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GEVREY CLASS SMOOTHING EFFECT FOR THE PRANDTL EQUATION

WEI-XI LI, DI WU AND CHAO-JIANG XU

Abstract. It is well know that the non linear Prandtl boundary layer equation is instable, and the well-posedness of the Cauchy problem in Sobolev space is an open problem. Recently, under the Oleiniks monotonicity assumption for the initial datum, [1] have proved the local well-posedness of Cauchy problem in Sobolev space. In this work, we study the Gevrey smoothing effects of the local solution obtained in [1]. We prove that the Soblev's class solution is belongs to some Gevrey class with respect to tangential variables at positive time. We get also the precise Gevery norms estimate with respect to time variable. This qualitative study of the solution for Prandtl's equation in Gevery class can help us to understand the Prandtl boundary layer theory which is justified in analytic frame.

1. Introduction

In this work, we study the regularity of solutions to the Prandtl equation which is the foundation of the boundary layer theory introduced by Prandtl in 1904, [25]. The inviscid limit of an incompressible viscous flow with the non-slip boundary condition is still a challenging problem due to the appearance of a boundary layer, where the tangential velocity adjusts rapidly from nonzero away from the boundary to zero on the boundary. Prandtl equation describes the behavior of the flow near the boundary in the small viscosity limit, and it reads

$$\begin{cases} u_t + uu_x + vu_y + p_x = u_{yy}, \quad t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = U(t,x), \\ u|_{t=0} = u_0(x,y)\,, \end{cases} \tag{1}$$

where $u(t,x,y)$ and $v(t,x,y)$ represent the tangential and normal velocities of the boundary layer, with y being the scaled normal variable to the boundary, while $U(t,x)$ and $p(t,x)$ are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U\partial_x U + \partial_x q = 0.$$

Because of the degeneracy in tangential variable, the well-posedness theories and the justification of the Prandtl equation remain as the challenging problems in the mathematical theory of fluid mechanics. Up to now, there are only a few rigorous mathematical results. Under a monotonic assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial-boundary value problems, and this result together with some of her works with collaborators were well presented in the monograph [24]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so called favorable condition on the pressure, Xin-Zhang [27] obtained the existence of global weak solutions to the Prandtl equation. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by degeneracy and mixed type of the equation. Very recently the well-posedness in the Sobolev space was explored by virtue of energy method instead of the Crocco transformation; see Alexandre et. all [1] and Masmoudi-Wong [22]. There is very few work concerned with the Prandtl equation without the monotonicity assumption; we refer [21, 26, 28] for the existence in the analytic frame, and [12, 18] for the recent works of concerning with the existence in Gevrey class. Recall Gevrey class, denoted by $G^s, s \geq 1$, is an intermediate space between analytic functions and C^∞ space. Given a domain Ω, the (global) Gevrey space $G^s(\Omega)$ is consist of such functions that $f \in C^\infty(\Omega)$ and that

$$\left\|\partial^\alpha f\right\|_{L^2(\Omega)} \leq L^{|\alpha|+1}(\alpha!)^s$$

for some constant L independent of α. The significant difference between Gevery and analytic classes is that there exist nontrivial Gevrey functions admitting compact support.

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We mention that due to the degeneracy in x it is natural to expect Gevrey regularity rather than analyticity for a subelliptic equation. We refer [6, 7, 8, 9] for the link between subellipticity and Gevrey reguality. In this paper we first study the intrinsic subelliptic structure due to the monotonicity condition, and then deduce, basing on the subelliptic estimate, the Gevrey smoothing effect; that is, given a general initial data belonging to some Sobolev space, the solution will lie in some Gevrey class at positive time, just as like heat equation.It is different from the Gevrey propagation property obtained in the aforementioned works, where the initial data is supposed to be of some Gevrey class, for instance $G^{7/4}$ in [12], and the well-posedness is obtained in the same Gevrey space.

Now we state our main result. Without loss of generality, we only consider here the case of an uniform outflow $U = 1$, and the conclusion will still hold for Gevrey class outflow U. We mention that the Gevrey regularity for outflow U is well developed (see [19] for instance). For the uniform outflow, we get the constant pressure p due to the Bernoulli law. Then the Prandtl equation can be rewritten as

$$\begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, \quad t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x, y)\,, \end{cases} \tag{2}$$

The main result concerned with the Gevrey class regularity can be stated as follows.

Theorem 1.1. *Let $u(t, x, y)$ be a classical local in time solution to Prandtl equation (2) on $]0, T[$ with the properties subsequently listed below:*

(i) there exist two constants $C_* > 1, \sigma > 1/2$ such that for any $(t, x, y) \in]0, T[\times\mathbb{R}^2_+$,

$$\begin{aligned} &C_*^{-1} \langle y\rangle^{-\sigma} \le \partial_y u(t, x, y) \le C_* \langle y\rangle^{-\sigma} \\ &\left|\partial_y^2 u(t, x, y)\right| + \left|\partial_y^3 u(t, x, y)\right| \le C_* \langle y\rangle^{-\sigma-1}\,, \end{aligned} \tag{3}$$

where $\langle y\rangle = (1 + |y|^2)^{1/2}$.

(ii) There exists $c > 0$ and integer $N_0 \ge 7$ such that

$$\|e^{2cy}\partial_x u\|_{L^\infty([0,T];\ H^{N_0}(\mathbb{R}^2_+))} + \|e^{2cy}\partial_x\partial_y u\|_{L^2([0,T];\ H^{N_0}(\mathbb{R}^2_+))} \le C_0. \tag{4}$$

Then for any $0 < T_1 < T$, there exists a constant A, depending only on C_0, T_1 and the above constants C_*, c and σ, such that for any $0 < t \le T_1$,

$$\forall\ m > 1 + N_0, \quad \left\|e^{\tilde{c}y}\partial_x^m u(t)\right\|_{L^2(\mathbb{R}^2_+)} \le t^{-3(m-1-N_0)}\, A^{m+1}\, (m!)^{3+3\sigma}\,, \tag{5}$$

where $\tilde{c}$ is an arbitrary number such that $\tilde{c} < c$. Therefore, the solution u is belong to the Gevery class of index $3 + 3\sigma$ with respect to x variable for any $0 < t \le T_1$.

Remark 1.2. *Such a solution in the theorem exists; for instance suppose the initial data u_0 can be written as*

$$u_0(x, y) = u_0^s(y) + u_0^\epsilon(x, y),$$

where u_0^s is a function of y but independent of x such that $C^{-1}\langle y\rangle^{-\sigma/2} \le u_0^s(y) \le C\langle y\rangle^{-\sigma/2}$ for some constant $C \ge 1$, and u_0^ϵ is a small perturbation such that its weighted Sobolev norm $\left\|e^{2cy}u_0^\epsilon\right\|_{H^{N_0}(\mathbb{R}^2_+)}$ is suitably small. Then using the arguments in [1]*, we can obtain the desired solution with the properties listed in Theorem 1.1 fulfilled. Precisely, the solution $u(x, y)$ is a perturbation of a shear flow $u^s(y)$ such that property (i) in the above theorem holds for u, and moreover $e^{2cy}(u - u^s) \in L^\infty\left([0, T];\ H^{N_0+1}(\mathbb{R}^2_+)\right)$. Recall the shear flow u^s is the solution to the following initial-boundary problem*

$$\begin{cases} \partial_t u^s - \partial_{yy} u^s = 0, \quad t > 0, \quad y > 0, \\ u^s|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u^s = 1, \\ u|_{t=0} = u_0^s(y), \end{cases}$$

where u_0^s is a function of y but independent of x.

Remark 1.3. *The integer N_0 in above theorem is fixed and large enough, and depends on the integer N given in (12) in the following section. Thus N_0 is a quantity depending only on the dimension of x (and therefore depending only on the dimension $n = 1$ here).*

The paper is organized as follows. We prove in Section 2 a subelliptic estimate for the linear Prandtl operator. Section 3 and Section 4 are devoted to presenting a crucial estimate for an auxilliary function. The proof of Theorem 1.1 is given in Section 5. The last section is an appendix, where the equation fulfilled by the auxilliary function is deduced.

2. Subelliptic estimate

In this section we will explore the subelliptic structure, due to the monotonic condition, for the linear Prandtl operator. We first introduce in the following subsection the notations used throughout the paper, and then recall in Subsection 2.2 some basic facts of pseudo-differential calculus. Subsection 2.3 is devoted to the estimates on commutator, and in Subsection 2.4 we prove the subelliptic estimate for the linear Prandtl operator.

2.1. **Notations.**

Throughout the paper, we use the notation

$$\langle\cdot\rangle = \left(1+|\cdot|^2\right)^{1/2},$$

which is the regularization of the modulus $|\cdot|$.

Let H^κ be the classical Sobolev space and let $L^2 = H^0$ be the classical space of complex-valued square-integrable functions, equipped with an inner product and a norm defined by

$$(f,\ g)_{L^2} = \int f(\tilde{x})\overline{g(\tilde{x})}d\tilde{x}, \quad \|f\|_{L^2} = (f,\ f)_{L^2}^{1/2},$$

where $\bar{g}$ is the complex conjugate of g. And we denote by Re $(f,g)_{L^2}$ the real part of $(f,g)_{L^2}$, i.e.,

$$\text{Re }(f,g)_{L^2} = \frac{1}{2}\Big(\,(f,g)_{L^2} + \overline{(f,g)_{L^2}}\Big) = \frac{1}{2}\Big(\,(f,g)_{L^2} + (g,f)_{L^2}\,\Big).$$

Let x be the tangent variable and ξ be its dual variable. Let $0<\delta<1$ be arbitrarily small and we denote by Λ^d and Λ_δ^d the Fourier multipliers associate to symbol $\langle\xi\rangle^d$ and $\langle\delta\xi\rangle^d$, respectively, i.e.,

$$\forall\ f\in\mathcal{S}(\mathbb{R}), \quad \left(\mathcal{F}_x\left(\Lambda^d f\right)\right)(\xi) = \langle\xi\rangle^d\left(\mathcal{F}_x f\right)(\xi)$$

and

$$\forall\ f\in\mathcal{S}(\mathbb{R}), \quad \left(\mathcal{F}_x\left(\Lambda_\delta^d f\right)\right)(\xi) = \langle\delta\xi\rangle^d\left(\mathcal{F}_x f\right)(\xi),$$

where $\mathcal{F}_x$ stands for the Fourier transform in x variable. It is clear that Λ^d is a self-adjoint operator in L^2, and moreover

$$\forall\ d_1,d_2\in\mathbb{R}, \quad \Lambda^{d_1}\Lambda^{d_1} = \Lambda^{d_1+d_2}. \tag{6}$$

