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Abstract: In this technical report, the capacity region of the two-user linear deterministic (LD) interference channel with noisy output feedback (IC-NOF) is fully characterized. This result allows the identification of several asymmetric scenarios in which implementing channel-output feedback in only one of the transmitter-receiver pairs is as beneficial as implementing it in both links, in terms of achievable individual rate and sum-rate improvements w.r.t. the case without feedback. In other scenarios, the use of channel-output feedback in any of the transmitter-receiver pairs benefits only one of the two pairs in terms of achievable individual rate improvements or simply, it turns out to be useless, i.e., the capacity regions with and without feedback turn out to be identical even in the full absence of noise in the feedback links.

Key-words: Capacity, Linear Deterministic Interference Channel, Noisy Channel-Output Feedback

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Capacité du Canal Linéaire Déterministe à Interférences avec Rétroalimentation Degradée par Bruit Additif.

Résumé : Dans ce rapport, la région de capacité du canal linéaire déterministe à interférences avec rétroalimentation degradée entre les récepteurs et leurs émetteurs correspondants est caractérisée. Ce résultat permet l'identification de plusieurs scenarios asymétriques dans lesquels la rétroalimentation dans un seul couple récepteur-émetteur montre autant de bénéfices que des rétroalimentations dans les deux couples récepteurs-émetteurs. Ces bénéfices sont mis en évidence par l'amélioration des taux de transmission individuels et de leur somme par rapport aux cas où il n'y a aucune rétroalimentation. D'autres scenarios montrent qu'une rétroalimentation dans un des couple émetteur-récepteur améliore le taux individuel d'un des deux couples émetteurs-récepteurs. D'ailleurs, il existe d'autres scenarios où l'utilisation d'un ou plusieurs liens de rétroalimentation ne montre aucun bénéfice ni pour les taux individuels ni pour leur somme. Dans ces scenarios, cela montre que les régions de capacité avec et sans rétroalimentation sont identiques.

Mots-clés : Région de Capacité, Modèle linéaire déterministe, canal à interférences, rétroalimentation degradée.

Contents

1 Introduction

Perfect channel-output feedback (POF) has been shown to dramatically enlarge the capacity region of the two-user interference channel (IC) [2, 3, 4, 5, 6]. More recently, the same observation has been made with a larger number of transmitter-receiver pairs in the IC [7]. In general, when a transmitter observes the channel-output at its intended receiver, it obtains a noisy version of the sum of its own transmitted signal and the interfering signals from other transmitters. This implies that, subject to a finite feedback delay, transmitters know at least partially the information transmitted by other transmitters in the network. This induces an implicit cooperation between transmitters that allows them to use interference as side-information [3, 6, 8, 9, 10]. A more explicit cooperation is also observed in the case in which one of the transmitter-receiver pairs acts as a relay for the other transmitter-receiver pair by providing an alternative path: transmitter $i \rightarrow$ receiver $j \rightarrow$ transmitter $j \rightarrow$ receiver i [4]. These types of cooperation, even when it is not explicitly desired by both transmitter-receiver pairs, play a fundamental role in enlarging the capacity region. Interestingly, this holds also in the case of fully decentralized networks in which each transmitter-receiver pair seeks exclusively to increase its individual rate. That is, channel-output feedback increases both the capacity region and the Nash equilibrium (NE) region [11].

Despite the vast existing literature, the benefits of feedback are unfortunately less well understood when the channel-output feedback links are impaired by additive noise. The capacity region of the LD-IC with noisy channel-output feedback (NOF) is known only in the two-user symmetric case, see [12]. The converse region in [12] inherits existing outer bounds from the case of POF, the cut-set outer bounds and includes two new outer bounds. The outer-bounds inherited from the POF are those of the individual rates and the sum-rate in [4]. The new outer-bounds are of the form $R_1 + R_2$ and $R_i + 2R_j$. The achievable region in [12] is obtained using a particularization of the achievability scheme presented in [2], which holds for a more general model, i.e., interference channel with generalized feedback.

In this technical report, the results presented in [12] are generalized for the asymmetric case and the corresponding capacity region of the two-user LD-IC-NOF is fully characterized. This generalization is achieved by using the same tools used in [12], however, it is far from trivial due to the number of parameters that describe this channel model: two forward signal to noise ratios (SNRs) $\overrightarrow{n}_{11}, \overrightarrow{n}_{22}$, two feedback SNRs $\overleftarrow{n}_{11}, \overleftarrow{n}_{22}$ and two forward interference to noise ratios (INRs) n_{12}, n_{21}. The new converse region also inherits existing outer bounds from the case of POF, the cut-set outer bounds and includes two new outer bounds. The new outer bounds are of the form $R_1 + R_2$ and $R_i + 2R_j$. These new bounds generalize those presented in [12]. The achievable region is obtained by using a coding scheme that combines a three-part message splitting, superposition coding and backward decoding. Despite the fact that this coding scheme is built using the exact number of required message-splitting parts for the IC-NOF, it can still be considered as a special case of the general scheme presented in [2].

Finally, this technical report is concluded by a discussion in which numerical examples are presented to highlight the benefits of NOF. At the same time, examples in which NOF is absolutely useless in terms of capacity region improvement are also presented.

2 Linear Deterministic Interference Channel with Noisy-Channel Output Feedback

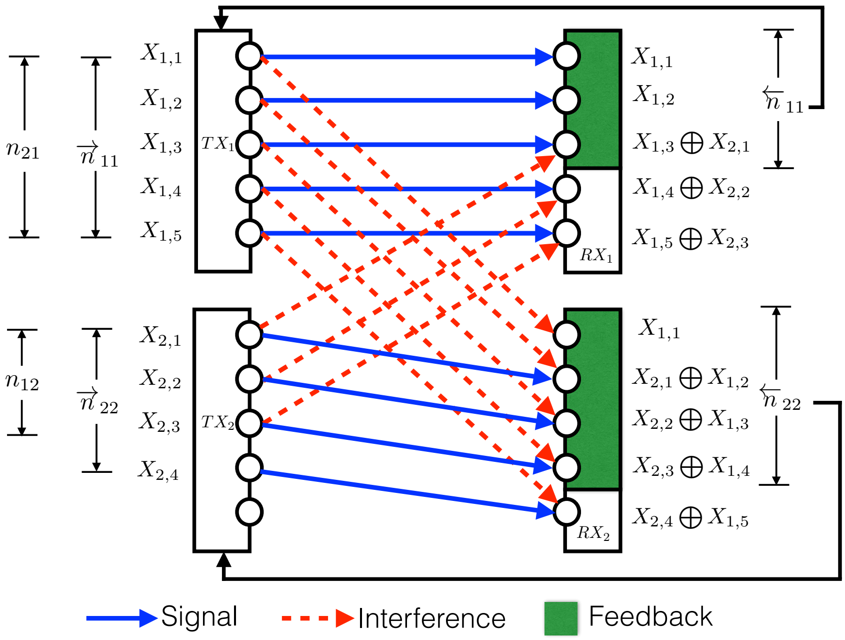

Figure 1: Two-user linear deterministic interference channel with noisy channel-output feedback (LD-IC-NOF).

Consider the two-user LD-IC-NOF, with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$ described in Fig. 1. $\overrightarrow{n}_{ii}$, $i \in \{1,2\}$, is a non-negative integer used to represent the signal-noise ratio (SNR) in receiver i; n_{ij}, $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, is a non-negative integer used to represent the interference-noise ratio (INR) in receiver i from transmitter j; and $\overleftarrow{n}_{ii}$, $i \in \{1,2\}$, is a non-negative integer used to represent the signal-noise ratio (SNR) in transmittter i in the feedback link from receiver i. At transmitter i, with $i \in \{1,2\}$, the channel-input $\boldsymbol{X}_i^{(n)}$ at channel use n, with $n \in \{1,\ldots,N\}$, is a q-dimensional binary vector $\boldsymbol{X}_i^{(n)} = \left(X_{i,1}^{(n)},\ldots,X_{i,q}^{(n)}\right)^{\mathsf{T}}$, with

$$q = \max\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}\right) \tag{1}$$

and N the block-length. At receiver i, the channel-output $\overrightarrow{\boldsymbol{Y}}_i^{(n)}$ at channel use n is also a q-dimensional binary vector $\overrightarrow{\boldsymbol{Y}}_i^{(n)} = \left(\overrightarrow{Y}_{i,1}^{(n)},\ldots,\overrightarrow{Y}_{i,q}^{(n)}\right)^{\mathsf{T}}$. The input-output relation during channel use n is given as follows

$$\overrightarrow{\boldsymbol{Y}}_i^{(n)}=\boldsymbol{S}^{q-\overrightarrow{n}_{ii}}\boldsymbol{X}_i^{(n)} + \boldsymbol{S}^{q-n_{ij}}\boldsymbol{X}_j^{(n)}, \tag{2}$$

and the feedback signal available at transmitter i at the end of channel use n is

$$\overleftarrow{\boldsymbol{Y}}_i^{(n)}=\boldsymbol{S}^{(q-\overleftarrow{n}_{ii})^+}\overrightarrow{\boldsymbol{Y}}_i^{(n-d)}, \tag{3}$$

where d is a finite delay, additions and multiplications are defined over the binary field, and $\boldsymbol{S}$ is a $q \times q$ lower shift matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

The parameters $\overrightarrow{n}_{ii}$, $\overleftarrow{n}_{ii}$ and n_{ij} correspond to $\left\lfloor \frac{1}{2}\log_2\left(\overrightarrow{\mathrm{SNR}}_i\right)\right\rfloor$, $\left\lfloor \frac{1}{2}\log_2\left(\overleftarrow{\mathrm{SNR}}_i\right)\right\rfloor$ and $\left\lfloor \frac{1}{2}\log_2\left(\mathrm{INR}_{ij}\right)\right\rfloor$ respectively, where $\overrightarrow{\mathrm{SNR}}_i$, $\overleftarrow{\mathrm{SNR}}_i$ and INR_{ij} are parameters of the Gaussian interference channel (G-IC).

Transmitter i sends M_i information bits $b_{i,1}, \ldots, b_{i,M_i}$ by sending the codeword $\left(\boldsymbol{X}_i^{(1)}, \ldots, \boldsymbol{X}_i^{(N)}\right)$. The encoder of transmitter i can be modeled as a set of deterministic mappings $f_i^{(1)}, \ldots, f_i^{(N)}$, with $f_i^{(1)} : \{0,1\}^{M_i} \rightarrow \{0,1\}^q$ and $\forall n \in \{2, \ldots, N\}$, $f_i^{(n)} : \{0,1\}^{M_i} \times \{0,1\}^{q(n-1)} \rightarrow \{0,1\}^q$, such that

$$\boldsymbol{X}_i^{(1)} = f_i^{(1)}\left(b_{i,1}, \ldots, b_{i,M_i}\right) \text{ and} \tag{5}$$

$$\boldsymbol{X}_i^{(n)} = f_i^{(n)}\left(b_{i,1}, \ldots, b_{i,M_i}, \overleftarrow{\boldsymbol{Y}}_i^{(1)}, \ldots, \overleftarrow{\boldsymbol{Y}}_i^{(n-1)}\right). \tag{6}$$

At the end of the block, receiver i uses the sequence $\overrightarrow{\boldsymbol{Y}}_i^{(1)}, \ldots, \overrightarrow{\boldsymbol{Y}}_i^{(N)}$ to generate the estimates $\hat{b}_{i,1}, \ldots, \hat{b}_{i,M_i}$. The average bit error probability at receiver i, denoted by p_i, is calculated as follows

$$p_i = \frac{1}{M_i} \sum_{\ell=1}^{M_i} \mathbb{1}_{\left\{\hat{b}_{i,\ell} \neq b_{i,\ell}\right\}}. \tag{7}$$

A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it satisfies the following definition.

Definition 1 (Achievable Rate Pairs) *The rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks $\mathcal{X}_1^N$ and $\mathcal{X}_2^N$ with codewords of length N, and the corresponding encoding functions $f_1^{(1)}, \ldots, f_1^{(N)}$ and $f_2^{(1)}, \ldots, f_2^{(N)}$ such that the average bit error probability can be made arbitrarily small by letting the block length N grow to infinity.*

The following section determines the set of all the rate pairs (R_1, R_2) that are achievable in the LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$.

3 Main Results

Denote by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ the capacity region of the LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. Theorem 1 fully characterizes the capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$.

Theorem 1 The capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the two-user LD-IC-NOF is the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:

$$
\begin{aligned}
R_i \ &\leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right), && \text{(8a)}\\
R_i \ &\leqslant \min\Big(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\Big(\overrightarrow{n}_{ii}, \overleftarrow{n}_{jj} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\Big)\Big), && \text{(8b)}\\
R_1 + R_2 &\leqslant \min\Big(\max\left(\overrightarrow{n}_{11}, n_{12}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{11} - n_{21}\right)^+\Big), && \text{(8c)}\\
R_1 + R_2 &\leqslant \max\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) + \max\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\Big) && \text{(8d)}\\
&\quad +\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right)\\
&\quad + \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big)\Big)^+ + \Big(\Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+\Big)^+\\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \min\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\Big)\Big)^+,\\
2R_i + R_j &\leqslant \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) + \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+ - \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\Big) && \text{(8e)}\\
&\quad +\Big(\Big(\min\left(\overleftarrow{n}_{jj}, \max\left(\overrightarrow{n}_{jj}, n_{ji}\right)\right) - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\Big)^+ - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^+ - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right)\\
&\quad + \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\Big)\Big)^+.
\end{aligned}
$$

Appendix E shows a simplified version of each of the expressions (8) in Theorem 1.

3.1 Proofs

Theorem 1 fully characterizes the capacity region of the LD-IC-NOF. That is, the converse and achievable regions are identical. In the converse region, the inequalities (8a) and (8c) are inherited from the converse region of the LD-IC-POF in [4]. The inequality in (8b) is a simple cut-set bound whose proof is presented in this technical report. The inequalities (8d) and (8e) are new and generalize those presented in [12]. The achievable region is obtained using a coding scheme that combines a three-part message splitting, superposition coding and backward decoding, as first suggested in [2, 4, 6]. This coding scheme is fully described in Appendix A and it is specially designed for the IC-NOF. However, it can also be obtained as a special case of the more general scheme, i.e., interference channel with generalized feedback, presented in [2]. The relevance of this new achievability scheme is that it plays a key role in the achievability of the NE region, subject to the inclusion of random messages, as suggested in [11, 13]. Nonetheless, the analysis of the achievability of the NE region [14] is out of the scope of this technical report. The outer bound region (converse region) is described in Appendix B. Appendix C makes connections to existing results on the interference channel.

3.2 Discussion

This section provides a set of examples in which particular scenarios are highlighted to show that channel-output feedback can be strongly beneficial for enlarging the capacity region of the

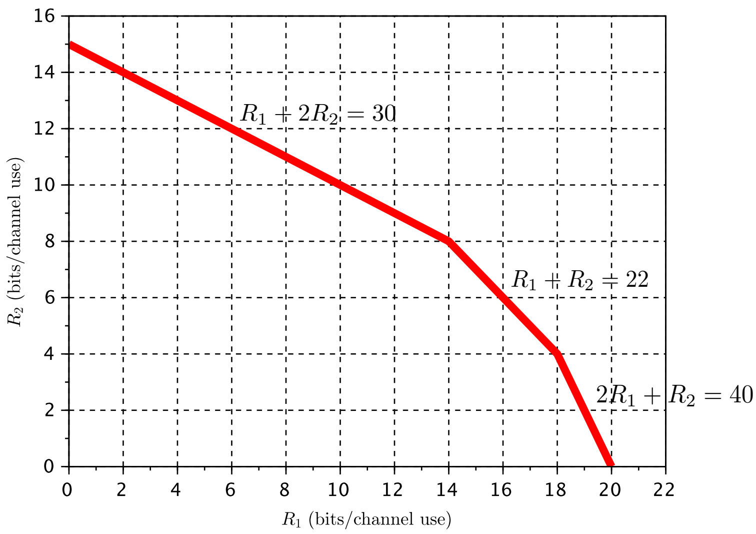

Figure 2: Capacity region $\mathcal{C}(20, 15, 12, 13, 0, 0)$ of the example in Sec. 3.2.1 without feedback

two-user LD-IC. At the same time, it also highlights other examples in which channel-output feedback does not bring any benefit in terms of the capacity region. These benefits are given in terms of the following metrics: (a) individual rate improvements Δ_1 and Δ_2; and (b) sum-rate improvement Σ.

In order to formally define Δ_1, Δ_2 and Σ, consider an LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. The maximum improvement $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the individual rate R_i due to the effect of channel-output feedback with respect to the case without feedback is

$$\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \max_{R_j > 0} \sup_{(R_i, R_j) \in \mathcal{C}_1} R_i - \sup_{(R_i^{\dagger}, R_j) \in \mathcal{C}_2} R_i^{\dagger}, \quad (9)$$

and the maximum sum rate improvement $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ with respect to the case without feedback is

$$\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \sup_{(R_1, R_2) \in \mathcal{C}_1} R_1 + R_2 - \sup_{(R_1^{\dagger}, R_2^{\dagger}) \in \mathcal{C}_2} R_1^{\dagger} + R_2^{\dagger}, \quad (10)$$

where $\mathcal{C}_1 = \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ and $\mathcal{C}_2 = \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ are the capacity region with noisy channel-output feedback and without feedback, respectively. The following describes particular scenarios that highlight some interesting observations.

3.2.1 Example 1: only one channel-output feedback link allows simultaneous maximum improvement of both individual rates

Consider the case in which transmitter-receiver pairs 1 and 2 are in weak and moderate interference regimes, with $\overrightarrow{n}_{11} = 20$, $\overrightarrow{n}_{22} = 15$, $n_{12} = 12$, $n_{21} = 13$. In Fig. 2, Fig. 3 and Fig.

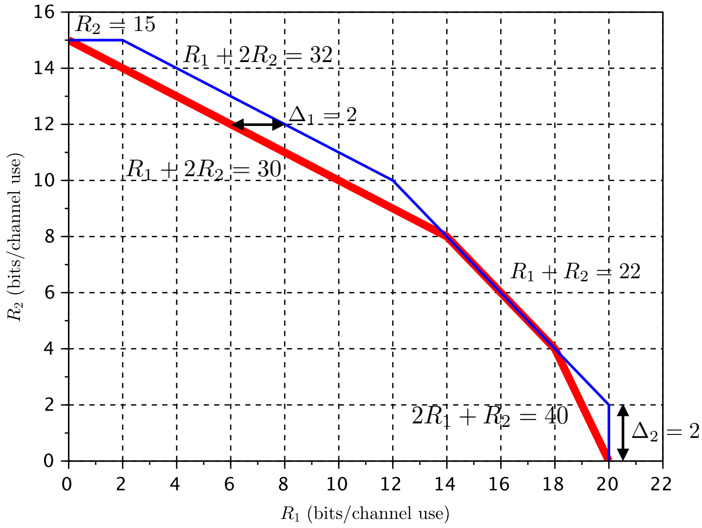

Figure 3: Capacity region $\mathcal{C}(20, 15, 12, 13, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(20, 15, 12, 13, 15, 14)$ with noisy channel-output feedback (thin blue line) of the example in Sec. 3.2.1. Note that $\Delta_1(20, 15, 12, 13, 15, 14) = 2$ bits/ch.use, $\Delta_2(20, 15, 12, 13, 15, 14) = 2$ bits/ch.use and $\Sigma(20, 15, 12, 13, 15, 14) = 0$ bits/ch.use.

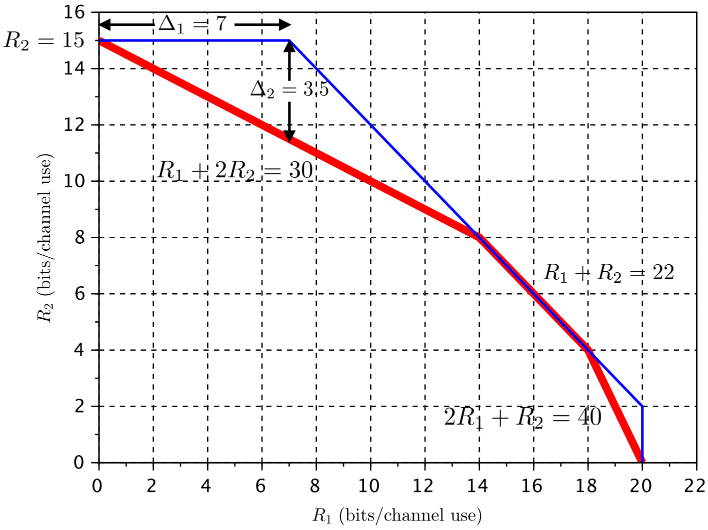

Figure 4: Capacity region $\mathcal{C}(20, 15, 12, 13, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(20, 15, 12, 13, 20, 15)$ with perfect channel-output feedback (thin blue line) of the example in Sec. 3.2.1. Note that $\Delta_1(20, 15, 12, 13, 20, 15) = 7$ bits/ch.use, $\Delta_2(20, 15, 12, 13, 20, 15) = 3.5$ bits/ch.use and $\Sigma(20, 15, 12, 13, 20, 15) = 0$ bits/ch.use.

4 the capacity region is plotted without channel-output feedback, without channel-output feedback and with noisy channel-output feedback, and without channel-output feedback and perfect

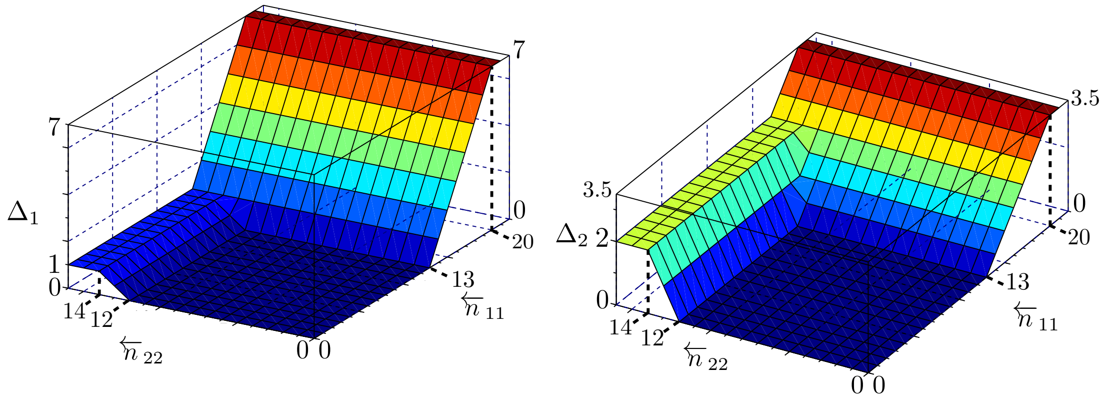

Figure 5: Maximum improvements $\Delta_1(20, 15, 12, 13, \cdot, \cdot)$ and $\Delta_2(20, 15, 12, 13, \cdot, \cdot)$ of individual rates of the example in Sec. 3.2.1

channel-output feedback respectively. In Fig. 5, $\Delta_i(20, 15, 12, 13, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ is plotted for both $i = 1$ and $i = 2$ as a function of $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. Therein, it is shown that: (a) Increasing parameter $\overleftarrow{n}_{11}$ beyond threshold $\overleftarrow{n}_{11}^{*} = 13$ allows simultaneous improvement of both individual rates independently of the value of $\overleftarrow{n}_{22}$. Note that in the case of perfect channel-output feedback, i.e., $\overleftarrow{n}_{11} = \max(\overrightarrow{n}_{11}, n_{12})$, the maximum improvement of both individual rates is simultaneously achieved even when $\overleftarrow{n}_{22} = 0$. (b) Increasing parameter $\overleftarrow{n}_{22}$ beyond threshold $\overleftarrow{n}_{22}^{*} = 12$ provides simultaneous improvement of both individual rates. However, the improvement on the individual rate R_2 strongly depends on the value of $\overleftarrow{n}_{11}$. (c) Finally, the sum rate does not increase by using channel-output feedback in this case.

3.2.2 Example 2: only one channel-output feedback link allows maximum improvement of one individual rate and the sum-rate

Consider the case in which transmitter-receiver pairs 1 and 2 are in very weak and moderate interference regimes, with $\overrightarrow{n}_{11} = 10$, $\overrightarrow{n}_{22} = 10$, $n_{12} = 3$, $n_{21} = 8$. In Fig. 6, Fig. 7 and Fig. 8 the capacity region is plotted without channel-output feedback, without channel-output feedback and with noisy channel-output feedback, and without channel-output feedback and perfect channel-output feedback respectively. In Fig. 9, $\Delta_i(10, 10, 3, 8, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ is plotted for both $i = 1$ and $i = 2$ as a function of $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. Therein, it is shown that: (a) Increasing $\overleftarrow{n}_{11}$ beyond threshold $\overleftarrow{n}_{11}^{*} = 8$ or increasing $\overleftarrow{n}_{22}$ beyond threshold $\overleftarrow{n}_{22}^{*} = 3$ allows simultaneous improvement of both individual rates. Nonetheless, maximum improvement on R_i is achieved by increasing $\overleftarrow{n}_{ii}$. (b) Increasing either $\overleftarrow{n}_{11}$ or $\overleftarrow{n}_{22}$ beyond thresholds $\overleftarrow{n}_{11}^{*}$ and $\overleftarrow{n}_{22}^{*}$, allows maximum improvement of the sum rate (see Fig. 9).

3.2.3 Example 3: at least one channel-output feedback link does not have any effect over the capacity region

Consider the case in which transmitter-receiver pairs 1 and 2 are in the weak interference regime, with $\overrightarrow{n}_{11} = 10$, $\overrightarrow{n}_{22} = 20$, $n_{12} = 6$, $n_{21} = 12$. In Fig. 10, Fig. 11 and Fig. 12 the capacity region is plotted without channel-output feedback, without channel-output feedback and with noisy channel-output feedback, and without channel-output feedback and perfect channel-output

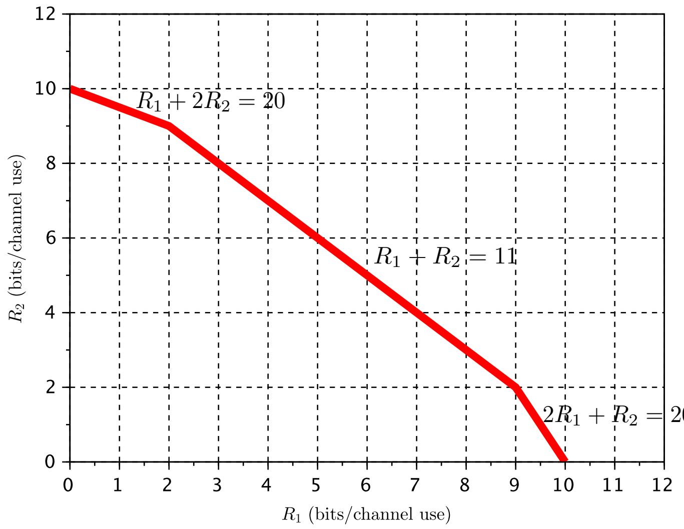

Figure 6: Capacity region of the example in Sec. 3.2.2 without feedback $\mathcal{C}(10, 10, 3, 8, 0, 0)$

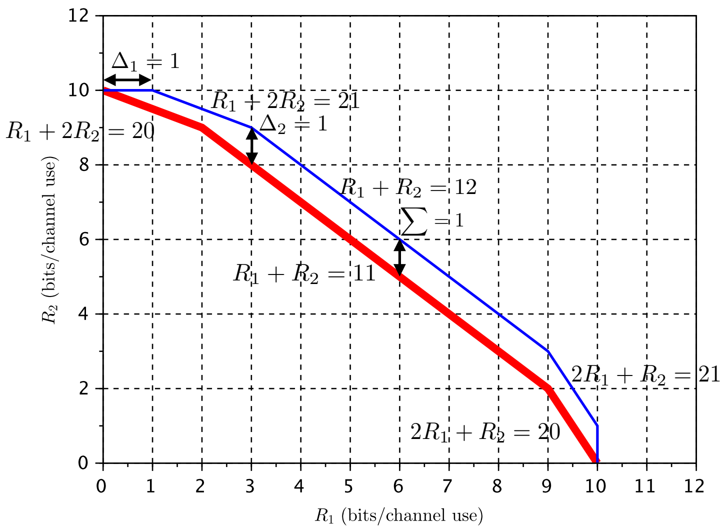

Figure 7: Capacity region $\mathcal{C}(10, 10, 3, 8, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(10, 10, 3, 8, 9, 4)$ with noisy channel-output feedback (thin blue line) of the example in Sec. 3.2.2. Note that $\Delta_1(10, 10, 3, 8, 9, 4) = 1$ bit/ch.use, $\Delta_2(10, 10, 3, 8, 9, 4) = 1$ bit/ch.use and $\Sigma(10, 10, 3, 8, 9, 4) = 1$ bit/ch.use.

feedback respectively. In Fig. 13, $\Delta_i(10, 20, 6, 12, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ is plotted for both $i = 1$ and $i = 2$ as a function of $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. Therein, it is shown that: (a) Increasing parameter $\overleftarrow{n}_{11}$ does not enlarge the capacity region, independently of the value of $\overleftarrow{n}_{22}$. (b) Increasing parameter $\overleftarrow{n}_{22}$ beyond threshold $\overleftarrow{n}_{22}^{*} = 8$ allows simultaneous improvement of both individual rates. (c) Finally, none of the parameters $\overleftarrow{n}_{11}$ or $\overleftarrow{n}_{22}$ increases the sum-rate in this case.

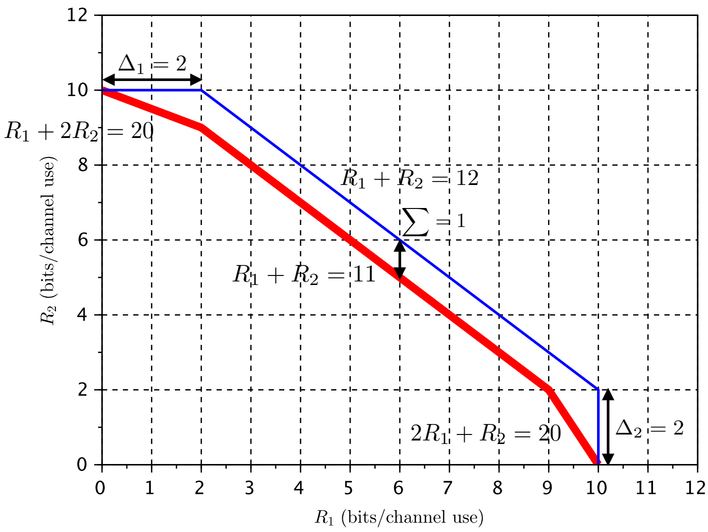

Figure 8: Capacity region $\mathcal{C}(10, 10, 3, 8, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(10, 10, 3, 8, 10, 10)$ with perfect channel-output feedback (thin blue line) of the example in Sec. 3.2.2. Note that $\Delta_1(10, 10, 3, 8, 10, 10) = 2$ bits/ch.use, $\Delta_2(10, 10, 3, 8, 10, 10) = 2$ bits/ch.use and $\Sigma(10, 10, 3, 8, 10, 10) = 1$ bit/ch.use.

