Macroscopic Modelling of Vibration and Stability Problems in Prestressed Microperiodic Elastic Solids

Woźniak Czesław[*1] and **Woźniak Margaret**[** 2]

[1] Technical University of Czestochowa, Institute of Mathematics and Computer Sciences, Al. Dabrowskiego 73, PL - 42 200 Czestochowa, Poland

[2] Technical University of Kielce, Departament of Road and Traffic Engineering, Al. 1000-lecia P. P. 7, PL - 25 314 Kielce, Poland

The aim of this contribution is to propose and apply a certain averaged (macroscopic) mathematical model for the analysis of vibration and stability problems of prestressed microperiodic elastic solids. In contrast to the known homogenized model [2], [3] the proposed model describes the effect of the period length on the macroscopic solid behaviour.

1 Formulation of the problem

The object of considerations is a description of dynamic behaviour of a microperiodic elastic composite solid subjected to initial stresses. This problem is not new. We can mention here approaches based on the known equation of the elasticity theory, [1] or on the homogenized model equations [2], [3]. However, the first approach can be applied only to some special periodic structures (like laminates) and the second approach neglects the effect of the period length on the overall solid behaviour. The main aim of this contribution is to formulate a new macroscopic model of the prestressed periodic solid describing the aforementioned effect. We also derive a 2D-model of a thin prestressed plate with the thickness of an order of the period length. Moreover, we apply the above 2D-model to a certain vibration and stability problem for a thin uniperiodic plate. The proposed approach is based on the tolerance averaging technique of PDEs with periodic coefficients and is a certain generalization of results given in [4]. In order to make this note selfconsistent we outline the basic concepts and assumptions of the applied modelling technique.

Let $\Delta(\mathbf{x}) \equiv \Delta + \mathbf{x}$ be a periodic cell with a center at point x in the physical space such that $\Delta(\mathbf{x}) \subset \Omega$ where Ω is a region occupied by a solid in this space. The solid under consideration is assumed to be microperiodic; it means that the diameter of Δ is negligibly small when compared to the smallest characteristic length dimension of Ω. The tolerance averaging technique is based on the well known definition of the averaging operation $\langle f\rangle(\mathbf{x})$ for an arbitrary integrable function f.

The basic concept is that of a slowly varying function of an argument $\mathbf{x}$. It is a function F satisfying the following tolerance averaging approximation

$$\langle fF\rangle(\mathbf{x}) \simeq \langle f\rangle(\mathbf{x})F(\mathbf{x})$$

which has to hold for every integrable function f, here $\simeq$ is a certain tolerance relation, [4]. Let $\mathbf{u}(\mathbf{x},t)$, $\mathbf{x} \in \Omega$, be a displacement field at time t from the reference configuration of the periodic solid. Define $\mathbf{w}(\mathbf{x},t) \equiv \langle \mathbf{u}\rangle(\mathbf{x},t)$. The first assumption of the modelling technique applied to the linear elasticity theory equations is that $\mathbf{w}(\cdot,t)$ is a slowly varying function together with all derivatives. Hence in the decomposition of displacement field $\mathbf{u} = \mathbf{w} + \mathbf{r}$ into the averaged $\mathbf{w}$ and the residual $\mathbf{r}$ part we obtain $\langle \mathbf{r}\rangle \simeq 0$. It follows that $\mathbf{r}$ can be interpreted as a fluctuation displacement field caused by the periodic nonhomogeneous structure of the solid. The second modelling assumption is that the fluctuation displacement field $\mathbf{r}$ can be approximated by means of the formula, summation over $A = 1, ..., N$ holds:

$$\mathbf{r}(\mathbf{x},t) = \mathbf{h}^A(\mathbf{x})\psi^A(\mathbf{x},t)$$

where $\mathbf{h}^A(\cdot)$ are the known *a priori* periodic shape functions, such that $\langle \mathbf{h}^A\rangle = 0$ and $\psi^A(\cdot,t)$ are slowly varying functions which are new unknowns. Functions ψ^A will be referred to as the fluctuation amplitudes. It follows that the kinematic averaged (macroscopic) description of a periodic solid will be given by functions $\mathbf{w}$, ψ^A which are assumed to be slowly varying together with all derivatives.