As for the symbol $\langle\delta\xi\rangle^d$ of Λ_δ^d, we will use the following fact:

$$\forall\ \beta\geq 1, \quad \left|\partial_\xi^\beta\left(\langle\delta\xi\rangle^d\right)\right| \leq C_{d,\beta}\langle\delta\xi\rangle^d\langle\xi\rangle^{-1}, \tag{7}$$

with $C_{d,\beta}$ a constant depending only on β and d, but independent of δ. To confirm this, we use Faà di Bruno's Formula to get, for any $\beta\geq 1$,

$$\begin{aligned}
&\partial_\xi^\beta\left(\langle\delta\xi\rangle^d\right)\\
=&\sum_{p_1+2p_2+\cdots+\beta p_\beta=\beta}\frac{\beta!}{p_1!p_2!\cdots p_\beta!}\left(\partial_z^{(p_1+\cdots+p_\beta)}\langle z\rangle^{d/2}\right)\Big|_{z=1+\delta^2\xi^2}\left(\frac{\partial_\xi\left(1+\delta^2\xi^2\right)}{1!}\right)^{p_1}\cdots\left(\frac{\partial_\xi^\beta\left(1+\delta^2\xi^2\right)}{\beta!}\right)^{p_\beta}\\
=&\sum_{p_1+2p_2=\beta}\frac{\beta!}{p_1!p_2!}\left(\partial_z^{(p_1+p_2)}\langle z\rangle^{d/2}\right)\Big|_{z=1+\delta^2\xi^2}\left(\frac{\partial_\xi\left(1+\delta^2\xi^2\right)}{1!}\right)^{p_1}\left(\frac{\partial_\xi^2\left(1+\delta^2\xi^2\right)}{2!}\right)^{p_2}.
\end{aligned}$$

Since $p_1+2p_2=\beta\geq 1$, then at least one of the two nonnegative integers is positive strictly, saying $p_1>0$ without loss of generality. Thus

$$\left|\left(\frac{\partial_\xi\left(1+\delta^2\xi^2\right)}{1!}\right)^{p_1}\right| \leq \left(3\left(1+\delta^2\xi^2\right)\langle\xi\rangle^{-1}\right)^{p_1} \leq 3^{p_1}\left(1+\delta^2\xi^2\right)^{p_1}\langle\xi\rangle^{-1}.$$

On the other hand, direct computation shows that

$$\left|\partial_z^{(p_1+p_2)}\langle z\rangle^{d/2}\right| \leq C_{p_1,p_2,d}\langle z\rangle^{\frac{d}{2}-p_1-p_2}$$

and thus

$$\left|\left(\partial_z^{(p_1+p_2)}\left\langle z\right\rangle^{d/2}\right)\Big|_{z=1+\delta^2\xi^2}\right| \leq C_{p_1,p_2,d}\left(1+\delta^2\xi^2\right)^{\frac{d}{2}-p_1-p_2}$$

with $C_{p_1,p_2,d}$ a constant depending only on p_1, p_2 and d. Moreover

$$\left|\left(\frac{\partial_\xi^2\left(1+\delta^2\xi^2\right)}{2!}\right)^{p_2}\right| \leq \left(1+\delta^2\xi^2\right)^{p_2}$$

since $0<\delta<1$. Combining these inequalities, we have

$$\left|\partial_\xi^\beta\left(\left\langle\delta\xi\right\rangle^d\right)\right| \leq C_{\beta,d}\left(1+\delta^2\xi^2\right)^{\frac{d}{2}-p_1-p_2}\left(1+\delta^2\xi^2\right)^{p_2}\left(1+\delta^2\xi^2\right)^{p_1}\left\langle\xi\right\rangle^{-1} = C_{\beta,d}\left\langle\delta\xi\right\rangle^d\left\langle\xi\right\rangle^{-1},$$

with $C_{\beta,d}$ a constant depending only on β and d. This is just the desired estimate (7).

2.2. **Weyl-Hörmander calculus.** We recall here some notations and basic facts of symbolic calculus, and refer to [17, Chapter 18] or [20] for detailed discussions on the pseudo-differential calculus.

Let $a(x,\xi)\in\mathcal{S}'(\mathbb{R}^2)$ be a tempered distribution and let $\kappa\in\mathbb{R}$. We define the operator $\mathrm{op}_\kappa a : \mathcal{S}(\mathbb{R}_x)\to\mathcal{S}^*(\mathbb{R}_x)$, with $\mathcal{S}^*(\mathbb{R}_x)$ the antidual of $\mathcal{S}(\mathbb{R}_x)$ (continuous antilinear forms), by the formula

$$\left\langle(\mathrm{op}_\kappa a)f,h\right\rangle_{\mathcal{S}^*,\,\mathcal{S}} = \left\langle a,\ \Omega_{f,h}(\kappa)\right\rangle_{\mathcal{S}',\,\mathcal{S}},$$

where

$$\Omega_{f,h}(\kappa)(x,\xi) = \int_{\mathbb{R}} e^{-2i\pi z\cdot\xi} f(x+(1-\kappa)z)\bar{h}(x-\kappa z)dz.$$

In particular we denote

$$a(x,D_x) = \mathrm{op}_0 a$$

and

$$a^w = \mathrm{op}_{1/2} a.$$

Here $a(x,D_x)$ is the classical quantization of symbol a and a^w is called the Weyl quantization of symbol a. We will write $a(D_x)$ instead of $a(x,D_x)$ when the symbol $a(x,\xi)$ is independent of x. Moreover $a(x,D_x)$ and a^w can be written as the following oscillatory integral

$$\left(a(x,D_x)f\right)(x) = \int\int e^{2i\pi(x-\tilde{x})\xi}a(x,\xi)f(\tilde{x})d\tilde{x}d\xi \tag{8}$$

and

$$\left(a^w f\right)(x) = \int\int e^{2i\pi(x-\tilde{x})\xi}a\left(\frac{x+\tilde{x}}{2},\xi\right)f(\tilde{x})d\tilde{x}d\xi. \tag{9}$$

We have the formula of changing quantization (see Proposition 1.1.10 and Lemma 4.1.2 of [20]):

$$a^w = \left(J^{1/2}a\right)(x,D_x), \tag{10}$$

where $J^{1/2}$ is defined by the oscillatory integrals

$$\left(J^{1/2}a\right)(x,\xi) = 2\iint e^{-4i\pi z\zeta}a(x+z,\xi+\zeta)dzd\zeta.$$

Lemma 2.1. *Let $a_1(x)$ be a function of x but independent of ξ, and $a_1(\xi)$ be a function of ξ but independent of x. Then*

(i) $a_1^w = a_1(x)$, i.e., a_1^w is the operator of multiplication by $a_1(x)$;
(ii) $a_2^w = a_2(D_x)$, and moreover $a_2(D_x)$ is the Fourier multiplier associated to symbol $a_2(\xi)$, i.e.,

$$\left(\mathcal{F}_x\left(a_2(D_x)f\right)\right)(\xi) = a_2(\xi)\left(\mathcal{F}_x f\right)(\xi).$$

Proof. We first prove Conclusion (ii). In view of (8) and (9), we have

$$a_2^w f = \int\int e^{2i\pi(x-\tilde{x})\xi}a_2\left(\xi\right)f(\tilde{x})d\tilde{x}d\xi = a_2(D_x)f.$$

Moreover observe

$$\int\int e^{2i\pi(x-\tilde{x})\xi}a_2\left(\xi\right)f(\tilde{x})d\tilde{x}d\xi = \int e^{2i\pi x\xi}a_2\left(\xi\right)\left(\mathcal{F}_x f\right)(\xi)d\xi,$$

which implies

$$\left(\mathcal{F}_x\left(a_2(D_x)f\right)\right)(\xi) = a_2(\xi)\left(\mathcal{F}_x f\right)(\xi).$$

Then the conclusion (ii) follows.

Now we prove (i). To do so, observe

$$2\iint e^{-4i\pi z\zeta}d\zeta = \delta_0(z)$$

with δ_0 being the Dirac delta function at origin, and thus

$$\left(J^{1/2}a_1\right)(x,\xi) = 2\iint e^{-4i\pi z\zeta}a_1(x+z)dzd\zeta = a_1(x).$$

As a result, using (8) and (10) gives

$$a_1^w f = \int\int e^{2i\pi(x-\tilde{x})\xi}\left(J^{1/2}a_1\right)(x,\xi)f(\tilde{x})d\tilde{x}d\xi = a_1(x)\int\int e^{2i\pi(x-\tilde{x})\xi}f(\tilde{x})d\tilde{x}d\xi = a_1(x)f(x),$$

that is $a_1^w = a_1(x)$, the operator of multiplication by $a_1(x)$. The proof is thus complete. □

In the following discussions, let

$$g = |dx|^2 + |d\xi|^2$$

be the flat metric. We denote by $S\left(\langle\xi\rangle^r, g\right), r \in \mathbb{R}$, the space of symbols $a(x,\xi)$ satisfying that

$$\forall\ \alpha,\beta \in \mathbb{Z}_+,\ \forall\ x,\xi \in \mathbb{R},\quad \left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^r$$

with $C_{\alpha,\beta}$ a constant depending only on α,β. The space $S(\langle\xi\rangle^r,\ g)$, endowed with the semi-norms

$$\left\|a\right\|_{k;S(\langle\xi\rangle^r,g)} = \max_{0\le\alpha+\beta\le k}\sup_{(x,\xi)\in\mathbb{R}^2}\langle\xi\rangle^{-r}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|, \tag{11}$$

becomes a Fréchet space.

An elementary property to be used frequently is the L^2 continuity theorem in the class $S(1,\ g)$, see [20, Theorem 2.5.1] for instance, which says that there exists a constant C and a positive integer N, *both depending only the dimension of x (and therefore depending only on the dimension $n = 1$ here),* such that if $a \in S(1,g)$ then

$$\forall\ u \in L^2(\mathbb{R}),\quad \left\|a^w u\right\|_{L^2} \le C\left\|a\right\|_{N;S(1,g)}\left\|u\right\|_{L^2} \tag{12}$$

with $\left\|a\right\|_{N;S(1,g)}$ is defined by (11).

From now on we alway let N be the integer given in (12), which is a fixed quantity depending only on dimension. We denote by $S_N(1,g)$ the space of functions satisfying

$$\forall\ \alpha+\beta \le N,\ \forall\ x,\xi \in \mathbb{R},\quad \left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta}.$$

which, equipped with the norm

$$\left\|a\right\|_{N;S(1,g)} = \max_{0\le\alpha+\beta\le N}\sup_{(x,\xi)\in\mathbb{R}^2}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|,$$

becomes a Banach space. Then the proof of [20, Theorem 2.5.1] also implies the following

Lemma 2.2. *Let N the positive integer given in (12) which depends only on the dimension of x. If $a \in S_N(1,g)$ then $a^w \in \mathcal{B}\left(L^2(\mathbb{R})\right)$, the space of bounded operators in $L^2(\mathbb{R})$. Moreover*

$$\forall\ u \in L^2(\mathbb{R}),\quad \left\|a^w u\right\|_{L^2} \le C\left\|a\right\|_{N;S(1,g)}\left\|u\right\|_{L^2}. \tag{13}$$

Now we introduce the composition formula of two pseudo-differential operators. Given $a_i \in S(\langle\xi\rangle^{r_i}, g)$ we have

$$a_1^w a_2^w = (a_1\sharp a_2)^w \tag{14}$$

where $a_1\sharp a_2 \in S\left(\langle\xi\rangle^{r_1+r_2},\ g\right)$ admits the expansion, denoting $Z = (x,\xi)$, $Z_i = (x_i,\xi_i), i = 1,2$,

$$a_1\sharp a_2(Z) = a_1a_2(Z) + \int_0^1\iint e^{-2i\mu(Z-Z_1,Z-Z_2)/\theta}\frac{1}{2i}\mu(\partial_{Z_1},\partial_{Z_2})a_1(Z_1)a_2(Z_2)dZ_1dZ_2d\theta/(\pi\theta)^2, \tag{15}$$

where μ is the symplectic form in $\mathbb{R}^2$ given by

$$\mu\left((z,\zeta),(\tilde{z},\tilde{\zeta})\right)=\zeta\cdot\tilde{z}-\tilde{\zeta}\cdot z.$$