3.2.4 Example 4: the channel-output feedback of link i exclusively improves R_j

Consider the case in which transmitter-receiver pairs 1 and 2 are in the very strong and strong interference regimes, with $\overrightarrow{n}_{11} = 7$, $\overrightarrow{n}_{22} = 8$, $n_{12} = 15$, $n_{21} = 13$. In Fig. 14, Fig. 15 and Fig. 16 the capacity region is plotted without channel-output feedback, without channel-output feedback and with noisy channel-output feedback, and without channel-output feedback and perfect channel-output feedback respectively. In Fig. 17, $\Delta_i(7, 8, 15, 13, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ is plotted for both $i = 1$ and $i = 2$ as a function of $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. Therein, it is shown that: (a) Increasing parameter $\overleftarrow{n}_{11}$ beyond threshold $\overleftarrow{n}_{11}^{*} = 8$ exclusively improves R_2. (b) Increasing parameter $\overleftarrow{n}_{22}$ beyond threshold $\overleftarrow{n}_{22}^{*} = 7$ exclusively improves R_1. (c) None of the parameters $\overleftarrow{n}_{11}$ or $\overleftarrow{n}_{22}$ has an impact over the sum rate in this case. Note that these observations are in line with the interpretation of channel-output feedback as an altruistic technique, as in [11, 14]. This is basically because the link implementing channel-output feedback provides an alternative path to the information sent by the other link, as first suggested in [4].

3.2.5 Example 5: none of the channel-output feedback links has any effect over the capacity region

Consider the case in which transmitter-receiver pairs 1 and 2 are in the very weak and strong interference regimes, with $\overrightarrow{n}_{11} = 10$, $\overrightarrow{n}_{22} = 9$, $n_{12} = 2$, $n_{21} = 15$. In Fig. 18 the capacity region is plotted without channel-output feedback and perfect channel-output feedback. Note that the capacity region of the LD-IC with and without channel-output feedback are identical, i.e., neither $\overleftarrow{n}_{11}$ nor $\overleftarrow{n}_{22}$ enlarges the capacity region.

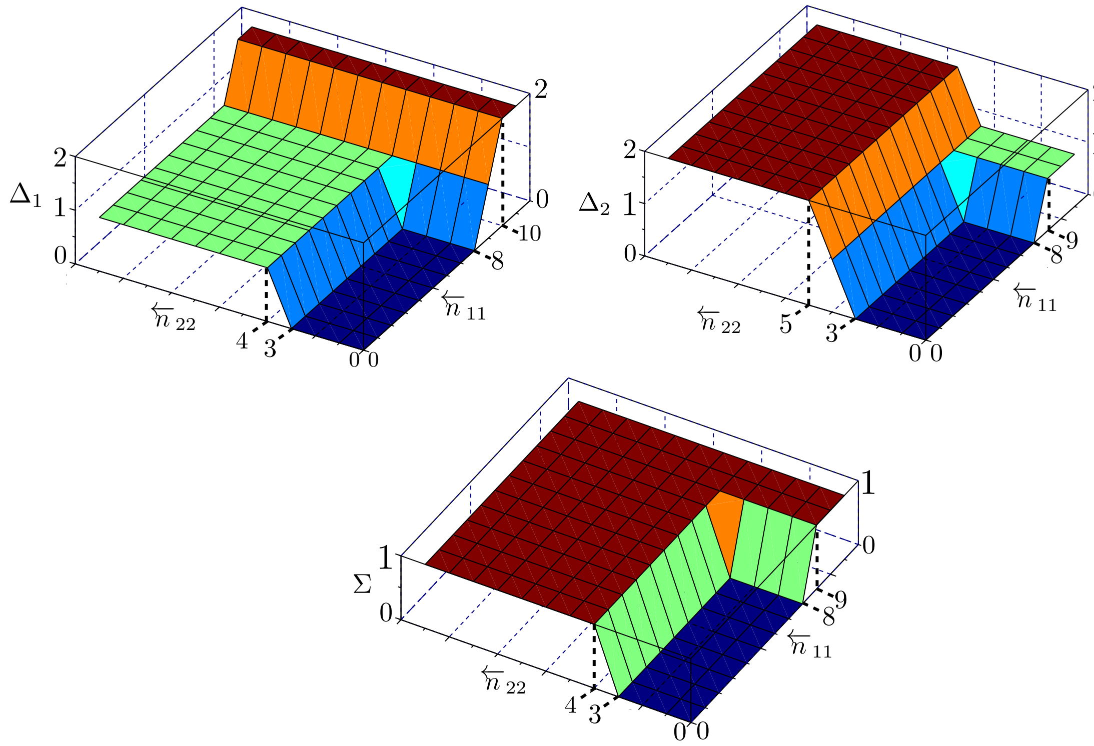

Figure 9: Maximum improvements $\Delta_1(10, 10, 3, 8, \cdot, \cdot)$ and $\Delta_2(10, 10, 3, 8, \cdot, \cdot)$ of one individual rate and $\Sigma(10, 10, 3, 8, \cdot, \cdot)$ of the sum rate of the example in Sec. 3.2.2.

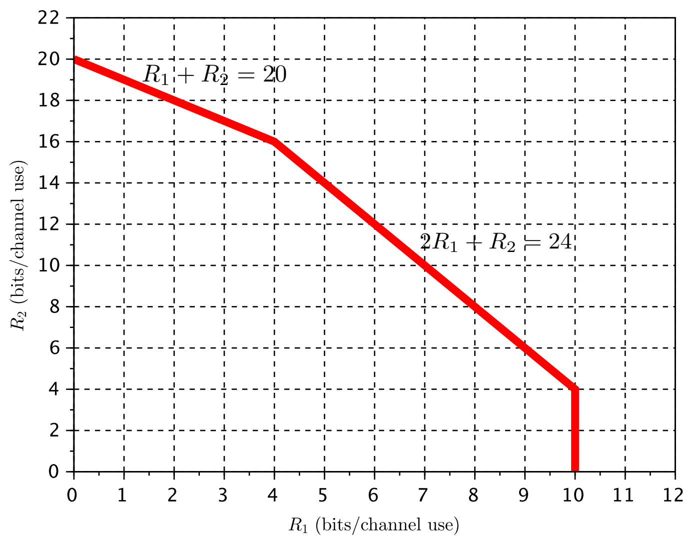

Figure 10: Capacity region of the example in Sec. 3.2.3 without feedback $\mathcal{C}(10, 20, 6, 12, 0, 0)$

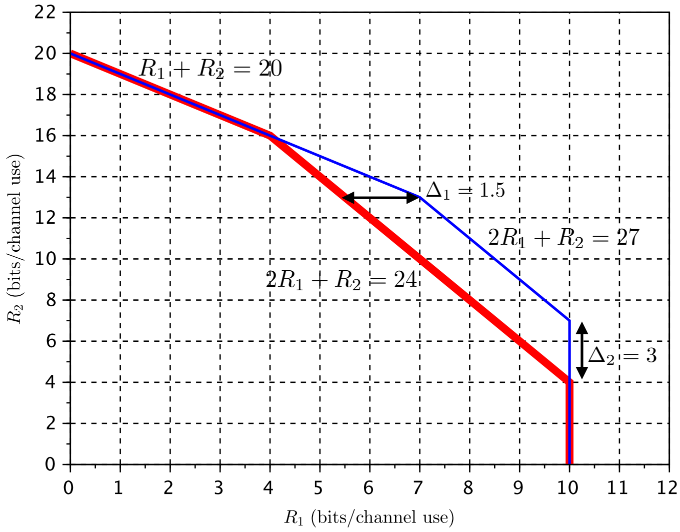

Figure 11: Capacity region $\mathcal{C}(10, 20, 6, 12, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(10, 20, 6, 12, 10, 11)$ with noisy channel-output feedback (thin blue line) of the example in Sec. 3.2.3. Note that $\Delta_1(10, 20, 6, 12, 10, 11) = 1.5$ bits/ch.use, $\Delta_2(10, 20, 6, 12, 10, 11) = 2$ bits/ch.use and $\Sigma(10, 20, 6, 12, 10, 11) = 0$ bits/ch.use.

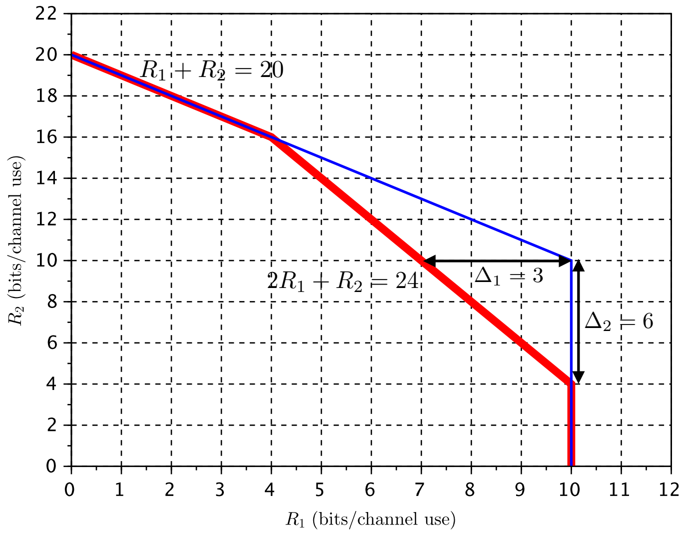

Figure 12: Capacity region $\mathcal{C}(10, 20, 6, 12, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(10, 20, 6, 12, 10, 20)$ with perfect channel-output feedback (thin blue line) of the example in Sec. 3.2.3. Note that $\Delta_1(10, 20, 6, 12, 10, 20) = 3$ bits/ch.use, $\Delta_2(10, 20, 6, 12, 10, 20) = 6$ bits/ch.use and $\Sigma(10, 20, 6, 12, 10, 20) = 0$ bits/ch.use.

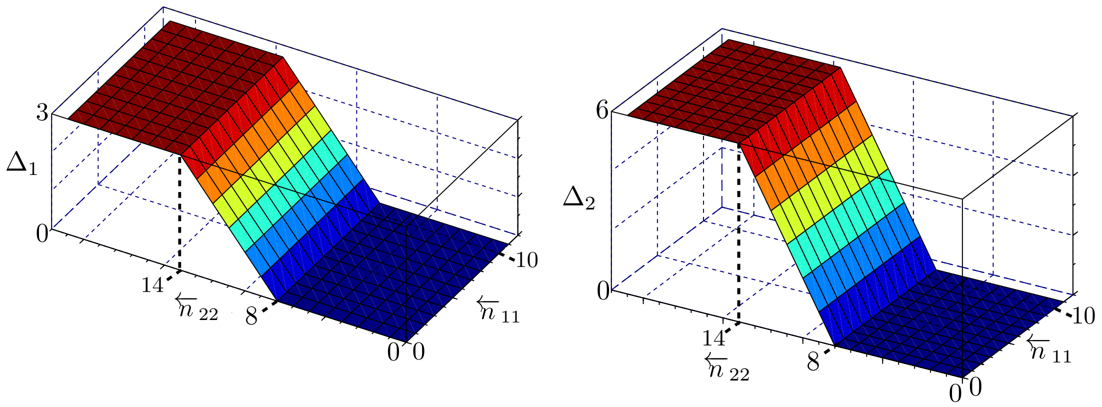

Figure 13: Maximum improvement $\Delta_1(10, 20, 6, 12, \cdot, \cdot)$ and $\Delta_2(10, 20, 6, 12, \cdot, \cdot)$ of one individual rate of the example in Sec. 3.2.3.

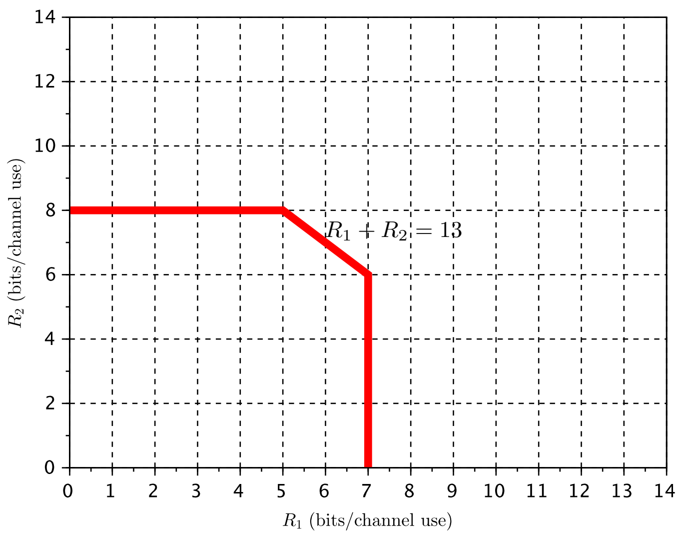

Figure 14: Capacity region of the example in Sec. 3.2.4 without feedback $\mathcal{C}(7, 8, 15, 13, 0, 0)$

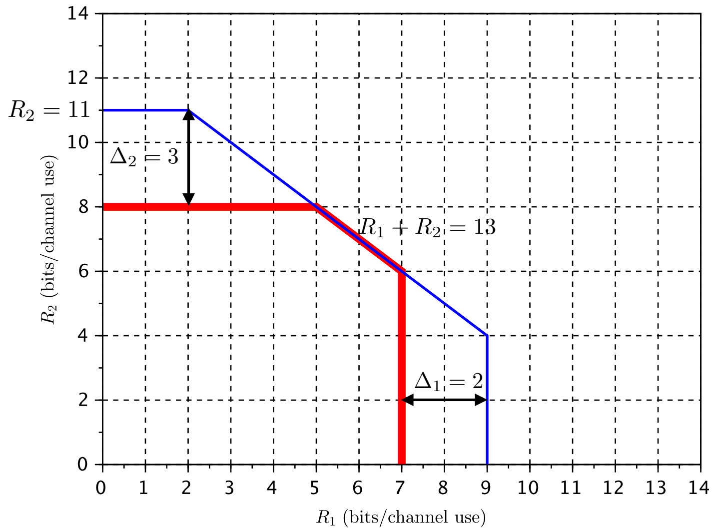

Figure 15: Capacity region $\mathcal{C}(7,8,15,13,0,0)$ without feedback (thick red line) and $\mathcal{C}(7,8,15,13,11,9)$ with noisy channel-output feedback (thin blue line) of the example in Sec. 3.2.4. Note that $\Delta_1(7,8,15,13,11,9) = 2$ bits/ch.use, $\Delta_2(7,8,15,13,11,9) = 3$ bits/ch.use and $\Sigma(7,8,15,13,11,9) = 0$ bits/ch.use.

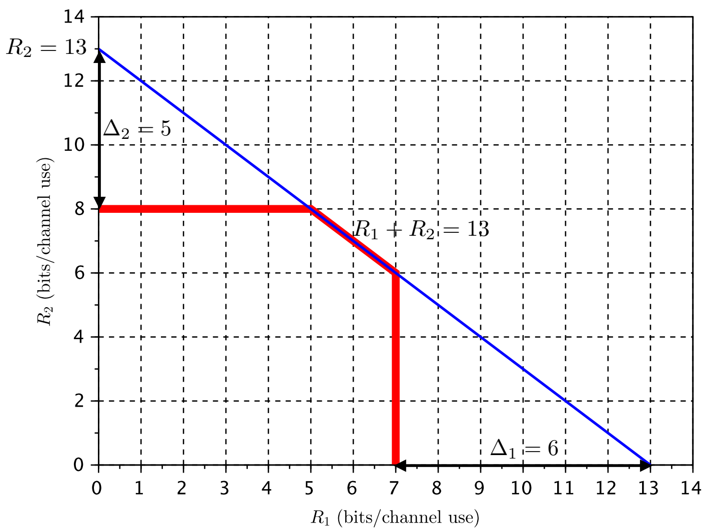

Figure 16: Capacity region of the example in Sec. 3.2.4 without feedback $\mathcal{C}(7,8,15,13,0,0)$ (thick red line) and with perfect channel-output feedback $\mathcal{C}(7,8,15,13,15,13)$ (thin blue line). Note that $\Delta_1(7,8,15,13,15,13) = 6$ bits/ch.use, $\Delta_2(7,8,15,13,15,13) = 5$ bits/ch.use and $\Sigma(7,8,15,13,15,13) = 0$ bits/ch.use.

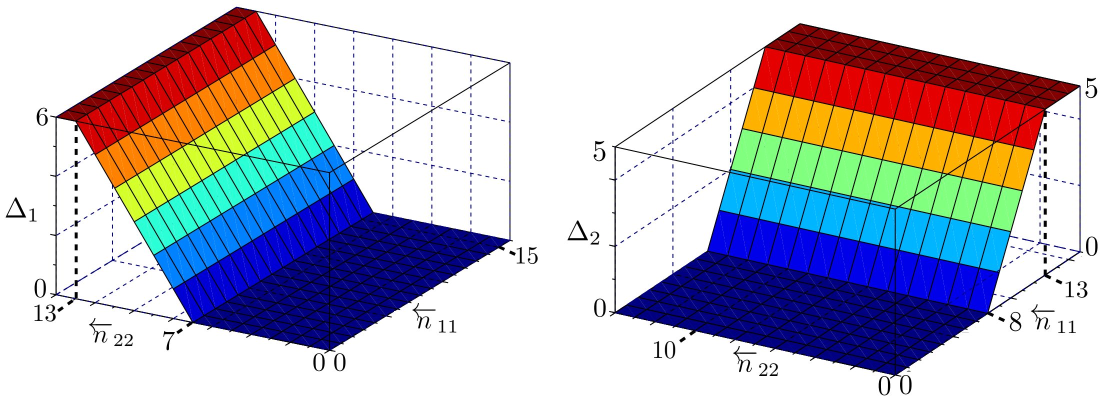

Figure 17: Maximum improvement $\Delta_1(7, 8, 15, 13, \cdot, \cdot)$ and $\Delta_2(7, 8, 15, 13, \cdot, \cdot)$ of one individual rate of the example in Sec. 3.2.4.

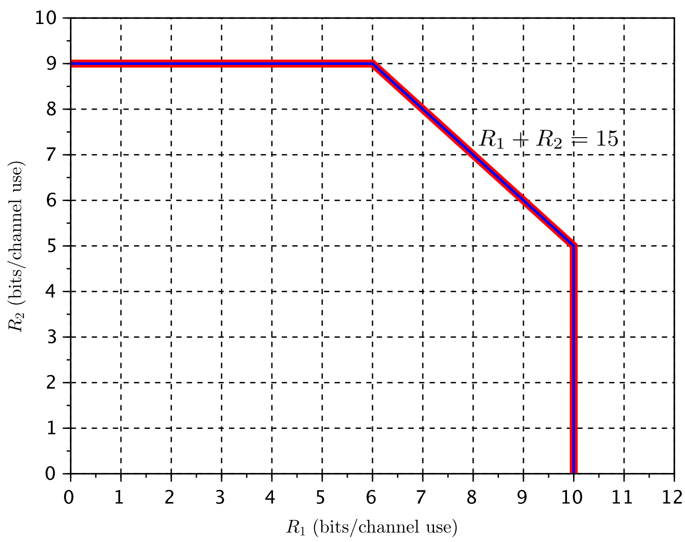

Figure 18: Capacity region $\mathcal{C}(10, 9, 2, 15, 0, 0)$ without feedback (thick red line) and $\mathcal{C}(10, 9, 2, 15, 10, 15)$ with perfect channel-output feedback (thin blue line) of the example in Sec. 3.2.5. Note that $\mathcal{C}(10, 9, 2, 15, 0, 0) = \mathcal{C}(10, 9, 2, 15, 10, 15)$.

4 Conclusions

In this technical report, the noisy channel-output feedback capacity of the linear deterministic interference channel has been fully characterized. Based on specific asymmetric examples, it is highlighted that even in the presence of noise, the benefits of channel-output feedback can be significantly relevant in terms of achievable individual rate and sum-rate improvements with respect to the case without feedback. Unfortunately, there also exist scenarios in which these benefits are totally inexistent.

Appendices

A Achievability Scheme

This appendix provides a description of the proposed achievability scheme, which is based on a three-part message splitting, superposition coding and backward decoding, as first suggested in [2, 4, 6]. The coding scheme is general and thus, it holds for other IC-NOF, i.e., Gaussian IC-NOF. However, the scope of this technical report is exclusively the case of the linear deterministic approximation.

A.1 Codebook Generation

Fix a joint probability distribution

$$P_{U\,U_1\,U_2\,V_1\,V_2\,X_1\,X_2}(u,u_1,u_2,v_1,v_2,x_1,x_2) = P_U(u)P_{U_1|U}(u_1|u)P_{U_2|U}(u_2|u)P_{V_1|U\,U_1}(v_1|u,u_1)$$
$$P_{V_2|U\,U_2}(v_2|u,u_2)P_{X_1|U\,U_1\,V_1}(x_1|u,u_1,v_1)P_{X_2|U\,U_2\,V_2}(x_2|u,u_2,v_2). \quad (11)$$

Let $R_{1,C1}$, $R_{1,C2}$, $R_{2,C1}$, $R_{2,C2}$, $R_{1,P}$ and $R_{2,P}$ be non-negative reals. Let also $R_{1,C} = R_{1,C1} + R_{1,C2}$, $R_{2,C} = R_{2,C1}+R_{2,C2}$, $R_1 = R_{1,C}+R_{1,P}$ and $R_2 = R_{2,C}+R_{2,P}$. Generate $2^{N(R_{1,C1}+R_{2,C1})}$ i.i.d. N-length codewords $\boldsymbol{u}(s,r) = \big(u_1(s,r),\ldots,u_N(s,r)\big)$ according to

$$P_{\boldsymbol{U}}\big(\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_U(u_i(s,r)), \quad (12)$$

with $s \in \{1,\ldots,2^{NR_{1,C1}}\}$ and $r \in \{1,\ldots,2^{NR_{2,C1}}\}$.
For encoder 1, generate for each codeword $\boldsymbol{u}(s,r)$, $2^{NR_{1,C1}}$ i.i.d. N-length codewords $\boldsymbol{u}_1(s,r,k) = \big(u_{1,1}(s,r,k),\ldots,u_{1,N}(s,r,k)\big)$ according to

$$P_{\boldsymbol{U}_1|\boldsymbol{U}}\big(\boldsymbol{u}_1(s,r,k)|\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_{U_1|U}\big(u_{1,i}(s,r,k)|u_i(s,r)\big), \quad (13)$$

with $k \in \{1,\ldots,2^{NR_{1,C1}}\}$. For each pair of codewords $\big(\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k)\big)$, generate $2^{NR_{1,C2}}$ i.i.d. N-length codewords $\boldsymbol{v}_1(s,r,k,l) = \big(v_{1,1}(s,r,k,l),\ldots,v_{1,N}(s,r,k,l)\big)$ according to

$$P_{\boldsymbol{V}_1|\boldsymbol{U}\,\boldsymbol{U}_1}\big(\boldsymbol{v}_1(s,r,k,l)|\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k)\big) = \prod_{i=1}^{N} P_{V_1|U\,U_1}\big(v_{1,i}(s,r,k,l)|u_i(s,r),u_{1,i}(s,r,k)\big), \quad (14)$$

with $l \in \{1,\ldots,2^{NR_{1,C2}}\}$. For each tuple of codewords $\big(\boldsymbol{u}(s,r),\ \boldsymbol{u}_1(s,r,k),\ \boldsymbol{v}_1(s,r,k,l)\big)$, generate $2^{NR_{1,P}}$ i.i.d. N-length codewords $\boldsymbol{x}_1(s,r,k,l,q) = \big(x_{1,1}(s,r,k,l,q),\ldots,\ x_{1,N}(s,r,k,l,q)\big)$ according to

$$P_{\boldsymbol{X}_1|\boldsymbol{U}\,\boldsymbol{U}_1\,\boldsymbol{V}_1}\big(\boldsymbol{x}_1(s,r,k,l,q)|\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k),\boldsymbol{v}_1(s,r,k,l)\big) =$$
$$\prod_{i=1}^{N} P_{X_1|U\,U_1\,V_1}\big(x_{1,i}(s,r,k,l,q)|u_i(s,r),u_{1,i}(s,r,k),v_{1,i}(s,r,k,l)\big), \quad (15)$$

with $q \in \{1,\ldots,2^{NR_{1,P}}\}$.

For encoder 2, generate for each codeword $\boldsymbol{u}(s,r)$, $2^{NR_{2,C1}}$ i.i.d. N-length codewords $\boldsymbol{u}_2(s,r,j) = \big(u_{2,1}(s,r,j),\ldots,u_{2,N}(s,r,j)\big)$ according to

$$P_{\boldsymbol{U}_2|\boldsymbol{U}}\big(\boldsymbol{u}_2(s,r,j)|\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_{U_2|U}\big(u_{2,i}(s,r,j)|u_i(s,r)\big), \tag{16}$$

with $j \in \{1,\ldots,2^{NR_{2,C1}}\}$. For each pair of codewords $\big(\boldsymbol{u}(s,r),\boldsymbol{u}_2(s,r,j)\big)$, generate $2^{NR_{2,C2}}$ i.i.d. length-N codewords $\boldsymbol{v}_2(s,r,j,m) = \big(v_{2,1}(s,r,j,m),\ldots,v_{2,N}(s,r,j,m)\big)$ according to

$$P_{\boldsymbol{V}_2|\boldsymbol{U}\,\boldsymbol{U}_2}\big(\boldsymbol{v}_2(s,r,j,m)|\boldsymbol{u}(s,r),\boldsymbol{u}_2(s,r,j)\big) = \prod_{i=1}^{N} P_{V_2|U\,U_2}(v_{2,i}(s,r,j,m)|u_i(s,r),u_{2,i}(s,r,j)), \tag{17}$$

with $m \in \{1,\ldots,2^{NR_{2,C2}}\}$. For each tuple of codewords $\big(\boldsymbol{u}(s,r),\ \boldsymbol{u}_2(s,r,j),\boldsymbol{v}_2(s,r,j,m)\big)$, generate $2^{NR_{2,P}}$ i.i.d. N-length codewords $\boldsymbol{x}_2(s,r,j,m,b) = \big(x_{2,1}(s,r,j,m,b),\ldots,\ x_{2,N}(s,r,j,m,b)\big)$ according to

$$\begin{aligned}P_{\boldsymbol{X}_2|\boldsymbol{U}\,\boldsymbol{U}_2\,\boldsymbol{V}_2}\big(\boldsymbol{x}_2(s,r,j,m,b)|\boldsymbol{u}(s,r),\boldsymbol{u}_2(s,r,j),\boldsymbol{v}_2(s,r,j,m)\big) = \\ \prod_{i=1}^{N} P_{X_2|U\,U_2\,V_2}(x_{2,i}(s,r,j,m,b)|u_i(s,r),u_{2,i}(s,r,j),v_{2,i}(s,r,j,m,b)),\end{aligned} \tag{18}$$

with $b \in \{1,\ldots,2^{NR_{2,P}}\}$.

A.2 Encoding

Denote by $W_i^{(t)} \in \{1,\ldots,2^{N(R_{i,C}+R_{i,P})}\}$ the message index of transmitter i during block t, respectively. Let $W_i^{(t)} = \Big(W_{i,C}^{(t)},W_{i,P}^{(t)}\Big)$ be the message index composed by the message index $W_{i,C}^{(t)} \in \{1,\ldots,2^{NR_{i,C}}\}$ and message index $W_{i,P}^{(t)} \in \{1,\ldots,2^{NR_{i,P}}\}$. The message index $W_{i,P}^{(t)}$ must be reliably decoded at receiver i. Let also $W_{i,C}^{(t)} = (W_{i,C1}^{(t)},W_{i,C2}^{(t)})$ be the message index composed by the message indices $W_{i,C1}^{(t)} \in \{1,\ldots,2^{NR_{i,C1}}\}$ and $W_{i,C2}^{(t)} \in \{1,\ldots,2^{NR_{i,C2}}\}$. The message index $W_{i,C1}^{(t)}$ must be reliably decoded at transmitter j (via feedback). The index message $W_{i,C2}^{(t)}$ must be reliably decoded at receiver j.

Consider Markov encoding with a length of T blocks. At encoding step t, with $t \in \{1,\ldots,T\}$, transmitter 1 sends the codeword $\boldsymbol{x}_1^{(t)} = \boldsymbol{x}_1\Big(W_{1,C1}^{(t-1)},W_{2,C1}^{(t-1)},W_{1,C1}^{(t)},W_{1,C2}^{(t)},W_{1,P}^{(t)}\Big)$, where $W_{1,C1}^{(0)} = W_{1,C1}^{(T)} = s^*$ and $W_{2,C1}^{(0)} = W_{2,C1}^{(T)} = r^*$. The pair $(s^*,r^*) \in \{1,\ldots,2^{N\,R_{1,C1}}\} \times \{1,\ldots,2^{NR_{2,C1}}\}$ is pre-defined and known at both receivers and transmitters. It is worth noting that the message index $W_{2,C1}^{(t-1)}$ is obtained by transmitter 1 from the feedback signal $\overleftarrow{\boldsymbol{y}}_1^{(t-1)}$ at the end of the previous encoding step $t-1$.

Transmitter 2 follows a similar encoding scheme.

A.3 Decoding

Both receivers decode their message indices at the end of block T in a backward decoding fashion. At each decoding step t, with $t \in \{1,\ldots,T\}$, receiver 1 obtains the message indices $\Big(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},\ \widehat{W}_{1,C2}^{(T-(t-1))},\ \widehat{W}_{1,P}^{(T-(t-1))},\widehat{W}_{2,C2}^{(T-(t-1))}\Big) \in \{1,\ldots,2^{NR_{1,C1}}\} \times$

$\{1,\ldots,2^{NR_{2,C1}}\}\times\{1,\ldots,2^{NR_{1,C2}}\}\times\{1,\ldots,2^{NR_{1,P}}\}\times\{1,\ldots,2^{NR_{2,C2}}\}$. The tuple $\Big(\widehat{W}_{1,C1}^{(T-t)}$, $\widehat{W}_{2,C1}^{(T-t)},\widehat{W}_{1,C2}^{(T-(t-1))},\widehat{W}_{1,P}^{(T-(t-1))},\widehat{W}_{2,C2}^{(T-(t-1))}\Big)$ is the unique tuple that satisfies

$$\begin{aligned}&\Big(\boldsymbol{u}\left(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)}\right),\boldsymbol{u}_1\left(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},W_{1,C1}^{(T-(t-1))}\right),\\&\boldsymbol{v}_1\left(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},W_{1,C1}^{(T-(t-1))},\widehat{W}_{1,C2}^{(T-(t-1))}\right),\\&\boldsymbol{x}_1\left(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},W_{1,C1}^{(T-(t-1))},\widehat{W}_{1,C2}^{(T-(t-1))},\widehat{W}_{1,P}^{(T-(t-1))}\right),\\&\boldsymbol{u}_2\left(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},W_{2,C1}^{(T-(t-1))}\right),\\&\boldsymbol{v}_2\left(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},W_{2,C1}^{(T-(t-1))},\widehat{W}_{2,C2}^{(T-(t-1))}\right),\overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}\Big)\in\mathcal{A}_e^{(N)},\end{aligned}\tag{19}$$

where $W_{1,C1}^{(T-(t-1))}$ and $W_{2,C1}^{(T-(t-1))}$ are assumed to be perfectly decoded in the previous decoding step $t-1$. The set $\mathcal{A}_e^{(N)}$ represents the set of jointly typical sequences. Finally, receiver 2 follows a similar decoding scheme.

