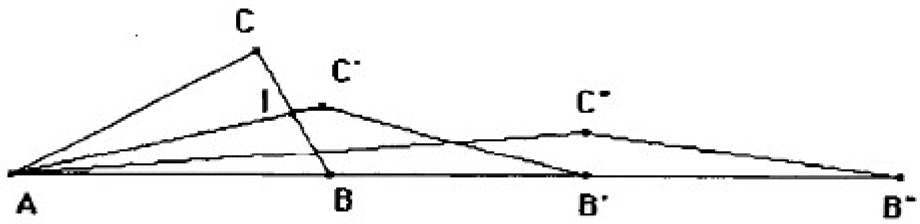
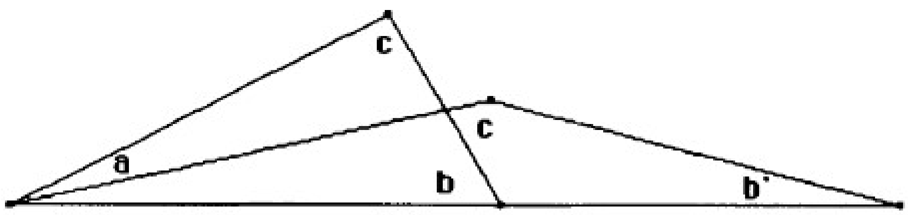
In the first edition of his elements Legendre (1794) gave a proof which was subsequently not used any more thereafter[14]. Here is Bkouche (1984)[15] outline of this first proof: *one gives oneself a segment and two angles, the*

[14] Note2020: Legendre had pedagogical reasons not to keep it: "This proof, of which one part is analytical and the other synthetic, leaves nothing to be desired in terms of geometric rigor; but to be accepted in the Elements, the study of geometry would have to be preceded by general notions on functions, which would require a fairly extensive knowledge of analysis that has not yet been introduced into the teaching of mathematics." (A. M. Legendre, 1833, p. 374)

[15] Note2020: This section was mainly based on the 1984's paper of Rudolph Bkouche that is now hardly possible to retrieve; most ideas can be found in an updated and more accessible version (Bkouche, 2007). Since then, Legendre elements and his réflexion on his efforts to prove

triangle which follows is then known and the third angle is a function of the two others (without being a function of the segment for reasons of homogeneity). To determine this function, Legendre shows that in a right-angled triangle, the sum of the three angles is two straight lines; since any triangle can be split by a height into two right-angled triangles, the result is immediate.

But Legendre recalled this proof in his note II to the proof he gave in the twelfth and ultimate edition of the Elements. Here are the key ideas of this last proof given in the twelfth edition[16] of the proposition XIX from book I: *in any triangle, the sum of the three angles is equal to two right angles*.

> *Legendre starts from a triangle* ABC *with* AB>AC>BC *then he constructs point* I *of* BC *such that* CI=IB*, point* C' *of* AI *such that* AC'=AB *and point* B' *of* AB *such that* AB'=2AI
>
> 
>
>
> *Let* A, B, C *be the angles of the triangle* ABC *and* A', B', C' *the angles of the triangle* AC'B'*. Legendre shows by* [reasoning on congruent triangles][17] *that*
>
> A+B+C=A'+B'+C' and that A'<(A/2)
>
> *Then he applies the same construction to the triangle AC'B' and obtains a triangle* AC'B' *whose angles are noted* A", B", C" *and verify:*
>
> A'+B'+C'=A "+B "+C" and A"< (A'/2)
>
> *By* "*continuing indefinitely the sequence of triangles*" *one arrives,* Legendre says*, at a triangle* abc *whose angle* a *is* "*less than any given angle*". If we construct from this triangle abc the following triangle a'b'c', the sum of the angles a'+b' being equal to the angle a" we see that the sum of the three angles of the triangle a'b'c' is reduced almost to the only angle c' *".*
>
> 
>
>
> *"*But it is conceivable that the triangle a'b'c' varies in its angles and sides so as to represent the successive triangles which later arise from the same construction and come closer and closer to the limit where the angles a' and b' would be zero*". At the limit the points a'b'c' are in a straight line and the sum of the angles is reduced to two straight lines. This very clever and visual demonstration implicitly assumes that any straight line can extend indefinitely and it stumbles on an assumption where infinity intervenes .... (Le Rest, 1982, pp. 145–146)*

---

the proposition XIX are available on-line, hence I restructured the section, but still I would like to pay tribute to Rudolph Bkouche whose comments on my work were always challenging.

[16] Note2020: https://gallica.bnf.fr/ark:/12148/bpt6k9771899n

[17] Note2020: Free translation of the French "*Legendre montre par les cas d'égalité des triangles*"

2.2.3 ANGLE AND ROTATION

Proof of the property of the sum of the angles of the triangle can be given by relying on a conception identifying the angle with the rotation. We give here the proof proposed by Gustave Choquet (1964, p. 100) whose definition we recalled above. This proof uses the Chasles theorem on oriented angles, it actually establishes a more general property of polygons; the theorem which interests us comes as a corollary.

59. Somme des angles d'un polygone fermé plan

Soit P un polygone fermé plan de n sommets, c'est-à-dire une suite $(a_1, a_2, ..., a_n)$ de points de Π, définie à une permutation circulaire près.

On supposera ici que, pour tout i, $a_i \neq a_{i+1}$ (donc aussi $a_n \neq a_1$); on pose alors δ_i = la demi-droite $D(a_i, a_{i+1})$.

On appelle *angle extérieur* de P au sommet a_i l'angle $\widehat{\delta_{i-1}\delta_i}$; on appelle *angle* de P en a_i l'angle $\widehat{a_{i-1}a_ia_{i+1}}$.

Proposition 59.1. La somme des angles extérieurs de tout polygone fermé plan est 0.

Démonstration. En effet, d'après la relation de Chasles

$\widehat{\delta_1\delta_2} + \widehat{\delta_2\delta_3} + ... + \widehat{\delta_n\delta_1} = \widehat{\delta_1\delta_1} = 0.$

Corollaire 59.2. La somme des angles de tout polygone fermé plan est 0 *ou* ϖ *suivant que le nombre n de ses sommets est pair ou impair.*

En effet soit δ'_{i-1} la demi-droite $D(a_i, a_{i-1})$; on a $\widehat{\delta_{i-1}\delta'_{i-1}} = \varpi$

Donc $\widehat{a_{i-1}a_ia_{i-1}} = \widehat{\delta'_{i-1}\delta_{i-1}} + \widehat{\delta_{i-1}\delta_i} = \varpi + \widehat{\delta_{i-1}\delta_i}$

La somme de ces angles vaut donc $n\varpi + 0 = 0$ ou ϖ suivant que n est pair ou impair (puisque $\varpi + \varpi = 0$).

En particulier la somme des angles d'un triplet (a_1, a_2, a_3) est ϖ; lorsqu'on aura défini la notion d'orientation, on précisera ce résultat en montrant que les trois angles d'un tel triplet ont même orientation.

A proof inspired by the same heuristic can be proposed at an elementary level, in the form of a thought experiment such as the one described in the excerpt below by the Geneva methodologist Grosgurin (1926).

58. Passons, dans cette question, de l'expérimentation pure au raisonnement.

On sait que deux angles opposés par le sommet (fig. 1) sont égaux (N° 37).

Prenons un triangle quelconque (fig. 2). Supposons une aiguille de montre en 1. Je la fais tourner de l'angle A (position 2). Puis je la fais glisser de 2 en 3 et tourner de l'angle B — ou plutôt de son égal, opposé par le sommet (position 4). Enfin je la fais glisser de 4 en 5 et tourner de l'angle C (position 6).

Au départ, en 1, l'aiguille pointait vers midi, à la fin vers six heures. Elle a donc tourné[1], tout en parcourant les trois angles, d'un demi-tour[2].

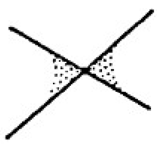


Fig. 1

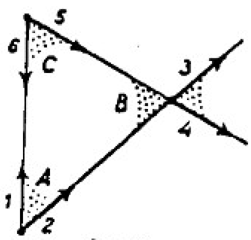


Fig. 2

2.2.4 REMARKS ON OBVIOUSNESS AND INTUITION

The theorem on the sum of the angles of a triangle is often taken as an example to support theses on the foundations of obviousness or on the intuitive nature of certain proofs. In relation to the conceptions of the angle mentioned above, I give below two examples that I will discuss in a final section.

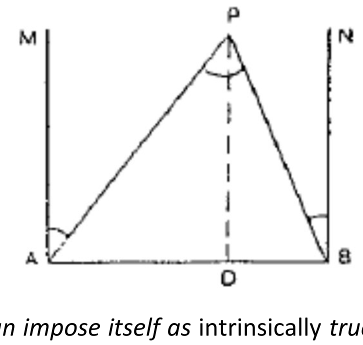
2.2.4.1 E. FISHBEIN, THE INTRINSIC TRUTH

The psychologist Ephraim Fishbein (1982) was interested in the role and meaning of intuition in mathematics and especially in its teaching and learning. "Intuition" has here not the mysterious connotation that it has in the common language, on the contrary it refers to a true knowledge of the subject:

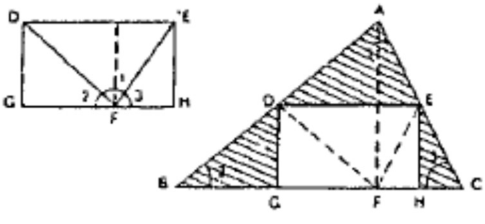
> *"an intuition is a form of knowledge. It has the role of a program of action -- but it is a cognition [...] An intuition cannot be built by mere verbal explorations nor by blindly practicing solving procedures. An intuition can be elaborated only in the frame of practical situations as a result of the personal involvement of the learner in solving genuine problems raised by the practical situations"* (Fishbein, 1982, p. 12).

Intuition in Ephraim Fishbein's sense thus appears to be a concept very close to those of "implicit model" or "model for action" forged by Guy Brousseau (1997, p. 10) or that of "concept-in-act" or "theorem-in-act" forged by Gérard Vergnaud (2009, pp. 85, 87). The relations between these different concepts have not been studied so far, it would be important that this be done; in any case this is not the place here, we limit ourselves to an analysis of Fishbein's use of the concept of intuition in a critical reflection on the notion of proof; in particular with regard to the proof of the theorem on the sum of the angles of a triangle:

> *"Let us consider again the theorem: "The sum of the angle of a triangle is equal to two right angles".*
>
> *Let us now draw a segment AB and the perpendiculars MA and NB to the segment. The angles MAB and NBA are right angles. We can "create" a triangle by "inclining" MA and NB. So, it can be seen that the angle APB "accumulates" what is « lost » by the angle MAB and NBA when "inclining" MA and NB.*
>
> *Of course, this is not mathematical language. It is rather a story about lines and angles, but a story which*



> *can catch the spirit, which can impose itself as* intrinsically *true. The same story can be translated into the form of a mathematical proof. Consequently, the formal necessity and the intrinsic necessity will coincide.*



> *I expose that procedure to a group of master's students. One of the students presented the following technique which, he said, is more strikingly intuitive and easier to understand. You cut a triangle out of a piece of paper. Let ABC be that triangle. Then you fold the triangle ADE, DBG*

and EHC so as to make the angles 1, 2 and 3 fit as in the Figure 14. You can see that their sum equals two right angles.

I do not agree with the above technique for the following reason. To « grasp intuitively » does not simply mean to « see ». In the example with the sum of the angles of a triangle what you have to grasp is not merely that, in a particular case, by joining the angles practically *you will get two right angles. The problem is to grasp intuitively why that constant effect is* necessarily *conserved, imposed, in the variable condition of a non-determined triangle. Intuitively it must be a problem of compensation. Therefore, the matter is not of showing practically that in a particular example the angles fit as the theorem predicted. What we have to "see" is that,* in variable conditions, by the way of compensation*, the sum must be conserved. And this I think is better suggested by my procedure because (a) a triangle appears as a particular case in a changeable situation; (b) the compensation leading to a constant result can directly be grasped; (c) we are behaviorally involved not in merely collecting angles but, rather, in a process of transformation which leaves constant the sum of the triangles; (d) this representation can be translated directly into a formal proof. The formal proof and the intuitive interpretation are perfectly congruent."* (ibid. pp. 17-18)

2.2.4.2 F. Halbwachs, the transfer of meaning

The physicist and epistemologist Francis Halbwachs (1981) searched for the articulation between "*meanings and reasons as an inferential organization of these meanings"* (ibid. p. 199). He points out that the reason (R) for an assertion (A) does not lie in its deductive or inductive proof; the reason for A is not its proof but what allows *"for its understanding in terms of the coordination between meanings. The reason R is not the truth of A, but its psychological status, in the sense that the meaning of A is going to be based on the meaning of R, and that the discovery of this type of relationship is going to found a new intelligibility, and in particular to give a new dimension to the necessity of A*" (ibid., p. 200).