In the following we use the notation $a_1\sharp_\theta a_2$ which is defined by

$$a_1\sharp_\theta a_2(Z)=\iint e^{-2i\mu(Z-Z_1,Z-Z_2)/\theta}\frac{1}{2i}a_1(Z_1)a_2(Z_2)dY_1dY_2/(\pi\theta)^2. \tag{16}$$

Proposition 2.3 (cf. Theorem 2.3.7 in [20] and Proposition 1.1 in [2]). *Let $\sharp$ and $\sharp_\theta$ be defined by (15) and (16), respectively, and let r_1,r_2 be two real numbers, and let N be the integer given in (12) which is fixed throughout the paper.*

(i) There exists a positive integer ℓ_N and a constant C_N, both depending only on the integer N, such that

$$\left\|a_1\sharp a_2\right\|_{N;S(\langle\xi\rangle^{r_1+r_2},g)}\leq C_N\left\|a_1\right\|_{\ell_N;S(\langle\xi\rangle^{r_1},g)}\left\|a_2\right\|_{\ell_N;S(\langle\xi\rangle^{r_2},g)}. \tag{17}$$

(ii) For any integer $k\geq 0$, there exists a positive integer $\tilde{\ell}_k$ and a constant $\tilde{C}_k$, both depending only on k, such that

$$\left\|a_1\sharp_\theta a_2\right\|_{k;S(\langle\xi\rangle^{r_1+r_2},g)}\leq \tilde{C}_k\left\|a_1\right\|_{\tilde{\ell}_k;S(\langle\xi\rangle^{r_1},g)}\left\|a_2\right\|_{\tilde{\ell}_k;S(\langle\xi\rangle^{r_2},g)}.$$

In particular, letting ℓ_N be the integer in (17), we can find an integer $k_0=k_0(N)$ and a constant C_N, depending only on ℓ_N and thus only on N, such that

$$\left\|a_1\sharp_\theta a_2\right\|_{\ell_N;S(\langle\xi\rangle^{r_1+r_2},g)}\leq C_N\left\|a_1\right\|_{k_0;S(\langle\xi\rangle^{r_1},g)}\left\|a_2\right\|_{k_0;S(\langle\xi\rangle^{r_2},g)}. \tag{18}$$

2.3. **Estimate on commutators.** Given two operators Q_1 and Q_2, the commutator between Q_1 and Q_2, denoting by $\left[Q_1,\ Q_2\right]$, is defined by

$$\left[Q_1,\ Q_2\right]=Q_1Q_2-Q_2Q_1=-\left[Q_2,\ Q_1\right].$$

Direct verification shows that

$$\left[Q_1,\ Q_2Q_3\right]=Q_2\left[Q_1,\ Q_3\right]+\left[Q_1,\ Q_2\right]Q_3. \tag{19}$$

This subsection is devoted to proving the following

Lemma 2.4. *Let N be given in (12), and let ℓ_N, $k_0=k_0(N)$ be the integers given in (17) and (18), respectively. Let q be a symbol such that $\partial_x q\in S_{k_0}(1,g)$, i.e.,*

$$\forall\ \alpha\leq k_0,\ \forall\ x\in\mathbb{R},\quad |\partial_x^\alpha(\partial_x q)(x)|\leq c_\alpha \tag{20}$$

with c_α a constant depending only on α. Let $\lambda(\xi)$ be function of ξ, such that $\partial_\xi\lambda\in S_{k_0}(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)$ uniformly with respect to δ, i.e.,

$$\forall\ \beta\leq k_0,\ \forall\ \xi\in\mathbb{R},\quad \left|\partial_\xi^\beta\left(\partial_\xi\lambda\right)(\xi)\right|\leq\tilde{c}_\beta\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2}, \tag{21}$$

where $\tilde{c}_\beta$ is a constant depending only on β but independent of δ. Then there exists a constant C_N, depending only on N but independent of δ, such that

$$\forall\ f\in\ H^{\tau_1+\tau_2}(\mathbb{R}_x),\quad \left\|\left[q,\ \lambda(D_x)\right]f\right\|_{L^2(\mathbb{R}_x)}\leq C_N\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}f\right\|_{L^2(\mathbb{R}_x)},$$

where $\Lambda^{\tau_1},\Lambda_\delta^{\tau_2}$ and $\lambda(D_x)$ stand for the Fourier multipliers associate to symbols $\langle\xi\rangle^{\tau_1},\langle\delta\xi\rangle^{\tau_2}$ and $\lambda(\xi)$, respectively.

Proof. To simplify the notations we will use in the proof C_N to denote different constants which depend only on N but independent of δ..

Step 1). By Lemma 2.1, we have

$$\left[q,\lambda(D_x)\right]=\left[q^w,\lambda^w\right]=q^w\lambda^w-\lambda^wq^w=\left(q\sharp\lambda\right)^w-\left(\lambda\sharp q\right)^w,$$

the last equation using (14). Moreover the expansion (15) gives

$$\left(q\sharp\lambda\right)^w-\left(\lambda\sharp q\right)^w=\left(q\sharp\lambda-\lambda\sharp q\right)^w=\left(r_1-r_2\right)^w,$$

where, recalling the notation $Z=(x,\xi)$ and $Z_i=(x_i,\xi_i),i=1,2$,

$$r_1(Z)=\int_0^1\iint e^{-2i\mu(Z-Z_1,Z-Z_2)/\theta}\frac{1}{2i}\mu(\partial_{Z_1},\partial_{Z_2})q(Z_1)\lambda(Z_2)dZ_1dZ_2d\theta/(\pi\theta)^2$$

and

$$r_2(Z)=\int_0^1\iint e^{-2i\mu(Z-Z_1,Z-Z_2)/\theta}\frac{1}{2i}\mu(\partial_{Z_1},\partial_{Z_2})\lambda(Z_1)q(Z_2)dZ_1dZ_2d\theta/(\pi\theta)^2.$$

Thus we get

$$\left[q,\lambda(D_x)\right]=\left(r_1-r_2\right)^w. \tag{22}$$

*Step 2).*In this step we will show that

$$r_1,r_2\in S_{\ell_N}(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)$$

uniformly with respect to δ, where ℓ_N is the positive number given in (17). For this purpose, observe $q(Z_1)=q(x_1)$ and $\lambda(Z_2)=\lambda(\xi_2)$, and thus

$$\begin{aligned}\mu(\partial_{Z_1},\partial_{Z_2})q(Z_1)\lambda(Z_2) &= \left(\frac{\partial}{\partial_{\xi_1}}\frac{\partial}{\partial_{x_2}}-\frac{\partial}{\partial_{\xi_2}}\frac{\partial}{\partial_{x_1}}\right)q(x_1)\lambda(\xi_2)\\ &= -\left(\partial_{x_1}q(x_1)\right)\partial_{\xi_2}\lambda(\xi_2)\\ &= -\left(\partial_x q\right)(Z_1)\left(\partial_\xi\lambda\right)(Z_2).\end{aligned}$$

This gives

$$\begin{aligned}r_1(Z) &= -\int_0^1\iint e^{-2i\sigma(Z-Z_1,Z-Z_2)/\theta}\frac{1}{2i}\left(\partial_x q\right)(Z_1)\left(\partial_\xi\lambda\right)(Z_2)dZ_1dZ_2d\theta/(\pi\theta)^2\\ &= -\int_0^1\Big[(\partial_x q)\sharp_\theta(\partial_\xi\lambda)\Big](Z)\,d\theta,\end{aligned}$$

recalling $(\partial_x q)\sharp_\theta(\partial_\xi\lambda)$ is defined in (16). Similarly,

$$r_2(Z)=\int_0^1\Big[(\partial_\xi\lambda)\sharp_\theta(\partial_x q)\Big](Z)\,d\theta.$$

In view of the assumptions (20) and (21), we use the estimate (18) in Proposition 2.3 to get

$$\big\|(\partial_x q)\sharp_\theta(\partial_\xi\lambda)\big\|_{\ell_N;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}\le C_N\big\|\partial_x q\big\|_{k_0;S(1,g)}\big\|\partial_\xi\lambda\big\|_{k_0;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}\le\tilde{C}_N,$$

where $\tilde{C}_N=C_N\left(\max_{\alpha\le k_0}c_\alpha\right)\left(\max_{\beta\le k_0}\tilde{c}_\beta\right)$ with c_α, $\tilde{c}_\beta$ be the constants given in (20) and (21). This implies $r_1\in S_{\ell_N}(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)$ and

$$\big\|r_1\big\|_{\ell_N;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}\le\int_0^1\big\|(\partial_x q)\sharp_\theta(\partial_\xi\lambda)\big\|_{\ell_N;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}d\theta\le\tilde{C}_N.$$

Similarly, $r_2\in S_{\ell_N}(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)$ and

$$\big\|r_2\big\|_{\ell_N;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}\le\tilde{C}_N.$$

Thus we have $r_1-r_2\in S_{\ell_N}(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)$ and

$$\big\|r_1-r_2\big\|_{\ell_N;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}\le C_N, \tag{23}$$

with C_N a constant independent of δ.

Step 3). In view of (22) and (6) we may write

$$[q,\lambda(D_x)]=\left(r_1-r_2\right)^w=\big(\left(r_1-r_2\right)^w\Lambda_\delta^{-\tau_2}\Lambda^{-\tau_1}\big)\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}=\Big(\left(r_1-r_2\right)\sharp\big(\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\big)\Big)^w\Lambda^{\tau_1}\Lambda_\delta^{\tau_2},$$

the last inequality using (14). On the other hand it follows from (7) that $\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\in S\left(\langle\xi\rangle^{-\tau_1}\langle\delta\xi\rangle^{-\tau_2},g\right)$ and moreover

$$\big\|\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\big\|_{\ell_N;S\left(\langle\xi\rangle^{-\tau_1}\langle\delta\xi\rangle^{-\tau_2},g\right)}\le C_N$$

with C_N a constant depending only on N but independent of δ. As a result we use (17) in Proposition 2.3 to conclude that,

$$\begin{aligned}&\big\|\left(r_1-r_2\right)\sharp\big(\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\big)\big\|_{N;S(1,g)}\\ \le\;&\big\|r_1-r_2\big\|_{\ell_N;S(\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2},g)}\big\|\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\big\|_{\ell_N;S\left(\langle\xi\rangle^{-\tau_1}\langle\delta\xi\rangle^{-\tau_2},g\right)}\le C_N\end{aligned}$$

with C_N independent of δ. Thus Lemma 2.2 yields

$$\begin{aligned}\big\|[q,\lambda(D_x)]f\big\|_{L^2(\mathbb{R}_x)} &= \Big\| \Big((r_1-r_2)\sharp\big(\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\big)\Big)^w \Lambda^{\tau_1}\Lambda_\delta^{\tau_2} f\big\|_{L^2(\mathbb{R}_x)}\\ &\le C_N\big\|(r_1-r_2)\sharp\big(\langle\delta\xi\rangle^{-\tau_2}\langle\xi\rangle^{-\tau_1}\big)\big\|_{N;S(1,g)}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2} f\big\|_{L^2(\mathbb{R}_x)}\\ &\le C_N\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2} f\big\|_{L^2(\mathbb{R}_x)},\end{aligned}$$

with C_N independent of δ. The proof is complete. □

Corollary 2.5. *Let k_0 be the fixed integer as in Lemma 2.4. Suppose that*

$$\forall\ 1\le\alpha\le k_0+2,\quad \big\|\partial_x^\alpha u\big\|_{L^\infty([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_x^\alpha\partial_y u\big\|_{L^\infty([0,T]\times\mathbb{R}^2_+)}+\sum_{j=0}^{3}\big\|\partial_x^{\alpha+j}v\big\|_{L^\infty([0,T]\times\mathbb{R}^2_+)}\le C_\alpha$$