A.4 Probability of Error Analysis

An error might occur during encoding step t if the message index $W_{2,C1}^{(t-1)}$ is not correctly decoded at transmitter 1. From the asymptotic equipartion property (AEP) [15], it follows that the message index $W_{2,C1}^{(t-1)}$ can be reliably decoded at transmitter 1 during encoding step t, under the condition:

$$\begin{aligned}R_{2,C1} &\leqslant I\left(\overleftarrow{Y}_1;U_2|U,U_1,V_1,X_1\right)\\&= I\left(\overleftarrow{Y}_1;U_2|U,X_1\right).\end{aligned}\tag{20}$$

An error might occur during the (backward) decoding step t if the message indices $W_{1,C1}^{(T-t)}$, $W_{2,C1}^{(T-t)}$, $W_{1,C2}^{(T-(t-1))}$, $W_{1,P}^{(T-(t-1))}$ and $W_{2,C2}^{(T-(t-1))}$ are not decoded correctly given that the message indices $W_{1,C1}^{(T-(t-1))}$ and $W_{2,C1}^{(T-(t-1))}$ were correctly decoded in the previous decoding step $t-1$. These errors might arise for two reasons: (i) there does not exist a tuple $\Big(\widehat{W}_{1,C1}^{(T-t)}$, $\widehat{W}_{2,C1}^{(T-t)},\widehat{W}_{1,C2}^{(T-(t-1))},\widehat{W}_{1,P}^{(T-(t-1))},\widehat{W}_{2,C2}^{(T-(t-1))}\Big)$ that satisfies (19), or (ii) there exist several tuples $\Big(\widehat{W}_{1,C1}^{(T-t)},\widehat{W}_{2,C1}^{(T-t)},\widehat{W}_{1,C2}^{(T-(t-1))},\widehat{W}_{1,P}^{(T-(t-1))},\widehat{W}_{2,C2}^{(T-(t-1))}\Big)$ that simultaneously satisfy (19). From the asymptotic equipartion property (AEP) [15], the probability of an error due to (i) tends to zero when N grows to infinity. Consider the error due to (ii) and define the event $E_{(s,r,l,q,m)}$ that describes the case in which the codewords $\big(\boldsymbol{u}(s,r)$, $\boldsymbol{u}_1(s,r,W_{1,C1}^{(T-(t-1))})$, $\boldsymbol{v}_1(s,r,W_{1,C1}^{(T-(t-1))},l)$, $\boldsymbol{x}_1(s,r,W_{1,C1}^{(T-(t-1))},l,q)$, $\boldsymbol{u}_2(s,r,W_{2,C1}^{(T-(t-1))})$ and $\boldsymbol{v}_2(s,r,W_{2,C1}^{(T-(t-1))},m)\big)$ are jointly typical with $\overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}$ during decoding step t. Assume now that the codeword to be decoded at decoding step t corresponds to the indices $(s,r,l,q,m)=(1,1,1,1,1)$, this is without loss of generality due to the symmetry of the code. Then, the probability of error due to (ii) during decoding step t, can be bounded as follows

$$
\begin{aligned}
P_e =& \Pr\left(\bigcup_{(s,r,l,q,m) \neq (1,1,1,1,1)} E_{(s,r,l,q,m)} \right) \\
\leqslant & \sum_{s=1,r=1,l=1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r=1,l=1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r=1,l=1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r=1,l\neq 1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r=1,l\neq 1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r=1,l\neq 1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r=1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq 1,l=1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r\neq 1,l=1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq 1,l=1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r\neq 1,l=1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq 1,l\neq 1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r\neq 1,l\neq 1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq 1,l\neq 1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s=1,r\neq 1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l=1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r=1,l=1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l=1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r=1,l=1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l\neq 1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r=1,l\neq 1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l\neq 1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r=1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l=1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r\neq 1,l=1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l=1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r\neq 1,l=1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l\neq 1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r\neq 1,l\neq 1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l\neq 1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\
& + \sum_{s\neq 1,r\neq 1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right). \qquad (21)
\end{aligned}
$$

From the asymptotic equipartion property (AEP) [15], it follows that:

$$
\begin{aligned}
P_e \leqslant\, & 2^{N(R_{2,C2}-I(\overrightarrow{Y}_1;V_2|U,U_1,U_2,V_1,X_1)+2\epsilon)} \\
& +2^{N(R_{1,P}-I(\overrightarrow{Y}_1;X_1|U,U_1,U_2,V_1,V_2)+2\epsilon)} \\
& +2^{N(R_{2,C2}+R_{1,P}-I(\overrightarrow{Y}_1;V_2,X_1|U,U_1,U_2,V_1)+2\epsilon)} \\
& +2^{N(R_{1,C2}-I(\overrightarrow{Y}_1;V_1,X_1|U,U_1,U_2,V_2)+2\epsilon)} \\
& +2^{N(R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;V_1,V_2,X_1|U,U_1,U_2)+2\epsilon)} \\
& +2^{N(R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;V_1,X_1|U,U_1,U_2,V_2)+2\epsilon)} \\
& +2^{N(R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;V_1,V_2,X_1|U,U_1,U_2)+2\epsilon)} \\
& +2^{N(R_{2,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{2,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\
& +2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}. \qquad (22)
\end{aligned}
$$

The same analysis of the probability of error holds for transmitter-receiver pair 2. Hence, in general, from (20) and (22), reliable decoding holds under the following conditions for transmitter $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$\begin{aligned}
R_{j,C1} &\leqslant I\left(\overleftarrow{Y}_i;U_j|U,U_i,V_i,X_i\right)\\
&= I\left(\overleftarrow{Y}_i;U_j|U,X_i\right), && \text{(23a)}\\
R_{i,C1}+R_{i,C2}+R_{i,P}+R_{j,C1}+R_{j,C2} &\leqslant I(\overrightarrow{Y}_i;U,U_i,U_j,V_i,V_j,X_i)\\
&= I(\overrightarrow{Y}_i;U,U_j,V_j,X_i), && \text{(23b)}\\
R_{j,C2} &\leqslant I(\overrightarrow{Y}_i;V_j|U,U_i,U_j,V_i,X_i)\\
&= I(\overrightarrow{Y}_i;V_j|U,U_j,X_i), && \text{(23c)}\\
R_{i,P} &\leqslant I(\overrightarrow{Y}_i;X_i|U,U_i,U_j,V_i,V_j), && \text{(23d)}\\
R_{i,P}+R_{j,C2} &\leqslant I(\overrightarrow{Y}_i;V_j,X_i|U,U_i,U_j,V_i), && \text{(23e)}\\
R_{i,C2}+R_{i,P} &\leqslant I(\overrightarrow{Y}_i;V_i,X_i|U,U_i,U_j,V_j)\\
&= I(\overrightarrow{Y}_i;X_i|U,U_i,U_j,V_j), && \text{(23f)}\\
R_{i,C2}+R_{i,P}+R_{j,C2} &\leqslant I(\overrightarrow{Y}_i;V_i,V_j,X_i|U,U_i,U_j)\\
&= I(\overrightarrow{Y}_i;V_j,X_i|U,U_i,U_j). && \text{(23g)}
\end{aligned}$$

More explicitly, the inequalities in (23) can be written as follows

$$\begin{aligned}
R_{2,C1} &\leqslant I\left(\overleftarrow{Y}_1;U_2|U,X_1\right) &&= a_1, && \text{(24a)}\\
R_{1,C1}+R_{1,C2}+R_{1,P}+R_{2,C1}+R_{2,C2} &\leqslant I(\overrightarrow{Y}_1;U,U_2,V_2,X_1) &&= a_2, && \text{(24b)}\\
R_{2,C2} &\leqslant I(\overrightarrow{Y}_1;V_2|U,U_2,X_1) &&= a_3, && \text{(24c)}\\
R_{1,P} &\leqslant I(\overrightarrow{Y}_1;X_1|U,U_1,U_2,V_1,V_2) &&= a_4, && \text{(24d)}\\
R_{1,P}+R_{2,C2} &\leqslant I(\overrightarrow{Y}_1;V_2,X_1|U,U_1,U_2,V_1) &&= a_5, && \text{(24e)}\\
R_{1,C2}+R_{1,P} &\leqslant I(\overrightarrow{Y}_1;X_1|U,U_1,U_2,V_2) &&= a_6, && \text{(24f)}\\
R_{1,C2}+R_{1,P}+R_{2,C2} &\leqslant I(\overrightarrow{Y}_1;V_2,X_1|U,U_1,U_2) &&= a_7, && \text{(24g)}\\
R_{1,C1} &\leqslant I\left(\overleftarrow{Y}_2;U_1|U,X_2\right) &&= b_1, && \text{(24h)}\\
R_{2,C1}+R_{2,C2}+R_{2,P}+R_{1,C1}+R_{1,C2} &\leqslant I(\overrightarrow{Y}_2;U,U_1,V_1,X_2) &&= b_2, && \text{(24i)}\\
R_{1,C2} &\leqslant I(\overrightarrow{Y}_2;V_1|U,U_1,X_2) &&= b_3, && \text{(24j)}\\
R_{2,P} &\leqslant I(\overrightarrow{Y}_2;X_2|U,U_1,U_2,V_1,V_2) &&= b_4, && \text{(24k)}\\
R_{2,P}+R_{1,C2} &\leqslant I(\overrightarrow{Y}_2;V_1,X_2|U,U_1,U_2,V_2) &&= b_5, && \text{(24l)}\\
R_{2,C2}+R_{2,P} &\leqslant I(\overrightarrow{Y}_2;X_2|U,U_1,U_2,V_1) &&= b_6, && \text{(24m)}\\
R_{2,C2}+R_{2,P}+R_{1,C2} &\leqslant I(\overrightarrow{Y}_2;V_1,X_2|U,U_1,U_2) &&= b_7. && \text{(24n)}
\end{aligned}$$

Taking into account that $R_1 = R_{1,C1} + R_{1,C2} + R_{1,P}$ and $R_2 = R_{2,C1} + R_{2,C2} + R_{2,P}$, a Fourier-Motzkin elimination process in (24) yields

$$
\begin{aligned}
R_1 &\leqslant \min\left(a_2, a_6 + b_1, a_4 + b_1 + b_3\right), & \text{(25a)}\\
R_2 &\leqslant \min\left(b_2, a_1 + b_6, a_1 + a_3 + b_4\right), & \text{(25b)}\\
R_1 + R_2 &\leqslant \min(a_2 + b_4, a_2 + b_6, a_4 + b_2, a_6 + b_2, a_1 + a_3 + a_4 + b_1 + b_5,\\
&\qquad a_1 + a_7 + b_1 + b_5, a_1 + a_4 + b_1 + b_7, a_1 + a_5 + b_1 + b_3 + b_4, a_1 + a_5 + b_1 + b_5,\\
&\qquad a_1 + a_7 + b_1 + b_4), & \text{(25c)}\\
2R_1 + R_2 &\leqslant \min(a_2 + a_4 + b_1 + b_7, a_1 + a_4 + a_7 + 2b_1 + b_5, a_2 + a_4 + b_1 + b_5), & \text{(25d)}\\
R_1 + 2R_2 &\leqslant \min(a_1 + a_5 + b_2 + b_4, a_1 + a_7 + b_2 + b_4, 2a_1 + a_5 + b_1 + b_4 + b_7), & \text{(25e)}
\end{aligned}
$$

where $a_1, \ldots a_7$ and $b_1, \ldots b_7$ are defined in (24).

A.5 Achievable Region

In the LD-IC-NOF model, the i-th channel input at each channel use is a q-dimensional vector $\boldsymbol{X}_i \in \{0,1\}^q$ with $i \in \{1,2\}$ and q as defined in (1). Following this observation, the random variables U, U_i, V_i and X_i described above in the codebook generation (Sec. A.1) must also be interpreted as vectors, and thus, in this subsection, they are denoted by $\boldsymbol{U}$, $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ and $\boldsymbol{X}_i$, respectively.

Assume that transmitter-receiver pair i uses the following coding scheme: the symbol $\boldsymbol{X}_i$ is obtained as the concatenation of three other symbols, i.e.,

$$\boldsymbol{X_i} = \left(\boldsymbol{U}_i, \boldsymbol{V}_i, \boldsymbol{X}_{i,P}, (0, \ldots, 0)\right), \tag{26}$$

where the null vector $(0, \ldots, 0)$ is used to meet the dimension q of $\boldsymbol{X}_i$. The symbols $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ and $\boldsymbol{X}_{i,P}$ are assumed to be mutually independent and uniformly distributed over the sets $\{0,1\}^{\left(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right)^+}$, $\{0,1\}^{\left(\min\left(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right)\right)}$ and $\{0,1\}^{\left(\overrightarrow{n}_{ii} - n_{ji}\right)^+}$, respectively. Note that the vectors $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ and $\boldsymbol{X}_{i,P}$ possess the following dimensions:

$$
\begin{aligned}
\dim \boldsymbol{U}_i &= \left(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right)^+, & (27)\\
\dim \boldsymbol{V}_i &= \min\left(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right), \text{ and} & (28)\\
\dim \boldsymbol{X}_{i,P} &= \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+. & (29)
\end{aligned}
$$

These dimensions satisfy the following condition:

$$\dim \boldsymbol{U}_i + \dim \boldsymbol{V}_i + \dim \boldsymbol{X}_{i,P} = \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) \leqslant q. \tag{30}$$

The intuition behind this choice follows from the following observations: (a) The vector $\boldsymbol{U}_i$ represents the signal levels in $\boldsymbol{X}_i$ that can be seen at least at transmitter j; (b) The vector $\boldsymbol{V}_i$ represents the signal levels in $\boldsymbol{X}_i$ that can be seen at least at receiver j; and finally, (c) The vector $\boldsymbol{X}_{i,P}$ is a notational artefact to denote the signal levels of $\boldsymbol{X}_i$ that are neither in $\boldsymbol{U}_i$ nor $\boldsymbol{V}_i$. In particular, the signal levels in $\boldsymbol{X}_{i,P}$ are only seen at receiver i.

This coding scheme can be obtained following the codebook generation described in Sec. A.1. Note that, in this particular case, the symbol $\boldsymbol{U}$ (denoted by U in Sec. A.1) is not used to generate the symbol $\boldsymbol{X}_i$ (denoted by X_i in Sec. A.1). These observations are shown in Figure 19 for different interference regimes.

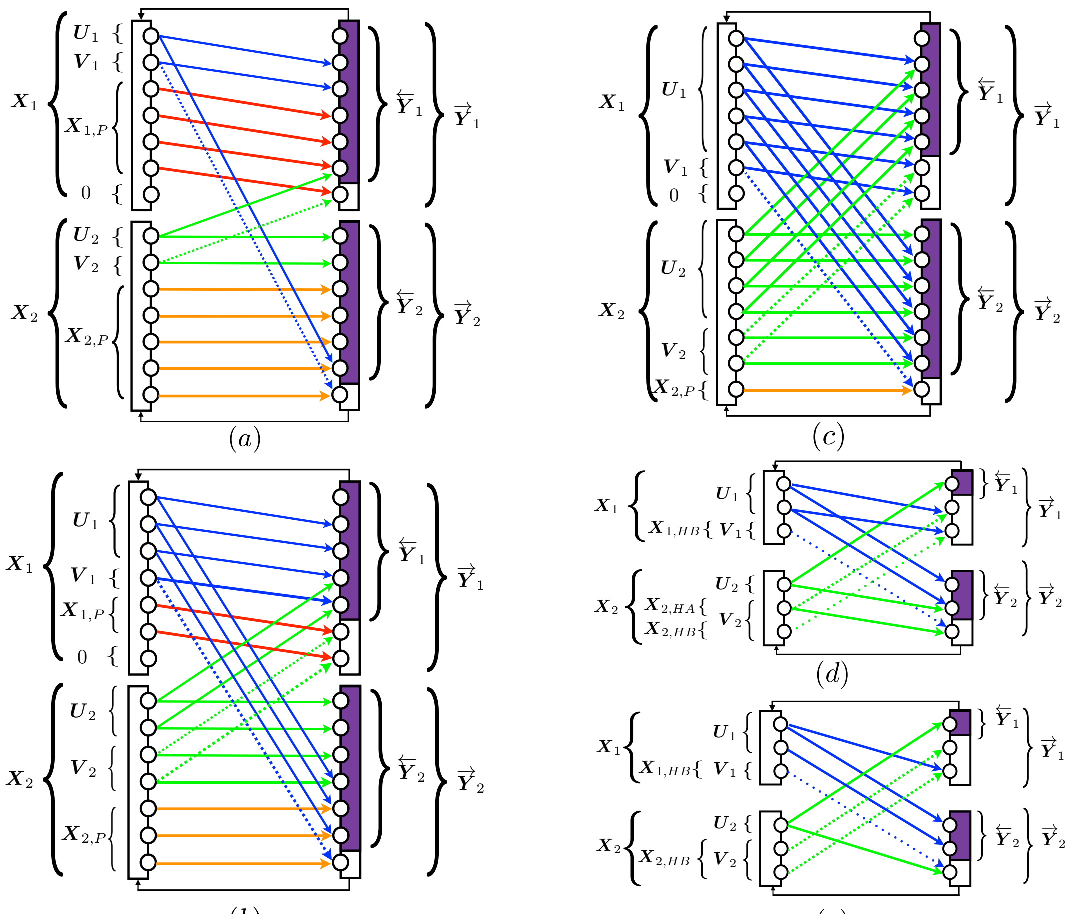

Figure 19: The auxiliary random variables and their relation with signals when channel-output feedback is considered in (a) very weak interference regime, (b) weak interference regime, (c) moderate interference regime, (d) strong interference regime and (e) very strong interference regime.

Considering this particular coding scheme, the following holds for the mutual information terms in inequalities (23a)-(23e):

$$
\begin{aligned}
I\left(\overleftarrow{\boldsymbol{Y}}_i;\boldsymbol{U}_j|\boldsymbol{U},\boldsymbol{X}_i\right) &= H\left(\overleftarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{X}_i\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right)\\
&\overset{(a)}{=} H\left(\overleftarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{X}_i\right)\\
&= H\left(\boldsymbol{U}_j\right)\\
&= \dim \boldsymbol{U}_j\\
&= \left(n_{ij} - \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^+\right)^+;
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{V}_j,\boldsymbol{X}_i\right) &= H\left(\overrightarrow{\boldsymbol{Y}}_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{V}_j,\boldsymbol{X}_i\right)\\
&\overset{(b)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i\right)\\
&= \dim \overrightarrow{\boldsymbol{Y}}_i\\
&= \max\left(\overrightarrow{n}_{ii}, n_{ij}\right); \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{V}_j|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right) &= H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{V}_j,\boldsymbol{X}_i\right)\\
&\overset{(b)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right)\\
&= H\left(\boldsymbol{V}_j\right)\\
&= \dim \boldsymbol{V}_j\\
&= \min\left(n_{ij}, \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^+\right); \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{X}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{V}_j\right) &= H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{V}_j\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{V}_j,\boldsymbol{X}_i\right)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{V}_j\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{V}_j,\boldsymbol{X}_i\right)\\
&\overset{(b)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{V}_j\right)\\
&= H\left(\boldsymbol{X}_{i,P}\right)\\
&= \dim \boldsymbol{X}_{i,P}\\
&= \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+; \text{ and} \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{V}_j,\boldsymbol{X}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i\right) &= I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{X}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i\right) + I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{V}_j|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{X}_i\right)\\
&= I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{X}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i\right) + I\left(\overrightarrow{\boldsymbol{Y}}_i;\boldsymbol{V}_j|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i,\boldsymbol{X}_i\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{V}_j,\boldsymbol{X}_i\right)\\
&\overset{(b)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_i,\boldsymbol{U}_j,\boldsymbol{V}_i\right)\\
&= \max\left(\dim \boldsymbol{X}_{i,P}, \dim \boldsymbol{V}_j\right)\\
&= \max\Big(\left(\overrightarrow{n}_{ii} - n_{ji}\right)^+,\\
&\quad \min\left(n_{ij}, \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^+\right)\Big), \qquad (35)
\end{aligned}
$$

where,
(a) follows from the fact that $H\left(\overleftarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{X}_i\right) = 0$; and
(b) follows from the fact that $H(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{U},\boldsymbol{U}_j,\boldsymbol{V}_j,\boldsymbol{X}_i) = 0$.

For the calculation of the last two mutual information terms in inequalities (23f) and (23g), special notation is used. Let for instance the vector $\boldsymbol{V}_i$ be the concatenation of the vectors $\boldsymbol{X}_{i,HA}$ and $\boldsymbol{X}_{i,HB}$, i.e., $\boldsymbol{V}_i = (\boldsymbol{X}_{i,HA}, \boldsymbol{X}_{i,HB})$. The vector $\boldsymbol{X}_{i,HA}$ is the part of $\boldsymbol{V}_i$ that is seen in both receivers. The vector $\boldsymbol{X}_{i,HB}$ is the part of $\boldsymbol{V}_i$ that is exclusively seen in receiver j (see Figure 19). Note also that $H(\boldsymbol{V}_i) = H(\boldsymbol{X}_{i,HA}) + H(\boldsymbol{X}_{i,HB})$. Using this notation, the following holds

$$
\begin{aligned}
I\left(\overrightarrow{\boldsymbol{Y}}_i; \boldsymbol{X}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{V}_j\right) =& H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{V}_j\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{V}_j, \boldsymbol{X}_i\right) \\
=& H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{V}_j\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{V}_j, \boldsymbol{X}_i\right) \\
\overset{(c)}{=}& H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{V}_j\right) \\
=& H\left(\boldsymbol{X}_{i,HA}, \boldsymbol{X}_{i,P}\right) \\
=& H\left(\boldsymbol{X}_{i,HA}\right) + H\left(\boldsymbol{X}_{i,P}\right) \\
=& \dim \boldsymbol{X}_{i,HA} + \dim \boldsymbol{X}_{i,P} \\
=& \min\left(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right) \\
& - \min\left(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^+, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right) \\
& + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+;
\end{aligned} \tag{36}
$$

and

$$
\begin{aligned}
I\left(\overrightarrow{\boldsymbol{Y}}_i; \boldsymbol{V}_j, \boldsymbol{X}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j\right) =& I\left(\overrightarrow{\boldsymbol{Y}}_i; \boldsymbol{X}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j\right) + I\left(\overrightarrow{\boldsymbol{Y}}_i; \boldsymbol{V}_j | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{X}_i\right) \\
=& I\left(\overrightarrow{\boldsymbol{Y}}_i; \boldsymbol{X}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j\right) + I\left(\overrightarrow{\boldsymbol{Y}}_i; \boldsymbol{V}_j | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{X}_i\right) \\
=& H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j, \boldsymbol{X}_i\right) \\
& + H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{X}_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{V}_j, \boldsymbol{X}_i\right) \\
\overset{(c)}{=}& H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{X}_i\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{X}_i\right) \\
=& H\left(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_i, \boldsymbol{U}_j\right) \\
=& \max\left(H\left(\boldsymbol{V}_j\right), H\left(\boldsymbol{X}_{i,HA}, \boldsymbol{X}_{i,P}\right)\right) \\
=& \max\left(H\left(\boldsymbol{V}_j\right), H\left(\boldsymbol{X}_{i,HA}\right) + H\left(\boldsymbol{X}_{i,P}\right)\right) \\
=& \max\left(\dim \boldsymbol{V}_j, \dim \boldsymbol{X}_{i,HA} + \dim \boldsymbol{X}_{i,P}\right) \\
=& \max\Big(\min\left(n_{ij}, \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^+\right), \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+ \\
& + \min\left(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right) \\
& - \min\left(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^+, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right)\Big);
\end{aligned} \tag{37}
$$

where,
(c) follows from the fact that $H(\overrightarrow{\boldsymbol{Y}}_i | \boldsymbol{U}, \boldsymbol{U}_j, \boldsymbol{V}_j, \boldsymbol{X}_i) = 0$.

Plugging (31) - (37) with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$ into (24a) - (24n) yields

$$
\begin{aligned}
a_1 &= \left(n_{12} - \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right)^+, && (38)\\
a_2 &= \max\left(\overrightarrow{n}_{11}, n_{12}\right), && (39)\\
a_3 &= \min\left(n_{12}, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right), && (40)\\
a_4 &= \left(\overrightarrow{n}_{11} - n_{21}\right)^+, && (41)\\
a_5 &= \max\left(\left(\overrightarrow{n}_{11} - n_{21}\right)^+, \min\left(n_{12}, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right)\right), && (42)\\
a_6 &= \min\left(n_{21}, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right) - \min\left(\left(n_{21} - \overrightarrow{n}_{11}\right)^+, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right) \\
&\quad + \left(\overrightarrow{n}_{11} - n_{21}\right)^+, && (43)\\
a_7 &= \max\Big(\min\left(n_{12}, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right), \left(\overrightarrow{n}_{11} - n_{21}\right)^+ && (44)\\
&\quad + \min\left(n_{21}, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right) - \min\left(\left(n_{21} - \overrightarrow{n}_{11}\right)^+, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right)\Big), \\
b_1 &= \left(n_{21} - \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right)^+, && (45)\\
b_2 &= \max\left(\overrightarrow{n}_{22}, n_{21}\right), && (46)\\
b_3 &= \min\left(n_{21}, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right), && (47)\\
b_4 &= \left(\overrightarrow{n}_{22} - n_{12}\right)^+, && (48)\\
b_5 &= \max\left(\left(\overrightarrow{n}_{22} - n_{12}\right)^+, \min\left(n_{21}, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right)\right), && (49)\\
b_6 &= \min\left(n_{12}, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right) - \min\left(\left(n_{12} - \overrightarrow{n}_{22}\right)^+, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right) \\
&\quad + \left(\overrightarrow{n}_{22} - n_{12}\right)^+, && (50)\\
b_7 &= \max\Big(\min\left(n_{21}, \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right), \left(\overrightarrow{n}_{22} - n_{12}\right)^+ && (51)\\
&\quad + \min\left(n_{12}, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right) - \min\left(\left(n_{12} - \overrightarrow{n}_{22}\right)^+, \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right)\Big).
\end{aligned}
$$

Finally, plugging (38) - (51) into (25a) - (25e) yields the system of inequalities in Theorem 1. These expressions might look different from those in Theorem 1. Thus, some manipulations are needed to show the equality. The rest of this section uses the identities

$$
\begin{aligned}
\min(A, B) &= A - (A - B)^+, \text{ and} && (52)\\
\max(A, B) &= A + (B - A)^+, && (53)
\end{aligned}
$$

for proving the equality of the regions determined by (8a)-(8e) and (25a)-(25e). This proof is divided into three parts.

Part I: The condition represented by (25a) is identical to the condition jointly represented by (8a) and (8b), with $i = 1$ and $j = 2$. To prove this, explicit expressions for a_2, $a_6 + b_1$ and

$a_4 + b_1 + b_3$ are needed. The term a_2 is given by (39) and $a_6 + b_1$ is as follows

$$
\begin{aligned}
a_6 + b_1 =& \min\Big(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big) - \min\Big((n_{21} - \overrightarrow{n}_{11})^+,\\
&(\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big) + (\overrightarrow{n}_{11} - n_{21})^+ + \Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+\\
=& n_{21} - \Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+ - \min\Big((n_{21} - \overrightarrow{n}_{11})^+,\\
&(\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big) + (\overrightarrow{n}_{11} - n_{21})^+ + \Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+\\
=& n_{21} - \min\Big((n_{21} - \overrightarrow{n}_{11})^+, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big) + (\overrightarrow{n}_{11} - n_{21})^+\\
=& \max(\overrightarrow{n}_{11}, n_{21}) - \min\Big((n_{21} - \overrightarrow{n}_{11})^+, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)\\
=& \begin{cases}
\max(\overrightarrow{n}_{11}, n_{21}) & \text{if } \overrightarrow{n}_{11} > n_{21};\\
\max(\overrightarrow{n}_{11}, n_{21}) & \text{if } \overrightarrow{n}_{11} < n_{21} \text{ and}\\
& \overleftarrow{n}_{22} > \max(\overrightarrow{n}_{22}, n_{21});\\
\max\Big(\overrightarrow{n}_{11}, \overleftarrow{n}_{22} - (\overrightarrow{n}_{22} - n_{21})^+\Big) & \text{if } \overrightarrow{n}_{11} < n_{21} \text{ and}\\
& \overleftarrow{n}_{22} < \max(\overrightarrow{n}_{22}, n_{21}).
\end{cases}
\end{aligned} \tag{54}
$$

Finally, the term $a_4 + b_1 + b_3$ is given as follows

$$
\begin{aligned}
a_4 + b_1 + b_3 =& (\overrightarrow{n}_{11} - n_{21})^+ + \Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+\\
&+ \min\Big(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)\\
=& (\overrightarrow{n}_{11} - n_{21})^+ + \Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+ + n_{21}\\
&- \Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+\\
=& (\overrightarrow{n}_{11} - n_{21})^+ + n_{21}\\
=& \max(\overrightarrow{n}_{11}, n_{21}).
\end{aligned} \tag{55}
$$

Plugging (39), (54) and (55) into (25a) yields

$$
R_1 \leqslant \min\Big(\max(\overrightarrow{n}_{11}, n_{21}), \max(\overrightarrow{n}_{11}, n_{12}), \max\Big(\overrightarrow{n}_{11}, \overleftarrow{n}_{22} - (\overrightarrow{n}_{22} - n_{21})^+\Big)\Big), \tag{56}
$$

which is the expression in Theorem 1.

The same procedure can be used prove that the condition (25b) is identical to the condition jointly determined by (8a) and (8b), with $i = 2$ and $j = 1$.

Part II: The condition represented by (25c) is identical to the condition jointly represented by (8c) and (8d). To prove this, explicit expressions for $(a_2 + b_4)$, $(a_4 + b_2)$, and $(a_1 + a_5 + b_1 + b_5)$ are needed. This is mainly because $\max(a_2 + b_4, a_4 + b_2, a_1 + a_5 + b_1 + b_5) \leqslant \min(a_2 + b_6, a_6 + b_2, a_1 + a_3 + a_4 + b_1 + b_5, a_1 + a_7 + b_1 + b_5, a_1 + a_4 + b_1 + b_7, a_1 + a_5 + b_1 + b_3 + b_4, a_1 + a_7 + b_1 + b_4)$.