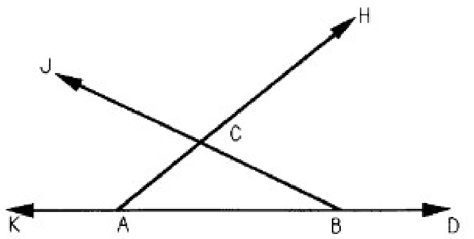
This theoretical reflection applied to the property of the sum of the angles of a triangle leads Halbwachs to the following criticisms and reflections:

Euclid's demonstration, which is usually given in teaching in this or that form, is static and artificial [...] It is the purely logical application of previous theorems, the sequence of which is perfectly convincing. But the raison d'être *of the property escapes, what shows that we are in the structural domain (sequence of composition and propositions). To show how to transcend this point of view, let's give what seems to be the "reason":*

The first step is to make the problem dynamic and constructive by specifying (or modifying) the notion of angle: an angle can be conceived as a portion of a plane swept by a half line pivoting around a fixed point, which puts a movement in the foreground. […]

Having said that thought, we start from the ABD half line, rotate at an angle Â around point A (counterclockwise) and come to ACH. Then we rotate at an angle C around point C (counterclockwise) and come to BCJ. Finally, we rotate by an angle B around point B (counterclockwise) and we come to BAK. So, we added the three angles taken in the same direction, and in total we went from the ABD half line to the BAK half line, i.e. we arrive on the starting line, but with a change of direction, which represents the flat angle (i.e. π or two right angles).

Here, contrary to Euclid's proof, the passage from the data to the conclusion took place by making the dynamic definition of the angle "work" [...] There is no longer only a logical anchoring of propositions (or as we say "transfer of truth"), but at the same time and in parallel transfer of meaning. (Halbwachs, 1981, pp. 201–202) [18].



« "Raisons" du théorème de la somme des angles »

2.2.4.3 About the unity of proofs and conceptions

Fishbein and Halbwachs *explanations* pretend reflecting the *fundamental* cognitive and epistemological nature of the theorem—as Halbwachs puts it, its *raison d'être*. They are quite different in that they are based on two different conceptions of the angle. In Fishbein's analysis one can retrieve Clairaut's explanation and the angle as the inclination to one another of two lines, in Halbwachs's analysis the angle is identified with rotation. It would be legitimate then to question which of these explanations is more intrinsically linked to the property.

In fact, one can imagine the answer, it flows directly from the theoretical foundations of each of these two authors. In other words, there is no inherently meaningful proof. Meaningfulness relates to the nature of the relationship between the explanation (or proof) and the subject's knowledge. Both explanations are meaningful, bearing reasons, for each of the authors because of their own *intuition* (in Fishbein's sense). The *explanatory nature* of each of these proofs is that they are clearly rooted in the conception of the angle of the writers, Fishbein and Halbwachs, which is evidenced by their analysis.

The analysis of both authors and their difference raise a fundamental questioning on the nature of the relations between the conceptions of the students and the foundations of the classical Euclidean proof. In my opinion, the objection is not of the static nature of Euclid's proof, as Halbwachs suggests, but that the conceptions—the knowledge as appropriated by the students—do not make it possible to constitute the reasons in the sense of an *inferential organization of meanings*. I formulate the hypothesis that it is in this discrepancy between proof and knowledge[19] that the obstacle to the constitution of reasons, and therefore to understanding, is laying.

As far as Euclid's proof is concerned, I would suggest that the auxiliary construction it requires is "understood" as soon as one "understands" that the theorem on the equality of certain angles determined by a secant on a pair of parallels allows a relation between the angles or their "transportation". Pragmatic proofs obtain this same transportation by cutting or folding.

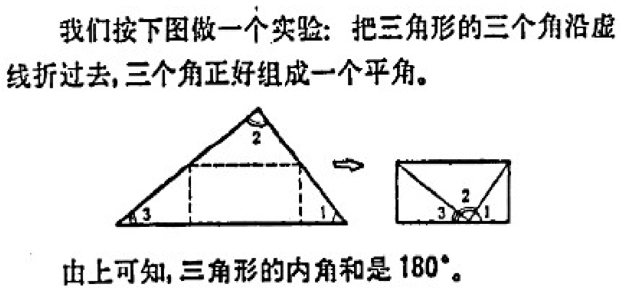
> It is very positive that the importance of the **operative form of knowledge** is recognized, today more than yesterday, the one that allows to do and to succeed. This does not devalue the **predicative form of knowledge**, that which takes the form of texts, statements, treatises and manuals, but it does more justice to the knowledge acquired through experience. (Vergnaud, 2001, p.287)

Halbwachs and Fishbein criticisms highlight by the discrepancy between the *predicative value* of the latter theorem—assertions on the equality of angle—and its *operative value* (Vergnaud, 2001). This operative value, or even this meaning of

[18] Note2020 : Edited DeepL-free translation
[19] Note2020: Taken here as a translation of the French "*connaissance*" and not "*savoir*".

the theorem, is in my opinion constructible by students only in a functioning of knowledge that is most often absent from school practices.

For these same reasons, one cannot, as Fishbein does, reject an explanation on the sole argument that it does not consider the intuitive nature of the property (which would mean that the property has a determined or intrinsic intuitive nature). I suggest that the explanation in question may not be meaningful to the author, but that it may be meaningful to the student proposing it depending on the meanings he or she attaches to the angle or to the property. I found a similar proof in a Chinese textbook; a familiar practices of paper folding might be the basis of its meaning, backed by a theorem-in-act according to which folding around the middle of the sides of a triangle allows bringing a vertex on its opposite side.


我们按下图做一个实验：把三角形的三个角沿虚线折过去，三个角正好组成一个平角。

由上可知，三角形的内角和是 180°。

However, I agree with Fishbein's remark that what brings meaning is not the use of action, which is not more intuitive in itself; contrary to what Halbwachs seems to suggest. Actually, it is a pedagogical hypothesis of this kind, which tends to replace the dogmatism of intellectual evidence with the dogmatism of pragmatic evidence that could be just as meaningless depending on students' conceptions.

3 DESIGN OF THE DIDACTICAL SEQUENCE

The first part of this text outlines the epistemic complexity of the concept of angle and of proving the theorem on the sum of the angles of a triangle. It provides some ground to envision what can be expected from students. The next part focuses on the design, the implementation and the analysis of a teaching experiment. The principles of design are those derived from the Theory of Didactical Situations (Brousseau 1997 esp. chap. 1 sect. 5 & 6), which constitute the foundation of a research methodology known as **didactic engineering** (Artigue 1992).

3.1 OVERALL PLAN OF THE SEQUENCE

Devolution is the act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility (Brousseau 1997 p.230)

Our objective is therefore to construct a didactic sequence that ensures the *devolution* to students of the responsibility of constructing the conjecture “the sum of the angles of a triangle is 180°” and to propose a proof establishing it.

It is expected that the initial dominant conception in a seventh-grade classroom will be that "the bigger a triangle, the greater the value of the sum of its angles". Such a theorem-in-act is likely to be quite resistant. It will serve as the basis for the construction of the sequence: the conjecture should stem from its destabilization. Furthermore, it will be necessary to ensure the disqualification of the use of measurement as a means of pragmatic proofs in order to allow the legitimate demand for intellectual proofs.

The devolution of the problem does not only mean its appropriation by each student, but also that the problem is recognized as such by the class, which takes *collective responsibility* for it. The sequence should therefore ensure this socialization.

Finally, it will be necessary to ensure the completion of the process in a duration compatible with the actual teaching constraints. In fact, this is a particularly difficult constraint to satisfy, both theoretically and practically; the problem raised is that of closure and institutionalization. I will examine it below (see section 3.4).

3.2 THE CONDITIONS FOR THE GENESIS OF THE CONJECTURE

3.2.1 CREATION AND STABILIZATION OF THE MILIEU

This section requires two tightly related concepts of the Theory of Didactical Situations, namely adidactical situation and milieu; both were not fully conceptualized at the beginning of the 80s. An **adidactical situation** is one in which the knowledge can be entirely justified by the internal logic of the situation and that the student can construct it without appealing to didactical reasoning. However, he or she cannot solve any adidactical situation immediately. In the teaching context, the teacher designs one which the student can handle. These adidactical situations arranged with didactical purpose determine the knowledge taught at a given moment and the particular meaning that this knowledge is going to have (summarizing Brousseau 1997, p.30). The knowledge taught is assumed to make it possible to read adidactical situations in the form of new "games" calling for new answers requiring and evolution of or a rupture with those he or she knows (ibid. p.57). To enable the implementation of such games, the didactical system must include and make explicit another system, distinct from the educational system, which will represent **the milieu** as a real or evoked reality allowing the emergence of these games of which the student is a protagonist. "*The milieu is the system opposing the taught system or, rather, the previously-taught system*." (ibid. p.57)

First, the challenge is to find the conditions in which the measurement of the angles of a triangle and the manipulation of these numbers can be introduced without the need to make explicit and explain the didactic objective that is finally aimed at.

At the chosen level – the seventh-grade – and at the time of this study – the second term of the 1983-84 school year – the continuation of lessons on angles provides a possible natural and favorable context.

Activities to measure the angles of a triangle can be easily introduced, as it is usual for the teacher to renew situations by making them a little more complex or by changing their context. Therefore, there is no need to justify this introduction in any other way. Furthermore, starting the sequence with measurement tasks helps to ensure that the instruments and corresponding techniques have been sufficiently mastered, and that they have been used beforehand: use of the protractor, technique for measuring the angles of a triangle (possible extension of the sides), manipulation of the numbers obtained. This starting activity will enable the teacher to check that students have *a robust basis for action*.

This initial situation is introduced as follows:

- *The teacher asks each of the student to draw a triangle, measure its angles, and then add up the results;*
- *The teacher justifies this activity by explaining that the aim is to continue the study of the use of the protractor and the measurement of angles;*
- *At the end of this activity, the teacher lists and notes the results on the blackboard, in the form of a histogram. He or she asks the students to comment.*

At this point, all results proposed by students are acceptable and can be accepted without distinction. As a matter of fact, they are the outcomes of measurements actually carried out and not the "true" value of the sum of the angles of a triangle. The diversity has no particular significance in this context: in the eyes of the students, it may result from the fact that different triangles have been drawn, yet some students could suggest that there are measurements uncertainty.

When entering an adidactical situation students mobilize conceptions which have proved efficient if not only useful in situations previously known to them. These conceptions work as implicit “theories”. The term **situation of action** denotes a didactical situation that allows the apparition of this theories, whose status in the classroom is not a priori definite (e.g. Brousseau p.162)

Comments are requested very openly. They are, in a way, only there to “complete” this first activity; it would indeed be very difficult to move without transition from the completion of the histogram to the next activity. These comments may relate to the dispersion of the values displayed by the graph, in particular by mentioning the numerical interval within which they are found.

With respect to the objective sought, the situation thus constructed does not constitute a *situation of action*, its function is only to create and stabilize the context in which the problem will be looming; it does not aim at mobilizing specific conceptions potentially related to it.

3.2.2 Knowing the sum of the angles of a triangle

Following the first activity, it is necessary to differentiate between what is due to measurement uncertainty and what is explained by students' conceptions in the diversity of the results obtained. To do this, the whole class must be confronted with the measurement of the angles of *a same triangle* with an incentive for students to engage their conceptions on the relationship between the size of a triangle and the sum of the measurements of its angles.

Hence, the second situation is introduced as follows:

- *The teacher gives each student a copy of the same triangle;*
- *he or she asks each student to make a bet on the sum of the measurements of the angles of this triangle and to write it down on a sheet of paper that will be collected;*
- *The teacher then asks each student to measure the three angles of the given triangle and then to calculate the sum of the outcomes of these measurements;*
- *At the end of this activity, the teacher lists and notes on the blackboard, in the form of a histogram, the results obtained;*
- *For each student the results are compared with the bets, and he or she is asked to comment on them;*
- *The teacher asks for comments on the histogram*.