with C_α a constant depending only on α. Then there exists a constant C independent of δ, such that for any $f\in L^2([0,T]\times\mathbb{R}_+;\ H^{\tau_1+\tau_2}(\mathbb{R}_x))$, we have

$$\sum_{k_1+k_2\le1}\big\|\big[\big(\partial_x^{k_1}\partial_y^{k_2}u\big),\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x\big]f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},$$

$$\big\|\big[u\partial_x,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}$$

and

$$\sum_{j=0}^{3}\big\|\big[(\partial_x^j v),\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le C\big\|\Lambda^{\tau_1-1}\Lambda_\delta^{\tau_2}f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Proof. It follows from (7) and Leibniz's formula that the condition (21) is fulfilled, with $\lambda(\xi)=\langle\xi\rangle^{\tau_1}\langle\delta\xi\rangle^{\tau_2}\xi$. Moreover (20) is fulfilled with $q=\partial_x^{k_1}\partial_y^{k_2}u, k_1+k_2\le1$. Thus the first estimate in Corollary 2.5 follows immediately from Lemma 2.4. Now we use using (19) to write,

$$\begin{aligned}\big[u\partial_x,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big] &= \big[u,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]\partial_x=\big[u,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x\big]-\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big[u,\ \partial_x\big]\\ &= \big[u,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x\big]+\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\left(\partial_x u\right)\\ &= \big[u,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x\big]+\left(\partial_x u\right)\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}-\big[\left(\partial_x u\right),\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big].\end{aligned}$$

Thus by the first estimate in Corollary 2.5 we have

$$\big\|\big[u\partial_x,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Similarly,

$$\sum_{j=0}^{3}\big\|\big[(\partial_x^j v),\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le C\big\|\Lambda^{\tau_1-1}\Lambda_\delta^{\tau_2}f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Then the proof of Corollary 2.5 is complete. □

2.4. **Subelliptic estimate for linear Prandtl operator.** In this subsection we consider the linear Prandtl operator which reads

$$\mathcal{P}=\partial_t+u\partial_x+v\partial_y-\partial_y^2, \tag{24}$$

where u,v are classical solution to Prandtl equation (2), and thus satisfy the following

$$\begin{cases} v(t,x,y)=-\int_0^y\partial_x u(t,x,\tilde{y})d\tilde{y}, & (t,x,y)\in]0,T[\times\mathbb{R}^2_+;\\ u(t,x,0)=v(t,x,0)=0, & (t,x)\in]0,T[\times\mathbb{R}\end{cases} \tag{25}$$

Moreover we suppose that u satisfies the monotonic condition and that $\partial_y u$ is bounded; that is,

$$\partial_y u(t,x,y)\ge C_*\langle y\rangle^{-\sigma},\qquad \partial_y u\in L^\infty([0,T]\times\mathbb{R}^2_+), \tag{26}$$

where $\sigma>1/2$ is a given number, and C_* is a constant independent of t,x. As to be seen later this monotonicity is crucial to investigate the subelliptic struction of Prandtl operator. In order to estimate the

commutators between $\mathcal{P}$ and Fourier multipliers defined in the previous subsection, we need the following assumption:

$$\forall\ 0\le\alpha\le k_0+2,\quad \left\|\partial_x^\alpha u\right\|_{L^\infty([0,T]\times\mathbb{R}^2_+)}+\sum_{j=0}^{3}\left\|\partial_x^{\alpha+j}v\right\|_{L^\infty([0,T]\times\mathbb{R}^2_+)}\le C_\alpha, \tag{27}$$

where k_0 is the integer given in (18) and C_α is a constant depending only on α.

The main result on subelliptic estimate for Prandtl operator can be stated as follows.

Proposition 2.6. *Let $\mathcal{P}$ be the linear Prandtl operator defined by (24) with the conditions (25)-(27) fulfilled, and let $h,\varphi\in L^2([0,T]\times\mathbb{R}^2_+)$ be given such that $\partial_y h,\partial_y\varphi\in L^2([0,T]\times\mathbb{R}^2_+)$. Suppose that $f\in L^2\left([0,T];\ H^2(\mathbb{R}^2_+)\right)$ with $\partial_y^3 f\in L^2\left([0,T]\times\mathbb{R}^2_+\right)$, is a classical solution to the equation*

$$\mathcal{P}f=\partial_t f+u\partial_x f+v\partial_y f-\partial_y^2 f=h,\qquad (t,x,y)\in]0,T[\times\mathbb{R}^2_+, \tag{28}$$

with the following boundary conditions fulfilled:

$$f(0,x,y)=f(T,x,y)=0,\quad (x,y)\in\mathbb{R}^2_+, \tag{29}$$

and

$$\partial_t f(t,x,0)=\left(\partial_y^2 f\right)(t,x,0)+\varphi(t,x,0),\quad \partial_y f(t,x,0)=0,\quad (t,x)\in]0,T[\times\mathbb{R}. \tag{30}$$

Then for any $\varepsilon>0$ there exists a constant C_ε, depending only on ε,σ and the constants C_* and C_α given respectively in (26) and (27), such that

$$\begin{aligned}
&\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|\partial_y^2\Lambda^{-1/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\varepsilon\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2(\mathbb{R}^2)}+C_\varepsilon\left(\left\|\Lambda^{-1/3}h\right\|_{L^2(\mathbb{R}^3_+)}+\left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_\varepsilon\left(\left\|\langle y\rangle^{-\frac{\sigma}{2}}\partial_y\Lambda^{1/6}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|\langle y\rangle\Lambda^{-1/3}\partial_y\varphi\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right),
\end{aligned}$$

and moreover

$$\begin{aligned}
&\left\|\partial_y^3\Lambda^{-2/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\left(\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2(\mathbb{R}^2)}+\left\|\Lambda^{-1/3}h\right\|_{L^2(\mathbb{R}^3_+)}+\left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C\left(\left\|\langle y\rangle^{-\frac{\sigma}{2}}\partial_y\Lambda^{1/6}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|\langle y\rangle\Lambda^{-1/3}\partial_y\varphi\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}$$

Recall Λ^d is the Fourier multiplier with symbol $\langle\xi\rangle^d$.

Proof. To simply the notation, in the proof we use C to denote different suitable constants which depend only on σ and the constants C_* and C_α given respectively in (26) and (27).

Taking the operator $\Lambda^{-2/3}$ on both sides of (28), we see the function $\Lambda^{-2/3}f$ satisfies the following equation in $]0,T[\times\mathbb{R}^2_+$:

$$\begin{aligned}
&\partial_t\Lambda^{-2/3}f+u\partial_x\Lambda^{-2/3}f+v\partial_y\Lambda^{-2/3}f-\partial_y^2\Lambda^{-2/3}f\\
&=\Lambda^{-2/3}h+\left[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\right]f,
\end{aligned} \tag{31}$$

and that

$$\Lambda^{-2/3}f\big|_{t=0}=\Lambda^{-2/3}f\big|_{t=T}=0,\quad \partial_y\Lambda^{-2/3}f\big|_{y=0}=0 \tag{32}$$

due to (29) and (30), since $\Lambda^{-2/3}$ is an operator acing only on x variable. Recall $\left[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\right]$ stands for the commutator between $u\partial_x+v\partial_y$ and $\Lambda^{-2/3}$.

Step 1). We will show in this step that

$$\begin{aligned}
&\left\|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le 2\left|\mathrm{Re}\ \left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|+\left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+C\left(\left\|\Lambda^{-1/3}h\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned} \tag{33}$$

To do so, we take $L^2([0,T]\times\mathbb{R}^2_+)$ inner product with the function $\partial_y\partial_x\Lambda^{-2/3}f \in L^2([0,T]\times\mathbb{R}^2_+)$ on the both sides of eq. (31), and then consider the real parts; this gives

$$
\begin{aligned}
&-\operatorname{Re}\ \Big(u\partial_x\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&=\operatorname{Re}\ \Big(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}-\operatorname{Re}\ \Big(\partial_y^2\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+\operatorname{Re}\ \Big(v\partial_y\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}-\operatorname{Re}\ \Big(\Lambda^{-2/3}h,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad-\operatorname{Re}\ \Big(\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
\tag{34}
$$

We will treat the terms on both sides. For the term on left hand side we integrate by parts to obtain, observing $u\big|_{y=0}=0$ due to (25),

$$
\begin{aligned}
&-\operatorname{Re}\ \Big(u\partial_x\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &-\frac{1}{2}\left\{\Big(u\partial_x\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}+\Big(\partial_y\partial_x\Lambda^{-2/3}f,\ u\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right\}\\
=\ &\frac{1}{2}\big\|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Next we estimate the terms on the right hand side and have, by Cauchy-Schwarz's inequality ,

$$
\begin{aligned}
\Big|-\operatorname{Re}\ \Big(\partial_y^2\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big|&\le\frac{1}{2}\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\frac{1}{2}\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},\\
\Big|\operatorname{Re}\ \Big(\Lambda^{-2/3}h,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big|&\le\big\|\Lambda^{-1/3}h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|-\operatorname{Re}\ \Big(v\partial_y\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big|\\
\le\ &\Big|\Big(\partial_y f,\ \big[\Lambda^{-2/3},\ v\big]\partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big|+\Big|\Big(v\partial_y f,\ \Lambda^{-2/3}\partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big|\\
\le\ &C\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
$$

the last inequality using Corollary 2.5. Finally

$$
\begin{aligned}
&\Big|-\operatorname{Re}\ \Big(\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big|\\
\le\ &\big\|\Lambda^{1/3}\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &2\left(\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{1/3}\Lambda^{-2/3}\big]f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{1/3}\big]\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+\big\|\partial_y\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\left(\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right),
\end{aligned}
$$

with the second inequality following from (19) and the last inequality using Corollary 2.5. These inequalities, together with (34), yields the desired (33).