The term $a_2 + b_4$ is

$$
a_2 + b_4 = \max(\overrightarrow{n}_{11}, n_{12}) + (\overrightarrow{n}_{22} - n_{12})^+, \tag{57}
$$

the term $a_4 + b_2$ is

$$
a_4 + b_2 = (\overrightarrow{n}_{11} - n_{21})^+ + \max(\overrightarrow{n}_{22}, n_{21}), \tag{58}
$$

and finally, the term $a_1 + a_5 + b_1 + b_5$ is

$$
\begin{aligned}
a_1{+}a_5 + b_1 + b_5 = &\left(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)^+ \\
&+ \max\left((\overrightarrow{n}_{11} - n_{21})^+, \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)\right) \\
&+ \left(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)^+ \\
&+ \max\left((\overrightarrow{n}_{22} - n_{12})^+, \min\left(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)\right) \\
=&\left(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)^+ + \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right) \\
&+ \left((\overrightarrow{n}_{11} - n_{21})^+ - \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)\right)^+ \\
&+ \left(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)^+ + \min\left(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right) \\
&+ \left((\overrightarrow{n}_{22} - n_{12})^+ - \min\left(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)\right)^+ \\
=&\left(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)^+ + n_{12} - \left(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)^+ \\
&+ \left((\overrightarrow{n}_{11} - n_{21})^+ - \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)\right)^+ \\
&+ \left(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)^+ + n_{21} - \left(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)^+ \\
&+ \left((\overrightarrow{n}_{22} - n_{12})^+ - \min\left(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)\right)^+ \\
=&n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)\right)^+ \\
&+n_{21} + \left((\overrightarrow{n}_{22} - n_{12})^+ - \min\left(n_{21}, (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right)\right)^+ \\
=&n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - \min(n_{12}, \max(\overrightarrow{n}_{11}, n_{12}) - \min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})))\right)^+ \\
&+n_{21} + \left((\overrightarrow{n}_{22} - n_{12})^+ - \min(n_{21}, \max(\overrightarrow{n}_{22}, n_{21}) - \min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})))\right)^+ \\
=&n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - n_{12} + (n_{12} - \max(\overrightarrow{n}_{11}, n_{12}) + \min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})))^+\right)^+ \\
&+n_{21} + \left((\overrightarrow{n}_{22} - n_{12})^+ - n_{21} + (n_{21} - \max(\overrightarrow{n}_{22}, n_{21}) + \min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})))^+\right)^+ \\
=&n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - n_{12} + \left(n_{12} - n_{12} - (\overrightarrow{n}_{11} - n_{12})^+ + \min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12}))\right)^+\right)^+ \\
&+n_{21} + \left((\overrightarrow{n}_{22} - n_{12})^+ - n_{21} + \left(n_{21} - n_{21} - (\overrightarrow{n}_{22} - n_{21})^+ + \min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21}))\right)^+\right)^+ \\
=&n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - n_{12} + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\right)^+ \\
&+n_{21} + \left((\overrightarrow{n}_{22} - n_{12})^+ - n_{21} + \left(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\right)^+\right)^+ \\
=&n_{12} + \left(\overrightarrow{n}_{11} - \min(\overrightarrow{n}_{11}, n_{21}) - n_{12} + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\right)^+ \\
&+n_{21} + \left(\overrightarrow{n}_{22} - \min(\overrightarrow{n}_{22}, n_{12}) - n_{21} + \left(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\right)^+\right)^+ \\
=&n_{12} + \left(\overrightarrow{n}_{11} + (n_{12} - \overrightarrow{n}_{11})^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) - n_{12}\right. \\
&\left.+ \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\right)^+ + n_{21} + \left(\overrightarrow{n}_{22} + (n_{21} - \overrightarrow{n}_{22})^+\right. \\
&\left.- (n_{21} - \overrightarrow{n}_{22})^+ - \min(\overrightarrow{n}_{22}, n_{12}) - n_{21} + \left(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\right)^+\right)^+
\end{aligned}
$$

$$
\begin{aligned}
&= n_{12} + \Big(\overrightarrow{n}_{11} - \min\left(\overrightarrow{n}_{11}, n_{12}\right) - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ \Big)^+ + n_{21} + \Big(\overrightarrow{n}_{22} - \min\left(\overrightarrow{n}_{22}, n_{21}\right) \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \Big)^+ \\
&= n_{12} + \Big(\overrightarrow{n}_{11} - \overrightarrow{n}_{11} + \left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ \Big)^+ + n_{21} + \Big(\overrightarrow{n}_{22} - \overrightarrow{n}_{22} + \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \Big)^+ \\
&= n_{12} + \Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ \Big)^+ + n_{21} + \Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \Big)^+ \\
&\overset{(d)}{=} n_{12} + \Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ \Big)^+ + n_{21} + \Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \Big)^+ \\
&\quad + \max\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21} \Big) + \max\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12} \Big) \\
&\quad + \Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21} \Big) \Big)^+ + \Big(\Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \min\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12} \Big) \Big)^+ \\
&\quad - \max\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21} \Big) - \max\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12} \Big) \\
&\quad - \Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21} \Big) \Big)^+ - \Big(\Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \min\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12} \Big) \Big)^+ \\
&= \Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ \Big)^+ + \Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \Big)^+ \\
&\quad + \max\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21} \Big) + \max\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12} \Big) \\
&\quad + \Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ \Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21} \Big) \Big)^+ + \Big(\Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ \Big)^+ \\
&\quad - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \min\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12} \Big) \Big)^+
\end{aligned}
$$

$$
\begin{aligned}
&-\left((\overrightarrow{n}_{11}-n_{12})^{+}-n_{21}\right)^{+}-\left((\overrightarrow{n}_{22}-n_{21})^{+}-n_{12}\right)\\
&-\Big(\left(\min\left(\overleftarrow{n}_{11},\max\left(\overrightarrow{n}_{11},n_{12}\right)\right)-(\overrightarrow{n}_{11}-n_{12})^{+}\right)^{+}-(n_{12}-\overrightarrow{n}_{11})^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)\\
&+\min\left((\overrightarrow{n}_{11}-n_{12})^{+},n_{21}\right)\Big)^{+}-\Big(\left(\min\left(\overleftarrow{n}_{22},\max\left(\overrightarrow{n}_{22},n_{21}\right)\right)-(\overrightarrow{n}_{22}-n_{21})^{+}\right)^{+}\\
&-(n_{21}-\overrightarrow{n}_{22})^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)+\min\left((\overrightarrow{n}_{22}-n_{21})^{+},n_{12}\right)\Big)^{+}\\
\overset{(e)}{=}&\max\left((\overrightarrow{n}_{11}-n_{12})^{+},n_{21}\right)+\max\left((\overrightarrow{n}_{22}-n_{21})^{+},n_{12}\right)\\
&+\Big(\left(\min\left(\overleftarrow{n}_{11},\max\left(\overrightarrow{n}_{11},n_{12}\right)\right)-(\overrightarrow{n}_{11}-n_{12})^{+}\right)^{+}-(n_{12}-\overrightarrow{n}_{11})^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)\\
&+\min\left((\overrightarrow{n}_{11}-n_{12})^{+},n_{21}\right)\Big)^{+}+\Big(\left(\min\left(\overleftarrow{n}_{22},\max\left(\overrightarrow{n}_{22},n_{21}\right)\right)-(\overrightarrow{n}_{22}-n_{21})^{+}\right)^{+}\\
&-(n_{21}-\overrightarrow{n}_{22})^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)+\min\left((\overrightarrow{n}_{22}-n_{21})^{+},n_{12}\right)\Big)^{+},
\end{aligned}
\tag{59}
$$

where,
(d) follows from adding and substracting the terms: $\Big(\left(\min\left(\overleftarrow{n}_{11},\max\left(\overrightarrow{n}_{11},n_{12}\right)\right)-(\overrightarrow{n}_{11}-n_{12})^{+}\right)^{+}-(n_{12}-\overrightarrow{n}_{11})^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)+\min\left((\overrightarrow{n}_{11}-n_{12})^{+},n_{21}\right)\Big)^{+}$, $\Big(\left(\min\left(\overleftarrow{n}_{22},\max\left(\overrightarrow{n}_{22},n_{21}\right)\right)-(\overrightarrow{n}_{22}-n_{21})^{+}\right)^{+}-(n_{21}-\overrightarrow{n}_{22})^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)+\min\left((\overrightarrow{n}_{22}-n_{21})^{+},n_{12}\right)\Big)^{+}$, $\max\left((\overrightarrow{n}_{11}-n_{12})^{+},n_{21}\right)$ and $\max\left((\overrightarrow{n}_{22}-n_{21})^{+},n_{12}\right)$;
(e) follows from the fact that $\Big((\overrightarrow{n}_{11}-n_{12})^{+}-(n_{12}-\overrightarrow{n}_{11})^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)+\left(\min\left(\overleftarrow{n}_{11},\max\left(\overrightarrow{n}_{11},n_{12}\right)\right)-(\overrightarrow{n}_{11}-n_{12})^{+}\right)^{+}\Big)^{+}+\Big((\overrightarrow{n}_{22}-n_{21})^{+}-(n_{21}-\overrightarrow{n}_{22})^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)+\left(\min\left(\overleftarrow{n}_{22},\max\left(\overrightarrow{n}_{22},n_{21}\right)\right)-(\overrightarrow{n}_{22}-n_{21})^{+}\right)^{+}\Big)^{+}-\left((\overrightarrow{n}_{11}-n_{12})^{+}-n_{21}\right)^{+}-\left((\overrightarrow{n}_{22}-n_{21})^{+}-n_{12}\right)-\Big(\left(\min\left(\overleftarrow{n}_{11},\max\left(\overrightarrow{n}_{11},n_{12}\right)\right)-(\overrightarrow{n}_{11}-n_{12})^{+}\right)^{+}-(n_{12}-\overrightarrow{n}_{11})^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)+\min\left((\overrightarrow{n}_{11}-n_{12})^{+},n_{21}\right)\Big)^{+}-\Big(\left(\min\left(\overleftarrow{n}_{22},\max\left(\overrightarrow{n}_{22},n_{21}\right)\right)-(\overrightarrow{n}_{22}-n_{21})^{+}\right)^{+}-(n_{21}-\overrightarrow{n}_{22})^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)+\min\left((\overrightarrow{n}_{22}-n_{21})^{+},n_{12}\right)\Big)^{+}=0.$

Plugging (57), (58) and (59) into (25c) yields (8c) and (8d) in Theorem 1.

Part III: The condition represented by (25e) is identical to the condition represented by (8e) with $i=2$ and $j=1$. In the following only the equality between $(a_1+a_5+b_2+b_4)$ and (8e) with $i=2$ and $j=1$ is proved as follows

$$
\begin{aligned}
a_1+a_5+b_2+b_4=&\left(n_{12}-\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)^{+}\\
&+\max\left((\overrightarrow{n}_{11}-n_{21})^{+},\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\right)+\max\left(\overrightarrow{n}_{22},n_{21}\right)+(\overrightarrow{n}_{22}-n_{12})^{+}\\
=&\left(n_{12}-\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)^{+}+\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\\
&+\left((\overrightarrow{n}_{11}-n_{21})^{+}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\right)^{+}+\max\left(\overrightarrow{n}_{22},n_{21}\right)+(\overrightarrow{n}_{22}-n_{12})^{+}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)^+ + n_{12} - \left(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)^+ \\
&\quad + \left((\overrightarrow{n}_{11} - n_{21})^+ - \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)\right)^+ + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - \min\left(n_{12}, (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right)\right)^+ + \max(\overrightarrow{n}_{22}, n_{21}) \\
&\quad + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - \min(n_{12}, \max(\overrightarrow{n}_{11}, n_{12}) - \min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})))\right)^+ \\
&\quad + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - n_{12} + (n_{12} - \max(\overrightarrow{n}_{11}, n_{12}) + \min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})))^+\right)^+ \\
&\quad + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - n_{12} + \left(n_{12} - n_{12} - (\overrightarrow{n}_{11} - n_{12})^+ + \min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12}))\right)^+\right)^+ \\
&\quad + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \left((\overrightarrow{n}_{11} - n_{21})^+ - n_{12} + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\right)^+ \\
&\quad + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \left(\overrightarrow{n}_{11} - \min(\overrightarrow{n}_{11}, n_{21}) - n_{12} + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\right)^+ \\
&\quad + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \Big(\overrightarrow{n}_{11} + (n_{12} - \overrightarrow{n}_{11})^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) - n_{12} \\
&\quad + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\Big)^+ + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \Big(\overrightarrow{n}_{11} - \min(\overrightarrow{n}_{11}, n_{12}) - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) \\
&\quad + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\Big)^+ + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \Big(\overrightarrow{n}_{11} - \overrightarrow{n}_{11} + (\overrightarrow{n}_{11} - n_{12})^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) \\
&\quad + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\Big)^+ + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&= n_{12} + \Big(\, (\overrightarrow{n}_{11} - n_{12})^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) \\
&\quad + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\Big)^+ + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&\overset{(f)}{=} n_{12} + \Big(\, (\overrightarrow{n}_{11} - n_{12})^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) \\
&\quad + \left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+\Big)^+ + \max(\overrightarrow{n}_{22}, n_{21}) + (\overrightarrow{n}_{22} - n_{12})^+ \\
&\quad + \max(\overrightarrow{n}_{11}, n_{12}) - \min\left((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\right) \\
&\quad + \Big(\left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) \\
&\quad + \min\left((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\right)\Big)^+ - \max(\overrightarrow{n}_{11}, n_{12}) + \min\left((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\right) \\
&\quad - \Big(\left(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\right)^+ - (n_{12} - \overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21}) \\
&\quad + \min\left((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\right)\Big)^+
\end{aligned}
$$

$$
\begin{aligned}
=&\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ \Big)^+ + \max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+ \\
&+ \max\left(\overrightarrow{n}_{11}, n_{12}\right) - \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \\
&+\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+ - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ + \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \\
&-\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+ \\
=&\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ \Big)^+ + \max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+ \\
&+ \max\left(\overrightarrow{n}_{11}, n_{12}\right) - \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \\
&+\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+ - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ + \left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - n_{21}\Big)^+ \\
&-\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+ \\
=&\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ \Big)^+ + \max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+ \\
&+ \max\left(\overrightarrow{n}_{11}, n_{12}\right) - \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \\
&+\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+ - \Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - n_{21}\Big)^+ \\
&-\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+ \\
=&\max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+ + \max\left(\overrightarrow{n}_{11}, n_{12}\right) - \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \\
&+\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
&+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) \Big)^+.
\end{aligned}
\tag{60}
$$

where
(f) follows from adding and substracting the terms: $\max\left(\overrightarrow{n}_{11}, n_{12}\right)$, $\min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big)$ and $\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) +$

$\min\left(\left(\overrightarrow{n}_{11}-n_{12}\right)^+, n_{21}\right)\Big)^+$;

(g) follows from the fact that $\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^+ - \left(n_{12}-\overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) + \Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11}-n_{12}\right)^+\Big)^+\Big)^+ - \Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^+ - n_{21}\Big)^+ - \Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11}-n_{12}\right)^+\Big)^+ - \left(n_{12}-\overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) + \min\left(\left(\overrightarrow{n}_{11}-n_{12}\right)^+, n_{21}\right)\Big)^+ = 0$.

The expression in (60) is equal to the expression (8e) with $i = 2$ and $j = 1$, and this completes the proof.

A similar procedure can be applied for the other two bounds in (25e).

By symmetry, a similar procedure can be used for bound on $2R_1 + R_2$.

B A New Outer Bound Region

This appendix provides a proof for the following Lemma.

Lemma 1 (Converse) The capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the two-user linear deterministic interference channel with noisy channel-output feedback is included in the set of non-negative rate pairs (R_1, R_2) that satisfy the following conditions for all $i \in \{1,2\}$ and for all $j \in \{1,2\} \setminus \{i\}$:

$$R_i \leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right), \quad (61)$$

$$R_i \leqslant \min\Big(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\Big(\overrightarrow{n}_{ii}, \overleftarrow{n}_{jj} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\Big)\Big), \quad (62)$$

$$R_1 + R_2 \leqslant \min\Big(\max\left(\overrightarrow{n}_{11}, n_{12}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{11} - n_{21}\right)^+\Big), \quad (63)$$

$$\begin{aligned} R_1 + R_2 \leqslant{}& \max\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big) + \max\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\Big) \\ &+\Big(\Big(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\Big)^+ - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\ &+ \min\Big(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\Big)\Big)^+ + \Big(\Big(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+\Big)^+ \\ &- \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \min\Big(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\Big)\Big)^+, \end{aligned} \quad (64)$$

$$\begin{aligned} 2R_i + R_j \leqslant{}& \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) + \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+ - \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\Big) \\ &+\Big(\Big(\min\left(\overleftarrow{n}_{jj}, \max\left(\overrightarrow{n}_{jj}, n_{ji}\right)\right) - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\Big)^+ - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^+ - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) \\ &+ \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\Big)\Big)^+. \end{aligned} \quad (65)$$

Proof of (61) **and** (63)**:** (61) and (63) correspond to the minimum cut-set bound [16] and the sum-rate bound from the case of the two-user interference channel with perfect channel-output feedback [4]. ■

The rest of the proof of Lemma 1 is presented using particular notation. For all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, the channel input $\boldsymbol{X}_i^{(n)}$ of the LD-IC-NOF in (2) for any channel use $n \in \{1, \ldots, N\}$ is a q-dimensional vector, with q in (1), that can be written as the concatenation of four vectors: $\boldsymbol{X}_{i,C}^{(n)}$, $\boldsymbol{X}_{i,P}^{(n)}$, $\boldsymbol{X}_{i,D}^{(n)}$ and $\boldsymbol{X}_{i,Q}^{(n)}$, i.e., $\boldsymbol{X}_i^{(n)} = \Big(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,P}^{(n)}, \boldsymbol{X}_{i,D}^{(n)}, \boldsymbol{X}_{i,Q}^{(n)}\Big)$, as shown in Fig. 20. Note that this notation is independent of the feedback parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$ and it holds for all $n \in \{1, \ldots, N\}$. More specifically,

- $\boldsymbol{X}_{i,C}^{(n)}$ contains the signal levels at transmitter i that are observed at both receivers and thus,

$$\dim \boldsymbol{X}_{i,C}^{(n)} = \min\left(\overrightarrow{n}_{ii}, n_{ji}\right); \quad (66)$$

- $\boldsymbol{X}_{i,P}^{(n)}$ contains the signal levels at transmitter i that are observed only at receiver i and thus,

$$\dim \boldsymbol{X}_{i,P}^{(n)} = \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+; \quad (67)$$

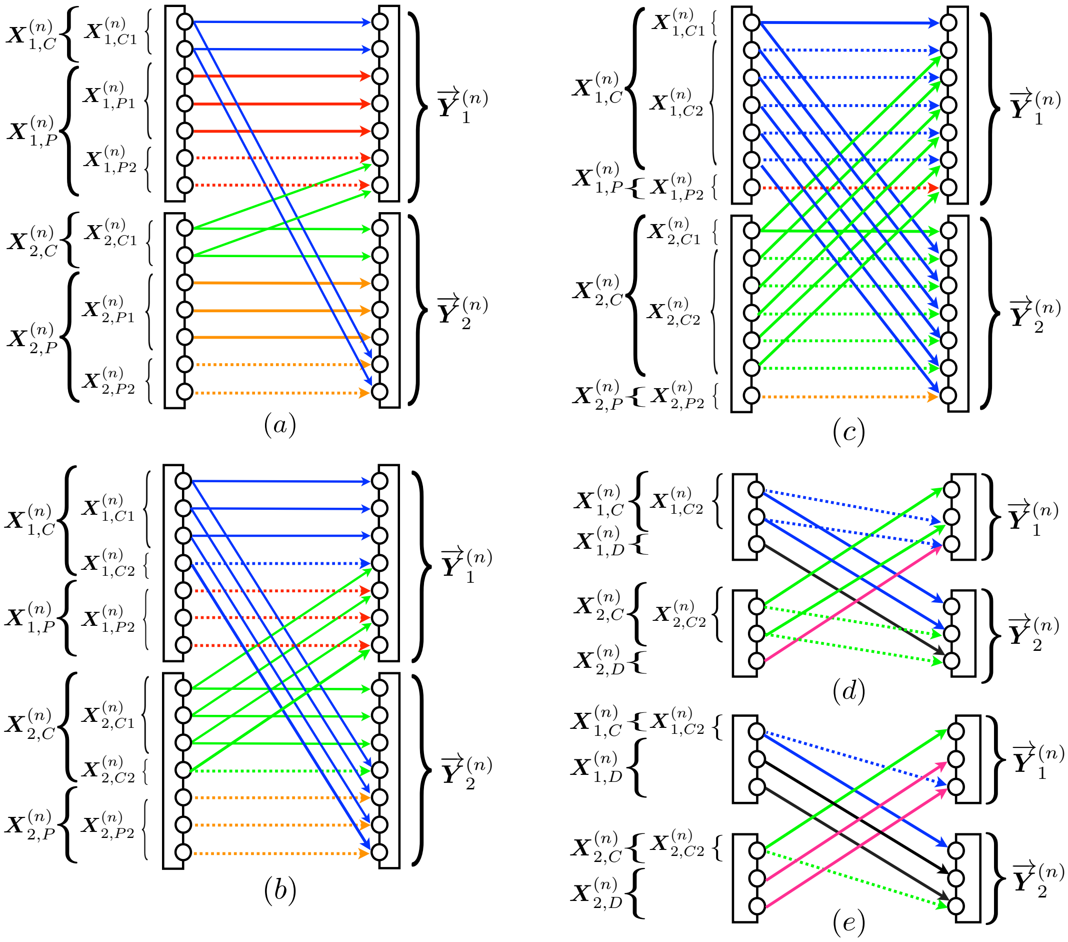

Figure 20: Examples of the channel inputs written in terms of $\boldsymbol{X}_{i,C}^{(n)}$, $\boldsymbol{X}_{i,P}^{(n)}$ and $\boldsymbol{X}_{i,D}^{(n)}$ in (a) very weak interference regime, (b) weak interference regime, (c) moderate interference regime, (d) strong interference regime and (e) very strong interference regime

- $\boldsymbol{X}_{i,D}^{(n)}$ contains the signal levels at transmitter i that are observed only at receiver j and thus,

$$\dim \boldsymbol{X}_{i,D}^{(n)} = (n_{ji} - \overrightarrow{n}_{ii})^+; \text{ and} \tag{68}$$

- $\boldsymbol{X}_{i,Q}^{(n)} = (0, \ldots, 0)$ is included for dimensional matching of the model in (3). Then,

$$\dim \boldsymbol{X}_{i,Q}^{(n)} = q - \max\left(\overrightarrow{n}_{ii}, n_{ji}\right). \tag{69}$$

These levels $\boldsymbol{X}_{i,Q}^{(n)}$ are not used for signal transmission by transmitter i and thus,

$$\begin{aligned} H\left(\boldsymbol{X}_i^{(n)}\right) =& H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,P}^{(n)}, \boldsymbol{X}_{i,D}^{(n)}, \boldsymbol{X}_{i,Q}^{(n)}\right) \\ =& H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,P}^{(n)}, \boldsymbol{X}_{i,D}^{(n)}\right) \\ \leqslant& \dim \boldsymbol{X}_{i,C}^{(n)} + \dim \boldsymbol{X}_{i,P}^{(n)} + \dim \boldsymbol{X}_{i,D}^{(n)}. \end{aligned} \tag{70}$$

Note that vectors $\boldsymbol{X}_{i,P}^{(n)}$ and $\boldsymbol{X}_{i,D}^{(n)}$ do not exist simultaneously. The former exists when $\overrightarrow{n}_{ii} > n_{ji}$, while the latter exists when $\overrightarrow{n}_{ii} < n_{ji}$. Moreover, the dimension of $\boldsymbol{X}_{i}^{(n)}$ satisfies

$$\begin{aligned}
\dim \boldsymbol{X}_{i}^{(n)} =& \dim \boldsymbol{X}_{i,C}^{(n)} + \dim \boldsymbol{X}_{i,P}^{(n)} + \dim \boldsymbol{X}_{i,D}^{(n)} + \dim \boldsymbol{X}_{i,Q}^{(n)} \\
=& \min\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+} + \left(n_{ji} - \overrightarrow{n}_{ii}\right)^{+} + q - \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) \\
=& \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + q - \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) \\
=& q. \qquad (71)
\end{aligned}$$

Vector $\boldsymbol{X}_{i,C}^{(n)}$ can be written as the concatenation of two vectors: $\boldsymbol{X}_{i,C1}^{(n)}$ and $\boldsymbol{X}_{i,C2}^{(n)}$, i.e., $\boldsymbol{X}_{i,C}^{(n)} = \left(\boldsymbol{X}_{i,C1}^{(n)}, \boldsymbol{X}_{i,C2}^{(n)}\right)$. Vector $\boldsymbol{X}_{i,C1}^{(n)}$ (resp. $\boldsymbol{X}_{i,C2}^{(n)}$) contains the levels of $\boldsymbol{X}_{i,C}^{(n)}$ that are seen at receiver i without any interference (resp. with interference of transmitter j). Hence,

$$\dim \boldsymbol{X}_{i,C1}^{(n)} = \min\left(\left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+}, n_{ji}\right) \text{ and} \qquad (72)$$

$$\dim \boldsymbol{X}_{i,C2}^{(n)} = \min\left(\overrightarrow{n}_{ii}, n_{ji}\right) - \min\left(\left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+}, n_{ji}\right). \qquad (73)$$

Vector $\boldsymbol{X}_{i,P}^{(n)}$ can also be written as the concatenation of two vectors: $\boldsymbol{X}_{i,P1}^{(n)}$ and $\boldsymbol{X}_{i,P2}^{(n)}$, i.e., $\boldsymbol{X}_{i,P}^{(n)} = \left(\boldsymbol{X}_{i,P1}^{(n)}, \boldsymbol{X}_{i,P2}^{(n)}\right)$. Vector $\boldsymbol{X}_{i,P1}^{(n)}$ (resp. $\boldsymbol{X}_{i,P2}^{(n)}$) contains the levels of $\boldsymbol{X}_{i,P}^{(n)}$ that are seen at receiver i without any interference (resp. with interference of transmitter j). Hence,

$$\dim \boldsymbol{X}_{i,P1}^{(n)} = \left(\left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+} - n_{ij}\right)^{+} \text{ and} \qquad (74)$$

$$\dim \boldsymbol{X}_{i,P2}^{(n)} = \min\left(\left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+}, n_{ij}\right). \qquad (75)$$

When feedback is taken into account, an alternative notation is needed. Let $\boldsymbol{X}_{i,C}^{(n)}$ be written in terms of $\boldsymbol{X}_{i,CF_j}^{(n)}$, $\boldsymbol{X}_{i,CG_j}^{(n)}$, with $j \in \{1,2\}$, i.e., $\boldsymbol{X}_{i,C}^{(n)} = \left(\boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,CG_j}^{(n)}\right)$. The vector $\boldsymbol{X}_{i,CF_j}^{(n)}$ represents the signal levels of $\boldsymbol{X}_{i,C}^{(n)}$ that can be fed back from receiver j to transmitter j; and $\boldsymbol{X}_{i,CG_j}^{(n)}$ represents the signal levels of $\boldsymbol{X}_{i,C}^{(n)}$ that can not be fed back from receiver j to transmitter j, as shown in Fig. 22. Let $\boldsymbol{X}_{i,D}^{(n)}$ be written in terms of $\boldsymbol{X}_{i,DF}^{(n)}$ and $\boldsymbol{X}_{i,DG}^{(n)}$, i.e., $\boldsymbol{X}_{i,D}^{(n)} = \left(\boldsymbol{X}_{i,DF}^{(n)}, \boldsymbol{X}_{i,DG}^{(n)}\right)$. The vector $\boldsymbol{X}_{i,DF}^{(n)}$ represents the signal levels of $\boldsymbol{X}_{i,D}^{(n)}$ that can be fed back from receiver j to transmitter j, with $j \in \{1,2\} \setminus \{i\}$. The dimension of vector $\boldsymbol{X}_{i,DF}^{(n)}$ is function of the dimensions of vectors $\boldsymbol{X}_{i,C}^{(n)}$, $\boldsymbol{X}_{i,D}^{(n)}$, $\boldsymbol{X}_{j,C1}^{(n)}$, and $\boldsymbol{X}_{j,P1}^{(n)}$, which were defined in (66), (69), (72) and (74) respectively. The vector $\boldsymbol{X}_{i,DG}^{(n)}$ represents the signal levels that are in $\boldsymbol{X}_{i,D}^{(n)}$ but not $\boldsymbol{X}_{i,DF}^{(n)}$, as shown in Fig. 22. The dimension of vectors $\boldsymbol{X}_{i,DF}^{(n)}$ and $\boldsymbol{X}_{i,DG}^{(n)}$ are defined as follows:

$$\begin{aligned}
\dim \boldsymbol{X}_{i,DF}^{(n)} =& \min\left(\dim \boldsymbol{X}_{i,D}^{(n)}, \left(\overleftarrow{n}_{jj} - \dim \boldsymbol{X}_{i,C}^{(n)} - \dim \boldsymbol{X}_{j,C1}^{(n)} - \dim \boldsymbol{X}_{j,P1}^{(n)}\right)^{+}\right) \\
=& \min\Big(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^{+}, \Big(\overleftarrow{n}_{jj} - \min\left(\overrightarrow{n}_{ii}, n_{ji}\right) - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right) \\
& - \left(\left(\overrightarrow{n}_{jj} - n_{ij}\right)^{+} - n_{ji}\right)^{+}\Big)^{+}\Big) \\
\stackrel{(a)}{=}& \min\Big(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^{+}, \Big(\overleftarrow{n}_{jj} - \overrightarrow{n}_{ii} - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right) \\
& - \left(\left(\overrightarrow{n}_{jj} - n_{ij}\right)^{+} - n_{ji}\right)^{+}\Big)^{+}\Big) \text{ and} \qquad (76) \\
\dim \boldsymbol{X}_{i,DG}^{(n)} =& \dim \boldsymbol{X}_{i,D}^{(n)} - \dim \boldsymbol{X}_{i,DF}^{(n)}, \qquad (77)
\end{aligned}$$

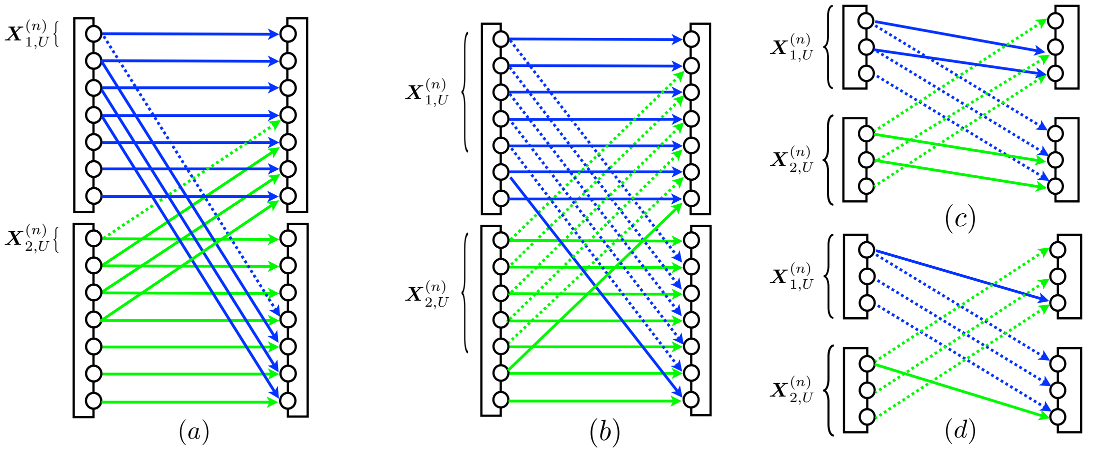

Figure 21: Vector $\boldsymbol{X}_{i,U}^{(n)}$ in (a) weak interference regime, (b) moderate interference regime, (c) strong interference regime and (d) very strong interference regime. Vector $\boldsymbol{X}_{i,U}^{(n)}$ does not exist in the very weak interference regime.

where (a) follows from the fact that vector $\boldsymbol{X}_{i,D}^{(n)}$ only exists if $n_{ji} > \overrightarrow{n}_{ii}$. Note that it is not necessary to include a subindex related to the receiver that implements the feedback, this is mainly because the signal levels $\boldsymbol{X}_{i,D}^{(n)}$ are seen only at receiver j.
More generally, when needed, the vector $\boldsymbol{X}_{iF_k}^{(n)}$ is used to represent the signal levels of $\boldsymbol{X}_{i}^{(n)}$ that can be fed back from receiver k to transmitter k, with $k \in \{1,2\}$. The vector $\boldsymbol{X}_{iG_k}^{(n)}$ is used to represent the signal levels of $\boldsymbol{X}_{i}^{(n)}$ that can not be fed back from receiver k to transmitter k.
In the proofs of (64) and (65), the vector $\boldsymbol{X}_{i,U}^{(n)}$ is used to represent the signal levels of vector $\boldsymbol{X}_{i}^{(n)}$ that interfere with signal levels of $\boldsymbol{X}_{j,C}^{(n)}$ at receiver j and those signal levels of $\boldsymbol{X}_{i}^{(n)}$ that are seen at receiver j and are not used by transmitter j. An example is shown in Fig. 21. Vector $\boldsymbol{X}_{i,U}^{(n)}$ was used for the first time in the proof of the converse in the symmetric case in [12].