The proposed triangle is large enough, occupying the entirety of a A4 sheet of paper, and “unremarkable” enough[20], to encourage the engagement of the expected theorem-in-act; indeed, it is expected that the first triangle students draw rarely occupies more than half of their sheet.

In order to decide on their bet, they may use simple visual perception to assess the measurement of the angles—something they may have practiced in class, particularly in recognizing 45° or 90° angles—or they may consider the proposed triangle as an object, considering its shape and its size, and compare it with their initial triangle. It is from the latter procedure that bets on numbers that are significantly larger than those obtained during the first activity could come out.

The commentary required from each student on the comparison of his or her bet with the result obtained, is intended to bring attention on the possible discrepancy between these two numbers.

Now, the situation thus achieved has the characteristics of a *situation of action*: it allows the mobilization of the *conceptions* that will intervene as *a model for action or decision* in the coming activities. No question of validation has yet been raised: the discrepancy between bet and measure

[20] Note2020: let’s say, a prototypical scalene.

can legitimately appear to be contingent, linked to the particular choice of triangles; it does not appear necessarily as a problem.

The comments on the histogram, requested from the class, should lead to making explicit and shared the requirement that *all students must have found the same result for the same triangle*. Differences that are bound to appear could be explained by measurement uncertainties; uncertainties that are instrument specific or due to practice. The teacher emphasizes this requirement collectively expressed.

3.2.3 A POSSIBLE CONJECTURE

At this stage of the sequence, the conjecture on the invariance of the sum of the measures of the angles of a triangle has no reason to have been formulated. It may be present in the minds of some students, but it is certainly not shared by the class.

A this point of the sequence, the conditions for its genesis are gathered:

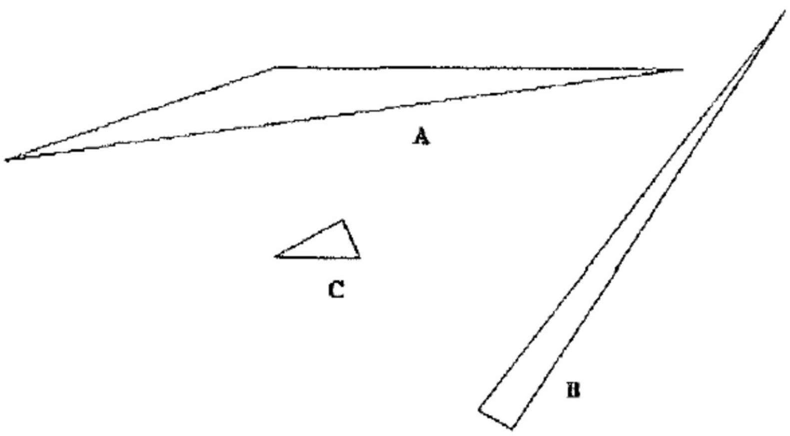
- *The conceptions of the students who will be at the origin of the conjecture have been solicited and the mathematical context in which this conjecture will emerge has been constituted;*
- *The fact of the uncertainty of the measurements has been recognized, the problem of knowing the sum of the angles of a given triangle is therefore posed.*
- *The disqualification of measurement as a means of knowing this sum will legitimize the demand for intellectual proof of the conjecture which formulation is expected as a consequence of the next situation.*

3.3 THE DEVOLUTION OF THE PROBLEM

3.3.1 THE EMERGENCE OF AN INVARIANT

The third situation aims at formulating the problem of the invariance of the sum of the measures of the angles of a triangle, and then at giving birth to the conjecture on the equality of this sum to 180°. The *devolution* of the problem to the class means not only that students appropriate it individually but also that it is placed under collective responsibility. This *socialization is a condition of a productive debate on proof* in the class community.

In order to induce the conjecture, indeed students must perform measurements and calculations for several triangles. But as the number of experiments that can be carried out during the class period is very limited, only a few cases can be scrutinized hence the choice of triangles is very important. Moreover, these experiments only make sense if students engage their conceptions: the conjecture and the problem of its proof will arise, at individual and class level, from the confrontation of conceptions according to which the sum of the angles of a triangle depends on the shape of the triangle versus the results of measurements which place this sum in the



neighborhood of 180°. For this, three types of triangles have been chosen (see figure below), with the strong expectation that their shape and size should favor the expected confrontation:

- *Triangles* A *and* B *were chosen for the provocative combination of the shape of the angles and the size of the triangle. In the case of triangle* A *both features (large triangle, big angle) point in the same direction, soliciting the idea that the sum of the angles should be large enough, for triangle* B *on the contrary these features (large triangle, sharp angle) point in a way which can make the evaluation of the sum problematic. The contrast of the shapes may be an obstacle to deciding that the triangles would have the same sum of angles. Moreover, the shapes of these triangles are unusual enough that one can expect that the bets that will be asked for are not self-evident and lead the students to non-trivial reasoning that engages their conceptions.*
- *Triangle* C *is very significantly smaller than the triangles* A *and* B*, so it may give rise to bets far below 180°.*

> I have called a **decision-making situation** a situation that calls for the implementation of validation processes without requiring the explicit and public production of evidence (c.f. *situation de décision* Balacheff 1988 p.33 sqq).

In order to stimulate making conceptions explicit and the emergence of epistemic conflicts, we organize the students' activity in teams: each team, freely decided by students on affinity criteria, is asked to agree on an *a priori* evaluation of the sum of the angles for each of the triangles. The requirement of a single bet for the group realizes the conditions of a *decision-making situation*. It is not the production of a common proof that is required but the production of a decision possibly based on a variety to criteria. The confrontation of points of view should lead to the proposal of explanations based on the conceptions of each one.

Hence, students working in teams of three or four (depending on class size), the third situation is introduced as follows:

- *The teacher gives each team a sheet of paper with the same drawings of the three triangles;*
- *The teacher asks each group to make a bet on the sum of the measurements of the angles for each of the triangles and then to write the bet on a paper that will be collected;*
- *Then, the teacher asks the groups to measure the angles for each triangle, and to work out the sum of the numbers obtained;*
- *At the end of this activity, the teacher lists and writes the proposed results on the blackboard, in the form of a histogram for each triangle;*
- *For each team, the results are compared with the bets, the teacher asks for comments.*

The census of the measurements, with the production of a histogram for each of the triangles, is the occasion for each of the groups to compare the bets with the outcomes of the measurements and computations. The commentary requested, as in the previous situations, is intended to help making conceptions explicit and to highlight the possible contradiction between bets and outcomes.

Following this, the teacher asks the students if they have any particular remarks to make after examining the three histograms and in the light of the comments that have just been made. Such an apparently rather open question is justified by the “common sense” requirement, possibly implicit, that each of the triangles must have a precise value for the sum of the measurements of its angles; that gives meaning to the teacher’s question.

Knowing what is the sum of the angles of a triangle is a problem which should impose itself to students.

3.3.2 FORMULATION OF THE CONJECTURE AND DEVOLUTION OF THE PROBLEM OF ITS PROOF

After the third situation, two cases may arise:

1. *The value 180° is remarkable, but students continue to argue that it is possible to find a triangle with a different sum of angles;*
2. *the conjecture is expressed by some students or groups of students, and is unambiguously shared by the whole class.*

In the latter case, the already recognized insufficiency of the use of instruments to determine the sum of the measurements of the angles should allow the teacher to verbalize the problem of proof however not having to tell at this stage whether the conjecture it true or false.

The most likely though is that the robustness of the initial conceptions will ensure the presence in the class of positions for and against the validity of the proposition. It is therefore expected that the class will find itself in the first case.

In order to get 180° to be considered a conjecture…

- *The teacher asks students who doubt it to try to construct a triangle whose sum of the measurements of the angles is very different from 180°;*
- *To those who argue that this is impossible, he or she asks to establish that 180° equality is necessary*.

Not taking sides, the teacher leaves the possibility for the proposition in question to constitute itself in conjecture under the responsibility of the class.

> Experimental *in vivo* class studies are difficult to carry out and manage. The main challenge is to understand to what extent the data gathered are (or are not) contingent to the specific time and context of the experiment. The role of the ***a priori* analysis** which is part of the didactic engineering is to anticipate the limits of its robustness to contingent events that may occur during the experiment itself. This section 3 is an example of such an analysis.

Of course, there is a risk[21] that as a result of such a process the students will welcome the fact that 180° is the sum of the angles of a triangle with sufficient conviction not to feel the need for proof. However, it is expected that the tensions that will have been maintained between the two "camps", for and against the conjecture, justifies the teacher's insistence that proof be sought. Anyhow, in my opinion, this risk seemed *a priori* to be rather limited.

In both cases, the students find themselves in a *situation of validation* where it is up to them to produce one (or more) proofs of the conjecture, or to refute it. At this stage, the teacher does not appear to the eyes of the students to be responsible for the validity of the conjecture, his or her responsibility in the sequence has never been defined otherwise than that of renewing the situations and ensuring the knowledge objective is maintained.

[21] Note2020: risk analysis of a didactical situations takes up the challenge of reproducibility of didactical engineering (e.g. Arsac et al., 1992; Artigue, 1986)

3.4 WRAPPING UP THE SITUATION

3.4.1 POSSIBLE SCENARIOS FOR A CONCLUSION

Devolution and institutionalization are two "negotiations" which mark, so to say, the opening and the ending of an adidactical situation. "In **institutionalization**, [the teacher] defines the relationships that can be allowed between the student's 'free' behaviour or production and the cultural or scientific knowledge and the didactical project; she provides a way of 'reading' these activities and gives them a status." (Brousseau 1997 p.56)

Making "devolution" a key feature of the design of the sequence raises the question of the consequences of transferring responsibility for the truth to the class. It is not possible to ensure *a priori* a definite response to this question, at least within the current status of the mathematics education knowledge-base[22]. However, one can envision possible "ends" and discuss them in the light of the knowledge constructed by students. A central issue in this discussion is that of *institutionalization* and its epistemic implication.

Three types of scenarios could be envisioned, the last offering an alternative:

i. The students have agreed on a proof of the conjecture. Then it remains for the teacher to endorse it. This assumes that the evidence is acceptable from his or her point of view. The "normal" relation between the teacher and the students which was suspended in order to allow the problem to devolve must be renewed: if the proof is not acceptable then the teacher must negotiate its rejection or modification;

ii. The students have not reached agreement, then the teacher must intervene to recognize the acceptable proofs and to reject the others. There will necessarily be a negotiation, possibly in two directions: one to have the rebuttal of false proofs accepted (and draw the consequences), and the other to have different proofs accepted;

iii. The students do not reach a solution, then the teacher must manage a way out. There are two possibilities for this:

    a. Either the teacher proposes a proof of the conjecture. To be acceptable to the class, this proof has to consider the conceptions of angle that students have mobilized and the directions of research that they have followed. Likewise, the level of proof should be consistent with the level at which students may have been attempting: generic example, thought experiment or classical geometrical deductive proofs. This rootedness in their efforts to solve the problem is a condition for the proposed proof to be accepted: the implicit counterpart of devolving the problem is that the solution of the problem is "reasonably" kept within the reach of the class.

    b. or the teacher proposes to accept the conjecture for true, leaving the problem of his proof open. This seems to be the classic position sometimes adopted in the classroom: "when more advanced, you will prove it ... this year, we will admit it ...". However, here the situation is radically different from what it usually is, in that the proposition in question has not only been serendipitously induced after a couple of observations, but has been *constructed* as a conjecture, *questioned* and *problematized*. Its potential truth has been debated. It is the production of a proof after an important effort to find one that is postponed until later. The proposal made by the teacher takes on a practical value (the research must have an end within an acceptable period of time) and not a principled one (as a claim that "the students cannot understand").

We believe that the latter option is rather unlikely unless there is a class that is, say, "very weak". From thought experiment to mathematical proof several types of proof are available, based on different conceptions of the angle or different meanings of the conjecture. This range of proofs, which we consider below, makes it possible to expect a sufficient number of positive ways to wrap up the sequence.

[22] Note2020: nor is it possible nowadays, a difficulty mentioned by Andreas J. Stylianides (2007, p. 15), and observed in reports of the like which I have studied

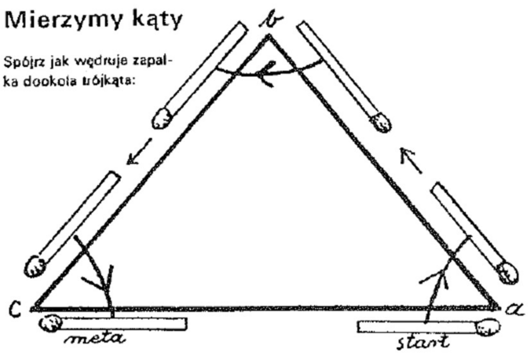
3.4.2 *A RANGE OF PROOFS*

At the time when this didactical sequence was designed there were few evidence-based elements to anticipate what proof could be constructed *by* the students. This state for the art of research in mathematics education was not a real challenge insofar as the range of possible proofs is quite wide both in terms of types, in the sense of the problem-solving strategies and with reference to the various conceptions of the angle, and in terms of levels. Moreover, the constraint of constructability of a proof by the students themselves is not a *sine qua non* condition for the "success" of the sequence.