Step 2). In this step we will estimate the second term on the right hand side of (33) and show that for any $\varepsilon>0$,

$$
\begin{aligned}
&\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le\varepsilon\ \big\|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+C_\varepsilon\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right),
\end{aligned}
\tag{35}
$$

with C_ε a constant depending on ε. To do so, in view of (28) - (30) we see the function $\Lambda^{-1/3}f$ satisfies the equation in $]0,T[\times\mathbb{R}^2_+$,

$$\begin{aligned}&\partial_t\Lambda^{-1/3}f+(u\partial_x+v\partial_y)\,\Lambda^{-1/3}f-\partial_y^2\Lambda^{-1/3}f\\&=\Lambda^{-1/3}h+\left[u\partial_x+v\partial_y,\ \Lambda^{-1/3}\right]f,\end{aligned}\tag{36}$$

with the boundary condition

$$\Lambda^{-1/3}f\big|_{t=0}=\Lambda^{-1/3}f\big|_{t=T}=0,\quad \partial_y\Lambda^{-1/3}f\big|_{y=0}=0.\tag{37}$$

Now we take $L^2([0,T]\times\mathbb{R}^2_+)$ inner product with the function $-\partial_y^2\Lambda^{-1/3}f\in L^2([0,T]\times\mathbb{R}^2_+)$ on both sides of (36), and then consider the real parts; this gives

$$\left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2(\mathbb{R}^3_+)}\le\sum_{p=1}^4 J_p,\tag{38}$$

where

$$\begin{aligned}J_1&=\left|\mathrm{Re}\ \left(\partial_t\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|,\\J_2&=\left|\mathrm{Re}\ \left((u\partial_x+v\partial_y)\,\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{[0,T]\times L^2([0,T]\times\mathbb{R}^2_+)}\right|,\\J_3&=\left|\mathrm{Re}\ \left(\Lambda^{-1/3}h,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|,\\J_4&=\left|\mathrm{Re}\ \left(\left[u\partial_x+v\partial_y,\ \Lambda^{-1/3}\right]f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|.\end{aligned}$$

Integrating by parts and observing the condition (37), we see

$$\left(\partial_t\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}=-\left(\partial_t\partial_y\Lambda^{-1/3}f,\ \partial_y\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)},$$

which along with the fact

$$\mathrm{Re}\ \left(\partial_t\partial_y\Lambda^{-1/3}f,\ \partial_y\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}=0$$

due to (37), implies

$$J_1=\left|\mathrm{Re}\ \left(\partial_t\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|=0.\tag{39}$$

About J_2 we integrate by parts again and observe the boundary condititon (37), to compute

$$\begin{aligned}&\mathrm{Re}\left(u\partial_x\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\=\ &-\mathrm{Re}\left(u\partial_x\Lambda^{-1/3}\partial_y f,\ \Lambda^{-1/3}\partial_y f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}-\mathrm{Re}\ \left((\partial_y u)\partial_x\Lambda^{-1/3}f,\ \Lambda^{-1/3}\partial_y f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\=\ &\frac{1}{2}\left((\partial_x u)\,\Lambda^{-1/3}\partial_y f,\ \Lambda^{-1/3}\partial_y f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}-\mathrm{Re}\ \left((\partial_y u)\partial_x\Lambda^{-1/3}f,\ \Lambda^{-1/3}\partial_y f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.\end{aligned}$$

This gives

$$\begin{aligned}&\left|\mathrm{Re}\left(u\partial_x\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\\le\ &\|\Lambda^{-1/3}\,(\partial_y u)\,\partial_x\Lambda^{-1/3}f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\\le\ &\left(\|\,(\partial_y u)\,\Lambda^{-1/3}\partial_x\Lambda^{-1/3}f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\|\left[\Lambda^{-1/3},\ \partial_y u\right]\partial_x\Lambda^{-1/3}f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\&+C\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\\le\ &C\left(\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\|f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\left\|\partial_y f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\\le\ &\varepsilon\,\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C_\varepsilon\left(\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).\end{aligned}$$

Moreover integrating by part, we obtain

$$\begin{aligned}\left|\mathrm{Re}\left(v\partial_y\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| &= \frac{1}{2}\left|\left((\partial_y v)\,\partial_y\Lambda^{-1/3}f,\ \partial_y\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\ &\le C\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.\end{aligned}$$

Thus

$$J_2 \le \varepsilon\,\big\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_\varepsilon\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right). \tag{40}$$

It remains to estimate J_3 and J_4. Let $\tilde\varepsilon > 0$ be an arbitrarily small number. Cauchy-Schwarz's inequality gives

$$\begin{aligned}J_3 &= \left|\mathrm{Re}\ \left(\Lambda^{-1/3}h,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\ &\le \tilde\varepsilon\,\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\tilde\varepsilon}\big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb{R}^3_+)},\end{aligned}$$

and for J_4, Corollary 2.5 implies

$$\begin{aligned}J_4 &= \left|\mathrm{Re}\ \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-1/3}\right]f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2(\mathbb{R}^2_+)}\right| \\ &\le \tilde\varepsilon\,\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\tilde\varepsilon}\left(\big\|f\big\|^2_{L^2(\mathbb{R}^3_+)} + \big\|\partial_y f\big\|^2_{L^2(\mathbb{R}^3_+)}\right),\end{aligned}$$

where $\tilde C_\varepsilon$ is constant depending on $\tilde\varepsilon$. Now the above two estimates for J_3 and J_4, along with (38) - (40), gives

$$\begin{aligned}\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \le\ & \tilde\varepsilon\,\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \varepsilon\,\big\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ &+C_\varepsilon\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\ &+C_{\tilde\varepsilon}\left(\big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb{R}^3_+)} + \big\|f\big\|^2_{L^2(\mathbb{R}^3_+)} + \big\|\partial_y f\big\|^2_{L^2(\mathbb{R}^3_+)}\right),\end{aligned}$$

and thus, letting $\tilde\varepsilon$ small sufficiently,

$$\begin{aligned}&\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le\ & \varepsilon\,\big\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_\varepsilon\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb{R}^3_+)}\right).\end{aligned}$$

This is just the desired estimate (35).

Step 3). Combining the estimates (33) and (35), we obtain, choosing ε sufficiently small,

$$\begin{aligned}&\big\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\le C\left|\mathrm{Re}\ \left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\ &\quad + C\left(\big\|\Lambda^{-1/3}h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).\end{aligned} \tag{41}$$

Step 4) It remains to treat the first term on the right hand side of (41). In this step we will prove that, for any $\varepsilon_1 > 0$,

$$\begin{aligned}&\left|\mathrm{Re}\ \left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\ &\le \varepsilon_1\int_0^T\int_{\mathbb{R}}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2 dxdt + C_{\varepsilon_1}\big\|\,\langle y\rangle\,\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad + \varepsilon_1^{-1}C\left(\big\|\,\langle y\rangle^{-\sigma/2}\,\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\,\langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).\end{aligned} \tag{42}$$

For this purpose we integrate by parts again and observe the boundary condition (32) , to compute

$$\begin{aligned}
&\left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &-\left(\Lambda^{-2/3}f,\ \partial_t\partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &\left(\partial_x\Lambda^{-2/3}f,\ \partial_t\partial_y\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &-\left(\partial_y\partial_x\Lambda^{-2/3}f,\ \partial_t\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}+\int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-2/3}f(t,x,0)\right)\left(\partial_x\Lambda^{-2/3}f(t,x,0)\right)dxdt,
\end{aligned}$$

which, along with the fact that

$$\begin{aligned}
&2\ \mathrm{Re}\ \left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &\left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}+\left(\partial_y\partial_x\Lambda^{-2/3}f,\ \partial_t\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

yields, for any $\varepsilon_1>0$,

$$\begin{aligned}
&\left|\mathrm{Re}\ \left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&=\frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-2/3}f(t,x,0)\right)\left(\partial_x\Lambda^{-2/3}f(t,x,0)\right)dxdt\right|\\
&=\frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}\left(\Lambda^{1/6}\partial_t\Lambda^{-2/3}f(t,x,0)\right)\left(\Lambda^{-1/6}\partial_x\Lambda^{-2/3}f(t,x,0)\right)dxdt\right|\\
&\le\varepsilon_1\int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-1/2}f(t,x,0)\right)^2dxdt+\varepsilon_1^{-1}\int_0^T\int_{\mathbb{R}}\left(\Lambda^{1/6}f(t,x,0)\right)^2dxdt.
\end{aligned}\tag{43}$$

Moreover observing

$$\Lambda^{1/6}f(t,x,0)=\left(\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right)(t,x,0),$$

it then follows from Sobolev inequality that

$$\begin{aligned}
\left|\Lambda^{1/6}f(t,x,0)\right|^2\ &\le\ C\left(\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right\|^2_{L^2(\mathbb{R}_+)}+\left\|\partial_y\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right\|^2_{L^2(\mathbb{R}_+)}\right)\\
&\le\ C\left(\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right\|^2_{L^2(\mathbb{R}_+)}+\left\|\langle y\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\right\|^2_{L^2(\mathbb{R}_+)}\right)
\end{aligned}$$

with C a constant independent of t,x. And thus

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\left(\Lambda^{1/6}f(t,x,0)\right)^2dxdt\\
&\le C\left(\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|\partial_y\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\le C\left(\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}\partial_yf\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}\tag{44}$$

Using the fact that

$$\partial_t\Lambda^{-1/2}f(t,x,0)=\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)+\Lambda^{-1/2}\varphi(t,x,0)$$

due to assumption (30), we conclude

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-1/2}f(t,x,0)\right)^2dxdt\\
\le\ &\int_0^T\int_{\mathbb{R}}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt+\int_0^T\int_{\mathbb{R}}\left|\Lambda^{-1/2}\varphi(t,x,0)\right|^2dxdt.
\end{aligned}$$

Moreover observe

$$\begin{aligned}\left|\Lambda^{-1/2}\varphi(t,x,0)\right| &= \left|-\int_0^{+\infty}\partial_{\tilde y}\Lambda^{-1/2}\varphi(t,x,\tilde y)d\tilde y\right| \\ &\le \left(\int_0^{+\infty}\langle\tilde y\rangle^{-2}\,d\tilde y\right)^{1/2}\left(\int_0^{+\infty}\langle\tilde y\rangle^{2}\left|\Lambda^{-1/2}\partial_{\tilde y}\varphi(t,x,\tilde y)\right|^2 d\tilde y\right)^{1/2},\end{aligned}$$

which implies

$$\int_0^T\int_{\mathbb R}\left|\Lambda^{-1/2}\varphi(t,x,0)\right|^2dxdt \le C\big\|\langle y\rangle\Lambda^{-1/2}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \le C\big\|\langle y\rangle\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb R^2_+)},$$

and thus

$$\begin{aligned}&\int_0^T\int_{\mathbb R}\left(\partial_t\Lambda^{-1/2}f(t,x,0)\right)^2dxdt \\ \le\ &\int_0^T\int_{\mathbb R}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt + C\big\|\langle y\rangle\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb R^2_+)}.\end{aligned}$$

This along with (43) and (44) yields the desired (42).

Step 5) Combining (41) and (42), we have, for any $\varepsilon_1>0$,

$$\begin{aligned}&\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \\ \le\ &\varepsilon_1\int_0^T\int_{\mathbb R}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt + C_{\varepsilon_1}\big\|\langle y\rangle\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \\ &+\varepsilon_1^{-1}C\left(\big\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\langle y\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)}\right) \\ &+C\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb R^3_+)}\right).\end{aligned}$$

Moreover we use the monotonicity condition and interpolation inequality to get, for any $\varepsilon_2>0$

$$\begin{aligned}\big\|\langle y\rangle^{-\sigma/2}\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} &\le \varepsilon_2\big\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \varepsilon_2^{-1}\big\|\langle y\rangle^{-\sigma/2}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \\ &\le \varepsilon_2\big\|\langle y\rangle^{-\sigma/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + C_{\varepsilon_2}\big\|\langle y\rangle^{-\sigma/2}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \\ &\le \varepsilon_2\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + C_{\varepsilon_2}\big\|f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)}.\end{aligned}$$

From the above inequalities, we infer that, choosing ε_2 small enough,

$$\begin{aligned}&\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \\ \le&\varepsilon_1\int_0^T\int_{\mathbb R}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt \\ &+C_{\varepsilon_1}\left(\big\|\langle y\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\langle y\rangle\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb R^2_+)}\right) \\ &+C_{\varepsilon_1}\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb R^3_+)}\right).\end{aligned}\tag{45}$$

Step 6) In this step we treat the first term on the right side of (45), and show that, for any $0<\varepsilon<1$,

$$\begin{aligned}&\int_0^T\int_{\mathbb R}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt \\ \le&C\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \varepsilon C\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} \\ &+C_\varepsilon\left(\big\|f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)} + \big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb R^2_+)}\right).\end{aligned}\tag{46}$$