2 Modelling technique

The starting point of considerations are the linearized equations of a prestressed microperiodic elastic composite solid

$$\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + \mathbf{N} : \nabla\nabla \mathbf{u} - \varrho \ddot{\mathbf{u}} + \varrho \mathbf{b} = \mathbf{0}$$

* Corresponding author: e-mail: wozniak@imi.pcz.czest.pl, Phone: +48 34 325 0324, Fax: +48 34 3250324

** e-mail: mwozniak@tu.kielce.pl, Phone: +48 41 3424560, Fax: +48 41 3443784

where $\mathbb{C}$, ϱ are the tensor of elastic moduli and mass density, respectively, which are periodic highly oscillating noncontinuous functions; $\mathbf{N}$ is a tensor of initial stress and $\mathbf{b}$ is a body force. Taking into account the modelling assumptions and the tolerance averaging formula, outlined in the previous section, after many manipulations we obtain the following system of equations for the averaged displacements $\mathbf{w}$ and fluctuation amplitudes ψ^A

$$\begin{aligned} &\langle\varrho\rangle\ddot{\mathbf{w}} - \nabla\cdot(\langle\mathbb{C}\rangle : \nabla\mathbf{w} + \langle\mathbb{C} : \nabla\mathbf{h}^A\rangle\psi^A) - \langle\mathbf{N}\rangle : \nabla\nabla\mathbf{w} = \varrho\mathbf{b} \\ &\langle\varrho\mathbf{h}^A\cdot\mathbf{h}^B\rangle\ddot{\psi}^B + \langle\nabla\mathbf{h}^A : \mathbb{C} : \nabla\mathbf{h}^B\rangle\psi^B + \langle\nabla\mathbf{h}^A : \mathbb{C}\rangle : \nabla\mathbf{w} = \mathbf{0} \end{aligned} \tag{1}$$

The above equations introduce into the equations obtained in [4] the prestressing tensor $\mathbf{N}$. It can be observed that for fluctuation amplitudes we have obtained a system of ordinary differential equations involving only time derivatives of ψ^A. Hence the boundary conditions can be imposed exclusively on the averaged displacement field w while the initial conditions have to be prescribed both for w and ψ^A. Moreover, coefficients $\langle\mathbf{r}\mathbf{h}^A\cdot\mathbf{h}^B\rangle$ are of an order of the period length and hence they describe the effect of the microstructure size on a dynamic solid behaviour. It has to be remembered that the aforementioned model equations have a physical sense only if $\mathbf{w}(\cdot,t)$, $\psi^A(\cdot,t)$ are slowly varying functions of a spatial coordinate.

Let Π be a region on the $0x_1x_2$-plane. Assume that equations (1) hold in a region $\Pi\times(-d/2,d/2)$ occupied by a thin plate with a constant thickness d . Let us also assume that the plate has a plane periodic structure and hence D is a $2D$-periodicity cell on the $0x_1x_2$-plane. Moreover, let the plate be homogeneous in the direction of the x_3-axis being made of periodically distributed materials along the midplane. Applying the Kirchhoff plate assumptions related to the averaged displacement field w and assuming that the fluctuation amplitudes are restricted by the condition $\psi^A(x,t) = x_3\varphi^A(x_1x_2,t)$ where φ^A are new unknowns, we obtain from (1) the following system of equations for the midplane deflection $v(x_1x_2,t)$ and $2D$-fluctuation amplitudes φ^A