Without repeating the analysis of the possible proofs already presented, I will here evoke some that could be used to wrap up the sequence.

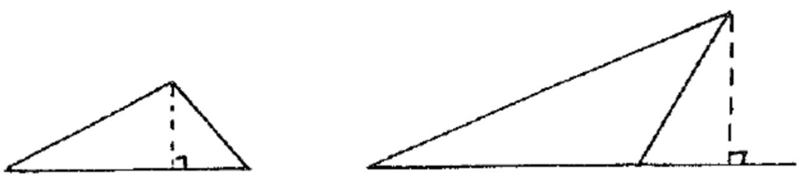
It is implausible that pragmatic evidence is proposed by the students: the use of measurement is disqualified by the construction of the sequence; as for cutting and folding, they are foreign to the usual practices of the French classes at the considered level. The crucial experiment does not appear to be a form adapted to the proof of this particular theorem; however, such a proof could be used to dismiss the claim that it would be possible to construct a triangle whose sum of the angles is very far from 180°.

It also seems unlikely, at the level of the French seventh-grade, that a "Euclidean" proof or a proof based on angle as rotation would be proposed by students. However, such proofs could be provided by the teacher, at an appropriate level, if it is consistent with students' conceptions of angle and the



types of research they have undertaken. For Euclidean proofs the classic presentations are known, as for reference to angle as rotation one can suggest a thought experiment of the type proposed by Grosgurin (see § 2.2.3.), or the version below from a Polish textbook:

In terms of the intellectual proofs that students could produce themselves, one can think of the intellectual proof based on the premise that a right-angled triangle is "half" of a rectangle. The source of this evidence, from a heuristic point of view, is the idea of looking for a link with the acquired fact that the sum of the angles of a rectangle is 4 times 90°, hence 360°. Any triangle can



then be considered “decomposable” into right-angled triangles by plotting one of its heights; students might consider both cases of a height inside outside the triangle.[23]

4 Experimental analysis and implementation

4.1 Preliminary Remarks

Carrying out experimental research in *didactique*, as well as in mathematics education in general, comes up against several difficulties in addition to constraints specific to the object of study.[24] I will not dwell at length on these aspects of research, but it is worth mentioning them as they may have conditioned the access to observation, and hence its very design.

There is no observatory or laboratory for didactical observation that allows the implementation of (sometimes important) experimental organization, and the collection of data at the junior high school level. Such a place existed, in France, for the Elementary School within the Jules Michelet School in Talence, associated with the IREM of Bordeaux[25]. This exceptional tool for experimental research in *didactique* is the product of the courageous will of a researcher: Guy Brousseau. Elsewhere, the possibility of experiments relies essentially on the generous collaboration of teachers, students and headmasters who are attentive and interested in the development of research.

This precarious situation has obvious material consequences, but also and more hidden, some at the theoretical level. It is not always possible for the teacher to be fully involved in the research project, which would be a normal functioning. This raises the essential problem of his or her introduction to the objectives and problems of the experiment, that of his or her appropriation of the different aspects of the didactical situations to be implemented and their motivation. This can only be the result of a negotiation that engages the teacher's own conceptions regarding the learning processes, the didactical functioning, but also the evaluation of the relevance of the proposed organization of the activities.

The implementation of an experiment in the classroom is in the first place the responsibility of the teacher. This is not only an ethical requirement but also a condition inherent to the very object of the research. Experimental research in *didactique* is therefore only possible if the experimental organization is compatible with the professional practices of the teacher and the functioning of the teaching institution. The experimental organization must be able to take its place in a teaching process that goes beyond it, that began before it and will continue after it. Moreover, its duration must be compatible with the school organization and the scheduling of the curriculum planned over the school year.

In this research, the experiment was to take up no more than two regular classroom sessions (about 55 minutes); it must be recognized that the theorem on the sum of the angles of a triangle, which was not included at that time in the seventh-grade curriculum, could not reasonably occupy more time. The observation was conducted in two junior high-school, D and E, in the Grenoble area, with volunteer teachers[26]. The whole sequence was video recorded, there were no other observers in the classroom than the camera operator. The analyses that we present have been made from the complete transcript of the video recordings.

[23] Note2020: interestingly Adrien Marie Legendre considered and criticized this idea, which he generalized with the following proposition: If there is only one triangle in which the sum of the angles is equal to two right angles, one must conclude that in any triangle, the sum of of the angles will be equal to two right angles. (A. M. Legendre, 1833, p. 375)

[24] Note2020 : since then didactic engineering (Artigue, 1992)

[25] Note2020 : see the “The center for observation: the École Jules Michelet at Talence” (Brousseau, 1997, Appendix)

[26] Note2020: the ethical approach in the 80s was respectful but not as formal as nowadays, then this translation adopts the current standards not mentioning the location of the schools nor the given names of the students.

4.2 AN OBSERVATION IN AN ORDINARY CLASSROOM

A first observation took place in an "ordinary" seventh-grade grade class of 25 students from junior high school D in March 1983. The scenario of the sequence had been discussed in detail with the teacher. I observed and filmed two sessions of about 50 minutes each.

4.2.1 CHRONICLE OF THE SEQUENCE

(D1) The student activity is introduced with reference to current activities.

> Teacher (1)[27]: *Today we will continue the work we have been doing up to now: measuring angles. Here's what I'm asking: you're going to draw a triangle, each one draws a triangle on his notebook ... then you measure each of the angles of this triangle. When you're finished, you'll add these three measurements together. Then I'll make a record of the results.*

When the students have started working, the teacher visits them. He completes the instructions with a few remarks addressed to the class, with the aim of specifying some practical points that may have been identified when visiting the students:

> Teacher (4): *You give the measurement in degrees, of each of the angles.*

> Teacher (9): *If you can't see very clearly, extend the sides*

Some of the interventions are intended to bring to the attention of the class points considered important in relation to the difficulties of some students. It is in fact the dialogue started with one of the students that is made public:

> "The size of the protractor":
>
> > Teacher (11): *If we don't have the same protractor, don't we have the same opening[28]?*
> > The class (12): *If, if ... obliged*
> > Teacher (13): *We measure in which unit* [...] *in degree. Is the degree on your ... it's an interesting question* [to the class] *is the degree on Kar's protractor the same as the degree on Murielle's protractor?*
> > The class (14): *But of course*
> > Kar (15): *Well, since one is bigger than the other, we won't get to the same place.*
> > Teacher (16): *I don't really understand what you mean. If one is taller than the other...?*
> > Kar (17): *If one is taller than the other one, it won't be the same as the one there.*
> > Teacher (18): *Is it the distance between the two half-rights that you measure Kar, when you measure with a protractor? Is it the length you measure?*
> > Kar (19): *No*
> > Teacher (20): *This is the angle opening. If I draw an angle like this* [on the blackboard] *does it change this angle when I extend the sides?*
> > The class (21): *No*
> > Teacher (22): *It's always the same angle. It's not the length here that you measure Kar, it's the opening of the angle* [the teacher illustrates with a gesture on the blackboard].

[27] Note2020: this number in bracket locate the utterance in the transcripts.
[28] In French: *écartement*

"Designation of angles":

> Teacher (23-24) [to X]*: You write AB equals 120°* [to the class] *if you have named the vertices of the triangle you can call them ... if you have named the vertices of the triangle A, B, C, how can you call the angles?*
> The class [with the teacher] (25): *A-B-C ...* [Teacher: like the…?] *... vertices*
> Teacher (26): *What is usually referred to as AB?*
> The class (27): *The segments*
> Teacher (28): *Segments rather, so it's not possible to designate an angle with this notation, isn't it?*

Interventions with the students aim to encourage them to work, or to re-determine the instructions; as for example for a student who has drawn three independent angles on his sheet of paper instead of a triangle.

(D2) The collection of results is displayed as a histogram.

The teacher accepts all the results in the same neutral manner, without remarks. However, the proposals 160° and 231° provoked strong reactions in the class, while the other results were received rather indifferently. The student who proposed 231° will change his result to 180°, which the teacher accepts with mixed feeling.

The teacher then invites the class to make remarks, simply repeating the sentences, “showing” them to the class as it were. In a few cases, a judgement or correction slip through:

> San (84): *It all happens between 150 and 200*
> Student (92): *There are many 180*
> Teacher (93): *That's an interesting point.*
>
> Student (97): *Every 180, they have a flat angle*
> Teacher (98): *180° corresponds to a flat angle*
>
> Student (99-100): *There are several triangles, they don't have the same angles but they have the same sum*
> Teacher (101): *Do you think they have the same sum?*
> Student (102): *In the 180's*

(D3) The transition to the next situation is based on this last remark.

> Teacher (103): *Among the students who found 180, had they all drawn the same triangle?*
> A student (104): *No, maybe not, eh?*

The teacher then introduces the following activity:

Teacher (105): *This is what I'm going to ask you to do now. You have chosen your triangle and measured the angles of this triangle. Well, this time I'm going to give you all the same triangle. You are going to do the exercise again, but with the same triangle for all of you.*
Student (106): *To see if we find the same one?*

Following this last remark, both views “yes” or “no” are expressed, in particular one student says: “*there are always one or two millimeters that escape*” (*Student* 117). The teacher then ends the instruction:

Teacher (124-140): *You're going to make a bet before you start... you bet on the sum of the angles of this triangle... on the little sheets I hand out, you write down the bet. And if possible, if you have a reason, you indicate it ... I pick up the sheets before you start measuring ... you do each one for yourself, not with your neighbor if possible ... you don't cheat, you give the measurements you find. You don't cheat to fit your bets*.

(D4) When the results are collected by the teacher, each measurement is confronted with the bet. Each student is asked to make a comment. Below are the pairs (bet, measure) followed by the student's comment (sometimes repeated by the teacher when the student is inaudible):

Isa (143): (180, 180) – *You had put at a glance*
Mur (150): (180, 180) – [she evaluated every angle]
Seb (152): (270, 180) – *The angle it seemed to me larger than when I measured*
Chr (158): (260, 180) – *It seemed bigger to you than when you measured*
San (162): (180, 180)
Flo (164): (180, 182) – *You're two degrees off.*
San’ (166): (250, 181) – *Already the large triangle and then an angle which was* large [Class laughs]
Fra (174): (180, 180)
Yan (174): (180, 180) – *At a glance*
Dav (175): (260, 163) – *This angle influenced you*
Jea (185): (220, 180) – *Because the angle ... it's a little bigger than a right angle.*
Bar (193): (160, 180) – *I saw that there were two small angles so I said it's less than 180.*
Car (197): (180, 180)
Kar (198): (180, 180)
Dom (199): (250, 180) – *The triangle it's too big ... and then the other one it was small*
Lud (207): (180, 180)
Kar' (209): (180, 180)
Phi (213): (180, 180) – *There is a right angle and two highs*
Phi' (217): (180, 180) – *You say it's got to be 180*. [Classroom fussing]
Val (221): (184, 181) – *The three angles do not have the same degree*
Eri (222): (440, 180) – *The triangle looked big to you.* [Class laughs]
Abd (225) (185, 180)
Kar" (227): (180, 180)
Rac (229): (180, 180)
Ald (231): (180, 180) – *But I thought it was a right angle and the other two were 45°.*

On the blackboard, the professor created the histogram as the measurements were collected:

Students are then invited to comment.

> Student (237): *If you put them together, it makes a flat angle.*
>
> Student (240): *163 is surprising.*
>
> Student (243): *440 is a big mistake.*
> Student (245): *Because one angle goes all the way to 360.*
> Student (246): *But the sum!*
> Student (251): *If it exceeds 360, the angles should be obtuse.*
> Isa (257): *There should be at least one reflex angle.*

These remarks do not consider that all students had the same triangle. The teacher intervenes to get the class to focus on this point:

> Teacher (258-260): *Someone said we should find the same thing. So, does everyone find the same thing?* [The class: *no!*] *Would it be normal to find the same thing or not?*

The class expresses mixed opinions *yes/no* with a few students sharing more precise thoughts:

> Dom (263-267): *The small triangle can be the same as the big one, but it's just the angles that you have to look at; it's not the length... what shocked me was the size.*
>
> Teacher (272): *Should we absolutely all find the same result?*
> Student (275): *With very precise measurements*
> Student (281): *They must have misplaced the protractor*
> Teacher (284): *They read to within 1 degree*

Following this exchange, and under pressure from some students, the teacher asks Dav (who found 163°) to redo his measurements: "*look at this, because some people think that you are placing wrongly the protractor*" (Teacher 287). The student confesses "*to having read badly*" (Dav 298).