To do so, we integrate by parts to get

$$\begin{aligned}\int_0^T\int_{\mathbb R}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt &= 2\mathrm{Re}\ \left(\partial_y^3\Lambda^{-1/2}f,\ \partial_y^2\Lambda^{-1/2}f\right)_{L^2([0,T]\times\mathbb R^2_+)} \\ &= 2\mathrm{Re}\ \left(\partial_y^3\Lambda^{-2/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb R^2_+)}.\end{aligned}$$

This yields

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dxdt \\
&\leq \frac{\varepsilon}{2} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + 2\varepsilon^{-1} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\leq \varepsilon \left\| \partial_y^3 \Lambda^{-2/3} f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_\varepsilon \left\| \partial_y f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned} \tag{47}
$$

the last inequality holding because we can use (30) to integrate by parts and then obtain

$$
\begin{aligned}
&\left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} = \left(\partial_y^2 \Lambda^{-2/3} f,\ \partial_y^2 f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq \left| \left(\partial_y^3 \Lambda^{-2/3} f,\ \partial_y f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right| \\
&\leq \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \left\| \partial_y f \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned} \tag{48}
$$

Thus in order to prove (46) it suffices to estimate $\left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}$. To do so, taking first ∂_y on both side of (28) and then taking $\Lambda^{-2/3}$, we have the equation

$$
\begin{aligned}
&\partial_t \Lambda^{-2/3} \partial_y f + u \partial_x \Lambda^{-2/3} \partial_y f + v \partial_y \Lambda^{-2/3} \partial_y f - \partial_y^3 \Lambda^{-2/3} f \\
= \quad & \Lambda^{-2/3} \partial_y h + \left[u\partial_x + v\partial_y,\ \Lambda^{-2/3} \right] \partial_y f - \Lambda^{-2/3} (\partial_y u) \partial_x f - \Lambda^{-2/3} (\partial_y v) \partial_y f,
\end{aligned}
$$

which implies, by taking L^2 inner product with $-\partial_y^3 \Lambda^{-2/3} f$,

$$
\begin{aligned}
&\left\| \partial_y^3 \Lambda^{-2/3} f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= -\mathrm{Re}\ \left(\partial_t \Lambda^{-2/3} \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \mathrm{Re}\ \left(u \partial_x \Lambda^{-2/3} \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \mathrm{Re}\ \left(v \partial_y \Lambda^{-2/3} \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad + \mathrm{Re}\ \left(\Lambda^{-2/3} \partial_y h,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad + \mathrm{Re}\ \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3} \right] \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \mathrm{Re}\ \left(\Lambda^{-2/3} (\partial_y u) \partial_x f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \mathrm{Re}\ \left(\Lambda^{-2/3} (\partial_y v) \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned} \tag{49}
$$

Next we will treat the terms on the right hand side. Observing

$$
\partial_t \Lambda^{-2/3} \partial_y f \big|_{y=0} = 0
$$

due to (30), we integrate by part to compute

$$
\begin{aligned}
&- \mathrm{Re}\ \left(\partial_t \Lambda^{-2/3} \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= - \mathrm{Re}\ \left(\partial_t \partial_y^2 \Lambda^{-2/3} f,\ \partial_y^2 \Lambda^{-2/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= 0,
\end{aligned} \tag{50}
$$

the last equality holding because

$$
\partial_y^2 \Lambda^{-2/3} f \big|_{t=0} = \partial_y^2 \Lambda^{-2/3} f \big|_{t=T} = 0
$$

due to (29). Since $u\big|_{y=0}$ then integrating by parts gives

$$\begin{aligned}
& - \text{ Re } \Big(u\partial_x\Lambda^{-2/3}\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
= & - \text{ Re } \Big(u\partial_x\Lambda^{-2/3}\partial_y^2 f,\ \partial_y^2\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& - \text{ Re } \Big((\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f,\ \partial_y^2\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
= & \ \frac{1}{2}\Big((\partial_x u)\Lambda^{-2/3}\partial_y^2 f,\ \Lambda^{-2/3}\partial_y^2 f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& - \text{ Re } \Big((\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f,\ \partial_y^2\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
\leq & \ \frac{1}{2}\big\|\partial_x u\big\|_{L^\infty}\big\|\Lambda^{-2/3}\partial_y^2 f\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& + \big\|\Lambda^{-1/3}(\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

On the other hand, using Corollary 2.5 gives

$$\begin{aligned}
& \big\|\Lambda^{-1/3}(\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
\leq & \ 2\big\|\Lambda^{-1/3}\partial_x\Lambda^{-2/3}(\partial_y u)\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + 2\big\|\Lambda^{-1/3}\big[\partial_y u,\ \partial_x\Lambda^{-2/3}\big]\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
\leq & \ C\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Thus

$$\begin{aligned}
& - \text{ Re } \Big(u\partial_x\Lambda^{-2/3}\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& \leq C\ \Big(\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\Big) \\
& \leq \tilde{\varepsilon}\ \big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_{\tilde{\varepsilon}}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned} \tag{51}$$

where the last inequality using (48). Using (48) we conclude

$$\begin{aligned}
& - \text{ Re } \Big(v\partial_y\Lambda^{-2/3}\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& \leq \ \frac{\tilde{\varepsilon}}{2}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_{\tilde{\varepsilon}}\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& \leq \ \tilde{\varepsilon}\ \big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_{\tilde{\varepsilon}}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned} \tag{52}$$

Cauchy-Schwarz inequality gives, for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned}
& \text{ Re } \Big(\Lambda^{-2/3}\partial_y h,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& \leq \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \tilde{\varepsilon}^{-1}\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned} \tag{53}$$

and

$$\begin{aligned}
& - \text{ Re } \Big(\Lambda^{-2/3}(\partial_y v)\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
& \leq \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \tilde{\varepsilon}^{-1}\big\|\partial_y v\big\|^2_{L^\infty}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}
\end{aligned} \tag{54}$$

and

$$
\begin{aligned}
&\mathrm{Re}\ \Big(\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le \frac{\tilde\varepsilon}{2}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+2\tilde\varepsilon^{-1}\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le \frac{\tilde\varepsilon}{2}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\varepsilon}\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\varepsilon}\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le \tilde\varepsilon\ \big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\varepsilon}\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
\tag{55}
$$

the second inequality using Corollary 2.5, while the last inequality following from (48). Finally,

$$
\begin{aligned}
&-\mathrm{Re}\ \Big(\Lambda^{-2/3}(\partial_y u)\partial_x f,\ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\tilde\varepsilon^{-1}\big\|\Lambda^{-2/3}(\partial_y u)\partial_x f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\tilde\varepsilon^{-1}\big\|(\partial_y u)\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\tilde\varepsilon^{-1}\big\|\big[\partial_y u,\ \Lambda^{-2/3}\big]\partial_x f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\varepsilon}\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C_{\tilde\varepsilon}\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

This, along with (49) -(55), yields, for any $\tilde\varepsilon>0$,

$$
\begin{aligned}
&\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\varepsilon}\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C_{\tilde\varepsilon}\left(\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}
$$

Thus letting $\tilde\varepsilon$ be small enough, we have

$$
\begin{aligned}
&\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & C\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\left(\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}
\tag{56}
$$

This along with (47) yields the desired estimate (46).

Step 7) Now we combine (45) and (46) to conclude for any $0<\varepsilon,\varepsilon_1<1$,

$$
\begin{aligned}
&\big\|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & \varepsilon_1 C\big\|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\varepsilon_1\varepsilon C\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C_{\varepsilon_1,\varepsilon}\left(\big\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\left\langle y\right\rangle\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_{\varepsilon_1,\varepsilon}\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb{R}^3_+)}\right),
\end{aligned}
$$

which implies, choosing $\varepsilon_1>0$ sufficiently small,

$$
\begin{aligned}
&\big\|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\,\varepsilon\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C_\varepsilon\left(\big\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\left\langle y\right\rangle\Lambda^{-1/3}\partial_y\varphi\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_\varepsilon\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb{R}^3_+)}\right),
\end{aligned}
\tag{57}
$$

with $\varepsilon > 0$ arbitrarily small. This, along with

$$\begin{aligned}\left\| \langle y\rangle^{-\sigma/2}\,\Lambda^{1/3} f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} &\le C\left\| (\partial_y u)^{1/2}\,\Lambda^{1/3} f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le C\left\| \langle y\rangle^{-\sigma/2}\,\partial_x\Lambda^{-2/3} f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C\left\| f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\end{aligned}$$

due to (26), implies, for any $\varepsilon > 0$,

$$\begin{aligned}&\left\| \langle y\rangle^{-\sigma/2}\,\Lambda^{1/3} f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3} f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ & \varepsilon\left\|\Lambda^{-2/3}\partial_y h\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &+C_\varepsilon\left(\left\| \langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6} f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y\rangle\,\Lambda^{-1/3}\partial_y\varphi\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\ &+C_\varepsilon\left(\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\Lambda^{-1/3} h\right\|^2_{L^2(\mathbb{R}^3_+)}\right).\end{aligned}$$

This is just the first estimate in Proposition 2.6. And the second estimate follows from (56) and (57). Thus the proof of Proposition 2.6 is complete. □

3. Auxiliary functions and weight functions

Here and throughout the paper we alway let u be the classical solution to the Prandtl equation (2). As in [22], in order to overcome the degeneracy in x direction we work with a family of auxilliary function instead of the original u. Precisely, with each $m \ge 1$ we associate a function

$$f_m = \partial_x^m\omega - \frac{\partial_y\omega}{\omega}\partial_x^m u = \omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right), \tag{58}$$

where and throughout the paper we use the notation $\omega = \partial_y u$. Now we introduce two weight functions to be used later: recalling c is the constant given in Theorem 1.1 and Λ^d is the multipliers associate to symbol $\langle\xi\rangle^d$,

$$W_m^i = e^{2cy}\left(1 + \frac{2cy}{(3m+i)\sigma}\right)^{-\frac{(3m+i)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{i}{3}} \tag{59}$$

for each pair $(m,i) \in \mathbb{Z}^2_+$ with $m \ge 1$ and $0 \le i \le 3$, and

$$\phi_m^i = \phi^{3(m-3)+i} \tag{60}$$

for each pair $(m,i) \in \mathbb{Z}^2_+$ with $m \ge 3$ and $0 \le i \le 3$, where ϕ is defined by

$$\phi(t) = t(T-t), \quad 0 \le t \le T.$$

Without loss of generality, we may assume the above $T \le 1$; otherwise we replace the above ϕ by

$$\tilde{\phi}(t) = \frac{t}{T}\left(1 - \frac{t}{T}\right), \quad 0 \le t \le T.$$

As a result we have $0 \le \phi \le 1$ and thus

$$\phi_{m_1}^{i_1} \le \phi_{m_2}^{i_2} \tag{61}$$

provided $3 \le m_2 \le m_1$ and $0 \le i_2 \le i_1 \le 3$.