$$\begin{aligned} &\langle\varrho\rangle\ddot{v} - j\langle\varrho\rangle\nabla\cdot\nabla\ddot{v} + j\nabla\nabla:(\langle\mathbb{D}\rangle : \nabla\nabla v) + j\langle\mathbb{C} : \nabla\mathbf{h}^B\rangle\varphi^B - \langle\mathbf{N}\rangle : \nabla\nabla v + j\langle\mathbf{N}\rangle : \nabla\nabla(\nabla\cdot\nabla v) = \mathbf{0} \\ &\langle\varrho\mathbf{h}^A\cdot\mathbf{h}^B\rangle\ddot{\varphi}^B + \langle\nabla\mathbf{h}^A : \mathbb{C} : \nabla\mathbf{h}^B\rangle\varphi^B + \langle\mathbb{C} : \nabla\mathbf{h}^A\rangle : \nabla\nabla v = 0 \end{aligned} \tag{2}$$

In the above equations $j = d^2/12$ and $\mathbf{D}$ is a $2D$-tensor of elastic moduli related to the plane stress; we have neglected in (2) the effect of a body force. The averaging is carried out over the $2D$-periodicity cell and shape functions $\mathbf{h}^A$ are independent of x_3. The above equations can be also applied to a thin plate which is made of two isotropic materials periodically spaced along the x_1-axis. Thus, we deal here with a $1D$-periodic structure; denote by l the period length and by l', l'' the length of intervals of the cell occupied by the pertinent constituent. In this case we can introduce only one saw-like shape function [4] which depends exclusively on the x_1-coordinate and hence introduce only one fluctuation amplitude. Let the isotropic constituents of the plate have mass densities ϱ', ϱ'', Young moduli E', E'' and Kirchhoff moduli μ', μ'', respectively. Denote $\zeta' = l'/l$, $\zeta'' = l''/l$, $\gamma = E/(1-\nu^2)$ and assume that $\mu' = \mu'' = \mu$. For a simply supported plate with $\Pi = (0,L)\times(0,L)$ subjected to the compression $\mathbf{N}$ along the x_1-axis, we obtain the following asymptotic formulae for the higher and lower free vibration frequencies

$$(\omega_-)^2 = \frac{4\langle\gamma\rangle(\mu+\langle\gamma\rangle)jk^4}{\langle\varrho\rangle(\mu+\zeta''\gamma'+\zeta'\gamma'')} - \frac{Nk^2}{\langle\varrho\rangle} + O(\varepsilon^2), \qquad (\omega_+)^2 = \frac{6\mu}{l'l''\langle\varrho\rangle}(1+2jk^2) + \frac{(\mu+\langle\gamma\rangle)k^2}{2\langle\varrho\rangle} + O(\varepsilon) \tag{3}$$

where $\varepsilon = (kl)^2 << 1$ and $k = 2\pi/L$. From the first of formulae (3) we can obtain the value of the static critical force $\mathbf{N}$, by setting $\omega_- = 0$.

The obtained model equations (1) describe the effect of the periodicity cell size on a macroscopic behaviour of a prestressed elastic solid. The 2D-model equations (2) describe plates with period length of an order of the plate thickness contrary to the known 2D-averaging procedures. In contrast to homogenized plate models also higher free vibration frequencies can be calculated from formulas (3). The effect of initial stresses on higher vibration frequency is small and can be neglected. The effect of a period length on a critical force is negligibly small.

References

[1] H.V. Bufler, H. Kennenknecht, Prestrained elastic laminates; deformations, stability and vibrations. Acta Mechanica, **48**, 1 - 30, (1983)

[2] A.G. Kolpakov, Homogenized model for plate periodic structure with initial stress. Int. J. Engng Sci., **38**, 2079 - 2094, (2000)

[3] A.G. Kolpakov, Euromech 444, Bremen, 2002

[4] Cz. Woźniak, E. Wierzbicki, Averaging techniques in thermomechanics of composite solids, Wydaw. Pol. Częst., Częstochowa, Poland, 2000