Based on its definition, the dimension of vector $\boldsymbol{X}_{i,U}^{(n)}$ is

$$\dim \boldsymbol{X}_{i,U}^{(n)}=\min\left(\overrightarrow{n}_{jj}, n_{ij}\right) - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right) + \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+}. \tag{78}$$

Finally, for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, the channel output $\overrightarrow{\boldsymbol{Y}}_{i}^{(n)}$ of the LD-IC-NOF in (2) for any channel use $n \in \{1,\ldots,N\}$ is a q-dimensional vector, with q in (1), that can be written as the concatenation of three vectors: $\overleftarrow{\boldsymbol{Y}}_{i}^{(n)}$, $\overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}$ and $\overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)}$, i.e., $\overrightarrow{\boldsymbol{Y}}_{i}^{(n)} = \left(\overleftarrow{\boldsymbol{Y}}_{i}^{(n)}, \overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}, \overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)}\right)$, as shown in Fig. 22. More specifically,

- $\overleftarrow{\boldsymbol{Y}}_{i}^{(n)}$ contains the signal levels at receiver i that are fed back to transmitter i and thus,

$$\dim \overleftarrow{\boldsymbol{Y}}_{i}^{(n)}=\min\left(\overleftarrow{n}_{ii}, \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right); \tag{79}$$

- $\overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}$ contains the signal levels at receiver i that are not fed back to transmitter i and thus,

$$\dim \overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}=\left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^{+}; \tag{80}$$

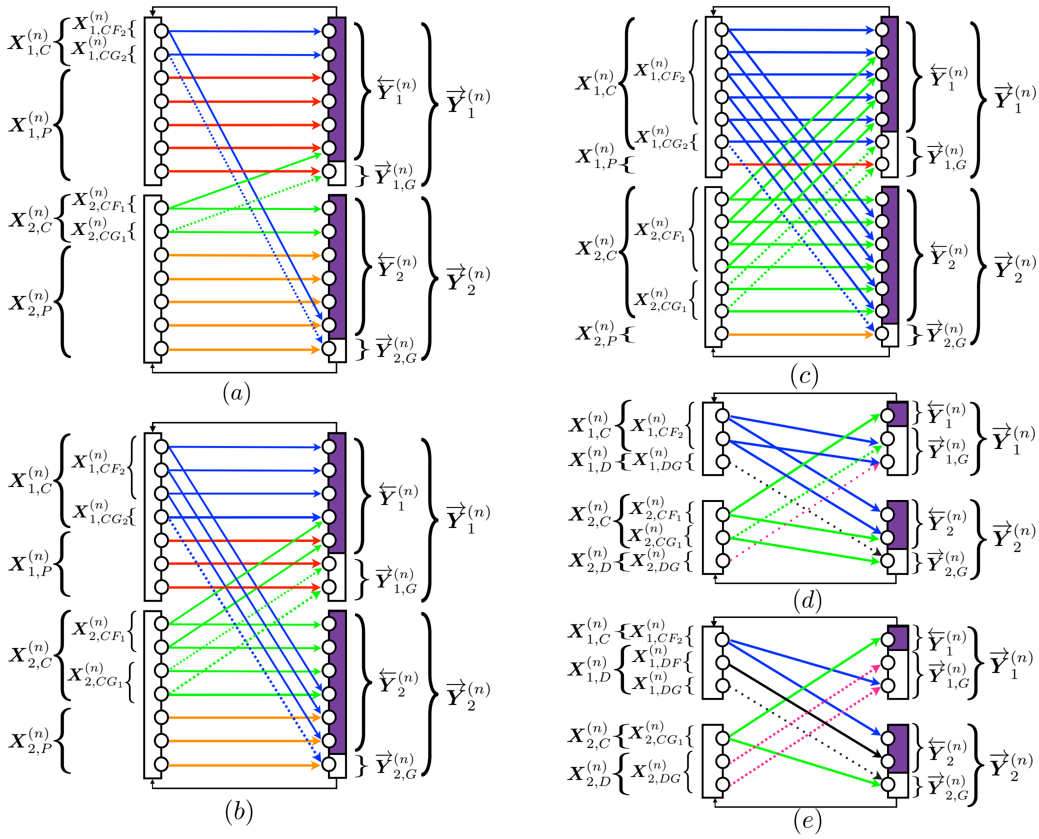

Figure 22: Different components of codewords when channel-output feedback is considered in (a) very weak interference regime, (b) weak interference regime, (c) moderate interference regime, (d) strong interference regime and (e) very strong interference regime.

- $\overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)}$ is included for dimensional matching of the model in (3). Then,

$$\dim \overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)} = q - \max\left(\overrightarrow{n}_{ii}, n_{ij}\right); \tag{81}$$

These levels $\overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)}$ do not represent an output of the channel and thus,

$$\begin{aligned} H\left(\overrightarrow{\boldsymbol{Y}}_{i}^{(n)}\right) &= H\left(\overleftarrow{\boldsymbol{Y}}_{i}^{(n)}, \overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}, \overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)}\right) \\ &= H\left(\overleftarrow{\boldsymbol{Y}}_{i}^{(n)}, \overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}\right) \\ &\leqslant \dim \overleftarrow{\boldsymbol{Y}}_{i}^{(n)} + \dim \overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}. \end{aligned} \tag{82}$$

The dimension of $\overrightarrow{\boldsymbol{Y}}_{i}^{(n)}$ satisfies

$$
\begin{aligned}
\dim \overrightarrow{\boldsymbol{Y}}_i^{(n)} =& \dim \overleftarrow{\boldsymbol{Y}}_i^{(n)} + \overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)} + \overrightarrow{\boldsymbol{Y}}_{i,Q}^{(n)} \\
=& \min\left(\overleftarrow{n}_{ii}, \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right) + \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^+ + q - \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) \\
=& \min\left(\overleftarrow{n}_{ii}, \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right) + \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \min\left(\overleftarrow{n}_{ii}, \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right) + q \\
& - \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) \\
=& q.
\end{aligned} \tag{83}
$$

In the proofs of (64) and (65), the vector $\left(\boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)}\right)$, $i \in \{1,2\}$ is used to represent the signal levels of vector $\boldsymbol{X}_i^{(n)}$ that are seen at the receiver j and are included into the feedback from the receiver j to transmitter j.
The dimension of vector $\left(\boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)}\right)$ is

$$
\dim\left(\boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)}\right) = \left(\min\left(\overleftarrow{n}_{jj}, \max\left(\overrightarrow{n}_{jj}, n_{ji}\right)\right) - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\right)^+. \tag{84}
$$

Proof of (62)**:** first, consider $n_{ji} \leqslant \overrightarrow{n}_{ii}$, i.e., vector $\boldsymbol{X}_{i,P}^{(n)}$ exists and vector $\boldsymbol{X}_{i,D}^{(n)}$ does not exist. Then, the following holds

$$
\begin{aligned}
NR_i =& H\left(W_i\right) \\
\stackrel{(a)}{=}& H\left(W_i | W_j\right) \\
=& I\left(W_i; \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(W_i | W_j, \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
\stackrel{(b)}{=}& I\left(W_i; \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(W_i | W_j, \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_j^{(1:N)}\right) \\
\stackrel{(c)}{=}& I\left(W_i; \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(W_i | W_j, \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_j^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) \\
\stackrel{(d)}{\leqslant}& I\left(W_i; \boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + N\delta(N) \\
=& H\left(\boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) - H\left(\boldsymbol{X}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_i, W_j\right) + N\delta(N) \\
=& \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_i^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_i^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) - H\left(\boldsymbol{X}_i^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_i, W_j, \boldsymbol{X}_i^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right)\right] \\
& + N\delta(N) \\
\stackrel{(e)}{=}& \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_i^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_i^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right)\right. \\
& \left. - H\left(\boldsymbol{X}_i^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_i, W_j, \boldsymbol{X}_i^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right)\right] + N\delta(N) \\
\stackrel{(f)}{\leqslant}& \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_i^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) - H\left(\boldsymbol{X}_i^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_i, W_j, \boldsymbol{X}_i^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right)\right] \\
& + N\delta(N) \\
\stackrel{(g)}{=}& \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_i^{(n)}, \boldsymbol{X}_{i,CF_j}^{(n)} | \boldsymbol{X}_j^{(1:n)}\right)\right. \\
& \left. - H\left(\boldsymbol{X}_i^{(n)}, \boldsymbol{X}_{i,CF_j}^{(n)} | W_i, W_j, \boldsymbol{X}_i^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right)\right] + N\delta(N)
\end{aligned}
$$

$$\begin{aligned}
&\stackrel{(h)}{=}\sum_{n=1}^{N}\Big[H\left(\boldsymbol{X}_i^{(n)}|\boldsymbol{X}_j^{(1:n)}\right)\\
&\quad -H\left(\boldsymbol{X}_i^{(n)}|W_i,W_j,\boldsymbol{X}_i^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)},\boldsymbol{X}_j^{(1:n)}\right)\Big]+N\delta(N)\\
&\stackrel{(i)}{=}\sum_{n=1}^{N}\Big[H\left(\boldsymbol{X}_i^{(n)}|\boldsymbol{X}_j^{(1:n)}\right)\\
&\quad -H\left(\boldsymbol{X}_i^{(n)}|W_i,W_j,\boldsymbol{X}_i^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)},\boldsymbol{X}_j^{(1:n)},\overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}\right)\Big]+N\delta(N)\\
&\stackrel{(j)}{=}\sum_{n=1}^{N}\Big[H\left(\boldsymbol{X}_i^{(n)}|\boldsymbol{X}_j^{(1:n)}\right)\\
&\quad -H\left(\boldsymbol{X}_i^{(n)}|W_i,W_j,\boldsymbol{X}_i^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)},\boldsymbol{X}_j^{(1:n)},\overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)},\boldsymbol{X}_i^{(1:n)}\right)\Big]+N\delta(N)\\
&=\sum_{n=1}^{N}H\left(\boldsymbol{X}_i^{(n)}|\boldsymbol{X}_j^{(1:n)}\right)+N\delta(N)\\
&=NH\left(\boldsymbol{X}_i^{(n)}|\boldsymbol{X}_j^{(n)}\right)+N\delta(N)\text{ for any } n\in\{1,\ldots,N\}\\
&\stackrel{(k)}{\leqslant}NH\left(\boldsymbol{X}_i^{(n)}\right)+N\delta(N),
\end{aligned}\tag{85}$$

where,
(a) follows from the fact that W_i and W_j are independent, $i\in\{1,2\}$ and $j\in\{1,2\}\backslash\{i\}$;
(b) follows from the fact that $\boldsymbol{X}_j^{(1:N)}=f\left(W_j,\overleftarrow{\boldsymbol{Y}}_j^{(1:N-1)}\right)$;
(c) follows from the fact that $\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}=f\left(\boldsymbol{X}_i^{(1:N)},\boldsymbol{X}_j^{(1:N)}\right)$;
(d) follows from Fano's inequality;
(e) follows from the fact that $\boldsymbol{X}_j^{(1:n)}=f\left(W_j,\overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right)$;
(f) follows from the fact that conditioning reduces the entropy;
(g) follows from the fact that $\overleftarrow{\boldsymbol{Y}}_j^{(n)}=f\left(\boldsymbol{X}_{i,CF_j}^{(n)},\boldsymbol{X}_{j,F_j}^{(n)}\right)$ where $\boldsymbol{X}_{j,F_j}^{(n)}$ is contained into $\boldsymbol{X}_j^{(n)}$;
(h) follows from the fact that $\boldsymbol{X}_{i,CF_j}^{(n)}$ and $\boldsymbol{X}_{i,DF}^{(n)}$ are contained into $\boldsymbol{X}_i^{(n)}$;
(i) follows from the fact that $\overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}=f\left(\boldsymbol{X}_i^{(1:n-1)},\boldsymbol{X}_j^{(1:n-1)}\right)$;
(j) follows from the fact that that $\boldsymbol{X}_i^{(1:n)}=f\left(W_i,\overleftarrow{\boldsymbol{Y}}_i^{(1:n-1)}\right)$ and $\overleftarrow{\boldsymbol{Y}}_i^{(1:n-1)}$ is included into $\overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}$;
(k) follows from the fact that conditioning reduces the entropy.

From (85), in the asymptotic regime, it holds that

$$\begin{aligned}
R_i\leqslant&H\left(\boldsymbol{X}_i^{(n)}\right)\\
=&H\left(\boldsymbol{X}_{i,C}^{(n)}\right)+H\left(\boldsymbol{X}_{i,P}^{(n)}\right)\\
\leqslant&\dim\boldsymbol{X}_{i,C}^{(n)}+\dim\boldsymbol{X}_{i,P}^{(n)}.
\end{aligned}\tag{86}$$

A tighter bound can be obtained for the case in which $n_{ji}>\overrightarrow{n}_{ii}$. In this case the vector $\boldsymbol{X}_{i,P}^{(n)}$ does not exist and the vector $\boldsymbol{X}_{i,D}^{(n)}$ exists. Hence, the following holds

$$
\begin{aligned}
NR_i =\, & H\left(W_i\right) \\
\overset{(a)}{=}\, & H\left(W_i | W_j\right) \\
=\, & I\left(W_i; \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(W_i | W_j, \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
\overset{(b)}{=}\, & I\left(W_i; \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(W_i | W_j, \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_j^{(1:N)}\right) \\
\overset{(c)}{=}\, & I\left(W_i; \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(W_i | W_j, \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_j^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) \\
\overset{(d)}{\leqslant}\, & I\left(W_i; \boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + N\delta(N) \\
=\, & H\left(\boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) - H\left(\boldsymbol{X}_{i,C}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_i, W_j\right) + N\delta(N) \\
=\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) - H\left(\boldsymbol{X}_{i,C}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) \right] \\
& + N\delta(N) \\
\overset{(e)}{=}\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \right. \\
& \left. - H\left(\boldsymbol{X}_{i,C}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \right] + N\delta(N) \\
\overset{(f)}{\leqslant}\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) - H\left(\boldsymbol{X}_{i,C}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \right] \\
& + N\delta(N) \\
\overset{(g)}{=}\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) \right. \\
& \left. - H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \right] + N\delta(N) \\
\overset{(h)}{=}\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) \right. \\
& \left. - H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \right] + N\delta(N) \\
\overset{(i)}{=}\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) \right. \\
& \left. - H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}\right) \right] + N\delta(N) \\
\overset{(j)}{=}\, & \sum_{n=1}^{N} \left[H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) \right. \\
& \left. - H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | W_i, W_j, \boldsymbol{X}_{i,C}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}, \boldsymbol{X}_i^{(1:n)}\right) \right] + N\delta(N) \\
=\, & \sum_{n=1}^{N} H\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | \boldsymbol{X}_j^{(1:n)}\right) + N\delta(N) \\
=\, & NH\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)} | \boldsymbol{X}_j^{(n)}\right) + N\delta(N), \text{ for any } n \in \{1, \ldots, N\}, \\
\overset{(k)}{\leqslant}\, & NH\left(\boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_{i,DF}^{(n)}\right) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&= NH\left(\boldsymbol{X}_{i,C}^{(n)}\right) + NH\left(\boldsymbol{X}_{i,DF}^{(n)}|\boldsymbol{X}_{i,C}^{(n)}\right) + N\delta(N)\\
&\leqslant NH\left(\boldsymbol{X}_{i,C}^{(n)}\right) + NH\left(\boldsymbol{X}_{i,DF}^{(n)}\right) + N\delta(N),
\end{aligned} \tag{87}
$$

where,
(a) follows from the fact that W_i and W_j are independent, $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$;
(b) follows from the fact that $\boldsymbol{X}_j^{(1:N)} = f\left(W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N-1)}\right)$;
(c) follows from the fact that $\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} = f\left(\boldsymbol{X}_{i,C}^{(1:N)}, \boldsymbol{X}_j^{(1:N)}\right)$ when $n_{ji} > \overrightarrow{n}_{ii}$;
(d) follows from Fano's inequality;
(e) follows from the fact that $\boldsymbol{X}_j^{(1:n)} = f\left(W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right)$;
(f) follows from the fact that conditioning reduces the entropy;
(g) follows from the fact that $\overleftarrow{\boldsymbol{Y}}_j^{(n)} = f\left(\boldsymbol{X}_{i,CF_j}^{(n)}, \boldsymbol{X}_{i,DF_j}^{(n)}, \boldsymbol{X}_{j,F_j}^{(n)}\right)$ where $\boldsymbol{X}_{j,F_j}^{(n)}$ is contained into $\boldsymbol{X}_j^{(n)}$;
(h) follows from the fact that $\boldsymbol{X}_{i,CF_j}^{(n)}$ is contained into $\boldsymbol{X}_{i,C}^{(n)}$;
(i) follows from the fact that $\overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)} = f\left(\boldsymbol{X}_{i,C}^{(1:n-1)}, \boldsymbol{X}_j^{(1:n-1)}\right)$ when $n_{ji} > \overrightarrow{n}_{ii}$;
(j) follows from the fact that that $\boldsymbol{X}_i^{(1:n)} = f\left(W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:n-1)}\right)$ and $\overleftarrow{\boldsymbol{Y}}_i^{(1:n-1)}$ is included into $\overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}$;
(k) follows from the fact that conditioning reduces the entropy.

From (87), , in the asymptotic regime, it holds that:

$$
\begin{aligned}
R_i &\leqslant H\left(\boldsymbol{X}_{i,C}^{(n)}\right) + H\left(\boldsymbol{X}_{i,DF}^{(n)}\right)\\
&\leqslant \dim \boldsymbol{X}_{i,C}^{(n)} + \dim \boldsymbol{X}_{i,DF}^{(n)}.
\end{aligned} \tag{88}
$$

Inequality (86) is valid when $n_{ji} \leqslant \overrightarrow{n}_{ii}$. Inequality (88) is valid when $n_{ji} > \overrightarrow{n}_{ii}$. It is worth noting that in the case in which $n_{ji} \leqslant \overrightarrow{n}_{ii}$, $H\left(\boldsymbol{X}_{i,D}^{(n)}\right) = 0$; and in the other case, i.e., $n_{ji} > \overrightarrow{n}_{ii}$, $H\left(\boldsymbol{X}_{i,P}^{(n)}\right) = 0$. Then, (86) and (88) can be expressed as one inequality as follows

$$
\begin{aligned}
R_i &\leqslant H\left(\boldsymbol{X}_{i,C}^{(n)}\right) + H\left(\boldsymbol{X}_{i,P}^{(n)}\right) + H\left(\boldsymbol{X}_{i,DF}^{(n)}\right)\\
&\leqslant \dim \boldsymbol{X}_{i,C}^{(n)} + \dim \boldsymbol{X}_{i,P}^{(n)} + \dim \boldsymbol{X}_{i,DF}^{(n)}.
\end{aligned} \tag{89}
$$

Plugging (66), (67) and (76) in (89), this yields

$$
\begin{aligned}
R_i \leqslant &\min\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^+ + \min\Big(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^+, \Big(\overleftarrow{n}_{jj} - \overrightarrow{n}_{ii} - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\right)\\
&- \left(\left(\overrightarrow{n}_{jj} - n_{ij}\right)^+ - n_{ji}\right)^+\Big)^+\Big)\\
=&\overrightarrow{n}_{ii} + \min\Big(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^+, \Big(\overleftarrow{n}_{jj} - \overrightarrow{n}_{ii} - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\right)\\
&- \left(\left(\overrightarrow{n}_{jj} - n_{ij}\right)^+ - n_{ji}\right)^+\Big)^+\Big)\\
\stackrel{(l)}{=}&\overrightarrow{n}_{ii} + \min\Big(\left(n_{ji} - \overrightarrow{n}_{ii}\right)^+, \Big(\overleftarrow{n}_{jj} - \overrightarrow{n}_{ii} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+ + \left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+ - n_{ij}\right)^+\\
&- \left(\left(\overrightarrow{n}_{jj} - n_{ij}\right)^+ - n_{ji}\right)^+\Big)^+\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(m)}{=}\overrightarrow{n}_{ii}+\min\Big((n_{ji}-\overrightarrow{n}_{ii})^+,\Big(\overleftarrow{n}_{jj}-\overrightarrow{n}_{ii}-(\overrightarrow{n}_{jj}-n_{ji})^+ +\Big((\overrightarrow{n}_{jj}-n_{ji})^+-n_{ij}\Big)^+\\
&\quad -\Big((\overrightarrow{n}_{jj}-n_{ji})^+-n_{ij}\Big)^+\Big)^+\Big)\\
&=\overrightarrow{n}_{ii}+\min\Big((n_{ji}-\overrightarrow{n}_{ii})^+,\Big(\overleftarrow{n}_{jj}-\overrightarrow{n}_{ii}-(\overrightarrow{n}_{jj}-n_{ji})^+\Big)^+\Big)\\
&=\min\Big(\overrightarrow{n}_{ii}+(n_{ji}-\overrightarrow{n}_{ii})^+,\overrightarrow{n}_{ii}+\Big(\overleftarrow{n}_{jj}-\overrightarrow{n}_{ii}-(\overrightarrow{n}_{jj}-n_{ji})^+\Big)^+\Big)\\
&\overset{(n)}{=}\min\Big(\max(\overrightarrow{n}_{ii},n_{ji}),\max\Big(\overrightarrow{n}_{ii},\overleftarrow{n}_{jj}-(\overrightarrow{n}_{jj}-n_{ji})^+\Big)\Big),
\end{aligned}
\tag{90}
$$

where,
(l) follows from the fact that $\min(A,B)=A-(A-B)^+$;
(m) follows follows from Remark 2 in appendix D;
(n) follows from the fact that $\max(A,B)=A+(B-A)^+$.

This completes the proof of (62). ■

Proof of (64)**:**

$$
\begin{aligned}
N(R_1+R_2)&=H(W_1)+H(W_2)\\
&=I\Big(W_1;\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)+H\Big(W_1|\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)+I\Big(W_2;\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)+H\Big(W_2|\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)\\
&\overset{(a)}{\leqslant}I\Big(W_1;\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)+I\Big(W_2;\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)+N\delta_1(N)+N\delta_2(N)\\
&\overset{(b)}{=}I\Big(W_1;\overrightarrow{\boldsymbol{Y}}_1^{(1:N)},\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)+I\Big(W_2;\overrightarrow{\boldsymbol{Y}}_2^{(1:N)},\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)+N\delta(N)\\
&=H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)},\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)},\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\Big)+H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)},\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)\\
&\quad-H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)},\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\Big)+N\delta(N)\\
&=H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)+H\Big(\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1,\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)\\
&\quad+H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)+H\Big(\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2,\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)\\
&\quad+N\delta(N)\\
&\overset{(b)}{=}H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1,\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)+H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)\\
&\quad-H\Big(\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2,\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)+N\delta(N)\\
&\overset{(c)}{=}H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1,\overleftarrow{\boldsymbol{Y}}_1^{(1:N)},\boldsymbol{X}_1^{(1:N)}\Big)+H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)\\
&\quad-H\Big(\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\Big)-H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2,\overleftarrow{\boldsymbol{Y}}_2^{(1:N)},\boldsymbol{X}_2^{(1:N)}\Big)+N\delta(N)\\
&=H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\Big)-H\Big(\boldsymbol{X}_{2,C}^{(1:N)}|W_1,\overleftarrow{\boldsymbol{Y}}_1^{(1:N)},\boldsymbol{X}_1^{(1:N)}\Big)+H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)\\
&\quad-H\Big(\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\Big)-H\Big(\boldsymbol{X}_{1,C}^{(1:N)}|W_2,\overleftarrow{\boldsymbol{Y}}_2^{(1:N)},\boldsymbol{X}_2^{(1:N)}\Big)+N\delta(N)\\
&\overset{(d)}{=}H\Big(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\Big)-H\Big(\boldsymbol{X}_{2,C}^{(1:N)},\boldsymbol{X}_{1,U}^{(1:N)}|W_1,\overleftarrow{\boldsymbol{Y}}_1^{(1:N)},\boldsymbol{X}_1^{(1:N)}\Big)\\
&\quad+H\Big(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}\Big)-H\Big(\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\Big)-H\Big(\boldsymbol{X}_{1,C}^{(1:N)},\boldsymbol{X}_{2,U}^{(1:N)}|W_2,\overleftarrow{\boldsymbol{Y}}_2^{(1:N)},\boldsymbol{X}_2^{(1:N)}\Big)\\
&\quad+N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&= H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}|W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) - H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}|W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) + \left[I\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}; W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) - H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right)\right]\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right) + \left[I\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}; W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right) - H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right)\right]\\
&\quad - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&\overset{(e)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) - H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}|\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right) - H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}|\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad + I\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}; W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) + I\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}; W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&\leqslant H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right)\\
&\quad + I\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}; W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) + I\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}; W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&\overset{(f)}{\leqslant} H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right)\\
&\quad + I\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}, W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}; W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) + I\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}, W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}; W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right) + I\left(W_2; W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right)\\
&\quad + I\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}; W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_2\right) + I\left(W_1; W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad + I\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}; W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right) + H\left(W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right)\\
&\quad - H\left(W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_2\right) + H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right)\\
&\quad - H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2, W_1, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}\right) + H\left(W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right) - H\left(W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_1\right)\\
&\quad + H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1, W_2, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}\right)\\
&\quad - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&\overset{(g)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right) + H\left(W_1\right) + H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right)\\
&\quad - H\left(W_1|W_2\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_2, W_1\right) + H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right)\\
&\quad + H\left(W_2\right) + H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) - H\left(W_2|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_1, W_2\right)\\
&\quad + H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) + N\delta(N)\\
&\overset{(h)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_{1}^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_{2}^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_2, W_1\right)\\
&\quad + H\left(\boldsymbol{X}_{2,C}^{(1:N)}, \boldsymbol{X}_{1,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_2\right) - H\left(\overleftarrow{\boldsymbol{Y}}_{2}^{(1:N)}|W_1, W_2\right) + H\left(\boldsymbol{X}_{1,C}^{(1:N)}, \boldsymbol{X}_{2,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_{1}^{(1:N)}|W_1\right)\\
&\quad + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(i)}{\leqslant} H\left(\overrightarrow{\boldsymbol{Y}}_1^{(1:N)}|\boldsymbol{X}_{1,C}^{(1:N)},\boldsymbol{X}_{2,U}^{(1:N)}\right)+H\left(\overrightarrow{\boldsymbol{Y}}_2^{(1:N)}|\boldsymbol{X}_{2,C}^{(1:N)},\boldsymbol{X}_{1,U}^{(1:N)}\right)+H\left(\boldsymbol{X}_{2,C}^{(1:N)},\boldsymbol{X}_{1,U}^{(1:N)},\overleftarrow{\boldsymbol{Y}}_2^{(1:N)}|W_2\right)\\
&\quad +H\left(\boldsymbol{X}_{1,C}^{(1:N)},\boldsymbol{X}_{2,U}^{(1:N)},\overleftarrow{\boldsymbol{Y}}_1^{(1:N)}|W_1\right)+N\delta(N)\\
&=\sum_{n=1}^{N}\Big[H\left(\overrightarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_{1,C}^{(1:N)},\boldsymbol{X}_{2,U}^{(1:N)},\overrightarrow{\boldsymbol{Y}}_1^{(1:n-1)}\right)+H\left(\overrightarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_{2,C}^{(1:N)},\boldsymbol{X}_{1,U}^{(1:N)},\overrightarrow{\boldsymbol{Y}}_2^{(1:n-1)}\right)\\
&\quad +H\left(\boldsymbol{X}_{2,C}^{(n)},\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|W_2,\boldsymbol{X}_{2,C}^{(1:n-1)},\boldsymbol{X}_{1,U}^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_2^{(1:n-1)}\right)\\
&\quad +H\left(\boldsymbol{X}_{1,C}^{(n)},\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|W_1,\boldsymbol{X}_{1,C}^{(1:n-1)},\boldsymbol{X}_{2,U}^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_1^{(1:n-1)}\right)\Big]+N\delta(N)\\
&\overset{(j)}{=}\sum_{n=1}^{N}\Big[H\left(\overrightarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_{1,C}^{(1:N)},\boldsymbol{X}_{2,U}^{(1:N)},\overrightarrow{\boldsymbol{Y}}_1^{(1:n-1)}\right)+H\left(\overrightarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_{2,C}^{(1:N)},\boldsymbol{X}_{1,U}^{(1:N)},\overrightarrow{\boldsymbol{Y}}_2^{(1:n-1)}\right)\\
&\quad +H\left(\boldsymbol{X}_{2,C}^{(n)},\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|W_2,\boldsymbol{X}_{2,C}^{(1:n-1)},\boldsymbol{X}_{1,U}^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_2^{(1:n-1)},\boldsymbol{X}_2^{(1:n)}\right)\\
&\quad +H\left(\boldsymbol{X}_{1,C}^{(n)},\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|W_1,\boldsymbol{X}_{1,C}^{(1:n-1)},\boldsymbol{X}_{2,U}^{(1:n-1)},\overleftarrow{\boldsymbol{Y}}_1^{(1:n-1)},\boldsymbol{X}_1^{(1:n)}\right)\Big]+N\delta(N)\\
&=\sum_{n=1}^{N}\Big[H\left(\overrightarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_{1,C}^{(1:N)},\boldsymbol{X}_{2,U}^{(1:N)},\overrightarrow{\boldsymbol{Y}}_1^{(1:n-1)}\right)+H\left(\overrightarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_{2,C}^{(1:N)},\boldsymbol{X}_{1,U}^{(1:N)},\overrightarrow{\boldsymbol{Y}}_2^{(1:n-1)}\right)\\
&\quad +H\left(\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_{1,U}^{(1:n-1)},\boldsymbol{X}_2^{(1:n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_{2,U}^{(1:n-1)},\boldsymbol{X}_1^{(1:n)}\right)\Big]+N\delta(N)\\
&\overset{(k)}{\leqslant}\sum_{n=1}^{N}\Big[H\left(\overrightarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_{1,C}^{(n)},\boldsymbol{X}_{2,U}^{(n)}\right)+H\left(\overrightarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_{2,C}^{(n)},\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_2^{(1:n)}\right)\\
&\quad +H\left(\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_1^{(1:n)}\right)\Big]+N\delta(N)\\
&=\sum_{n=1}^{N}\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}|\boldsymbol{X}_{1,C}^{(n)},\boldsymbol{X}_{2,U}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}|\boldsymbol{X}_{2,C}^{(n)},\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_2^{(1:n)}\right)\\
&\quad +H\left(\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_1^{(1:n)}\right)\Big]+N\delta(N)\\
&\overset{(k)}{\leqslant}\sum_{n=1}^{N}\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_2^{(1:n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_1^{(1:n)}\right)\Big]\\
&\quad +N\delta(N)\\
&=N\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_2^{(n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)},\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_1^{(n)}\right)\Big]\\
&\quad +N\delta(N),\ \text{for any } n\in\{1,\ldots,N\}\\
&=N\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)}|\boldsymbol{X}_2^{(n)}\right)+H\left(\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_2^{(n)},\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)}|\boldsymbol{X}_1^{(n)}\right)\\
&\quad +H\left(\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_1^{(n)},\boldsymbol{X}_{2,U}^{(n)}\right)\Big]+N\delta(N)\\
&\overset{(k)}{\leqslant}N\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\overleftarrow{\boldsymbol{Y}}_2^{(n)}|\boldsymbol{X}_2^{(n)},\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)}\right)\\
&\quad +H\left(\overleftarrow{\boldsymbol{Y}}_1^{(n)}|\boldsymbol{X}_1^{(n)},\boldsymbol{X}_{2,U}^{(n)}\right)\Big]+N\delta(N)\\
&=N\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{1,CF_2}^{(n)},\boldsymbol{X}_{1,DF}^{(n)}|\boldsymbol{X}_2^{(n)},\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)}\right)\\
&\quad +H\left(\boldsymbol{X}_{2,CF_1}^{(n)},\boldsymbol{X}_{2,DF}^{(n)}|\boldsymbol{X}_1^{(n)},\boldsymbol{X}_{2,U}^{(n)}\right)\Big]+N\delta(N)\\
&=N\Big[H\left(\boldsymbol{X}_{1,P}^{(n)}\right)+H\left(\boldsymbol{X}_{2,P}^{(n)}\right)+H\left(\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{1,CF_2}^{(n)},\boldsymbol{X}_{1,DF}^{(n)}|\boldsymbol{X}_{1,U}^{(n)}\right)+H\left(\boldsymbol{X}_{2,U}^{(n)}\right)\\
&\quad +H\left(\boldsymbol{X}_{2,CF_1}^{(n)},\boldsymbol{X}_{2,DF}^{(n)}|\boldsymbol{X}_{2,U}^{(n)}\right)\Big]+N\delta(N),
\end{aligned}
\tag{91}
$$