(D5) The first session ends here with the teacher indicating that the work will be continued in the next session, two days later.

(D6) The second session opens with the announcement of the continuation of the activity of measuring the angles of a triangle.

> Teacher (300-5): *We will continue the work of measuring the angles of a triangle and the sum of these three measurements ... this time - this time I'll give you three ... you will have the same three triangles, and you will have to measure the angles and add them up. You will also, before, like last time, have to make a bet on the sum of the measurements for each of the triangles. And another little peculiarity is that you won't work alone, you'll work in groups of three* [...] *I'll give you the three triangles first and you have to make a bet, so don't start taking the protractors right away* [...] *you make a bet for each triangle, for the group, if you don't agree, you discuss it.*

The teacher follows the activity of each group visiting them but without intervening, especially in their debates. Each group gives the results of its measurements, then is confronted with the bets it has made.

(D7) Comments are requested on the matching of bets to measurements. The account below indicate the pairs (*bet*, *measure*) for each of the triangles A, B, C, followed by the comment of the group of students (sometimes repeated by the teacher when the student is inaudible):

Group (311) – *(130, 180) (130, 177) (130, 180) (130, 180)*
We had to bet, we bet... big opening... there were some little ones, so it could be equal to the others...

Group (335) – *(200, 180) (180, 180) (180, 180) (180, 180)*
At this angle, the big one there that cheated us... I thought it was already 180 degrees.

Group (349) – *(175, 184) (130, 177) (30, 180)*
The triangle was small... so you thought 30 for this triangle...

Group (370) – *(180, 180) (180, 180) (180, 180) (180, 180)*

Group (376) – *(180, 180) (180, 181) (180, 180) (180, 180)*

Group (386) – *(180, 182) (180, 177) (180, 180) (180, 180)*

Group (390) – *(180, 180) (180, 180) (60, 180)*
It was small.

Group (398) *(180, 180) (180, 180) (180, 180) (180, 180)*

While collecting the results, the teacher constructs the histograms on the blackboard:

Triangle A :

180 □□□□□□
182 □
184 □

Triangle B :

177 □□□
180 □□□□
181 □

Triangle C :

180 □□□□□□□□

(D8) The teacher calls on the students to comment on the results obtained which are displayed on the blackboard:

Kar (406): *Many of them found 180 for C*
Student (408): *Well, everyone...*

Student (410): *In all the triangles that we've measured, it's all around 180...*
Student (411): *170 and 180*

Student (414): *For the B there are differences like this because it is smaller and you have to extend the straight lines and ...*
[the teacher points out that it's the C that's the little triangle]

Students (419-421): *The C is the same as the large figure, they are the same angles* [students' comments summarized by the teacher],

As no other comments were forthcoming, the professor reopened the debate by encouraging interventions that would lead to the formulation of the conjecture:

Teacher (425): *Now, if I gave you a triangle of my own choosing... would you be willing to bet on the sum of the measures of the angles of that triangle...*
The class (426): *yes, yes*
Teacher (427): *Can someone draw me a triangle whose sum of angles is very far from 180°?*
Class (428): *no, no*
Teacher (429): *for example, 150° ... who could draw one?*
Student: *maybe*

Some students claim that 187° was found, but others point out that this is related to measurement errors (Student 434-5). One of them (Student 436) expresses that "*if we are very precise, it is obligatory* [that the measurement is 180°]". The teacher exerts a little more pressure to encourage the appearance of the controversy of which, in the confusion of the interventions, one can see some clues:

Teacher (437): *You say "it's obligatory". You say it's obligatory, but are you sure, sure, sure that it's perfectly 180° when you see the results?*
Students express mixt opinions: *yes-no*

Acknowledging this uncertainty, the professor recalls the request for a triangle whose sum of angles would be 150° or, failing that, a proof that the sum is always 180°. A student then proposes a triangle whose sum of angles is 4°:

Student (450): *... of 4 ...*
Teacher (451): *I don't understand what you're talking about, did you draw a triangle?*
Same student (452): *Yes, and in total I have 4 ...*
Teacher (453): *4 degrees, the sum of the angles 4 degrees, I will see.*

Student (463): *I found 6 degrees*
Teacher (464): *6 degrees, let me see... yeah... measure me that one there... that one there... it's almost an angle...*
Karen (465): *Right ...*
Teacher (466): *Right-angle as Karen says... that on, there... that one...*
Same Student (467): *Oh, yeah...*

Teacher (468, taking a student back): *in your opinion a triangle cannot exceed 180°.*
Student (469-71): *a little more ... when you got the measure wrong.*
Student (476-78): *you can draw anything ... it will be equal to ... 180 ... a little more a little less ...*
Student (480): *you can put less even, you can make triangles that are smaller ... I think*
Student (487): *it's not the triangle, it's the angles that count.*

Lud offers an explanation that he has prepared. He's misunderstood, but he tries to explain:

Lud (500): *if we find different measurements* [than 180°] *it's because... the angles are wrongly measured. The sectors of the angles continue to infinity and we will always find 180° at the sum of the angles of the triangles*

The noisy class is split between "yes" and "no", when the bell marking the end of the session rings. Within some confusion, one student claims to have found a triangle whose angles add up to 130°.

(D9) The teacher concludes by reviewing the activity and announcing that it will continue during the next session (which could only take place two weeks later because of the Easter holidays):

> Teacher (511): *we found triangles whose angles summed up to 177° and 187°. Well try to find me triangles whose sum of angles is very far from that; if it's possible. If you can't, see if you can conclude something, or try to show me something. Try to continue this work.*

4.2.2 ANALYSIS OF THE SEQUENCE

Because it is a natural follow-up to similar activities on angle measurement, the phase of creation of the geometrical context (D1) plays its role well. Students engage in the activity without comment. On the other hand, the teacher's interventions are numerous, they aims at avoiding errors or difficulties which are potential hindrance to the development of the sequence. They consist in collectively sharing the observation of certain of the students' difficulties, highlighting and treating them as information of general interest (remarks on the size of the protractor, the designation of angles).

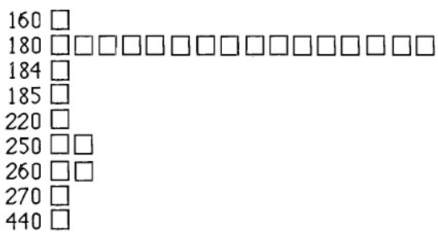
When the results are proposed (D2), the teacher receives them without comments. In fact, these results could be considered accepted, if one appreciates the teacher neutrality in the context of ordinary relations in the classroom: the task is just a technical task, the teacher is in charge of his usual responsibilities. But when students are invited to comment, the teacher attitude is less neutral; some of his remarks could even be clues for further activities (e.g. Teacher 93). However, this did not have consequences, as shown in the following observation.

The notable fact is that "the class" reacted strongly to certain values proposed as the sum of the angles of a triangle: 160° and 231°. However, we note that 7 students proposed values very far apart from 180° (more than 200°). These apparently strong reactions may come from the group of students who obtained a result close to 180°; this group is in the majority (there are 18 students between 176° and 182°) and may consider itself to represent normality.

> These ideas of "norm" and "conformism" were burrowed from the Palo Alto School, especially from my readings of the French translation of "How real is real?" from P. Watzlawick (1976)

A *behavior of conformism*, linked to the production of a "norm" from the first measurements collected during this first activity, may explain why from 7 results equal to 180° during the first phase, we pass to 18 bets on 180° during the second phase (D3 and D4). This does not mean that this value made sense in terms of its substance—i.e. the recognition of the necessary character of the invariant.

The diagram below shows the distribution of bets in the second activity:



Bets above 180° are either based on the presence of a "large" angle in the proposed triangle, or on a comparison of the size of the triangle with the triangle previously drawn by the students. On the other hand, not all bets over 180° mean that this value has been recognized as the *a priori* value of the sum of the angles of a triangle (cf. for example Isa 143, Yan 174, Ald 231). The necessary character of 180° is claimed by only one student (Phi' 217). It causes agitation in the class, the origin of which is not identifiable, but which is an indication that "the question of 180°" could be the subject of a debate.

Only one value, that is 440°, is rejected by noisy manifestations of "the class". It is not the value of the sum of the angles of a triangle that is at stake here, but the consideration of what is a possible value of the measurement of an angle. As shown by the exchange which followed (Ele 245-257), in the eyes of some students this value could not exceed 360°; and as it were the same applies to the sum of the angles that has been defined as the juxtaposition without overlapping of angular sectors.

The problem of deciding on *the value* of the sum of the angles of the given triangle is not raised, although it was mentioned by one student as a plausible goal of the activity (Ele 106). It is the teacher who brought it up again (Teacher 272), the few remarks made by the students show that the idea of measurement uncertainty is present in the class, but it is not institutionalized; the imminent end of the session may have prevented it.

At the end of this session (D5), the teacher indicates that the work will be continued; but without specifying which work it is. Implicitly and clearly enough, it is a work on measuring the angles of a triangle. *The ground is ready, but the conjecture is not yet formulated*.

The third phase (D6 and D7) shows that the conception that links the size of a triangle to the value of the sum of its angles is strong enough to resist collective debate in the face of rather "disturbing" cases. The behavior of conformism possibly adopted during the previous phase is strongly shaken, one can notice in particular a group which bets 180° for A and B but not for C considered too small (Group 390), or another which does not bet 180° for A because it has an obtuse angle (Group 335).

In this situation the class splits in two groups:

- 4 teams (i.e. about 12 students) did not reach the conjecture that 180° is the sum of the angles of a triangle. However, for two of them 180° is in some way a privileged value (Group 335 and 390); for another one there is a certain idea of invariance combined with considering the size of the triangles, a compromise that led to the bet of 130° for each of the three triangles (Group 311).
- 4 other teams seemed to be convinced that the sum of the angles of a triangle is 180°.

The fact that 180° is the result actually obtained in the majority of cases (18 measurements out of the 24 made) is an indication of a possible consensus on this value; it is quite likely that the students first found values in the vicinity of 180° and then corrected them.

If the precursor of a conjecture is present, witnessed to by statements of several students, it has not yet been identified as such and the problem of its proof has not been raised.

The teacher's incentives (D8) led to an impasse. The intervention (Prof 437) aimed at getting at least one student to raise the problem of proof that could be then taken over and institutionalized. In fact, the obstacle at this point may have been a strong consensus in the class on the validity of the outcome in question; students do not understand from the outset what is expected from them. The challenge to find a 150° triangle has the weight of the teacher's authority and thus shakes the most fragile conceptions. The success of this incentive—some students looked for triangles whose sum of the angles is very far from 180°—justifies the teacher introducing the requirement of proof as part of an alternative—providing an example or proof that this is impossible—which can preserve the devolution to the class of responsibility for formulating the conjecture and the problem of its proof

(D9). In this way, *the teacher's authority becomes a tool for successful devolution*, which however implies a (momentary) abdication. But I shall return to this point, which seems contradictory, in the conclusion of this study.

Wrapping up the situation could not take place during the observed sequence. When the students returned from Easter holidays, they had all given up building a triangle whose sum of angles would be different from 180°. The teacher then closed the sequence by bringing the “Pythagorean proof”, the study of angles with parallel sides having been carried out in previous lessons. Without observation, there is no sufficient elements to evaluate the status of this proof for the students, so the problem of institutionalization remains.

4.3 AN OBSERVATION IN AN EXPERIMENTAL CLASSROOM

> The experiment was carried out in a junior high school which followed the principles of an innovative approached coined by of Marc Legrand (2002, section 2.): “the classroom-based scientific debate”, which *basic assumption [is] that students can neither enunciate conjectures nor propose proofs unless they have their own opinion about what is scientifically reasonable and what is not. The student who wants to participate* […] *is invited to speak directly to the other students in the following somewhat iconoclastic way*: *“I do think that this idea is valid … that this argument proves or contradicts an idea defended by me or another … and here are my reasons.”* … *During such a debate, the role of the teacher is to answer for the scientific aspect of the debate but not to vouch for the truth nor for the validity of the results and arguments that are proposed. Only at the end of the debate does the teacher institutionalize the results by giving the appropriate definitions and theorems, and by identifying appealing but wrong results, the kind of recurrent mistakes which need to be addressed over and over again*. These rules are made explicit; hence they are known from the students since the beginning of their involvement in this innovative practice.