Next we list some inequalities for the weight W_m^i. Observe the function

$$\gamma \longrightarrow \left(1 + \frac{cy}{\gamma}\right)^{-\gamma}$$

is a monotonically decreasing function as γ varies in the interval $[1, +\infty[$ for $y \ge 0$. Thus

$$\forall\ i \ge 0, \quad \left\|W_{m_1}^i f\right\|_{L^2(\mathbb{R}_x)} \le \left\|W_{m_2}^i f\right\|_{L^2(\mathbb{R}_x)} \tag{62}$$

and

$$\forall\ 0 \le \ell \le i \le 3, \quad \left\|W_{m_1}^i f\right\|_{L^2(\mathbb{R}_x)} \le \left\|W_{m_2}^{i-\ell}\Lambda^{\ell/3} f\right\|_{L^2(\mathbb{R}_x)} \le \left\|W_{m_3}^i f\right\|_{L^2(\mathbb{R}_x)}, \tag{63}$$

provided that $m_1 \ge m_2 \ge 1$, and that $3m_2 + i - \ell \ge 3m_3 + i$. Moreover, since

$$\forall\ 0 \le \alpha \le 3,\ \forall\ \gamma \ge 1, \quad \left|\partial_y^\alpha e^{2cy}\left(1 + \frac{cy}{\gamma}\right)^{-\gamma}(1+cy)^{-1}\right| \le C_\alpha e^{2cy}\left(1 + \frac{cy}{\gamma}\right)^{-\gamma}(1+cy)^{-1},$$

with C_α a constant independent of γ, then the following estimates:

$$\big\|[\partial_y,\ W_m^i]f\big\|_{L^2(\mathbb{R}^2_+)} \le C\big\|W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}, \tag{64}$$

$$\begin{aligned}\big\|[\partial_y^2,\ W_m^i]f\big\|_{L^2(\mathbb{R}^2_+)} &\le C\left(\big\|W_m^i f\big\|_{L^2(\mathbb{R}^2_+)} + \big\|W_m^i\partial_y f\big\|_{L^2(\mathbb{R}^2_+)}\right)\\ &\le \tilde{C}\left(\big\|W_m^i f\big\|_{L^2(\mathbb{R}^2_+)} + \big\|\partial_y W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}\right)\end{aligned} \tag{65}$$

$$\begin{aligned}\big\|[\partial_y^3,\ W_m^i]f\big\|_{L^2(\mathbb{R}^2_+)} &\le C\left(\big\|W_m^i f\big\|_{L^2(\mathbb{R}^2_+)} + \big\|W_m^i\partial_y f\big\|_{L^2(\mathbb{R}^2_+)} + \big\|W_m^i\partial_y^2 f\big\|_{L^2(\mathbb{R}^2_+)}\right)\\ &\le \tilde{C}\left(\big\|W_m^i f\big\|_{L^2(\mathbb{R}^2_+)} + \big\|\partial_y W_m^i f\big\|_{L^2(\mathbb{R}^2_+)} + \big\|\partial_y^2 W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}\right)\end{aligned} \tag{66}$$

hold for all integers m, i with $m \ge 1$ and $0 \le i \le 3$, where $C, \tilde{C}$ are two constants independent of m.

Assumption 3.1. *Let u, v be the classical solution to Prandtl equation (2) with (25), (27) and the following two conditions fulfilled:*

(i) for some $\sigma > 1/2$ and some constant $C_* \ge 1$, we have

$$C_*^{-1}(1+y)^{-\sigma} \le \omega = \partial_y u(y) \le C_*(1+y)^{-\sigma}$$

and

$$\big|\partial_y^2 u(y)\big| + \big|\partial_y^3 u(y)\big| \le C_*(1+y)^{-\sigma-1};$$

(ii) we have

$$\forall\ 1 \le \alpha \le k_0+2, \quad \sum_{j=1}^{4}\big\|\partial_x^\alpha e^{2cy}\partial_y^j u\big\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} \le C_\alpha$$

with C_α a constant depending only on α.

Lemma 3.2. *Let Assumption 3.1 be fulfilled. Let c be the constant given in (59), and let $\Lambda^{\tau_1}, \Lambda_\delta^{\tau_2}$ be the Fourier multiplier associate with the symbols $\langle\xi\rangle^{\tau_1}$ and $\langle\delta\xi\rangle^{\tau_2}$, respectively. Then there exists a constant C, such that for any $m, n \ge 1, 0 \le \ell \le 3$, and for any $\tilde{c} > 0$ with $\tilde{c} < c$, we have*

$$\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)} \le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)}, \tag{67}$$

and

$$\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m v\big\|_{L^\infty(\mathbb{R}_+;\ L^2(\mathbb{R}_x))} \le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_{m+1}\big\|_{L^2(\mathbb{R}^2_+)}. \tag{68}$$

Proof. In the proof we use C to denote different constants which are independent of m. Observe $\omega \in L^\infty$ and $\omega > 0$ then

$$\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)} \le C\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}$$

On the other hand, integrating by parts we have

$$\begin{aligned}\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\big\|^2_{L^2(\mathbb{R}^2_+)} &= \int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &= \frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty\left(\partial_y e^{2\tilde{c}y}\right)\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &= \frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left[\partial_y\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\right]\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &\quad+\frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\overline{\partial_y\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &\le \frac{1}{\tilde{c}}\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)},\end{aligned}$$

which implies

$$\begin{aligned}\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)} &\le \big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\ &= \big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}e^{\tilde{c}y}\omega^{-1}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\ &\le \big\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}+\big\|\big[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

Thus we have, by the above inequalities,

$$\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\le C\big\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}+C\big\|\big[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.$$

On the other hand, Assumption 3.1 enables us to use Lemma 2.4 with $\lambda(D_x)=\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}$ and $q=\omega^{-1}e^{\tilde{c}y}$, to obtain

$$\begin{aligned}\big\|\big[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\big]\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)} &\le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\ &\le C\big\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

As a result,

$$\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\le C\big\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)},$$

the last inequality using the fact that $f_m=\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)$, and that

$$e^{\tilde{c}y}\omega^{-1}\le Ce^{\tilde{c}y}(1+y)^\sigma\le Ce^{2cy}\left(1+\frac{2cy}{\gamma}\right)^{-\gamma/2}$$

for any $\gamma\ge 1$. This is just the desired (67). Now we prove (68). Recall $v(t,x,y)=-\int_0^y\partial_x u(t,x,y')dy'$. Then we have

$$\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m v=-\int_0^y\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^{m+1}u(x,y')dy'$$

Therefore

$$\begin{aligned}\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m v\|_{L^\infty(\mathbb{R}_+;\,L^2(\mathbb{R}_x))} &\le \big\|e^{-\tilde{c}y}\big\|_{L^2(\mathbb{R}_+)}\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^{m+1}u\big\|_{L^2(\mathbb{R}^2_+)}\\ &\le C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_{m+1}\big\|_{L^2(\mathbb{R}^2_+)},\end{aligned}$$

the last inequality using (67). Thus the desired (68) follows and the proof of Lemma 3.2 is complete. □

Lemma 3.3. *Suppose Assumption 3.1 is fulfilled. Then there exists a constant C, depending only on σ, c, and the constants C_α, C_* in Assumption 3.1, such that for any $m\ge 1$ and any $0\le\ell\le 3$, we have*

$$\big\|\langle y\rangle^{-1}W_m^\ell\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}+\big\|\langle y\rangle^{-1}W_m^\ell\partial_x^m\omega\big\|_{L^2(\mathbb{R}^2_+)}\le C\big\|W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)}. \tag{69}$$

Proof. In the proof we use C to denote different constants which depend only on σ, c, and the constants C_α, C_* in Assumption 3.1 and are independent of m. By Assumption 3.1 we see

$$\begin{aligned}\omega\langle y\rangle^{-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(1+cy)^{-1} &\le C(1+y)^{-\sigma-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}\\ &\le CR^{\sigma+1}(R+y)^{-\sigma-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2},\end{aligned}$$

where $R\ge 1$ is a large number to be determined later. Thus using the notation

$$b_{m,\ell}^R(y)=\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(R+y)^{-\sigma-1},$$

we have

$$\big\|\langle y\rangle^{-1}W_m^\ell\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}=\big\|\omega\langle y\rangle^{-1}W_m^\ell\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}\le CR^{\sigma+1}\big\|e^{2cy}b_{m,\ell}^R\frac{\Lambda^{\ell/3}\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}.$$

On the other hand, Assumption 3.1 enables us to use Lemma 2.4 with $\lambda(D_x)=\Lambda^{\ell/3}$ and $q=\omega^{-1}$, to obtain

$$\big\|\big[\Lambda^{\ell/3},\ \omega^{-1}\big]e^{2cy}b^R_{m,\ell}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\le C\big\|e^{2cy}b^R_{m,\ell}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\le C\big\|e^{2cy}b^R_{m,\ell}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)},$$

and thus

$$\begin{aligned}\big\|e^{2cy}b^R_{m,\ell}\frac{\Lambda^{\ell/3}\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)} &\le \big\|\Lambda^{\ell/3}e^{2cy}b^R_{m,\ell}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}+\big\|\big[\Lambda^{\ell/3},\ \omega^{-1}\big]e^{2cy}b^R_{m,\ell}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\\ &\le C\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

Combining these inequalities we conclude

$$\big\|\langle y\rangle^{-1}W^\ell_m\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\le CR^{\sigma+1}\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}. \tag{70}$$

Moreover, observe $u\big|_{y=0}=0$ and thus we have, by integrating by parts,

$$\begin{aligned}\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|^2_{L^2(\mathbb{R}^2_+)} &= \int_{\mathbb{R}}\int_0^\infty e^{4cy}\left(b^R_{m,\ell}(y)\right)^2\left(\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}}dydx\\ &= \frac{1}{4c}\int_{\mathbb{R}}\int_0^\infty\left(\partial_y e^{4cy}\right)\left(b^R_{m,\ell}(y)\right)^2\left(\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}}dydx\\ &= -\frac{1}{2c}\int_{\mathbb{R}}\int_0^\infty e^{4cy}b^R_{m,\ell}(y)\left(\partial_y b^R_{m,\ell}(y)\right)\left(\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}}dydx\\ &\quad -\frac{1}{4c}\int_{\mathbb{R}}\int_0^\infty e^{4cy}\left(b^R_{m,\ell}(y)\right)^2\left[\partial_y\left(\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\right)\right]\overline{\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}}dydx\\ &\quad -\frac{1}{4c}\int_{\mathbb{R}}\int_0^\infty e^{4cy}\left(b^R_{m,\ell}(y)\right)^2\left(\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\right)\overline{\partial_y\Lambda^{\ell/3}\left(\frac{\partial_x^m u}{\omega}\right)}dydx,\end{aligned}$$

which, along with the estimate

$$\left|\partial_y b^R_{m,\ell}\right|\le\left(c+(\sigma+1)R^{-1}\right)b^R_{m,\ell},$$

gives

$$\begin{aligned}\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|^2_{L^2(\mathbb{R}^2_+)} &\le \frac{c+(\sigma+1)R^{-1}}{2c}\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|^2_{L^2(\mathbb{R}^2_+)}\\ &\quad+\frac{1}{2c}\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}\big\|e^{2cy}b^R_{m,\ell}\partial_y\left(\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

Now we choose $R=1+2(\sigma+1)c^{-1}$, which gives $R\ge 1$ and

$$(\sigma+1)R^{-1}\le\frac{c}{2}.$$

Then we deduce, from the above inequalities,

$$\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}\le\frac{2}{c}\big\|e^{2cy}b^R_{m,\ell}\partial_y\Lambda^{\ell/3}\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.$$

Moreover, observe $R\ge c^{-1}+1$ and the monotonicity assumption $\omega\ge C_*^{-1}(1+y)^{-\sigma}$, and thus

$$\begin{aligned}b^R_{m,\ell} &\le c(1+y)^{-\sigma}(1+cy)^{-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}\\ &\le c\,C_*\omega(1+cy)^{-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}.\end{aligned}$$