where,
(a) follows from Fano's inequality;
(b) follows from the fact that $\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}$ is included into $\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}$;
(c) follows from the fact that $\boldsymbol{X}_i^{(1:N)} = f\left(W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N-1)}\right)$;
(d) follows from the fact that $\boldsymbol{X}_{i,U}^{(1:N)}$ is included into $\boldsymbol{X}_i^{(1:N)}$;
(e) follows from the fact that $H(Y) - H(X) = H(Y|X) - H(X|Y)$;
(f) follows from the fact that injecting information increases the mutual information;
(g) follows from the fact that $H\left(\boldsymbol{X}_{i,C}^{(1:N)}, \boldsymbol{X}_{j,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) = 0$;
(h) follows from the fact that W_i is independent of W_j, then $H(W_i|W_j) = H(W_i)$;
(i) follows from the fact that conditioning reduces the entropy;
(j) follows from the fact that $\boldsymbol{X}_i^{(1:n)} = f\left(W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:n-1)}\right)$;
(k) follows from the fact that conditioning reduces the entropy.

From (91), in the asymptotic regime, it holds that:

$$\begin{aligned}
R_1 + R_2 \leqslant & H\left(\boldsymbol{X}_{1,P}^{(n)}\right) + H\left(\boldsymbol{X}_{2,P}^{(n)}\right) + H\left(\boldsymbol{X}_{1,U}^{(n)}\right) + H\left(\boldsymbol{X}_{1,CF_2}^{(n)}, \boldsymbol{X}_{1,DF}^{(n)} | \boldsymbol{X}_{1,U}^{(n)}\right) + H\left(\boldsymbol{X}_{2,U}^{(n)}\right) \\
& + H\left(\boldsymbol{X}_{2,CF_1}^{(n)}, \boldsymbol{X}_{2,DF}^{(n)} | \boldsymbol{X}_{2,U}^{(n)}\right) \\
= & \dim \boldsymbol{X}_{1,P}^{(n)} + \dim \boldsymbol{X}_{2,P}^{(n)} + \dim \boldsymbol{X}_{1,U}^{(n)} + \left(\dim\left(\boldsymbol{X}_{1,CF_2}^{(n)}, \boldsymbol{X}_{1,DF}^{(n)}\right) - \dim \boldsymbol{X}_{1,U}^{(n)}\right)^+ \\
& + \dim \boldsymbol{X}_{2,U}^{(n)} + \left(\dim\left(\boldsymbol{X}_{2,CF_1}^{(n)}, \boldsymbol{X}_{2,DF}^{(n)}\right) - \dim \boldsymbol{X}_{2,U}^{(n)}\right)^+ .
\end{aligned} \tag{92}$$

Plugging (67), (78), and (84) in (92), this yields

$$\begin{aligned}
R_1 + R_2 \leqslant & \left(\overrightarrow{n}_{11} - n_{21}\right)^+ + \left(\overrightarrow{n}_{22} - n_{12}\right)^+ + \min\left(\overrightarrow{n}_{22}, n_{12}\right) - \min\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\right) \\
& + \left(n_{21} - \overrightarrow{n}_{22}\right)^+ + \Big(\left(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right)^+ - \min\left(\overrightarrow{n}_{22}, n_{12}\right) \\
& + \min\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\right) - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ \Big)^+ + \min\left(\overrightarrow{n}_{11}, n_{21}\right) - \min\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\right) \\
& + \left(n_{12} - \overrightarrow{n}_{11}\right)^+ + \Big(\left(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
& + \min\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\right) - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ \Big)^+ \\
= & \overrightarrow{n}_{11} + \overrightarrow{n}_{22} + \left(n_{12} - \overrightarrow{n}_{11}\right)^+ + \left(n_{21} - \overrightarrow{n}_{22}\right)^+ - \min\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\right) \\
& - \min\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\right) + \Big(\left(\min\left(\overleftarrow{n}_{22}, \max\left(\overrightarrow{n}_{22}, n_{21}\right)\right) - \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right)^+ \\
& - \min\left(\overrightarrow{n}_{22}, n_{12}\right) + \min\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}\right) - \left(n_{21} - \overrightarrow{n}_{22}\right)^+ \Big)^+ \\
& + \Big(\left(\min\left(\overleftarrow{n}_{11}, \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right) - \left(\overrightarrow{n}_{11} - n_{12}\right)^+\right)^+ - \min\left(\overrightarrow{n}_{11}, n_{21}\right) \\
& + \min\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}\right) - \left(n_{12} - \overrightarrow{n}_{11}\right)^+ \Big)^+
\end{aligned}$$

$$
\begin{aligned}
=&\max(\overrightarrow{n}_{11}, n_{12}) + \max(\overrightarrow{n}_{22}, n_{21}) - \min\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big) - \min\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big)\\
&+\Big(\Big(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\Big)^+ - \min(\overrightarrow{n}_{22}, n_{12})\\
&+\min\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big) - (n_{21} - \overrightarrow{n}_{22})^+\Big)^+\\
&+\Big(\Big(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\Big)^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
&+\min\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big) - (n_{12} - \overrightarrow{n}_{11})^+\Big)^+\\
=&\max(\overrightarrow{n}_{11}, n_{12}) + \max(\overrightarrow{n}_{22}, n_{21}) - (\overrightarrow{n}_{22} - n_{21})^+ + \Big((\overrightarrow{n}_{22} - n_{21})^+ - n_{12}\Big)^+\\
&-(\overrightarrow{n}_{11} - n_{12})^+ + \Big((\overrightarrow{n}_{11} - n_{12})^+ - n_{21}\Big)^+ + \Big(\Big(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\Big)^+\\
&-\min(\overrightarrow{n}_{22}, n_{12}) + \min\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big) - (n_{21} - \overrightarrow{n}_{22})^+\Big)^+\\
&+\Big(\Big(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\Big)^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
&+\min\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big) - (n_{12} - \overrightarrow{n}_{11})^+\Big)^+\\
=&(\overrightarrow{n}_{11} - n_{12})^+ + n_{12} + (\overrightarrow{n}_{22} - n_{21})^+ + n_{21} - (\overrightarrow{n}_{22} - n_{21})^+ + \Big((\overrightarrow{n}_{22} - n_{21})^+ - n_{12}\Big)^+\\
&-(\overrightarrow{n}_{11} - n_{12})^+ + \Big((\overrightarrow{n}_{11} - n_{12})^+ - n_{21}\Big)^+ + \Big(\Big(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\Big)^+\\
&-\min(\overrightarrow{n}_{22}, n_{12}) + \min\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big) - (n_{21} - \overrightarrow{n}_{22})^+\Big)^+\\
&+\Big(\Big(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\Big)^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
&+\min\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big) - (n_{12} - \overrightarrow{n}_{11})^+\Big)^+\\
=&n_{12} + n_{21} + \Big((\overrightarrow{n}_{22} - n_{21})^+ - n_{12}\Big)^+ + \Big((\overrightarrow{n}_{11} - n_{12})^+ - n_{21}\Big)^+\\
&+\Big(\Big(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\Big)^+ - \min(\overrightarrow{n}_{22}, n_{12})\\
&+\min\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big) - (n_{21} - \overrightarrow{n}_{22})^+\Big)^+\\
&+\Big(\Big(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\Big)^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
&+\min\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big) - (n_{12} - \overrightarrow{n}_{11})^+\Big)^+\\
=&\max\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big) + \max\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big)\\
&+\Big(\Big(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22} - n_{21})^+\Big)^+ - \min(\overrightarrow{n}_{22}, n_{12})\\
&+\min\Big((\overrightarrow{n}_{22} - n_{21})^+, n_{12}\Big) - (n_{21} - \overrightarrow{n}_{22})^+\Big)^+\\
&+\Big(\Big(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11} - n_{12})^+\Big)^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
&+\min\Big((\overrightarrow{n}_{11} - n_{12})^+, n_{21}\Big) - (n_{12} - \overrightarrow{n}_{11})^+\Big)^+. \qquad (93)
\end{aligned}
$$

This completes the proof of (64). ■

Proof of (65)**:** for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, it follows that

$$
\begin{aligned}
N\big(2R_i + R_j\big) &= 2H\left(W_i\right) + H\left(W_j\right) \\
&= 2I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + 2H\left(W_i | \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + H\left(W_j | \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\overset{(a)}{\leqslant} 2I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + 2N\delta_i(N) + N\delta_j(N) \\
&= 2I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&= I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&= I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\quad + N\delta(N) \\
&\overset{(b)}{\leqslant} I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) \\
&\quad + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) \\
&\quad - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_i, W_j\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) \\
&\quad + N\delta(N) \\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) \\
&\quad + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_i, W_j\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) \\
&\quad - H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | \overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_i, W_j\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\quad + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | \overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&\overset{(c)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\quad - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_i, W_j\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\quad + N\delta(N) \\
&\overset{(d)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) - H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&\overset{(e)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)} | W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\boldsymbol{X}_{i,C}^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N) \\
&\overset{(f)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)} | W_i\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)} | W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\boldsymbol{X}_{i,C}^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(g)}{\leqslant} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}|W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) - H\left(\boldsymbol{X}_{i,C}^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + \big[I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right)\big] + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right)\\
&\quad - H\left(\boldsymbol{X}_{i,C}^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + \left[I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right)\right]\\
&\quad + H\left(\boldsymbol{X}_{i,C}^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right)\\
&\quad - H\left(\boldsymbol{X}_{i,C}^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + \left[I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right)\right]\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}\right) + N\delta(N)\\
&\overset{(g)}{\leqslant} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + \left[I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right)\right]\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&\overset{(h)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&\overset{(i)}{\leqslant} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + I\left(W_j; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad + I\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}; W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_i\right) + H\left(W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j\right)\\
&\quad + H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j, W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&= H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(W_i\right) - H\left(W_i|W_j\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, W_i,\right)\\
&\quad + H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j\right) - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j, W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&\overset{(j)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, W_i,\right) + H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j\right)\\
&\quad - H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j, W_i, \overleftarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)\\
&\overset{(k)}{=} H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) - H\left(\overleftarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, W_i,\right) + H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}|W_j\right)\\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}|W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)}|\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)}\right) + H\left(\boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)} | W_j\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(1:N)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(1:N)} | \boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}\right) + N\delta(N) \\
&\overset{(l)}{\leqslant} \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_{j,C}^{(1:n-1)}, \boldsymbol{X}_{i,U}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) \Big] \\
&\quad + N\delta(N) \\
&\overset{(m)}{=} \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_{j,C}^{(1:n-1)}, \boldsymbol{X}_{i,U}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}, \boldsymbol{X}_j^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) \Big] \\
&\quad + N\delta(N) \\
&= \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | W_j, \boldsymbol{X}_{i,U}^{(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_j^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | W_j, \overleftarrow{\boldsymbol{Y}}_j^{(1:N)}, \boldsymbol{X}_{i,C}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}, \boldsymbol{X}_j^{(1:N)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) \Big] \\
&\quad + N\delta(N) \\
&= \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{i,U}^{(1:n-1)}, \boldsymbol{X}_j^{(1:n)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | \boldsymbol{X}_{i,C}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_i^{(1:n-1)}, \boldsymbol{X}_j^{(1:N)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(1:N)}, \boldsymbol{X}_{i,U}^{(1:N)}, \overrightarrow{\boldsymbol{Y}}_j^{(1:n-1)}\right) \Big] + N\delta(N) \\
&\overset{(n)}{\leqslant} \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}, \overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(n)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | \boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_j^{(n)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) \Big] + N\delta(N) \\
&= \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)} | \boldsymbol{X}_j^{(n)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | \boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_j^{(n)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) \Big] + N\delta(N) \\
&\overset{(n)}{\leqslant} \sum_{n=1}^{N} \Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | \boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_j^{(n)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) \Big] + N\delta(N) \\
&= N\Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)} | \boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_j^{(n)}\right) \\
&\quad + H\left(\overrightarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) \Big] + N\delta(N) \text{ for any } n \in \{1, \ldots, N\} \\
&= N\Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\boldsymbol{X}_{i,P}^{(n)} | \boldsymbol{X}_{i,C}^{(n)}, \boldsymbol{X}_j^{(n)}\right) \\
&\quad + H\left(\boldsymbol{X}_{j,P}^{(n)} | \boldsymbol{X}_{j,C}^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) \Big] + N\delta(N) \\
&\overset{(n)}{\leqslant} N\Big[H\left(\overrightarrow{\boldsymbol{Y}}_i^{(n)}\right) + H\left(\boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\overleftarrow{\boldsymbol{Y}}_j^{(n)} | \boldsymbol{X}_j^{(n)}, \boldsymbol{X}_{i,U}^{(n)}\right) + H\left(\boldsymbol{X}_{i,P}^{(n)}\right) + H\left(\boldsymbol{X}_{j,P}^{(n)}\right) \Big] \\
&\quad + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
=&N\left[H\left(\overrightarrow{\boldsymbol{Y}}_{i}^{(n)}\right)+H\left(\boldsymbol{X}_{i,U}^{(n)}\right)+H\left(\boldsymbol{X}_{i,CF_j}^{(n)},\boldsymbol{X}_{i,DF}^{(n)}|\boldsymbol{X}_{j}^{(n)},\boldsymbol{X}_{i,U}^{(n)}\right)+H\left(\boldsymbol{X}_{i,P}^{(n)}\right)+H\left(\boldsymbol{X}_{j,P}^{(n)}\right)\right]\\
&+N\delta(N)\\
=&N\left[H\left(\overrightarrow{\boldsymbol{Y}}_{i}^{(n)}\right)+H\left(\boldsymbol{X}_{i,U}^{(n)}\right)+H\left(\boldsymbol{X}_{i,CF_j}^{(n)},\boldsymbol{X}_{i,DF}^{(n)}|\boldsymbol{X}_{i,U}^{(n)}\right)+H\left(\boldsymbol{X}_{i,P}^{(n)}\right)+H\left(\boldsymbol{X}_{j,P}^{(n)}\right)\right]\\
&+N\delta(N),
\end{aligned}
\tag{94}
$$

where,
(a) follows from Fano's inequality;
(b) follows from the fact that $I\left(W_i;\overrightarrow{\boldsymbol{Y}}_{i}^{(1:N)}\right)\leqslant I\left(W_i;\overrightarrow{\boldsymbol{Y}}_{i}^{(1:N)},\overleftarrow{\boldsymbol{Y}}_{j}^{(1:N)}|W_j\right)$;
(c) follows from the fact that $\overleftarrow{\boldsymbol{Y}}_{i}^{(1:N)}$ is included into $\overrightarrow{\boldsymbol{Y}}_{i}^{(1:N)}$;
(d) follows from the fact that $H\left(\overrightarrow{\boldsymbol{Y}}_{i}^{(1:N)},\overleftarrow{\boldsymbol{Y}}_{j}^{(1:N)}|W_i,W_j\right)=0$;
(e) follows from the fact that $\boldsymbol{X}_{i}^{(1:N)}=f\left(W_i,\overleftarrow{\boldsymbol{Y}}_{i}^{(1:N-1)}\right)$;
(f) follows from the fact that $\boldsymbol{X}_{i,U}^{(1:N)}$ is included into $\boldsymbol{X}_{i}^{(1:N)}$;
(g) follows from the fact that including another random variable increases the mutual information;
(h) follows from the fact that $H(Y|X)=H(X,Y)-H(X)$;
(i) follows from the fact that injecting information increases the mutual information;
(j) follows from the fact that W_i is independent of W_j, then $H(W_i|W_j)=H(W_i)$;
(k) follows from the fact that $H\left(\boldsymbol{X}_{j,C}^{(1:N)},\boldsymbol{X}_{i,U}^{(1:N)},\overleftarrow{\boldsymbol{Y}}_{j}^{(1:N)}|W_j,W_i,\overleftarrow{\boldsymbol{Y}}_{i}^{(1:N)}\right)=0$;
(l) follows from the fact that conditioning reduces the entropy;
(m) follows from the fact that $\boldsymbol{X}_{j}^{(1:n)}=f\left(W_j,\overleftarrow{\boldsymbol{Y}}_{j}^{(1:n-1)}\right)$ and $\boldsymbol{X}_{j}^{(1:N)}=f\left(W_j,\overleftarrow{\boldsymbol{Y}}_{j}^{(1:N-1)}\right)$;
(n) follows from the fact that conditioning reduces the entropy.

From (94), in the asymptotic regime, it holds that:

$$
\begin{aligned}
2R_i+R_j\leqslant& H\left(\overrightarrow{\boldsymbol{Y}}_{i}^{(n)}\right)+H\left(\boldsymbol{X}_{i,U}^{(n)}\right)+H\left(\boldsymbol{X}_{i,CF_j}^{(n)},\boldsymbol{X}_{i,DF}^{(n)}|\boldsymbol{X}_{i,U}^{(n)}\right)+H\left(\boldsymbol{X}_{i,P}^{(n)}\right)+H\left(\boldsymbol{X}_{j,P}^{(n)}\right)\\
\overset{(o)}{\leqslant}&\dim\overleftarrow{\boldsymbol{Y}}_{i}^{(n)}+\dim\overrightarrow{\boldsymbol{Y}}_{i,G}^{(n)}+\dim\boldsymbol{X}_{i,U}^{(n)}+\left(\dim\left(\boldsymbol{X}_{i,CF_j}^{(n)},\boldsymbol{X}_{i,DF}^{(n)}\right)-\dim\boldsymbol{X}_{i,U}^{(n)}\right)^{+}\\
&+\dim\boldsymbol{X}_{i,P}^{(n)}+\dim\boldsymbol{X}_{j,P}^{(n)},
\end{aligned}
\tag{95}
$$

where,
(o) follows from (82).

Plugging (67), (78), (79), (80) and (84) in (95), this yields

$$
\begin{aligned}
2R_i+R_j\leqslant&\min\left(\overleftarrow{n}_{ii},\max\left(\overrightarrow{n}_{ii},n_{ij}\right)\right)+\left(\max\left(\overrightarrow{n}_{ii},n_{ij}\right)-\overleftarrow{n}_{ii}\right)^{+}+\min\left(\overrightarrow{n}_{jj},n_{ij}\right)\\
&-\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)+\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}+\Bigg(\bigg(\min\left(\overleftarrow{n}_{jj},\max\left(\overrightarrow{n}_{jj},n_{ji}\right)\right)\\
&-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\bigg)^{+}-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}\Bigg)^{+}\\
&+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}+\left(\overrightarrow{n}_{jj}-n_{ij}\right)^{+}\\
=&\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\min\left(\overrightarrow{n}_{jj},n_{ij}\right)-\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)+\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}\\
&+\Bigg(\bigg(\min\left(\overleftarrow{n}_{jj},\max\left(\overrightarrow{n}_{jj},n_{ji}\right)\right)-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\bigg)^{+}-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)\\
&+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}\Bigg)^{+}+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}+\left(\overrightarrow{n}_{jj}-n_{ij}\right)^{+}
\end{aligned}
$$

$$
\begin{aligned}
=&\overrightarrow{n}_{jj} + \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right) + \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+}\\
&+\Big(\left(\min\left(\overleftarrow{n}_{jj}, \max\left(\overrightarrow{n}_{jj}, n_{ji}\right)\right) - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) + \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right)\\
&-\left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+}\Big)^{+} + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+}\\
=&\max\left(\overrightarrow{n}_{jj}, n_{ji}\right)^{+} + \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right)\\
&+\Big(\left(\min\left(\overleftarrow{n}_{jj}, \max\left(\overrightarrow{n}_{jj}, n_{ji}\right)\right) - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) + \min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\right)\\
&-\left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+}\Big)^{+} + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+}. \qquad (96)
\end{aligned}
$$

This completes the proof of (65). ■

C Connections to Existing Results

This appendix provides connections between the capacity region of the two-user LD-IC-NOF and existing results. These results are described using the notation adopted in this technical report and it might be different from the notation used by other authors. Often the identities (52) and (53) are used.

C.1 Capacity of the LD-IC without Channel-Output Feedback

Theorem 1 simplifies to the following lemma for the case in which channel-output feedback is not available, i.e., $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$.

Lemma 2 (Lemma 4 in [17]) The capacity region $\mathcal{C}(n_{11}, n_{22}, n_{12}, n_{21}, 0, 0)$ of the two-user LD-IC without channel-output feedback (LD-IC-WOFB) is included in the set of non-negative rate pairs (R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\leqslant n_{11}, & (97)\\
R_2 &\leqslant n_{22}, & (98)\\
R_1 + R_2 &\leqslant (n_{11} - n_{12})^+ + \max(n_{22}, n_{12}), & (99)\\
R_1 + R_2 &\leqslant (n_{22} - n_{21})^+ + \max(n_{11}, n_{21}), & (100)\\
R_1 + R_2 &\leqslant \max\left(n_{21}, (n_{11} - n_{12})^+\right) + \max\left(n_{12}, (n_{22} - n_{21})^+\right), & (101)\\
2R_1 + R_2 &\leqslant \max(n_{11}, n_{21}) + (n_{11} - n_{12})^+ + \max\left(n_{12}, (n_{22} - n_{21})^+\right), & (102)\\
R_1 + 2R_2 &\leqslant \max(n_{22}, n_{12}) + (n_{22} - n_{21})^+ + \max\left(n_{21}, (n_{11} - n_{12})^+\right). & (103)
\end{aligned}$$

In the following, an alternative proof of Lemma 4 in [17] is presented using Theorem 1.

Proof of (97) **and** (98)**:** Consider (8a) and (8b) with $i = 1$ and $j = 2$, and let $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}
R_1 &\leqslant \min\left(\max(n_{11}, n_{21}), \max(n_{11}, n_{12}), \min\left(\max(n_{11}, n_{21}), \max\left(n_{11}, -(n_{22} - n_{21})^+\right)\right)\right)\\
&\leqslant \min\left(\max(n_{11}, n_{21}), \max(n_{11}, n_{12}), n_{11}\right)\\
&= n_{11}.
\end{aligned}$$

This completes the proof of (97). The same procedure can be applied to prove (98) considering (8a) and (8b) with $i = 2$ and $j = 1$, and letting $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$. ■

Proof of (99)**,** (100) **and** (101)**:** Consider (8c) with $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}
R_1 + R_2 &\leqslant \min\left(\max(n_{11}, n_{12}) + (n_{22} - n_{12})^+, \max(n_{22}, n_{21}) + (n_{11} - n_{21})^+\right)\\
&= \min\left((n_{11} - n_{12})^+ + n_{12} + (n_{22} - n_{12})^+, (n_{22} - n_{21})^+ + n_{21} + (n_{11} - n_{21})^+\right)\\
&= \min\left((n_{11} - n_{12})^+ + \max(n_{22}, n_{12}), (n_{22} - n_{21})^+ + \max(n_{11}, n_{21})\right).
\end{aligned}$$

This completes the proof of (99) and (100). Consider (8d) with $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}R_1 + R_2 \leqslant & \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right)\\ &+\left(-(n_{12}-n_{11})^+ - \min(n_{11}, n_{21}) + \min\left((n_{11}-n_{12})^+, n_{21}\right)\right)^+\\ &+\left(-(n_{21}-n_{22})^+ - \min(n_{22}, n_{12}) + \min\left((n_{22}-n_{21})^+, n_{12}\right)\right)^+\\ \stackrel{(a)}{=} & \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right),\end{aligned}$$

where,
(a) follows from Remark 1 in appendix D.

This completes the proof of (101). ■

Proof of (102) **and** (103)**:** Consider (8e) with $i = 1$ and $j = 2$, and let $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}2R_1 + R_2 \leqslant & \max(n_{22}, n_{21}) + \max(n_{11}, n_{12}) + (n_{11}-n_{21})^+ - \min\left((n_{22}-n_{21})^+, n_{12}\right)\\ &+\left(-(n_{21}-n_{22})^+ - \min(n_{22}, n_{12}) + \min\left((n_{22}-n_{21})^+, n_{12}\right)\right)^+\\ \stackrel{(b)}{=} & \max(n_{22}, n_{21}) + \max(n_{11}, n_{12}) + (n_{11}-n_{21})^+ - \min\left((n_{22}-n_{21})^+, n_{12}\right),\end{aligned}$$

where,
(b) follows from Remark 1 in appendix D.

This completes the proof of (102). The same procedure can be applied to prove (103) considering (8e) with $i = 2$ and $j = 1$, and letting $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$.

C.2 Capacity of the LD-IC with Perfect Channel-Output Feedback

Theorem 1 simplifies to the following lemma for the case in which there exists perfect channel-output feedback, i.e., $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = \max(n_{11}, n_{12})$ and $\overleftarrow{n}_{22} = \max(n_{22}, n_{21})$.

Lemma 3 (Corollary 1 **in [4])** The capacity region $\mathcal{C}\Big(n_{11},\ n_{22},\ n_{12},\ n_{21},\ \max\left(n_{11},\ n_{12}\right),$ $\max\left(n_{22},\ n_{21}\right)\Big)$ of the two-user LD-IC with perfect channel-output feedback (LD-IC-FB) is included in the set of non-negative rate pairs (R_1, R_2) satisfying

$$R_1 \leqslant \min\left(\max(n_{11}, n_{21}), \max(n_{11}, n_{12})\right), \tag{104}$$

$$R_2 \leqslant \min\left(\max(n_{22}, n_{12}), \max(n_{22}, n_{21})\right), \tag{105}$$

$$R_1 + R_2 \leqslant \min\left(\max(n_{22}, n_{21}) + (n_{11}-n_{21})^+, \max(n_{11}, n_{12}) + (n_{22}-n_{12})^+\right). \tag{106}$$

In the following, an alternative proof of Corollary 1 in [4] is presented using Theorem 1.

Proof of (104) **and** (105)**:** Consider (8a) with $i = 1$ and $j = 2$, and let $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = \max(n_{11}, n_{12})$ and $\overleftarrow{n}_{22} = \max(n_{22}, n_{21})$. This yields

$$R_1 \leqslant \min\left(\max(n_{11}, n_{21}), \max(n_{11}, n_{12})\right).$$

Consider also (8b) with $i = 1$ and $j = 2$, and let $\overrightarrow{n}_{11} = n_{11}$, $\overrightarrow{n}_{22} = n_{22}$, $\overleftarrow{n}_{11} = \max(n_{11}, n_{12})$ and $\overleftarrow{n}_{22} = \max(n_{22}, n_{21})$. This yields

$$\begin{aligned}R_1 \leqslant & \min\left(\max(n_{11}, n_{21}), \max\left(n_{11}, \max(n_{22}, n_{21}) - (n_{22}-n_{21})^+\right)\right)\\ =&\min\left(\max(n_{11}, n_{21}), \max\left(n_{11}, n_{21} + (n_{22}-n_{21})^+ - (n_{22}-n_{21})^+\right)\right)\\ =&\min\left(\max(n_{11}, n_{21}), \max(n_{11}, n_{21})\right)\\ =&\max(n_{11}, n_{21}).\end{aligned} \tag{107}$$

This completes the proof of (104). ■

The same procedure can be applied to (8a) and (8b) with $i=2$ and $j=1$ to prove (105).

Proof of (106)**:** Consider (8c) with $\overrightarrow{n}_{11}=n_{11}$, $\overrightarrow{n}_{22}=n_{22}$, $\overleftarrow{n}_{11}=\max(n_{11},n_{12})$ and $\overleftarrow{n}_{22}=\max(n_{22},n_{21})$. This yields (106).