A second observation took place in a seventh-grade grade class of 23 students from junior high school E in January 1984, which was already engaged in an innovative project aimed at renewing relations between teacher and students, and between the students themselves, in order to create “the conditions for scientific debate in the classroom” (Legrand, 1986)[29]. This project was conducted by the research group “Learning and Reasoning” of the IREM of Grenoble (1985). The scenario of the sequence had been precisely discussed with the teacher. We observed and filmed two sessions of about 50 minutes each.

The teacher in charge of the class, member of the IREM research group, taught the sequence with another member of this group. The students therefore had simultaneously two teachers that day. This situation was not exceptional, it corresponded to a form of collective work of the IREM research group to which the students were accustomed. In fact, everything happened as if there was only one teacher in the classroom, but two voices: each one fading away in the face of the other's interventions. The specific phenomena of our study did not seem to be impacted by this two-headed presence, so I took the decision to present this observation and its analysis by “merging” the voices of both into one voice: *the* teacher.

On the other hand, the original pedagogical approach of the IREM research group, through its constant references to *the rules of “scientific debate”* (Legrand, 1993, pp. 124–142) makes this class a special place. Our hypothesis was that the difference with an “ordinary class” mainly concerns the conditions of devolution, which were here potentially facilitated, but that *the epistemic conditions of the construction of the conjecture* would not be modified.

[29] For an English presentation of these ideas and work, see (Legrand, 2002). Marc Legrand inspired the project of the IREM research-group and was an influential thinker for the evolution of teaching mathematics, with a special interest for higher education.

4.3.1 *CHRONICLE OF THE SEQUENCE*

(E1) The activity is introduced at the outset, without any particular reference to activities of previously given classes. However, it follows on from sessions devoted to the study of angles and their measurement:

> Teacher (1): *you are going to draw a triangle on your notebook ... very different from each other ... the size you want, the shape you want. And big enough, because you are going to measure the angles of the triangle and it must not be too small [...] I have drawn several shapes* [on the blackboard]*, but you make a single triangle [...] Once you have drawn them you call the vertices A, B, and C.*

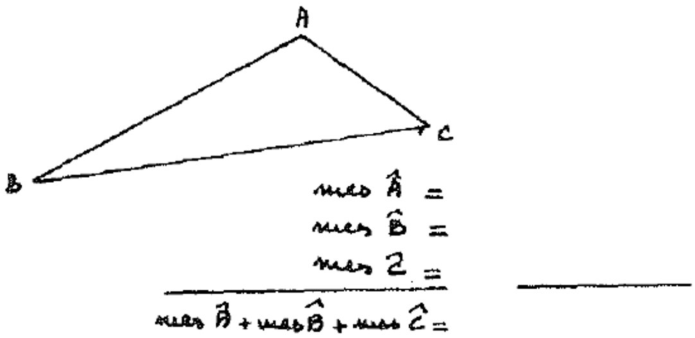
The students make the shapes asked for, but with too marked a tendency, for some, to conform to the those drawn on the blackboard. The teacher intervenes so that the students distance themselves from these examples:

> Teacher (2-3): *I drew two triangles on the blackboard, there are many who feel obliged to draw two triangles... you draw one triangle but with the shape you want. I put two shapes on the blackboard but you can …*

Then the teacher completes the task instruction:

> Teacher (3-11): *Once you've done that* [drawing] *you call A, B, and C ... the three vertices of the triangle ... and then you measure them and then next to your triangle you write down the measurements ... well, when you've measured the three angles ... I think everyone has found ... you add the three measurements together.*

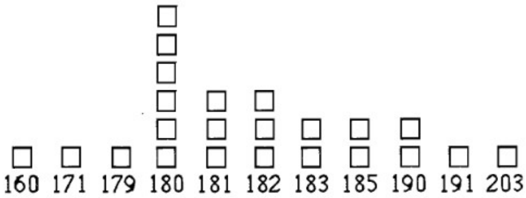
A sample presentation is displayed on the blackboard:



Some students have difficulties using the protractor, the teacher gives them advice and then makes a presentation to the class:

> Teacher (10): *Look carefully at the measurements you find. If they correspond to acute or obtuse angles. Yesterday we learned to recognize them by looking at them* [...] *so you look at your measurements.*

(E2) A histogram shows the results



The professor received these results without any particular comments, except for two proposals which were rejected:

Student (15): *I, sir, had 185.5...*
Teacher (16): *0.5 close, you know ... with the mistakes of ... eh, it's difficult ... so ...*
Student (18): *180.5*
Teacher (19): *180,5 you gave me what? 180, what do you think between 180,5 and 180 ... or 185,5 and 185*
Student (20): *rounding up*
Teacher (21): *and rounding up why? Because it's simpler or is there a ...*
Student (22): *to do the computation*
Student (23): *it's a bit complicated to do it*
Teacher (24): *It's a bit too complicated, it's true that it would complicate our lives a bit. You don't have any other reason to, um... round up, no...*

In addition, two results provoke reactions in the class: 163° and 203°. A student asks:

Student (27-31): *How did she find 203* [...] *because it is almost always within the 180s.*

A debate begins without the challenged student having responded directly to the question:

Student (32): *he may have taken a large triangle...*

A student (35): *I also made a big triangle* [and obtained a result close to 180°]

Student (36): *it's not that, he must have made a mistake because if you make a big triangle at the top, automatically at the bottom it will be smaller*
Student (42): *he says that if the triangle is big at the top, it must be small at the bottom. I don't know if it's true ... It can be perfectly big enough because the bigger it is, the bigger the measure increases*
The class (43) [protesting]: *no!*
A student *(45-46): all the triangles we did there, they're 180 of ... when you add them up ... 203 is too much of an exaggeration*
Student (49): *I think it's wrong because he says the triangle is made big; but if you make it big at the top, at the bottom it will be small*
Student (53): *I think he's got a sharp angle and I think he measured it wrong. When he took* [the protractor] *from the other side, so instead of giving him, I don't know ... 30, it gives him 150 ...*

Teacher (58): *that's not impossible, because when I passed in ... a lot of you, you put ... your zero, you put it on the 180, indeed it gave you an angle ...*

Student (59): *I'm saying that what he said ... that the triangle is bigger, it makes a bigger number, it's not true because I ... he made a triangle bigger than Dav's, and I think it's smaller ...*

Teacher (62): *So, it wouldn't have any relation* [i.e. the size and the result].

(E3) The teacher takes stock of the debate and then moves on to the next activity

Teacher (64): *Some of you think it should always be the same [...] but there are others who don't agree; well, some of you say if he found something different it might be because he made a mistake, and then there are others who say, well, depending on whether you make a bigger or smaller triangle, well, the angles will change. And at the moment we don't all agree, we don't all say the same thing.*

One student asks that the 203° result be checked, but the teacher resists, invoking the contract of “scientific debate” explicitly established in this class:

Teacher (68): *I've already explained that if two or three of you were thinking something and the whole class was thinking the opposite, you didn't have to crash and say: no, no, it must be like the others. You may well be right three against the others. So, the fact that he's the only one who found 203 doesn't automatically prove that he's the one who was wrong.*

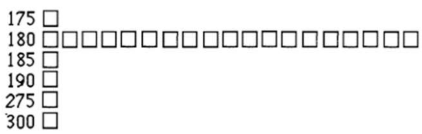
The teacher presents the next activity:

Teacher (79 - 81): *Okay, so now we'll give you all the same triangle. We have photocopied them, you will do the same thing again, measure the three angles again, and add up the three measurements again* [...] *you try it again at a glance ... I ask that you don't measure it ... I give you a piece of paper on which you write your name and your first name* [...] *on the paper there is written "I bet that the sum of the angles of a triangle is worth: ...", and you complete it. You don't say it out loud* [...] *for this triangle* [...] *you look closely at the triangle in front of you ... at first sight you try to find out how much each angle makes and how much the sum is worth.*

(E4) When the results were collected, two histograms were plotted on the blackboard, one for the bets, the other for the results of the measurements and the sum.

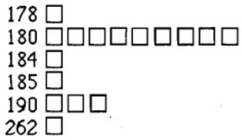
The teacher did not compare the bet with the result for each of the students.

Histogram of bet:



This histogram does not account for some additional comments written by students on their sheet of paper at the time of their bet. Seven students bet 180°, but three of them add the comment “at most 180°” and three others add the comment “180° on average”; the bet for 175° was justified, as it were, by additional information: “the first is 35°, the second 90 and the third 40”.

Histogram of measurements:



The students laugh at the result 262°, the teacher then intervenes to remind them again of the rule under which the class is working:

> Teacher (85): *You know that it has been said several times that when someone says something strange you don't laugh because it's not interesting ... she is right, if it happens.*

At this point the bell rings the end of the session, the next session takes place after a break of about ten minutes.

(E5) When the class is resumed, the students are assembled in teams of three or four.

On the blackboard are the three histograms of the "*first experiment*", as the teacher says, the betting and measurement histogram of the "*second experiment*". The new task is then given, the teacher does do not call for just mere remarks on the results displayed but for elaborated comments:

> Teacher (86 - 88): *In each group you discuss silently so that we don't hear ... what you want to say about it and then in two minutes we share what you have discussed among yourselves* [...] *so you discuss gently saying "well, with these experiences we want to say something else". You remember we didn't quite agree ... you try to discuss it among yourselves and if you agree, you try to think of a sentence that will allow you to explain it to the others.*

Some groups provide the following sentences that the teacher writes on the blackboard:

> (A) Looking at the blackboard we can see that the average is almost 180°.
> (B) We disagree because two think 190° and two think 180°.
> (C) Regardless of the triangle, the sum of the three vertices is always 180°.
> (D) Whatever the triangle, the sum never exceeds 180°.
> (E) Whatever the size of the triangle, the sum of the angles is 180°.

Although not all groups provide a sentence, the teacher opens the discussion:

> Teacher (115): *Are there sentences written on the blackboard that you disagree with at all ... you can give a reason why you disagree.*

Reactions had already taken place when the sentences were proposed, about sentence (A) one student had said: "*What do you know about it?*" (Student 104). The authors of (E) rejects sentence (B) in the name of their own statement, but the teacher finds this insufficient and passes the floor to students who claim to have a proof:

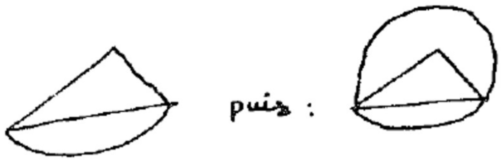
Student (122): *Yes, I have proof because a triangle... is actually a U-turn*
Student (126) [drawing on the blackboard]: *I think that a triangle there, it's 180°, it's a half turn, and if we also have this one, it's a full turn*

The class accepts this explanation, but the authors of (B) resist. Their opponents respond:

> Student (134): *if we can't convince them, well, they... convince us that that's 190*

Another attempt at an explanation:

> Student (138): *That's 180... and it'll be 360... it forms a circle.*



The professor intervenes again, but this time to get out of the debate on sentences (B) and (E) on which there was no unanimity:

> Teacher (139): *Well, at the moment there is a proposal from some people who say it's 180 and they gave us an explanation, which they called a proof: because in a triangle it makes a half turn and if I make two half turns or a half circle underneath it makes a full turn. Are there any other sentences on the blackboard that you disagree with ... you can explain better?*
> Class (140): *C*
> Teacher (141): *What does the sentence C say?*

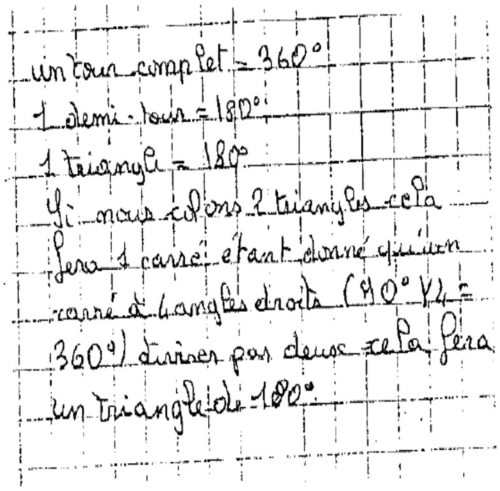
A student reads the text of (C) on the blackboard.