Consider (8d) with $\overrightarrow{n}_{11}=n_{11}$, $\overrightarrow{n}_{22}=n_{22}$, $\overleftarrow{n}_{11}=\max(n_{11},n_{12})$ and $\overleftarrow{n}_{22}=\max(n_{22},n_{21})$. This yields

$$
\begin{aligned}
R_1+R_2 \leqslant & \max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Bigg(\Big(\min\big(\max(n_{11},n_{12}),\\
&\max(n_{11},n_{12})\big)-(n_{11}-n_{12})^+\Big)^+-(n_{12}-n_{11})^+-\min(n_{11},n_{21})\\
&+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Bigg)^++\Bigg(\Big(\min(\max(n_{22},n_{21}),\max(n_{22},n_{21}))-(n_{22}-n_{21})^+\Big)^+\\
&-(n_{21}-n_{22})^+-\min(n_{22},n_{12})+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Bigg)^+\\
=&\max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Bigg(\Big(\max(n_{11},n_{12})-(n_{11}-n_{12})^+\Big)^+\\
&-(n_{12}-n_{11})^+-\min(n_{11},n_{21})+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Bigg)^+\\
&+\Bigg(\Big(\max(n_{22},n_{21})-(n_{22}-n_{21})^+\Big)^+-(n_{21}-n_{22})^+-\min(n_{22},n_{12})\\
&+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Bigg)^+\\
=&\max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Bigg(\Big((n_{11}-n_{12})^++n_{12}\\
&-(n_{11}-n_{12})^+\Big)^+-(n_{12}-n_{11})^+-\min(n_{11},n_{21})+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Bigg)^+\\
&+\Bigg(\Big((n_{22}-n_{21})^++n_{21}-(n_{22}-n_{21})^+\Big)^+-(n_{21}-n_{22})^+-\min(n_{22},n_{12})\\
&+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Bigg)^+\\
=&\max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Big((n_{12})^+-(n_{12}-n_{11})^+\\
&-\min(n_{11},n_{21})+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Big)^++\Big((n_{21})^+-(n_{21}-n_{22})^+\\
&-\min(n_{22},n_{12})+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Big)^+\\
=&\max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Big(n_{12}-(n_{12}-n_{11})^+-\min(n_{11},n_{21})\\
&+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Big)^++\Big(n_{21}-(n_{21}-n_{22})^+-\min(n_{22},n_{12})\\
&+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Big)^+\\
=&\max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Big(\min(n_{11},n_{12})-\min(n_{11},n_{21})\\
&+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Big)^++\Big(\min(n_{22},n_{21})-\min(n_{22},n_{12})+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Big)^+\\
=&\max\Big((n_{11}-n_{12})^+,n_{21}\Big)+\max\Big((n_{22}-n_{21})^+,n_{12}\Big)+\Big(\min(n_{11},n_{12})-n_{11}+(n_{11}-n_{21})^+\\
&+\min\Big((n_{11}-n_{12})^+,n_{21}\Big)\Big)^++\Big(\min(n_{22},n_{21})-n_{22}+(n_{22}-n_{12})^+\\
&+\min\Big((n_{22}-n_{21})^+,n_{12}\Big)\Big)^+
\end{aligned}
$$

$$
\begin{aligned}
&= \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \Big(n_{11} - (n_{11}-n_{12})^+ - n_{11} \\
&\quad + (n_{11}-n_{21})^+ + \min\left((n_{11}-n_{12})^+, n_{21}\right)\Big)^+ + \Big(n_{22} - (n_{22}-n_{21})^+ - n_{22} + (n_{22}-n_{12})^+ \\
&\quad + \min\left((n_{22}-n_{21})^+, n_{12}\right)\Big)^+ \\
&= \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \Big(-(n_{11}-n_{12})^+ + (n_{11}-n_{21})^+ \\
&\quad + \min\left((n_{11}-n_{12})^+, n_{21}\right)\Big)^+ + \Big(-(n_{22}-n_{21})^+ + (n_{22}-n_{12})^+ \\
&\quad + \min\left((n_{22}-n_{21})^+, n_{12}\right)\Big)^+ \\
&= \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \Big(-(n_{11}-n_{12})^+ + (n_{11}-n_{21})^+ \\
&\quad + (n_{11}-n_{12})^+ - \left((n_{11}-n_{12})^+ - n_{21}\right)^+\Big)^+ + \Big(-(n_{22}-n_{21})^+ + (n_{22}-n_{12})^+ \\
&\quad + (n_{22}-n_{21})^+ - \left((n_{22}-n_{21})^+ - n_{12}\right)^+\Big)^+ \\
&= \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \Big((n_{11}-n_{21})^+ \\
&\quad - \left((n_{11}-n_{12})^+ - n_{21}\right)^+\Big)^+ + \Big((n_{22}-n_{12})^+ - \left((n_{22}-n_{21})^+ - n_{12}\right)^+\Big)^+ \\
&\overset{(c)}{=} \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \Big((n_{11}-n_{21})^+ \\
&\quad - \left((n_{11}-n_{21})^+ - n_{12}\right)^+\Big)^+ + \Big((n_{22}-n_{12})^+ - \left((n_{22}-n_{12})^+ - n_{21}\right)^+\Big)^+ \\
&= \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \left(\min\left((n_{11}-n_{21})^+, n_{12}\right)\right)^+ \\
&\quad + \left(\min\left((n_{22}-n_{12})^+, n_{21}\right)\right)^+ \\
&= \max\left((n_{11}-n_{12})^+, n_{21}\right) + \max\left((n_{22}-n_{21})^+, n_{12}\right) + \min\left((n_{11}-n_{21})^+, n_{12}\right) \\
&\quad + \min\left((n_{22}-n_{12})^+, n_{21}\right) \\
&= \left((n_{11}-n_{12})^+ - n_{21}\right)^+ + n_{21} + \left((n_{22}-n_{21})^+ - n_{12}\right)^+ + n_{12} + \min\left((n_{11}-n_{21})^+, n_{12}\right) \\
&\quad + \min\left((n_{22}-n_{12})^+, n_{21}\right) \\
&= \left((n_{11}-n_{12})^+ - n_{21}\right)^+ + n_{21} + \left((n_{22}-n_{21})^+ - n_{12}\right)^+ + n_{12} + \min\left((n_{11}-n_{21})^+, n_{12}\right) \\
&\quad + \min\left((n_{22}-n_{12})^+, n_{21}\right) + (n_{11}-n_{12})^+ - (n_{11}-n_{12})^+ + (n_{22}-n_{21})^+ - (n_{22}-n_{21})^+ \\
&= \max(n_{11}, n_{12}) + \max(n_{22}, n_{21}) + \left((n_{11}-n_{12})^+ - n_{21}\right)^+ + \left((n_{22}-n_{21})^+ - n_{12}\right)^+ \\
&\quad + \min\left((n_{11}-n_{21})^+, n_{12}\right) + \min\left((n_{22}-n_{12})^+, n_{21}\right) - (n_{11}-n_{12})^+ - (n_{22}-n_{21})^+ \\
&= \max(n_{11}, n_{12}) + \max(n_{22}, n_{21}) - \min\left((n_{11}-n_{12})^+, n_{21}\right) - \min\left((n_{22}-n_{21})^+, n_{12}\right) \\
&\quad + \min\left((n_{11}-n_{21})^+, n_{12}\right) + \min\left((n_{22}-n_{12})^+, n_{21}\right) \\
&= \max(n_{11}, n_{12}) + (n_{22}-n_{21})^+ + n_{21} - \min\left((n_{11}-n_{12})^+, n_{21}\right) - (n_{22}-n_{21})^+ \\
&\quad + \left((n_{22}-n_{21})^+ - n_{12}\right)^+ + \min\left((n_{11}-n_{21})^+, n_{12}\right) + \min\left((n_{22}-n_{12})^+, n_{21}\right) \\
&= \max(n_{11}, n_{12}) + n_{21} - \min\left((n_{11}-n_{12})^+, n_{21}\right) + \left((n_{22}-n_{21})^+ - n_{12}\right)^+ \\
&\quad + \min\left((n_{11}-n_{21})^+, n_{12}\right) + \min\left((n_{22}-n_{12})^+, n_{21}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \max(n_{11}, n_{12}) + n_{21} - n_{21} + \left(n_{21} - (n_{11} - n_{12})^+\right)^+ + \left((n_{22} - n_{21})^+ - n_{12}\right)^+ \\
&\quad + \min\left((n_{11} - n_{21})^+, n_{12}\right) + \min\left((n_{22} - n_{12})^+, n_{21}\right) \\
&= \max(n_{11}, n_{12}) + \left(n_{21} - (n_{11} - n_{12})^+\right)^+ + \left((n_{22} - n_{21})^+ - n_{12}\right)^+ \\
&\quad + \min\left((n_{11} - n_{21})^+, n_{12}\right) + \min\left((n_{22} - n_{12})^+, n_{21}\right) \\
&\overset{(c)}{=} \max(n_{11}, n_{12}) + \left(n_{21} - (n_{11} - n_{12})^+\right)^+ + \left((n_{22} - n_{21})^+ - n_{12}\right)^+ \\
&\quad + \min\left((n_{11} - n_{21})^+, n_{12}\right) + (n_{22} - n_{12})^+ - \left((n_{22} - n_{21})^+ - n_{12}\right)^+ \\
&= \max(n_{11}, n_{12}) + \left(n_{21} - (n_{11} - n_{12})^+\right)^+ + \min\left((n_{11} - n_{21})^+, n_{12}\right) \\
&\quad + (n_{22} - n_{12})^+, \qquad (108)
\end{aligned}
$$

where (c) follows from Remark 2 in appendix D.

Note that the upper bound on $R_1 + R_2$ in (108) is not active given the upper bound on $R_1 + R_2$ in (106). This completes the proof of (106). ■.

In this case it can be verified that the upper bound on $2R_1 + R_2$ and $R_1 + 2R_2$ are not active.

C.3 Capacity of the Symmetric LD-IC with Noisy Channel-Output Feedback

Theorem 1 simplifies to the following lemma for the case in which the interference channel is symmetric and there exists noisy channel-output feedback, i.e., $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$ and $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$.

Lemma 4 (Theorem 1 in [12]) The capacity region $\mathcal{C}(n, n, m, m, \ell, \ell)$ of the two-user symmetric LD-IC with noisy channel-output feedback (S-LD-IC-NFB) is the set of non-negative rate pairs (R_1, R_2) satisfying

$$
\begin{aligned}
R_1 &\leqslant \max(n, m), && (109)\\
R_2 &\leqslant \max(n, m), && (110)\\
R_1 &\leqslant n + (\ell - n)^+, && (111)\\
R_2 &\leqslant n + (\ell - n)^+, && (112)\\
R_1 + R_2 &\leqslant \max(n, m) + (n - m)^+, && (113)\\
R_1 + R_2 &\leqslant 2\max\left((n - m)^+, m\right) + 2\min\left((n - m)^+, \left(\ell - \max\left(m, (n - m)^+\right)\right)^+\right), && (114)\\
2R_1 + R_2 &\leqslant \max(n, m) + (n - m)^+ + \max\left((n - m)^+, m\right) + \min\big((n - m)^+, \\
&\qquad \left(\ell - \max\left(m, (n - m)^+\right)\right)^+\big), && (115)\\
R_1 + 2R_2 &\leqslant \max(n, m) + (n - m)^+ + \max\left((n - m)^+, m\right) + \min\big((n - m)^+, \\
&\qquad \left(\ell - \max\left(m, (n - m)^+\right)\right)^+\big). && (116)
\end{aligned}
$$

In the following, an alternative proof of Theorem 1 in [12] is presented using Theorem 1.

Proof of (109)**,** (110)**,** (111) **and** (112) **:** Consider (8a) and (8b) with $i = 1$ and $j = 2$, and let $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$ and $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$. This yields

$$
\begin{aligned}
R_1 &\leqslant \min\left(\max(n, m), \max(n, m)\right) \\
&= \max(n, m), \qquad (117)
\end{aligned}
$$

and

$$
\begin{aligned}
R_1 \leqslant & \min\Big(\max(n,m), \max\Big(n, \ell-(n-m)^+\Big)\Big),\\
= & \begin{cases} \max(n,m) & \text{if } n>m;\\ \max(n,m) & \text{if } n<m \text{ and } (\ell-n)^+ > (m-n)^+;\\ n+(\ell-n)^+ & \text{if } n<m \text{ and } (\ell-n)^+ < (m-n)^+. \end{cases}
\end{aligned} \tag{118}
$$

Inequalities (117) and (118) correspond to (109) and (111). The same procedure can be applied to prove (110) and (112) considering (8a) and (8b) with $i=2$ and $j=1$, and letting $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$ and $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$. ■

Proof of (113) **and** (114)**:** Consider (8c) with $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$ and $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$. This yields

$$
\begin{aligned}
R_1 + R_2 \leqslant & \min\Big(\max(n,m) + (n-m)^+, \max(n,m) + (n-m)^+\Big)\\
= & \max(n,m) + (n-m)^+.
\end{aligned} \tag{119}
$$

This completes the proof of (113).

Consider (8d) with $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$ and $\ell \leqslant \max(n,m)$. This yields

$$
\begin{aligned}
R_1 + R_2 \leqslant & \max\Big((n-m)^+, m\Big) + \max\Big((n-m)^+, m\Big)\\
& + \Big(\Big(\ell-(n-m)^+\Big)^+ - (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\
& + \Big(\Big(\ell-(n-m)^+\Big)^+ - (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\
= & 2\max\Big((n-m)^+, m\Big) + 2\Big(\Big(\ell-(n-m)^+\Big)^+ - (m-n)^+ - \min(n,m)\\
& + \min\Big((n-m)^+, m\Big)\Big)^+\\
= & 2\max\Big((n-m)^+, m\Big) + 2\Big(\Big(\ell-(n-m)^+\Big)^+ - (m-n)^+ - m + (m-n)^+\\
& + \min\Big((n-m)^+, m\Big)\Big)^+\\
= & 2\max\Big((n-m)^+, m\Big) + 2\Big(\Big(\ell-(n-m)^+\Big)^+ - m + \min\Big((n-m)^+, m\Big)\Big)^+\\
\overset{(d)}{=} & 2\max\Big((n-m)^+, m\Big) + 2\Big(\ell - \max\Big((n-m)^+, m\Big)\Big)^+\\
\overset{(e)}{=} & 2\max\Big((n-m)^+, m\Big) + 2\min\Big((n-m)^+, \Big(\ell - \max\Big(m, (n-m)^+\Big)\Big)^+\Big),
\end{aligned} \tag{120}
$$

where,
(d) follows from Remark 3 in appendix D,
(e) follows from Remark 4 in appendix D.

This completes the proof of (114). ■

Proof of (115) **and** (116)**:**

Consider (8e) with $i=1$ and $j=2$, and let $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$ and $\ell \leqslant \max(n,m)$. This yields

$$
\begin{aligned}
2R_1 + R_2 \leqslant & \max(n,m) + \max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big)\\
& + \Big(\Big(\ell-(n-m)^+\Big)^+ - (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+
\end{aligned}
$$

$$
\begin{aligned}
&= 2\max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big) + \bigg(\Big(\ell - (n-m)^+\Big)^+ - (m-n)^+ \\
&\quad - \min(n,m) + \min\Big((n-m)^+, m\Big)\bigg)^+ \\
&= 2\max(n,m) + (n-m)^+ - (n-m)^+ + \Big((n-m)^+ - m\Big)^+ + \bigg(\Big(\ell - (n-m)^+\Big)^+ \\
&\quad - (m-n)^+ - m + (m-n)^+ + \min\Big((n-m)^+, m\Big)\bigg)^+ \\
&= 2\max(n,m) + \Big((n-m)^+ - m\Big)^+ + \bigg(\Big(\ell - (n-m)^+\Big)^+ - m + \min\Big((n-m)^+, m\Big)\bigg)^+ \\
&= \max(n,m) + \max(n,m) + \Big((n-m)^+ - m\Big)^+ + \bigg(\Big(\ell - (n-m)^+\Big)^+ - m \\
&\quad + \min\Big((n-m)^+, m\Big)\bigg)^+ \\
&= \max(n,m) + (n-m)^+ + m + \Big((n-m)^+ - m\Big)^+ + \bigg(\Big(\ell - (n-m)^+\Big)^+ - m \\
&\quad + \min\Big((n-m)^+, m\Big)\bigg)^+ \\
&= \max(n,m) + (n-m)^+ + \max\Big((n-m)^+, m\Big) + \bigg(\Big(\ell - (n-m)^+\Big)^+ - m \\
&\quad + \min\Big((n-m)^+, m\Big)\bigg)^+ \\
&\overset{(f)}{=} \max(n,m) + (n-m)^+ + \max\Big((n-m)^+, m\Big) + \Big(\ell - \max\Big((n-m)^+, m\Big)\Big)^+ \\
&\overset{(g)}{=} \max(n,m) + (n-m)^+ + \max\Big((n-m)^+, m\Big) + \min\big((n-m)^+ \\
&\quad \Big(\ell - \max\Big(m, (n-m)^+\Big)\Big)^+\big),
\end{aligned}
$$

where,
(f) follows from Remark 3 in appendix D,
(g) follows from Remark 4 in appendix D.

This completes the proof of (115).

The same procedure can be applied to prove (8e) with $i = 2$ and $j = 1$, and letting $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$ and $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = \ell$.

C.4 Capacity of the Symmetric LD-IC with only one Perfect Channel-Output Feedback

Theorem 1 simplifies to the following lemma for the case in which the interference channel is symmetric and there exists only one perfect channel-output feedback, i.e., $\overrightarrow{n}_{11} = n$, $\overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \max(\overrightarrow{n}_{11}, n_{12})$, and $\overleftarrow{n}_{22} = 0$.

Lemma 5 (Theorem 4.1 **model** 1000 **in [10])** The capacity region $\mathcal{C}(n, n, m, m, \max(n,m), 0)$ of the two-user symmetric LD-IC with only one perfect channel-output feedback (S-LD-IC-1FB) between receiver 1 and transmitter 1 is included in the set of non-negative rate pairs (R_1, R_2) satisfying

$$
\begin{aligned}
R_1 &\leqslant n, && (121)\\
R_2 &\leqslant \max(n,m), && (122)\\
R_1 + R_2 &\leqslant \max(n,m) + (n-m)^+, && (123)\\
2R_1 + R_2 &\leqslant \max(n,m) + (n-m)^+ + \max(n-m, m). && (124)
\end{aligned}
$$

In the following, an alternative proof of Theorem 4.1 model 1000 in [10] is presented using Theorem 1.

Proof of (121)**:** Consider (8a) and (8b) with $i = 1$ and $j = 2$, and let $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \max(n, m)$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}R_1 \leqslant &\min\Big(\max(n,m), \max(n,m), \min\Big(\max(n,m), \max\Big(n, -(n-m)^+\Big)\Big)\Big)\\ &= \min(\max(n,m), \min(\max(n,m), n))\\ &= \min(\max(n,m), n)\\ &= n.\end{aligned}$$

This completes the proof of (121). ■

Proof of (122)**:** Consider (8a) and (8b) with $i = 2$ and $j = 1$, and let $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \max(n, m)$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}R_2 \leqslant &\min\Big(\max(n,m), \max(n,m), n + \min\Big((m-n)^+, \Big(\max(n,m) - n - \min\Big((n-m)^+, m\Big)\\ &- \Big((n-m)^+ - m\Big)^+ \Big)^+\Big)\Big)\\ &= \min\Big(\max(n,m), \max(n,m) \Big)\\ &= \max(n,m).\end{aligned}$$

This completes the proof of (122). ■

Proof of (123)**:** Consider (8c) with $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \max(n, m)$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}R_1 + R_2 \leqslant &\min\Big(\max(n,m) + (n-m)^+, \max(n,m) + (n-m)^+\Big)\\ &= \max(n,m) + (n-m)^+, \end{aligned} \tag{125}$$

and this is equal to (123). Now, It will be proved that the bound (8d) is not active.

Consider (8d) with $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \max(n, m)$ and $\overleftarrow{n}_{22} = 0$. This yields

$$\begin{aligned}R_1 + R_2 \leqslant &\max\Big((n-m)^+, m\Big) + \max\Big((n-m)^+, m\Big) + \Big(\Big(\max(n,m) - (n-m)^+\Big)^+\\ &- (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big) \Big)^+ + \Big(\Big(-(n-m)^+\Big)^+\\ &- (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big) \Big)^+\\ &= 2\max\Big((n-m)^+, m\Big) + \Big(\Big((n-m)^+ + m - (n-m)^+\Big)^+ - (m-n)^+ - \min(n,m)\\ &+ \min\Big((n-m)^+, m\Big) \Big)^+ + \Big(-(m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\ &= 2\max\Big((n-m)^+, m\Big) + \Big(m - (m-n)^+ - m + (m-n)^+ + \min\Big((n-m)^+, m\Big)\Big)^+\\ &+ \Big(-(m-n)^+ - m + (m-n)^+ + \min\Big((n-m)^+, m\Big)\Big)^+\\ &= 2\max\Big((n-m)^+, m\Big) + \min\Big((n-m)^+, m\Big) + \Big(-m + \min\Big((n-m)^+, m\Big)\Big)^+\\ &= 2\max\Big((n-m)^+, m\Big) + \min\Big((n-m)^+, m\Big) + \Big(-m + m - \Big(m - (n-m)^+\Big)^+\Big)^+\\ &= 2\max\Big((n-m)^+, m\Big) + \min\Big((n-m)^+, m\Big) + \Big(-\Big(m - (n-m)^+\Big)^+\Big)^+\end{aligned}$$

$$
\begin{aligned}
&=2\max\left((n-m)^+,m\right)+\min\left((n-m)^+,m\right)\\
&=2\left(\left(m-(n-m)^+\right)^++(n-m)^+\right)+m-\left(m-(n-m)^+\right)^+\\
&=\left(m-(n-m)^+\right)^++2\,(n-m)^++m\\
&=\max(n,m)+(n-m)^++\left(m-(n-m)^+\right)^+. \qquad (126)
\end{aligned}
$$

note that the upper bound (126) on R_1+R_2 is not active given the upper bound (125). This completes the proof of (123). ■

Proof of (124)**:**

Consider (8e) with $i=1$ and $j=2$, and let $\overrightarrow{n}_{11}=\overrightarrow{n}_{22}=n$, $n_{12}=n_{21}=m$, $\overleftarrow{n}_{11}=\max(n,m)$ and $\overleftarrow{n}_{22}=0$. This yields

$$
\begin{aligned}
2R_1+R_2\leqslant&\max(n,m)+\max(n,m)+(n-m)^+-\min\left((n-m)^+,m\right)\\
&+\left(\left(-(n-m)^+\right)^+-(m-n)^+-\min(n,m)+\min\left((n-m)^+,m\right)\right)^+\\
=&\max(n,m)+(n-m)^++m+(n-m)^+-(n-m)^++\left((n-m)^+-m\right)^+\\
&+\left(-(m-n)^+-\min(n,m)+\min\left((n-m)^+,m\right)\right)^+\\
=&\max(n,m)+(n-m)^++m+\left((n-m)^+-m\right)^+\\
&+\left(-(m-n)^+-m+(m-n)^++m-\left(m-(n-m)^+\right)\right)^+\\
=&\max(n,m)+(n-m)^++\max\left((n-m)^+,m\right)^++\left(-\left(m-(n-m)^+\right)\right)^+\\
=&\max(n,m)+(n-m)^++\max\left((n-m)^+,m\right)^+\\
=&\max(n,m)+(n-m)^++\max(n-m+,m)^+.
\end{aligned}
$$

This completes the proof of (124). ■

Finally, it will be proved that the bound on R_1+2R_2 is not active. From (122) and (123) it is possible to obtain a bound for R_1+2R_2 as follows

$$
R_1+2R_2\leqslant 2\max(n,m)+(n-m)^+. \qquad (127)
$$

Consider (8e) with $i = 2$ and $j = 1$, and let $\overrightarrow{n}_{11} = \overrightarrow{n}_{22} = n$, $n_{12} = n_{21} = m$, $\overleftarrow{n}_{11} = \max(n, m)$ and $\overleftarrow{n}_{22} = 0$. This yields

$$
\begin{aligned}
R_1 + 2R_2 \leqslant & \max(n,m) + \max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big)\\
&+ \Big(\Big(\max(n,m) - (n-m)^+\Big)^+ - (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\
=& 2\max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big)\\
&+ \Big(\Big((n-m)^+ + m - (n-m)^+\Big)^+ - (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\
=& 2\max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big)\\
&+ \Big(m - (m-n)^+ - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\
=& 2\max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big)\\
&+ \Big(\min(n,m) - \min(n,m) + \min\Big((n-m)^+, m\Big)\Big)^+\\
=& 2\max(n,m) + (n-m)^+ - \min\Big((n-m)^+, m\Big) + \min\Big((n-m)^+, m\Big)\\
=& 2\max(n,m) + (n-m)^+ .
\end{aligned} \tag{128}
$$

Note that the upper bounds (127) and (128) are identical, therefore the bound $R_1 + 2R_2$ is not active.

It is worth noting than the same procedure can be applied to prove the bounds for the model 0001 (symmetric LD-IC with only one perfect channel-output feedback between receiver 2 and transmitter 2 [10]. It is also worth noting that the model 1001 (symmetric LD-IC with perfect channel-output feedback) [10] corresponds to the Lemma 3 for symmetric parameters of the interference channel.

C.5 Sum-Capacity of the LD-IC with Source Cooperation

In the two-user IC-NOF, a transmitter sees a noisy version of the sum of its own transmitted signal and the interfering signal from the other transmitter. Hence, subject to a finite delay, one transmitter knows at least partially the information transmitted by the other transmitter in the network. This observation highlights the connections between the IC with feedback and the IC with source cooperation studied in [8]. These two channel models are related but they are not the same. There are two main differences between the two channel models. First, the channel-output signal observed by the transmitter in the case of IC-NOF is impaired by the noise in the feedback link and the noise in the forward channel. In the case of source cooperation, the cooperation signal is only affected by the noise in the cooperative link. Second, the cooperation between transmitters is direct and symmetric in the case of source cooperation. Conversely, in the case of IC-NOF, the signal that is observed by the transmitter is affected by the delay in the feedback link, and the part of the signal that was transmitted by the other transmitter is obtained from the substraction between the signal observed by the transmitter and the own signal that was transmitted previously. Then, the cooperation is not direct [12].

The two-user interference channel with source cooperation has two transmitters, i.e., 1 and 2, two receivers, i.e., 3 and 4, and it also has noisy connections between the two transmitters [8].

The sum-capacity region $\mathcal{C}(n_{1,3},\ n_{2,4},\ n_{2,3},\ n_{1,4},\ n_c,\ n_c)$ of the two-user symmetric LD-IC with source cooperation (LD-IC-SC) is the minimum of the following [8]:

$$
\begin{aligned}
R_1 + R_2 &= \max\left(n_{1,3} - n_{1,4} + n_c, n_{2,3}, n_c\right) + \max\left(n_{2,4} - n_{2,3} + n_c, n_{1,4}, n_c\right), && \text{(129a)}\\
R_1 + R_2 &= \max\left(n_{1,3}, n_{2,3}\right) + \left(\max\left(n_{2,4}, n_{2,3}, n_c\right) - n_{2,3}\right), && \text{(129b)}\\
R_1 + R_2 &= \max\left(n_{2,4}, n_{1,4}\right) + \left(\max\left(n_{1,3}, n_{1,4}, n_c\right) - n_{1,4}\right), && \text{(129c)}\\
R_1 + R_2 &= \max\left(n_{1,3}, n_c\right) + \max\left(n_{2,4}, n_c\right), && \text{(129d)}\\
R_1 + R_2 &= \max\left(n_{1,3} + n_{2,4}, n_{1,4} + n_{2,3}\right), \text{if } n_{1,3} - n_{2,3} \neq n_{1,4} - n_{2,4}; \text{or} && \\
&\qquad \max\left(n_{1,3}, n_{2,4}, n_{1,4}, n_{2,3}\right), \text{otherwise}, && \text{(129e)}
\end{aligned}
$$

In order to establish a connection between (129) and the sum-rate capacity in Theorem 1, consider the following notation: $n_{1,3} = \overrightarrow{n}_{11}$, $n_{2,4} = \overrightarrow{n}_{22}$, $n_{2,3} = n_{12}$, $n_{1,4} = n_{21}$, and $n_c = \overleftarrow{n}_{11} - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ = \overleftarrow{n}_{22} - \left(\overrightarrow{n}_{22} - n_{21}\right)^+$. The last equality implies that the feedback must include the signal levels that contain information about the non-intended source in order to allow cooperation between the sources (see Figure 23). Consider the following two examples.

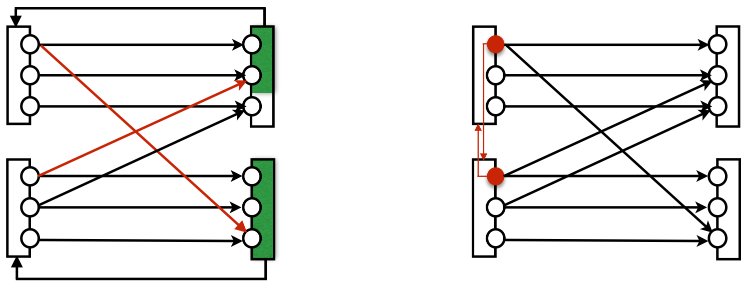

Figure 23: Source Cooperation vs Noisy Channel-Output Feedback I.

Example 1: Let $\overrightarrow{n}_{11} = n_{1,3} = 3$, $\overrightarrow{n}_{22} = n_{2,4} = 3$, $n_{12} = n_{2,3} = 2$, $n_{21} = n_{1,4} = 1$ and $n_{1,2} = n_{2,1} = n_c = \overleftarrow{n}_{11} - \left(\overrightarrow{n}_{11} - n_{12}\right)^+ = \overleftarrow{n}_{22} - \left(\overrightarrow{n}_{22} - n_{21}\right)^+ = 1$ (see Figure 23). Plugging these values in (129a)-(129e) and (8c)-(8d), it yields

$$
\begin{aligned}
R_1 + R_2 &= \max\left(3 - 1 + 1, 2, 1\right) + \max\left(3 - 2 + 1, 1, 1\right) = 5, && \text{(130a)}\\
R_1 + R_2 &= \max\left(3, 2\right) + \left(\max\left(3, 2, 1\right) - 2\right) = 4, && \text{(130b)}\\
R_1 + R_2 &= \max\left(3, 1\right) + \left(\max\left(3, 1, 1\right) - 1\right) = 5, && \text{(130c)}\\
R_1 + R_2 &= \max\left(3, 1\right) + \max\left(3, 1\right) = 6, && \text{(130d)}\\
R_1 + R_2 &= \max\left(3 + 3, 1 + 2\right) = 6, && \text{(130e)}
\end{aligned}
$$

and

$$
\begin{aligned}
R_1 + R_2 &\leqslant \min\left(\max\left(3, 2\right) + \left(3 - 2\right)^+, \max\left(3, 1\right) + \left(3 - 1\right)^+\right) = 4, && \text{(131a)}\\
R_1 + R_2 &\leqslant \max\left(\left(3 - 2\right)^+, 1\right) + \max\left(\left(3 - 1\right)^+, 2\right)\\
&\quad + \Big(\left(\min\left(3, \max\left(3, 2\right)\right) - \left(3 - 2\right)^+\right)^+ - \left(2 - 3\right)^+ - \min\left(3, 1\right) + \min\left(\left(3 - 2\right)^+, 1\right)\Big)^+\\
&\quad + \Big(\left(\min\left(3, \max\left(3, 1\right)\right) - \left(3 - 1\right)^+\right)^+ - \left(1 - 3\right)^+ - \min\left(3, 2\right) + \min\left(\left(3 - 1\right)^+, 2\right)\Big)^+,\\
&= 6. && \text{(131b)}
\end{aligned}
$$

The minimum value among (130a)-(130e) is equal to 4 and the minimum value between (131a) and (131b) is equal to 4. Then, the sum-capacity in [8] is equal than the sum-rate in Theorem 1 for this specific example. It is worth noting that the bound (131a) corresponds to the sum-rate in [4].