> Teacher (143): *What do you think of this sentence? You say it's not always 180, it's an average. So, do you have something to explain that ... can you convince ... there's a team that says on average, that's you, there are those who always say. You say, I don't agree with those who always say it. Can you ... something to try to convince them, what do you rely on to ...*
> Student (144): *The protractor is not very precise. We don't have to find 180 ...*

Another student proposes a proof for 180, the teacher asks him to present it on the blackboard:

Student (150): A square, if you count the angles, 4 of 90° is 360°, that's it. And a triangle 180° [he draws the two halves of the rectangle as shown opposite and shows them], *so a triangle is 180°.*
Student (151): ah yes that's what we wanted to explain

The students write down the proof:


un tour complet = 360°
1 demi-tour = 180°
1 triangle = 180°
si nous collons 2 triangles cela
fera 1 carré étant donné qu'un
carré à 4 angles droits (90° X 4 =
360°) diviser par deux cela fera
un triangle de 180°.

Transcription:

a complete turn = 360°
1 Half-turn=180°
1 triangle=180°
If we paste 2 triangles this will make 1 square. Given that a square has 4 right angles (90°x4=360°) divided by two this will make a triangle of 180°.

But some students still don't agree, the teacher invites them to try to convince their classmates, to which they reply:

Student (159): *we found 190 because we measured*
Teacher (161): *And each time you found 190 ... in the first triangle and in the triangle which you were given?*
Student (162): *Yeah, we measured...*

The professor does not insist but takes stock of the situation at that time and proposes to go further:

Teacher (163): *There are arguments that have been developed, for example: we've taken the measurement several times... came up with 190, so he believes it. And then there are a certain number of you who came to in front of the class to give explanations to convince the others* [...] *We're going to try to go a little further, we're going to do some more experiments to get to the bottom of this, because we don't all agree at the moment.*

(E6) The teacher presents the next activity; students remain in teams as in the previous phase:

Teacher (163-168): *So, we're going to distribute the triangles and again you don't take measurements* [...] *you don't use your protractors* [...] *you're still asked to put your names and then, what you think of the sum of each of these triangles. Then you will respond telling what you think of it... when you have reached a good agreement among yourselves, then you give this answer [...] if you can't agree, then you say: we can't agree, there are two who think* *this, there are two who think that* [...] *and you don't measure, eh* [...] *you can give names to the triangles, call them A, B, C to differentiate them.*

It takes students quite a long time to decide on a bet and carry out the measures.

(E7) Then each group is asked to give the result of their measurements.

This time the results are compared with the bets. Below are the pairs (bet, measurement) for each of the triangles A, B, C, followed by the comment of the group of students (sometimes repeated by the teacher when students are inaudible):

Group (169) – (190, 190) (180, 180) (165, 165)
I mean... the measurements of the big, small and pointy... they're very close.

Group (195) – (180, 180) (180, 180) (180, 180) (180, 180)
they're all 180, whatever shape they're in... is that confirmed by your measurements? […] *Yeah.*

Group (180) – (180, 180) (180, 180) (180, 180) (180, 180)
they don't agree, they say it's 190 to 180...it's the same position as before [...] *so you agreed on 180, your measurements what do they say* [...] *they confirm what you said*

Group (184) – (180, 180) (180, 180) (180, 180) (180, 180)

Group (185) – (180, 180) (180, 180) (180, 180) (180, 180)

Only the results of the measurements are shown in the table, they are noted without a histogram being drawn. The professor did not question all the groups, two are not listed on the blackboard; their bet was found on their sheet; for one "*The sum is 190°. 180°*", for the other "*They are 180°*".

The teacher calls on the students to comment on the results that have been obtained:

Students (192): *If we compare them ... the E* [these are the measurements shown above] *... they say the triangles are 180° and compared to the A ... we think they must rely on the size.*

This position, reworded, is submitted to the class. The class seemed to accept it, but the professor then pointed out that there was no unanimity:

Teacher (197): *it seems that there are different opinions here, there are some who say 180, there are some who say 190 ... have they found ways to convince others of what they think?*

Teacher (199): *it's 180, it's always the same thing, you say it's 180: it's half of a geometrical shape*

Student (200): *mister, a triangle is already half a geometric shape ... why do they say it's half*
Teacher (203): *so, you say, what's that got to do with talking about geometric shapes since a triangle is already a geometric shape?*

Teacher (205): *You look carefully to see if the first drawing that is made on the blackboard. If it is the same as what we said earlier*

Then the teacher recalls the previous proof.

Student (206): *Yes! ...it's different because there are three instead of two...*
Student (211): *this one's bigger than those two. If we put them next to each other it will be the same as this*

(E8) End of the session

The teacher intervenes with a long commentary including quoting students' attempts to provide a proof.

Teacher (214): *I don't know... I'll give you a little bit of my opinion. I see a difference between both things that are there, it's that there is a rectangle, he cut it in half and then he got a triangle. Doesn't this triangle have something a little peculiar?*
Student (215): *Yes, sir, it has a right angle.*
Teacher (216): *it has a 90° angle. And here, does he have something a little bit different, in his method ... here, what does it show in this particular case? If there's a triangle that has an angle of 90°, then you can always make a kind of rectangle, and then what has been done here seems to show that it works for that rectangle there, and it's quite particular because it has an angle of 90° and you can't do the same thing here, so here what does it show what you've just done ... listen here there's a proposal on the blackboard that's different from what's up there, you try to show what it shows. You can explain it better...*

[inaudible]

Teacher (216): *Now you come back to your idea, it's half a geometric shape. You said earlier that a triangle itself is a geometrical shape, so maybe that doesn't help us much ...*

[inaudible]

Teacher (218): *what I think is happening here is that her drawing... look at the drawing here... she drew a triangle that was like the one you were given... so she didn't have to have a right angle... she drew it the way you were given it... and then she built around that... so what she wants to say here, is that these four angles here are right angles, so what can you deduce about your starting triangle. Can you say something about your starting triangle... because this time we cut the triangle into three pieces?*

[inaudible]

(E9) The teacher then closes the sequence, leaving the problem of the decision on the validity of the conjecture open.

Teacher (218): *So, it seems to me that at this point, for the moment, nobody has really managed to convince the others... that is to say that everyone has brought their arguments which go in one direction... we'll come back to that next time.*

4.3.2 ANALYSIS OF THE SEQUENCE

The first phase (E1) is introduced without any particular justification of the activity, this is certainly acceptable to the students as it follows angle measurement activities; the teacher does not have to justify what may appear to be a simple change of "packaging". The activity instruction is presented in two parts, first the drawing of the triangle, then the measurement of its angles and the computation of the sum of these numbers. Each of these phases is accompanied by interventions that anticipate errors that are deemed *a priori* irrelevant to the objectives of the situation and that aim avoiding them.

> The didactical contract confronts the teacher and the students with a **didactic paradox**. Everything that the teacher undertakes to obtain the behaviors expected of the students tends to deprive them of the conditions necessary for learning the targeted notion: *if the teacher says what he or she wants, he or she can no longer obtain it*. Conversely, if the students accept that the teacher teaches them the result, they do not establish it themselves and therefore do not learn mathematics; they do not appropriate it. If, on the other hand, they refuse any intervention by the teacher, then the didactic relationship is broken: *learning implies that the students accept the didactic relationship while considering it temporary*. (abridged version of Brousseau 1997 p.41)

One of these interventions (Teacher 1-3) illustrates a *paradox of the didactical contract* (here in the devolution process): for the activity to "succeed", the students must produce some kind of triangles, for which the teacher shows them examples in an effort to free them from the prototypical models, but the students tend to reproduce these examples. Moreover, this idea of a model to be followed, and given by the teacher, is reinforced by the layout proposed on the blackboard (see § 4.3.2.1, E2) to which the students should stick. Indeed, this intervention intended only to avoid contingent errors that would be due to poor data management.

Collecting the results (E2) takes place without any particular teacher intervention, nor students' reactions; except in the case of the proposed extreme values: 160° and 203°. This last value is discussed. The students immediately engage in a debate on its acceptance (Student 32, Student 42) or rejection (Student 35 and following). The arguments for rejection are of two types: arguments on the substance (Student 36), arguments referring to a norm that is being emerging (Student 45-46). This does not mean that 180° is recognized as the sum of the angles of any triangle, but rather that the shape that emerges from the histogram plot marginalizes the extreme results.

The teacher makes few interventions, but they are quite clearly concerned with the validity of the assertions (Teacher 58 and 62). If the debate were to continue, it could lead to the rejection of the erroneous conception (Student 59), thus taking the first phase of this sequence beyond what was planned. This risk is recognized by the teacher who takes control of the situation and acknowledges a disagreement that preserves the validation debates expected later. But the students resist this rupture, and it is by recalling the rules of conduct in force in the class (Teacher 68) that the teacher regains control of the situation and allows proceeding to the next activity. In this context, it may be thought that on the students' side, everything happens as if results very different from 180° were to remain plausible.

The second phase (E3) is introduced by insisting on the individual character of the activity, moreover the teacher gives a rather heavy indication on a possible strategy to achieve the bet on the sum of the angles of the triangle, which consists in evaluating the value of each of the angles (Teacher 79-81). This suggestion could conceal the erroneous conception we were expecting to mobilize, in fact the confrontation of the bet and the measurement may then simply lead the students to think of an error in their evaluation of each individual angle.

We note (E4) that the bets clearly favor 180°, which may be an indication of conformist behaviors; as for bets other than 180°, their origin is undetermined and remains so since these students are not requested to comment on the possible discrepancy between their bet and the result of their measurements. However, the rejection by the teacher of the ironical reaction of “the class” when the result 262° is proposed, reminds students that the established “scientific debate” rules implies respect for all mathematical production, unless it can be rejected by intellectual arguments.

The implementation that follows (E5) is very different from what was planned in the initial scenario, but is consistent with the collective work habits of this class. This divergence should not significantly alter the sequence. On the other hand, the same does not apply to the comments asked of the students as was planned. These comments here fall within the framework of the "scientific debate" rules in force in this class as carried out by the project of the IREM research group. They are understood as the production of statements that will then have to be refuted or proven within the framework of an agreed organization of the validation debate.

Not all groups produced statements, this is a consequence of both time management and student activity, all students have to be busy: it is difficult to keep those who have reached a production early enough waiting.

Students find it difficult to formulate their proof, but also to accept or retain for effective examination, the arguments that are produced. The teacher then makes decisions in the management of the debate, eluding exchange that is deemed unsuccessful to making room for another one. In a way, ensuring that attempts at proof or rebuttal are scrutinized, without decision being made on their validity.

This second phase of the situation eventually did not perform the function that was expected:

- Statements which are potential conjectures whose construction was only foreseen in the next stage, are discussed from the outset, without their status being made clear; they may be mere speculations.
- The problem of knowing the sum of the angles of the triangle under examination is not raised. As for the considerations on the accuracy of the measurements, they were set aside by the professor's remarks during the first phase.

The third phase (E6) may therefore have lost part of its function, especially if one considers that it stops a debate on proof which it was meant to provoke. We observe (E7) the *de facto* adoption of the conjecture by the whole class, since 5 groups out of 7 (i.e. 17 students) bet that the sum of the angles is 180°, then "find" 180° as the result of measurements and calculation; as for the other two groups, they seem to adopt the idea that the result obtained must be close to 180° (e.g. Group 169). In this context, where the class seems to have reached a consensus on the invariance of the sum of the angles of a triangle, the production of a proof is a response to requirements from the teacher who relies on the rules explicitly in force and who emphasizes that there is not strict unanimity.

Proposals are made but there is no real debate *among* the students: the responsibility for the validity of the conjecture is not devolved to the class, or possibly the students' convictions are too strong for them to engage in a validation process. Proofs are awkwardly or incompletely formulated, without significant work on their formulation. The students are deprived of it by the teacher who takes on this task and ultimately empties the desired interaction of its meaning. However, it is possible to think that the proposal of proof based on the division of the rectangle into two right-angled triangles has the support of “the class”. The impact of the teacher's criticism is not clear.

As in the previous case, the situation could not be closed at the end of the two class sessions. At the next class session—which I did not attend—the teacher, considering the proposed proof too unsatisfactory, then closed the sequence by asking the students to admit the conjecture and postponing the production of a proof. He reported, however, that the students had generally found the proof of the decomposition of a rectangle to be convincing. But this was not precise enough information to conclude on the epistemic status of his choice of conclusion for that sequence.

4.4 COMPARING BOTH OBSERVATIONS

The observations took place in the context of very different practices: that of a class, let us say, ordinary—in the literal sense of the term—and that of a class that is the field for an action-research whose aims are the intellectual autonomy of students and their initiation to scientific debate[30].

We had to negotiate the implementation of the experiment in these two contexts, making it acceptable and giving it meaning in relation to the professional practices of the teachers who implemented it. In the "ordinary" class (class D) the main problem was the introduction of a new type of interaction, both with the teacher and among the students, whose crucial point was the successful devolution to the class of the problem of proving a conjecture. As I later pointed out, the novelty of such a situation alone could be a source of failure in the experiment. On the contrary, the second class (class E), may appear to offer a more favorable context, since the students were systematically introduced to the notion of conjecture and the validation debate, in particular to the mathematical and social need for proof.

The initialization of the sequence took place in a similar way in both classes, whether or not the link with previous activities was made explicit. Both teachers initiated interventions to ensure a solid practical foundation. They helped *to build and stabilize the mathematical context* in which the problem would later be constructed.

After the results had been collected, the students in class D made a few remarks. This activity was only intended to complete the first phase, albeit in a rather artificial way; it may have appeared to the students only as a moment of the school ritual. In class E, on the other hand, a debate began in which a result that seemed too different from the others was rejected. In this way, the student shown that they actually adhered to the operating rules established in the class, which the teacher recalled precisely to end the discussion (§ 4.3.1., E3). In fact, the teacher wanted to preserve the sequence by avoiding premature formulation of the conjecture. It may be noted that nothing would prevent the case of this result from being totally settled. Indeed, this would only have the effect of reinforcing a phenomenon that appeared in both classes and that we had not anticipated: the creation of a norm as soon as the first measurements were produced: one must find around 180°. The students in both classes shown their amusement when one deviated too far from it. But this is by no means, as far as “the class” is concerned, the conjecture whose construction is sought. Extreme results could have been checked and corrected without risk, if this was at the request of the students.

This remarkable fact of the creation of a norm as soon as the first measurements are examined is revealed by the events just recalled, and confirmed by the reinforcement of bets on 180° in the following activity and even more by the overwhelming majority of 180° as the calculated value, both in class D and in class E. This last indicator is the most significant because it implies that there have been corrections “towards 180°” of the results initially obtained. This does not mean, however, that the students have acquired the idea that the sum of the angles of any triangle is 180°. Thus, in class D, 21 out of the 25 students “find” 180° for the proposed triangle, but during the third activity and after a discussion some of them agree on a different value for one of the three triangles (such bets are made by 12 students). The conformism at the origin of these first answers is essentially fragile because it is not based on knowledge. Conformism will not be sufficient to deal with the following situations where it will give way to the students’ conceptions which, although erroneous, are *their* “true” knowledge.

[30] Note2020: in the sense of Marc Legrand (2001).

At the end of this second phase, the objectives set were achieved in unequal ways:

- The comments the students are invited to share do not lead in a "natural" way to the explicit requirement that everyone should have found the same result for the same triangle, even in the class where this question has been raised *a priori* (D3, Student 106). It is raised by the teacher in class D to lead essentially to remarks on the accuracy of the measurements, but it is not raised in class E. In class E, the problem of measurement uncertainty is "solved" from the first activity under the teacher's authority. In fact, when comments are requested, students in both classes first deal with a set of numbers without really relating it to the specific task that has been proposed. In class E, the explicit rules of debate require the students to produce "a fairly precise sentence tending to identify 'the general fact' observed" (IREM de Grenoble, 1985, p. 30), which contributes to obscuring the object of the proposed activity: to measure and compute the sum of the angles of a given triangle.
- On the other hand, the objective of mobilizing conceptions, due to the necessity of an *a priori* decision in order to bet on the sum of the angles of a triangle, has been achieved. This is evidenced in class D where each student is explicitly confronted with the possible discrepancy between his or her bet and the result of the measurement and the computation. In class E, the examination of the histogram of bets shows that erroneous conceptions had been mobilized, their presence had moreover been attested from the first phase (§ 4.3.1. E2, Student 32). However, this mobilization has not the occasion to be made explicit since the students are not individually confronted with the possible discrepancy between their bet and the measure they proposed.

The third phase witnesses the strength of the conceptions that links the sum of the angles of a triangle to its size. Especially in class D. In class E this resistance seems to be the fact of only one group of students, although for the others there is no guarantee that it is not the production of an agreed answer: 180° is particularly highlighted in this class and elements to prove that this is the sum of the angles of any triangle have been proposed. In both classes this third phase is the object of important work on the part of the teacher aiming at the construction and recognition of the conjecture, and then posing the problem of its proof. But both classes evolve in a very different way from each other mainly because of the nature of the *didactical contract* and its evolution more than because of the nature and evolution of the students' conceptions:

- *In class D:* the teacher's work focuses on the construction of the conjecture, for which he gives strong support to students who are not convinced that 180° is the value of the sum of the angles of a triangle. This work aims at the same time to give the students the responsibility and the duty to give a proof, since the alternative is to produce a triangle whose sum of angles would be very far from 180° *or* to produce proof that this is impossible. To do this, the teacher uses his position in relation to knowledge to give legitimacy to students who doubt the validity of the conjecture: taking their side preserves the idea of equal plausibility of both positions. To use Brousseau's expression (1984b, pp. 189–190)[31]: the teacher "*conceals his knowledge"* in order to legitimize the search for a triangle whose angles add up to 150° (§ 4.2.1 D8 Teacher 429).
- *In class E:* the problem of proof has imposed itself prematurely with respect to the planned strategy. It may only appear as an agreed activity in line with the validation practices explicitly in force in this class. Moreover, the explicit right *to be right alone* against all allows some students to remain on their positions; the "stakes to be accepted" seem to be void. Be it bets in the third phase or the measurements obviously and massively corrected towards 180°, both observations lead us to the hypothesis that the statement about the sum of the angles of a triangle has been defused as conjecture: the problem of proof is not rooted in a problem of the validity of a true conjecture. Moreover, students

[31] Note2020: This is a reference to the text of a seminar Guy Brousseau gave in Grenoble, it is nowadays difficult to access. The English reader will find this idea developed in Brousseau's collected translations in relation to the concept of *devolution* (Brousseau, 1997, p. 229 sqq)

do not engage in a debate about the various proofs proposed; they are convinced and show little interest in comparing or criticizing them. It is the teacher who is going to carry out this necessary work.

5 Conclusion

5.1 An Element of Robustness: Student's Conceptions

The observations confirm the presence and strength of a conception of a monotonous relation between shapes and measures[32], which includes the theorem-in-action: “the sum of the angles of a triangle is an increasing function of its ‘size’”. It is this presence, not necessarily massive, and this resistance that gives the situation its robustness. These case studies come in support to the hypothesis that the strength of this conception, "against" which the situations are constructed, is the fundamental guarantor of the reproducibility of the proposed sequence. In fact, it is not so much the resistance of this conception as the fact that it is built around a true (false) theorem-in-action present as such in the decisions at the time of the bets, and engaged in the validation debate. The construction of the conjecture against this theorem-in-action ensures the authenticity of its conjectural character.

Indeed, the existence, based on this conception, of a doubt about the invariance of the sum of the angles of any triangle and its equality to 180° in the face of a contrary conviction, which is becoming more and more widespread, allows the constitution of the fact of the invariance into a true conjecture and legitimizes that the problem of its proof be posed. On the other hand, the resistance of the erroneous conception limits the behavior of conformism: such behavior is strongly destabilized by a situation that appears sufficiently different from the ones that gave rise to it. Thus, when betting on atypical triangles, the erroneous conception takes over. This is because conformist behaviors are primarily that of the production of answers "expected" or assumed as such by the students; they do not reflect authentic knowledge.

5.2 The Achilles' heel of the situation: the didactic contract

As the observation made in class E shows quite clearly, the situations proposed, and therefore the very meaning of the students’ behaviors and productions, are highly sensitive to the teacher's interventions. They are also highly sensitive to the *modus vivendi* of the class, the characteristics of which were not known beforehand.

For example, the idea that any class activity implies behaviors or responses expected by the teacher is most likely the source of the creation of a norm, or of “conforming” responses as of the first situation, whatever the class observed. This is reinforced by the reluctance that students may have to remain on the fringes of what would appear to be the dominant behavior; what Watzlawick links to the desire to be in agreement with the group which may lead to a renunciation to reality for “the satisfaction of feeling in harmony with the group” (Watzlawick, 1976, p. 87).

Here is the main risk of failure, because if the conjecture is produced as an agreed statement then it loses most of its meaning, and thus the authenticity of the debate on proof is severely compromised. We have already discussed how this precludes the teacher from being the producer of this statement, but it also places strong constraints on the institutionalization of the conjecture constructed by the students. This construction must in fact lead to a true consensus of the problematic validity of the statement resulting from the tension between the "pros" and "cons", and then to the designation by the class of the search for a proof as a common task; institutionalization is limited to marking this state, in a way endorsing it for the class community. The fact that the constructed sequence leads to a situation having the characteristics of a situation of proof rests essentially on the creation of this *collective desire to know* to which each of the students would adhere. The requirement of

[32] Note2020: it could be sketched by the adage: the biggest the shape, the largest its measures.

authenticity of this desire constitutes the fragile link in this design approach due to its sensitivity to the teacher's interventions.

Thus, the rules adopted in class E, which explicitly govern the debate, if they allow the regulation of social interaction, also contain in themselves the sources of difficulties in establishing a true situation of validation. In particular, the contradictory debate based largely on the obligation to convince, combined with the legitimacy of solitude in the truth, opens the possibility of refusing to be convinced. The debate on proof, the stake of which is knowledge, is replaced by a social game, the stake of which is to be convinced or not to be convinced. Making rules explicit has the possible, and sometimes essential, consequence of creating a legal vacuum: here, the absence of rules that require one to admit that one is convinced. Moreover, this legislation is only operative if it is monitored: laws make judges necessary. Satisfying the law may mean first of all satisfying the judge. In our case, the existence of explicit rules in class E, although specifying a “scientific” way of debating and deciding the validity of a statement, can provoke a shift towards the production of evidence or arguments according to their capacity to satisfy the teacher before the requirements intrinsic to the problem of validity.[33]

This does not mean that no rules should exist, if only because “rules are bound to emerge, and especially in human interaction, any exchange invariably reduces the possibilities which until that moment were open to the partners” (Watzlawick, 1976, p. 94). The problem is that of the possibility, the conditions and the meaning of making them explicit.

Nor is it a question of the teacher disengaging: the situation cannot function without him or her. The situation needs what the teacher is, his or her position in relation to knowledge. As Brousseau points out, *the teacher is and remains responsible* (1984a, p. 46). As observed, students can have very different positions on the validity of the property in question: those who already know, those who reject, those who doubt, those who would willingly adopt a majority position, those who are not interested. Faced with students who maintain that the sum of the angles of a triangle is invariant, other students may claim the existence of a particular case even if they are unable to produce one. We are in a situation of essentially intellectual conflict; no material feedback can contradict any of the positions taken. On the other hand, this inability to produce a counter-example to the property may give interest to the search for intellectual proofs that would settle the debate. This problem of *proof and refutation* can only be developed if the property (or thesis) and its contradiction (antithesis) have an equal recognized potentiality. They must be defended in a sufficiently balanced way in the class so that an authentic dialectic of validation can be established. This is what the teacher allows by giving legitimacy to the search for a triangle whose sum of angles would be different from 180°.

The “truth” becomes problematic not because the professor does not decide, but because he or she endorses that it is legitimate to support or to give credit to the thesis and the antithesis. The teacher gives a kind of status to uncertainty, and at the same time marks the interest that there would be in “knowing”. This ensures that the statement in question is not mere speculation, but conjecture.

If, therefore, conjecture has its roots in the questioning of the knowledge and conceptions of each student, this case study shows that it only achieves its full status in social negotiation, which confirms both its interest and its problematic nature. This negotiation includes both processes of devolution and institutionalization in which *the teacher plays a central role*.

[33] Note2020: shaping this perspective I proposed the notion of *didactical custom* “understood […] as a set of obligatory practices […] established as such by their use, and which, in the majority of the cases, is established implicitly. Custom regulates the way in which the social group expects to establish relationships and interactions among its members and, therefore, it is initially characterized as a product of social practices.” (Balacheff, 1999, p. 25). It is in the context of a given didactical custom that the didactical contract is negotiated to obtain the devolution of an adidactical situation.

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