Example 2: Let $\overrightarrow{n}_{11} = n_{1,3} = 1$, $\overrightarrow{n}_{22} = n_{2,4} = 5$, $n_{12} = n_{2,3} = 3$, $n_{21} = n_{1,4} = 6$ and $n_{1,2} = n_{2,1} = n_c = \overleftarrow{n}_{11} - (\overrightarrow{n}_{11} - n_{12})^+ = \overleftarrow{n}_{22} - (\overrightarrow{n}_{22} - n_{21})^+ = 6$ (see Figure 24). Plugging these values in (129a)-(129e) and (8c)-(8d), it yields

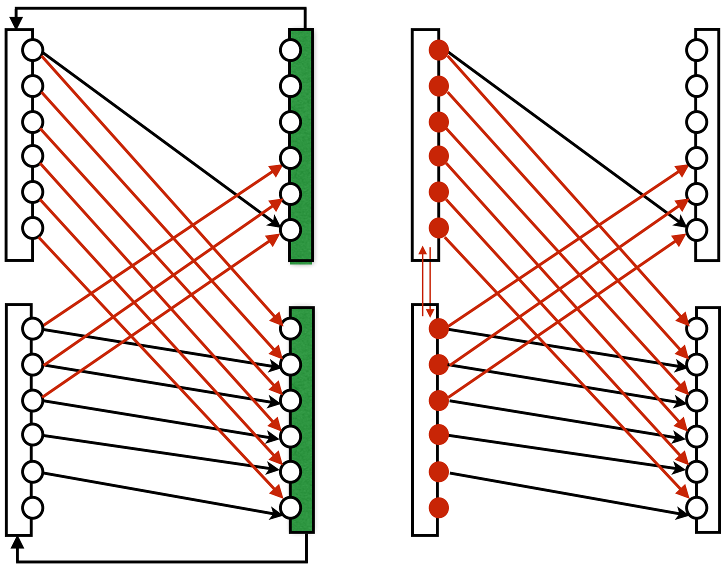

Figure 24: Source Cooperation vs Noisy Channel-Output Feedback II.

$$R_1 + R_2 = \max(1-6+6, 3, 6) + \max(5-3+6, 6, 6) = 14, \quad (132a)$$
$$R_1 + R_2 = \max(1, 3) + (\max(5, 3, 6) - 3) = 6, \quad (132b)$$
$$R_1 + R_2 = \max(5, 6) + (\max(1, 6, 6) - 6) = 6, \quad (132c)$$
$$R_1 + R_2 = \max(1, 6) + \max(5, 6) = 12, \quad (132d)$$
$$R_1 + R_2 = \max(1+5, 3+6) = 9, \quad (132e)$$

and

$$R_1+R_2 \leqslant \min\Big(\max(1,3) + (5-3)^+, \max(5,6) + (1-6)^+\Big) = 5, \quad (133a)$$
$$\begin{aligned}R_1+R_2 \leqslant\ & \max\Big((1-3)^+, 6\Big) + \max\Big((5-6)^+, 3\Big)\\ &+\Big(\Big(\min(6, \max(1,3)) - (1-3)^+\Big)^+ - (3-1)^+ - \min(1,6) + \min\Big((1-3)^+, 6\Big)\Big)^+\\ &+\Big(\Big(\min(6, \max(5,6)) - (5-6)^+\Big)^+ - (6-5)^+ - \min(5,3) + \min\Big((5-6)^+, 3\Big)\Big)^+,\\ =\ &12. \end{aligned} \quad (133b)$$

The minimum value among (130a)-(130e) is equal to 6 and the minimum value between (131a) and (131b) is equal to 5. Then, the sum-capacity in [8] is different than the sum-rate in Theorem 1 for this specific example. In the case of LD-IC with source cooperation, each transmitter knows what is transmitted by the other for transmitter in every channel-use. In the case of LD-IC-NOF, transmitter 1 can receive partially what was transmitted by transmitter 2, because the cross connection is not strong enough, in spite of the strong feedback which include all the signal levels in receiver 1. The last two examples allow to conclude that the two channel models are related but they are not the same.

D Auxiliary Results

This appendix provides auxiliary results that are used in the different proofs of this technical report.

Remark 1 The expressions $\Big(-(n_{12}-n_{11})^{+}-\min(n_{11},n_{21})+\min\big((n_{11}-n_{12})^{+},n_{21}\big)\Big)^{+}$ and $\big(-(n_{21}-n_{22})^{+}-\min(n_{22},n_{12})+\min\big((n_{22}-n_{21})^{+},n_{12}\big)\big)^{+}$ are both equal to zero.

The Remark 1 is proved considering all the possible cases given the parameters of the linear deterministic interference channel as follows

Let $A=\Big(-(n_{12}-n_{11})^{+}-\min(n_{11},n_{21})+\min\big((n_{11}-n_{12})^{+},n_{21}\big)\Big)^{+}$. There are two cases: $n_{11}>n_{12}$ and $n_{11}<n_{12}$.

Case 1: If $n_{11}>n_{12}$ then

$$A=(-\min(n_{11},n_{21})+\min(n_{11}-n_{12},n_{21}))^{+},$$

and there are two possible subcases: $n_{11}-n_{12}>n_{21}$ and $n_{11}-n_{12}<n_{21}$.

Case 1.1: if $n_{11}-n_{12}>n_{21}$ $(n_{11}>n_{21})$ then

$$\begin{aligned}A=&(-n_{21}+n_{21})^{+}\\=&0.\end{aligned}$$

Case 1.2: if $n_{11}-n_{12}<n_{21}$ then

$$\begin{aligned}A=&(-\min(n_{11},n_{21})+n_{11}-n_{12})^{+}\\=&\Big(-n_{11}+(n_{11}-n_{21})^{+}+n_{11}-n_{12}\Big)^{+}\\=&\Big((n_{11}-n_{21})^{+}-n_{12}\Big)^{+}\\=&0,\end{aligned}$$

given the condition in this case that $n_{11}-n_{21}<n_{12}$.

Case 2: If $n_{11}<n_{12}$ then

$$\begin{aligned}A=&(-n_{12}+n_{11}-\min(n_{11},n_{21}))^{+},\\=&\Big(-n_{12}+n_{11}-n_{11}+(n_{11}-n_{21})^{+}\Big)^{+}\\=&\Big(-n_{12}+(n_{11}-n_{21})^{+}\Big)^{+},\end{aligned}$$

and there are two possible subcases: $n_{11}>n_{21}$ and $n_{11}<n_{21}$.

Case 2.1: If $n_{11}>n_{21}$ then

$$\begin{aligned}A=&(-n_{12}+n_{11}-n_{21})^{+}\\=&0,\end{aligned}$$

given the condition in case 2 that $n_{11}<n_{12}$.

Case 2.2: If $n_{11}<n_{21}$ then

$$\begin{aligned}A=&(-n_{12})^{+}\\=&0.\end{aligned}$$

Therefore, the expression $\Big(-\left(n_{12}-n_{11}\right)^{+}-\min\left(n_{11},n_{21}\right)+\min\Big(\left(n_{11}-n_{12}\right)^{+},n_{21}\Big)\Big)^{+}$ is equal to zero. A similar procedure can be applied to prove that $\Big(-\left(n_{21}-n_{22}\right)^{+}-\min\left(n_{22},n_{12}\right)+\min\Big(\left(n_{22}-n_{21}\right)^{+},n_{12}\Big)\Big)^{+}$ is equal to zero.

Remark 2 For all A, B and $C \in \mathbb{Z}^{+}$, the expression $\Big((A-B)^{+}-C\Big)^{+}=\Big((A-C)^{+}-B\Big)^{+}$ is an identity.

The Remark 2 is proved as follows

$$\begin{aligned}\Big((A-B)^{+}-C\Big)^{+}=&(\max\left(A-B,0\right)-C)^{+}\\=&(\max\left(A-B-C,-C\right))^{+}\\=&\max\left(A-B-C,0\right)\\=&\max\left(A-C,B\right)-B\\=&\max\left(A-C,B,0\right)-B\\=&\max\left(\max\left(A-C,0\right),B\right)-B\\=&\max\Big((A-C)^{+},B\Big)-B\\=&\max\Big((A-C)^{+}-B,0\Big)\\=&\Big((A-C)^{+}-B\Big)^{+},\end{aligned} \tag{134}$$

and this proves Remark 2.

Remark 3 The expression $\Big(\Big(\ell-(n-m)^{+}\Big)^{+}-m+\min\Big((n-m)^{+},m\Big)\Big)^{+}$ is equal to the expression $\Big(\ell-\max\Big((n-m)^{+},m\Big)\Big)^{+}$.

The Remark 3 is proved as follows

Let $B=\Big(\Big(\ell-(n-m)^{+}\Big)^{+}-m+\min\Big((n-m)^{+},m\Big)\Big)^{+}$. There are two cases: $\ell<(n-m)^{+}$ and $\ell>(n-m)^{+}$.

Case 1: If $\ell<(n-m)^{+}$ then

$$\begin{aligned}B=&\Big(-m+\min\Big((n-m)^{+},m\Big)\Big)^{+}\\=&0.\end{aligned} \tag{135}$$

Case 2: If $\ell>(n-m)^{+}$ then

$$\begin{aligned}B=&\Big(\ell-(n-m)^{+}-m+\min\Big((n-m)^{+},m\Big)\Big)^{+}\\=&\Big(\ell-(n-m)^{+}-m+(n-m)^{+}-\Big((n-m)^{+}-m\Big)^{+}\Big)^{+}\\=&\Big(\ell-m-\Big((n-m)^{+}-m\Big)^{+}\Big)^{+}\\=&\Big(\ell-\max\Big((n-m)^{+},m\Big)\Big)^{+}.\end{aligned} \tag{136}$$

Equality (136) is equal to equality (135) when $\ell<(n-m)^{+}$.

Therefore, $\Big(\Big(\ell-(n-m)^{+}\Big)^{+}-m+\min\Big((n-m)^{+},m\Big)\Big)^{+}=\Big(\ell-\max\Big((n-m)^{+},m\Big)\Big)^{+}$.

Remark 4 The expression $\min\Big((n-m)^+, \big(\ell - \max\big((n-m)^+, m\big)\big)^+\Big)$ is equal to the expression $\big(\ell - \max\big((n-m)^+, m\big)\big)^+$.

The Remark 4 is proved determining the possible conditions under which the expression $\min\Big((n-m)^+, \big(\ell - \max\big((n-m)^+, m\big)\big)^+\Big)$ is equal to $(n-m)^+$.

Therefore, $\min\Big((n-m)^+, \big(\ell - \max\big((n-m)^+, m\big)\big)^+\Big) = (n-m)^+$ if

$$\begin{aligned}(n-m)^+ &< \big(\ell - \max\big((n-m)^+, m\big)\big)^+ \\ &= \ell - \max\big((n-m)^+, m\big) \\ &= \ell - \big(m - (n-m)^+\big)^+ - (n-m)^+ . \end{aligned} \tag{137}$$

The condition (138) is equivalent to

$$\ell > 2\,(n-m)^+ + \big(m - (n-m)^+\big)^+ , \tag{138}$$

and there are two possible cases: $n < m$ and $n > m$.

Case 1: if $n < m$ then

$$\ell > m, \tag{139}$$

and the condition (139) is not possible because $\ell \leqslant \max(n, m) = m$.

Case 2: if $n > m$ then

$$\ell > 2\,(n-m) + (2m-n)^+ , \tag{140}$$

and there are two possible subcases: $2m > n$ and $2m < n$.

Case 2.1: if $2m > n$ then

$$\begin{aligned}\ell &> 2\,(n-m) + 2m - n \\ &= n, \end{aligned} \tag{141}$$

and the condition (141) is not possible because $\ell \leqslant \max(n, m) = n$.

Case 2.2: if $2m < n$ then

$$\ell > 2\,(n-m) \tag{142}$$

$$= n + n - 2m$$

$$> n, \tag{143}$$

$$\tag{144}$$

and the condition (143) is not possible because $\ell \leqslant \max(n, m) = n$. All these cases allow to prove that $\min\Big((n-m)^+, \big(\ell - \max\big((n-m)^+, m\big)\big)^+\Big) = (n-m)^+$.

E Simplification on the bounds in the capacity region of the two-user linear deterministic interference channel with noisy-channel output feedback

Lemma 6 provides a simplification on the capacity region of the two-user linear deterministic interference channel with noisy channel output feedback presented in Theorem 1.

Lemma 6 The capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the two-user LD-IC-NOF is the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$:

$$
\begin{aligned}
R_i &\leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right), && \text{(145a)}\\
R_i &\leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, \overleftarrow{n}_{jj} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\right)\right), && \text{(145b)}\\
R_1 + R_2 &\leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \left(\overrightarrow{n}_{11} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right), && \text{(145c)}\\
R_1 + R_2 &\leqslant \max\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}, \overrightarrow{n}_{11} - \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right)\\
&\quad + \max\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}, \overrightarrow{n}_{22} - \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right), && \text{(145d)}\\
2R_i + R_j &\leqslant \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+\\
&\quad + \max\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}, \overrightarrow{n}_{jj} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right). && \text{(145e)}
\end{aligned}
$$

Proof: The bound (8a) corresponds to a min-cut set, and it comes from the case of perfect channel-output feedback [4]. For $i = 1$ and $j = 2$, the bound (8a) can be expressed as follows

$$R_1 \leqslant \min\left(\max\left(\overrightarrow{n}_{11}, n_{21}\right), \max\left(\overrightarrow{n}_{11}, n_{12}\right)\right). \tag{146}$$

The bound (146) implies that the information from transmitter 1 to receiver 1 can not exceed the maximum amount of information that gets out from transmitter 1 (source) and the maximum amount of information that gets into receiver 1 (destination). The first term is related to the maximum amount of information generated by transmitter 1, and the second term is the maximum amount of information that arrives to receiver 1. The same interpretation can be done for $i = 2$ and $j = 1$ in (8a).

The bound (8b) corresponds to an "improved" min-cut set. For $i = 1$ and $j = 2$, the bound (8b) can be expressed as follows

$$R_1 \leqslant \min\left(\max\left(\overrightarrow{n}_{11}, n_{21}\right), \max\left(\overrightarrow{n}_{11}, \overleftarrow{n}_{22} - \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right)\right). \tag{147}$$

The term n_{12} in (146) is replaced by $\overleftarrow{n}_{22} - \left(\overrightarrow{n}_{22} - n_{21}\right)^+$ in (147). The term $\overleftarrow{n}_{22} - \left(\overrightarrow{n}_{22} - n_{21}\right)^+$ represents the number of signal levels that are included in the feedback and are affected by interference (signal levels that contain information from the opposite source). The same interpretation can be done for $i = 2$ and $j = 1$ in (8b).

The bound (8c) also corresponds to a min-cut set, and it comes from the case of perfect channel-output feedback [4]. The bound (8c) can be rewritten as follows

$$
\begin{aligned}
R_1 + R_2 &\leqslant \min\left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) + \left(\overrightarrow{n}_{22} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{22}, n_{21}\right) + \left(\overrightarrow{n}_{11} - n_{21}\right)^+\right)\\
&= \min\left(n_{12} + \left(\overrightarrow{n}_{11} - n_{12}\right)^+ + \left(\overrightarrow{n}_{22} - n_{12}\right)^+, n_{21} + \left(\overrightarrow{n}_{22} - n_{21}\right)^+ + \left(\overrightarrow{n}_{11} - n_{21}\right)^+\right)\\
&= \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \left(\overrightarrow{n}_{11} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right). && \text{(148)}
\end{aligned}
$$

The terms $\max(\overrightarrow{n}_{22}, n_{12})$ and $\max(\overrightarrow{n}_{11}, n_{21})$ in bound (**??**) correspond to the maximum information that gets out from transmitters 2 and 1 respectively. The terms $(\overrightarrow{n}_{11} - n_{12})^+$ and $(\overrightarrow{n}_{22} - n_{21})^+$ represent the signal levels that are not affected by interference in receivers 1 and 2 respectively. The bound (148) corresponds to the bound (145c) in Lemma 6.

The bound (8d) can be further simplified as follows

$$
\begin{aligned}
R_1+R_2 \leqslant\ & \max\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big) + \max\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\\
& +\Big(\Big(\min(\overleftarrow{n}_{11}, \max(\overrightarrow{n}_{11}, n_{12})) - (\overrightarrow{n}_{11}-n_{12})^+\Big)^+ - (n_{12}-\overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
& +\min\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big)\Big)^+ + \Big(\Big(\min(\overleftarrow{n}_{22}, \max(\overrightarrow{n}_{22}, n_{21})) - (\overrightarrow{n}_{22}-n_{21})^+\Big)^+\\
& -(n_{21}-\overrightarrow{n}_{22})^+ - \min(\overrightarrow{n}_{22}, n_{12}) + \min\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\Big)^+\\
=\ & \max\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big) + \max\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\\
& +\Big(\Big(\max(\overrightarrow{n}_{11}, n_{12}) - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+ - (\overrightarrow{n}_{11}-n_{12})^+\Big)^+ - (n_{12}-\overrightarrow{n}_{11})^+\\
& -\min(\overrightarrow{n}_{11}, n_{21}) + \min\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big)\Big)^+ + \Big(\Big(\max(\overrightarrow{n}_{22}, n_{21})\\
& -(\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+ - (\overrightarrow{n}_{22}-n_{21})^+\Big)^+ - (n_{21}-\overrightarrow{n}_{22})^+ - \min(\overrightarrow{n}_{22}, n_{12})\\
& +\min\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\Big)^+\\
=\ & \max\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big) + \max\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\\
& +\Big(\Big(n_{12} + (\overrightarrow{n}_{11}-n_{12})^+ - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+ - (\overrightarrow{n}_{11}-n_{12})^+\Big)^+ - (n_{12}-\overrightarrow{n}_{11})^+\\
& -\min(\overrightarrow{n}_{11}, n_{21}) + \min\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big)\Big)^+ + \Big(\Big(n_{21} + (\overrightarrow{n}_{22}-n_{21})^+\\
& -(\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+ - (\overrightarrow{n}_{22}-n_{21})^+\Big)^+ - (n_{21}-\overrightarrow{n}_{22})^+ - \min(\overrightarrow{n}_{22}, n_{12})\\
& +\min\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\Big)^+\\
=\ & \max\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big) + \max\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\\
& +\Big(\Big(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\Big)^+ - (n_{12}-\overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21})\\
& +\min\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}\Big)\Big)^+ + \Big(\Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+ - (n_{21}-\overrightarrow{n}_{22})^+\\
& -\min(\overrightarrow{n}_{22}, n_{12}) + \min\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}\Big)\Big)^+\\
\stackrel{(a)}{=}\ & \max\Big((\overrightarrow{n}_{11}-n_{12})^+, n_{21}, n_{21} + (\overrightarrow{n}_{11}-n_{12})^+ + \Big(n_{12} - (\max(\overrightarrow{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\Big)^+\\
& -(n_{12}-\overrightarrow{n}_{11})^+ - \min(\overrightarrow{n}_{11}, n_{21})\Big) + \max\Big((\overrightarrow{n}_{22}-n_{21})^+, n_{12}, n_{12} + (\overrightarrow{n}_{22}-n_{21})^+\\
& +\Big(n_{21} - (\max(\overrightarrow{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\Big)^+ - (n_{21}-\overrightarrow{n}_{22})^+ - \min(\overrightarrow{n}_{22}, n_{12})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(b)}{=}\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},n_{21}+\left(\overrightarrow{n}_{11}-n_{12}\right)^{+}+n_{12}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\\
&\quad-\left(n_{12}-\overrightarrow{n}_{11}\right)^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)\Big)+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},n_{12}+\left(\overrightarrow{n}_{22}-n_{21}\right)^{+}\\
&\quad+n_{21}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)-\left(n_{21}-\overrightarrow{n}_{22}\right)^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},n_{21}+\max\left(\overrightarrow{n}_{11},n_{12}\right)-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\\
&\quad-\left(n_{12}-\overrightarrow{n}_{11}\right)^{+}-\min\left(\overrightarrow{n}_{11},n_{21}\right)\Big)+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},n_{12}+\max\left(\overrightarrow{n}_{22},n_{21}\right)\\
&\quad-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)-\left(n_{21}-\overrightarrow{n}_{22}\right)^{+}-\min\left(\overrightarrow{n}_{22},n_{12}\right)\Big)\\
&\overset{(c)}{=}\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},n_{21}+\max\left(\overrightarrow{n}_{11},n_{12}\right)-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\\
&\quad-\max\left(\overrightarrow{n}_{11},n_{12}\right)+\left(\overrightarrow{n}_{11}-n_{21}\right)^{+}\Big)+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},n_{12}+\max\left(\overrightarrow{n}_{22},n_{21}\right)\\
&\quad-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)-\max\left(\overrightarrow{n}_{22},n_{21}\right)+\left(\overrightarrow{n}_{22}-n_{12}\right)^{+}\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},n_{21}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)+\left(\overrightarrow{n}_{11}-n_{21}\right)^{+}\Big)\\
&\quad+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},n_{12}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)+\left(\overrightarrow{n}_{22}-n_{12}\right)^{+}\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},\max\left(\overrightarrow{n}_{11},n_{21}\right)-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\Big)\\
&\quad+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},\max\left(\overrightarrow{n}_{22},n_{12}\right)-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},\max\Big(\overrightarrow{n}_{11}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right),n_{21}\\
&\quad-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\Big)\Big)+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},\\
&\quad\max\Big(\overrightarrow{n}_{22}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right),n_{12}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)\Big)\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},\overrightarrow{n}_{11}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right),n_{21}\\
&\quad-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\Big)+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},\\
&\quad\overrightarrow{n}_{22}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right),n_{12}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},\overrightarrow{n}_{11}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)\Big)\\
&\quad+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},\overrightarrow{n}_{22}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)\Big)\\
&=\max\Big(\left(\overrightarrow{n}_{11}-n_{12}\right)^{+},n_{21},\overrightarrow{n}_{11}-\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\Big)\\
&\quad+\max\Big(\left(\overrightarrow{n}_{22}-n_{21}\right)^{+},n_{12},\overrightarrow{n}_{22}-\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\Big),
\end{aligned}
\tag{149}
$$

where,
(a) follows from the fact that

$$
\begin{aligned}
\max\left(A,B\right)+\left(C+\min\left(A,B\right)\right)^{+}&=\max\left(A,B\right)+\max\left(0,C+\min\left(A,B\right)\right)\\
&=\max\left(\max\left(A,B\right),C+\max\left(A,B\right)+\min\left(A,B\right)\right)\\
&=\max\left(A,B,A+B+C\right),
\end{aligned}
$$

(b) follows from the fact that

$$\begin{aligned}\left(n_{12}-\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)^{+}=&n_{12}-n_{12}+\left(n_{12}-\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right)^{+}\\=&n_{12}-\min\left(n_{12},\left(\max\left(\overrightarrow{n}_{11},n_{12}\right)-\overleftarrow{n}_{11}\right)^{+}\right),\end{aligned}$$

and

$$\left(n_{21}-\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right)^{+}=n_{21}-\min\left(n_{21},\left(\max\left(\overrightarrow{n}_{22},n_{21}\right)-\overleftarrow{n}_{22}\right)^{+}\right),$$

(c) follows from the fact that

$$\begin{aligned}\left(n_{12}-\overrightarrow{n}_{11}\right)^{+}+\min\left(\overrightarrow{n}_{11},n_{21}\right)=&\left(n_{12}-\overrightarrow{n}_{11}\right)^{+}+\overrightarrow{n}_{11}-\left(\overrightarrow{n}_{11}-n_{21}\right)^{+}\\=&\max\left(\overrightarrow{n}_{11},n_{12}\right)-\left(\overrightarrow{n}_{11}-n_{21}\right)^{+}\end{aligned}$$

and

$$\left(n_{21}-\overrightarrow{n}_{22}\right)^{+}+\min\left(\overrightarrow{n}_{22},n_{12}\right)=\max\left(\overrightarrow{n}_{22},n_{21}\right)-\left(\overrightarrow{n}_{22}-n_{12}\right)^{+}.$$

The bound (149) corresponds to the bound (145d) in Lemma 6.

The bound (8e) can be further simplified as follows

$$\begin{aligned}2R_i+R_j\leqslant&\max\left(\overrightarrow{n}_{jj},n_{ji}\right)+\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}-\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\\&+\Big(\left(\min\left(\overleftarrow{n}_{jj},\max\left(\overrightarrow{n}_{jj},n_{ji}\right)\right)-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\right)^{+}-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)\\&+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\Big)^{+}\\=&\max\left(\overrightarrow{n}_{jj},n_{ji}\right)+\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}-\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\\&+\Big(\left(\max\left(\overrightarrow{n}_{jj},n_{ji}\right)-\left(\max\left(\overrightarrow{n}_{jj},n_{ji}\right)-\overleftarrow{n}_{jj}\right)^{+}-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\right)^{+}-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}\\&-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\Big)^{+}\\=&\max\left(\overrightarrow{n}_{jj},n_{ji}\right)+\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}-\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\\&+\Big(\left(n_{ji}+\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}-\left(\max\left(\overrightarrow{n}_{jj},n_{ji}\right)-\overleftarrow{n}_{jj}\right)^{+}-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\right)^{+}-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}\\&-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\Big)^{+}\\=&\max\left(\overrightarrow{n}_{jj},n_{ji}\right)+\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}-\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\\&+\Big(\left(n_{ji}-\left(\max\left(\overrightarrow{n}_{jj},n_{ji}\right)-\overleftarrow{n}_{jj}\right)^{+}\right)^{+}-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)\\&+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\Big)^{+}\\=&\max\left(\overrightarrow{n}_{jj},n_{ji}\right)+\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}-n_{ij}+\left(n_{ij}-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\right)^{+}\\&+\Big(\left(n_{ji}-\left(\max\left(\overrightarrow{n}_{jj},n_{ji}\right)-\overleftarrow{n}_{jj}\right)^{+}\right)^{+}-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)\\&+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\Big)^{+},\\=&\max\left(\overrightarrow{n}_{jj},n_{ji}\right)+\max\left(\overrightarrow{n}_{ii},n_{ij}\right)+\left(\overrightarrow{n}_{ii}-n_{ji}\right)^{+}-n_{ij}+\left(n_{ij}-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}\right)^{+}\\&+\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}-\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+}+\Big(\left(n_{ji}-\left(\max\left(\overrightarrow{n}_{jj},n_{ji}\right)-\overleftarrow{n}_{jj}\right)^{+}\right)^{+}-\left(n_{ji}-\overrightarrow{n}_{jj}\right)^{+}\\&-\min\left(\overrightarrow{n}_{jj},n_{ij}\right)+\min\left(\left(\overrightarrow{n}_{jj}-n_{ji}\right)^{+},n_{ij}\right)\Big)^{+}\end{aligned}$$

$$
\begin{aligned}
&= \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) + \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+} - n_{ij} + \max\Big(n_{ij}, \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}\Big) \\
&\quad - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+} + \bigg(\Big(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)^{+} - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\Big)\bigg)^{+} \\
&= n_{ji} + \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+} + n_{ij} + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+} - n_{ij} + \max\Big(n_{ij}, \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}\Big) \\
&\quad - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+} + \bigg(\Big(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)^{+} - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\Big)\bigg)^{+} \\
&= n_{ji} + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+} + \max\Big(n_{ij}, \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}\Big) \\
&\quad + \bigg(\Big(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)^{+} - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\Big)\bigg)^{+} \\
&= \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\Big(n_{ij}, \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}\Big) \\
&\quad + \bigg(\Big(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)^{+} - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) \\
&\quad + \min\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}\Big)\bigg)^{+} \\
&\overset{(d)}{=} \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, n_{ij} + \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+} \\
&\quad + \Big(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)^{+} - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right)\bigg), \\
&\overset{(e)}{=} \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, n_{ij} + \left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+} \\
&\quad n_{ji} - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big) - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right)\bigg) \\
&= \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, n_{ij} + \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) \\
&\quad - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big) - \left(n_{ji} - \overrightarrow{n}_{jj}\right)^{+} - \min\left(\overrightarrow{n}_{jj}, n_{ij}\right)\bigg) \\
&= \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, n_{ij} + \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) \\
&\quad - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big) - \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) + \left(\overrightarrow{n}_{jj} - n_{ij}\right)^{+}\bigg) \\
&= \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, n_{ij} \\
&\quad - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big) + \left(\overrightarrow{n}_{jj} - n_{ij}\right)^{+}\bigg) \\
&= \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, \max\left(\overrightarrow{n}_{jj}, n_{ij}\right) \\
&\quad - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)\bigg) \\
&= \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^{+} + \max\bigg(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^{+}, n_{ij}, \\
&\quad \max\Big(\overrightarrow{n}_{jj} - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big), n_{ij} - \min\Big(n_{ji}, \\
&\quad \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big)\Big)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
=&\max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ + \max\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij},\\
&\overrightarrow{n}_{jj} - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big), n_{ij} - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big)\Big)\\
=&\max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ + \max\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\\
&\overrightarrow{n}_{jj} - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big)\Big)\\
=&\max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ + \max\Big(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}, \overrightarrow{n}_{jj} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big),
\end{aligned} \tag{150}
$$

where,
(d) follows from the fact that $\max(A, B) + (C + \min(A, B))^+ = \max(A, B, A + B + C)$;
(e) follows from the fact that

$$
\Big(n_{ji} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big)^+ = n_{ji} - \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\Big),
$$

(f) follows form the fact that $\left(n_{ji} - \overrightarrow{n}_{jj}\right)^+ + \min\left(\overrightarrow{n}_{jj}, n_{ij}\right) = \max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \left(\overrightarrow{n}_{jj} - n_{ij}\right)^+$. The bound (150) corresponds to the bound (145e) in Lemma 6 and this completes the proof of Lemma 6

■

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