As a result, we obtain

$$\big\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\big\|_{L^2(\mathbb{R}^2_+)}\le C_*\big\|\omega W^\ell_m\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)},$$

which along with (70) gives

$$\begin{aligned}
&\big\| \langle y\rangle^{-1} W_m^\ell \partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\\
\leq\ & C\big\|\omega W_m^\ell \partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\
\leq\ & C\big\|W_m^\ell \omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)} + C\big\|\big[\omega,\ W_m^\ell\big]\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.
\end{aligned}$$

On the other hand, Assumption 3.1 enables us to use Lemma 2.4 with $\lambda(D_x)=\Lambda^{\ell/3}$ and $q=\omega e^{2cy}$, to obtain, using the notation $\rho_{m,\ell}(y)=e^{2cy}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(1+cy)^{-1}$,

$$\begin{aligned}
\big\|\big[\omega,\ W_m^\ell\big]\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)} &= \big\|\big[\omega,\ \Lambda^{\ell/3}\big]\rho_{m,\ell}(y)\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\
&= \big\|\big[\omega e^{2cy},\ \Lambda^{\ell/3}\big]e^{-2cy}\rho_{m,\ell}(y)\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq C\big\|e^{-2cy}\rho_{m,\ell}(y)\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq C\big\|\rho_{m,\ell}(y)\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq C\big\|W_m^\ell\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)}.
\end{aligned}$$

Then, combining these inequalities we conclude, recalling $f_m=\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)$,

$$\big\| \langle y\rangle^{-1} W_m^\ell \partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)} \leq C\big\|W_m^\ell\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\big\|_{L^2(\mathbb{R}^2_+)} \leq C\big\|W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)}.$$

This along with (58), yields

$$\begin{aligned}
&\big\| \langle y\rangle^{-1} W_m^\ell \partial_x^m \omega\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq \big\| \langle y\rangle^{-1} W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)} + \big\| \langle y\rangle^{-1} W_m^\ell\big((\partial_y\omega)/\omega\big)\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq \big\| \langle y\rangle^{-1} W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)} + \big\|\big((\partial_y\omega)/\omega\big) \langle y\rangle^{-1} W_m^\ell\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\\
&\quad + \big\|\big[(\partial_y\omega)/\omega,\ W_m^\ell\big] \langle y\rangle^{-1}\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq \big\| \langle y\rangle^{-1} W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)} + C\big\| \langle y\rangle^{-1} W_m^\ell\partial_x^m u\big\|_{L^2(\mathbb{R}^2_+)}\\
&\leq C\big\|W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)},
\end{aligned}\tag{71}$$

where the third inequality holds because Assumption 3.1 enables us to use Lemma 2.4 with $\lambda(D_x)=\Lambda^{\ell/3}$ and $q=(\partial_y\omega)/\omega$. Then the first inequality (69) follows and thus the proof of Lemma 3.3 is complete. □

Lemma 3.4. *Let m and i be given with $m\geq 1$ and $1\leq i\leq 3$, and let $\tau\in\mathbb{R}$. There exists a constant C such that*

$$\forall\ f\in H^2\left(\mathbb{R}^2_+\right),\quad \big\| \langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/3}\Lambda_\delta^\tau W_m^{i-1} f\big\|_{L^2(\mathbb{R}^2_+)} \leq C\big\|\partial_y\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)} + C\big\|\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}.$$

Proof. In the proof we use C to denote different constants which are independent of m. By the definition of W_k^{i-1}, we can write

$$\Lambda^{1/3}\Lambda_\delta^\tau W_m^{i-1} = e^{2cy}\left(1+\frac{2cy}{(3m+i-1)\sigma}\right)^{-\frac{(3m+i-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{i}{3}}\Lambda_\delta^\tau = a_{m,i}(y)\Lambda_\delta^\tau W_m^i,$$

where

$$a_{m,i}(y)=\left(1+\frac{2cy}{(3m+i-1)\sigma}\right)^{-\frac{(3m+i-1)\sigma}{2}}\left(1+\frac{2cy}{(3m+i)\sigma}\right)^{\frac{(3m+i)\sigma}{2}}.$$

Direct computation gives

$$\begin{aligned}|a_{m,i}(y)| &= \left(1+\frac{2cy}{(3m+i-1)\sigma}\right)^{\sigma/2}\left(1+\frac{2cy}{(3m+i-1)\sigma}\right)^{-\frac{(3m+i)\sigma}{2}}\left(1+\frac{2cy}{(3m+i)\sigma}\right)^{\frac{(3m+i)\sigma}{2}}\\ &\le \left(1+\frac{2cy}{(3m+i-1)\sigma}\right)^{\sigma/2}\le C\langle y\rangle^{\sigma/2}.\end{aligned}$$

Moreover observe $|\partial_y a_{m,i}(y)|\le 2c\,|a_{m,i}(y)|$, and thus

$$|\partial_y a_{m,i}(y)|\le C\langle y\rangle^{\sigma/2}.$$

As a result,

$$\begin{aligned}\big\|\langle y\rangle^{-\sigma/2}\partial_y\Lambda^{1/3}\Lambda_\delta^\tau W_m^{i-1}f\big\|_{L^2(\mathbb{R}^2_+)} &= \big\|\langle y\rangle^{-\sigma/2}\partial_y a_{m,i}\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}\\ &\le \big\|\langle y\rangle^{-\sigma/2}a_{m,i}\partial_y\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}+\big\|\langle y\rangle^{-\sigma/2}(\partial_y a_{m,i})\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}\\ &\le C\Big(\big\|\partial_y\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}+\big\|\Lambda_\delta^\tau W_m^i f\big\|_{L^2(\mathbb{R}^2_+)}\Big).\end{aligned}$$

The proof of Lemma 3.4 is thus complete. □

Lemma 3.5. *Suppose Assumption 3.1 is fulfilled. There exists a constant L_1, depending only on σ, c, and the constants C_α, C_* in Assumption 3.1, such that for any integers $k\ge 2$ and $1\le i\le 3$, we have*

$$\begin{aligned}&\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le L_1\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+L_1\big\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+L_1\big\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\end{aligned}\tag{72}$$

$$\begin{aligned}&\big\|\phi_k^0 W_k^0 f_k\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}+\big\|\partial_y\phi_k^0 W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y^2\Lambda^{-2/3}\phi_k^0 W_k^0 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le L_1\big\|\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}+L_1\big\|\partial_y\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+L_1\big\|\partial_y^2\Lambda^{-2/3}\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\end{aligned}\tag{73}$$

$$\begin{aligned}&\big\|\partial_y^3\Lambda^{-1}\phi_k^0 W_k^0 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le L_1\big\|\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}+L_1\big\|\partial_y\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+L_1\big\|\partial_y^2\Lambda^{-2/3}\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+L_1\big\|\partial_y^3\Lambda^{-1}\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\end{aligned}\tag{74}$$

$$\big\|\Lambda^{-2/3}\phi_{k-1}^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le L_1\big\|\phi_{k-1}^i W_{k-1}^i f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\tag{75}$$

and

$$\begin{aligned}&\big\|\Lambda^{-2/3}\partial_y\phi_{k-1}^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le L_1\big\|\partial_y\phi_{k-1}^i W_{k-1}^i f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+L_1\big\|\phi_{k-1}^i W_{k-1}^i f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.\end{aligned}\tag{76}$$

Proof. In the proof we use C to denote different constants which are independent of k.

(i) To prove (72), we use (65) and (66), respectively, to compute

$$\begin{aligned}&\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le \big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^2 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\big[\partial_y^2,\ W_k^i\big]\phi_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le \big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^2 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\end{aligned}$$

and

$$\begin{aligned}
&\big\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &\big\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^3 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1}\Lambda_\delta^{-2}\big[\partial_y^3,\ W_k^i\big]\phi_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &\big\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^3 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^2 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^3 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y^2 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i\partial_y f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1}\partial_y^3 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1}\partial_y^2 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1}\partial_y f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

the last inequality following from (63). Moreover, using (64), (65) and (66) again implies

$$\begin{aligned}
&\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1}\partial_y^3 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1}\partial_y^2 f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad+\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1}\partial_y f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\partial_y\phi_k^{i-1} W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\phi_k^{i-1} W_k^{i-1} f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Combining these inequalities, we obtain (72).

(ii) The proof of (73) and (74) is quite similar as in (72). So we omit it here for brevity.

(iii) We now prove (75). Recall that

$$\Lambda^{-2/3}W_k^{i-1}f_k=e^{2cy}\left(1+\frac{2cy}{(3k+i-1)\,\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{i}{3}}\Lambda^{-1}f_k.$$

Observe

$$\left(1+\frac{2cy}{(3k+i-1)\,\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}}\leq\left(1+\frac{2cy}{(3(k-1)+i)\,\sigma}\right)^{-\frac{(3(k-1)+i)\sigma}{2}}$$

since

$$\gamma\longrightarrow\left(1+\frac{cy}{\gamma}\right)^{-\gamma}$$

is a monotonically decreasing. Thus

$$\big\|\Lambda^{-2/3}\phi_{k-1}^i W_k^{i-1} f_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\leq\big\|\Lambda^{-1}\phi_{k-1}^i W_{k-1}^i f_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.\tag{77}$$

Moreover, in view of (58),

$$\begin{aligned}
&\big\|\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\big\|\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}\partial_x\big(\partial_x^{k-1}\omega-\frac{\partial_y\omega}{\omega}\partial_x^{k-1}u\big)\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\big\|\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}\big[\partial_x\left((\partial_y\omega)/\omega\right)\big]\partial_x^{k-1}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\big\|\big[\partial_x\left((\partial_y\omega)/\omega\right)\big]\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}\partial_x^{k-1}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\big\|\big[\partial_x\left((\partial_y\omega)/\omega\right),\ \Lambda^{-1}W^i_{k-1}\big]\phi^i_{k-1}\partial_x^{k-1}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\big\|\big[\partial_x\left((\partial_y\omega)/\omega\right)\big]\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}\partial_x^{k-1}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\big\|\big[\partial_x\left((\partial_y\omega)/\omega\right)\langle y\rangle,\ \Lambda^{-1}W^i_{k-1}\big]\langle y\rangle^{-1}\phi^i_{k-1}\partial_x^{k-1}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\big\|\langle y\rangle^{-1}\phi^i_{k-1}W^i_{k-1}\partial_x^{k-1}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

the last inequality using Assumption 3.1 and Lemma 2.4 with $\lambda(D_x)=\Lambda^{\ell/3-1}$ and $q=\langle y\rangle\,\partial_x\left((\partial_y\omega)/\omega\right)$. This along with Lemma 3.3 gives

$$\big\|\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le C\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},$$

and thus, combining (77),

$$\big\|\Lambda^{-2/3}\phi^i_{k-1}W^{i-1}_kf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\le C\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.$$

Then the desired estimate (75) follows.

(iv) Now we prove (76). The proof is quite similar as above. In fact we have, making use of (64),

$$\begin{aligned}
&\big\|\Lambda^{-2/3}\partial_y\phi^i_{k-1}W^{i-1}_kf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\Lambda^{-2/3}\phi^i_{k-1}W^{i-1}_k\partial_yf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\big\|\big[\partial_y,\ W^{i-1}_k\big]\Lambda^{-2/3}\phi^i_{k-1}f_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\Lambda^{-2/3}\phi^i_{k-1}W^{i-1}_k\partial_yf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\big\|\Lambda^{-2/3}\phi^i_{k-1}W^{i-1}_kf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\Lambda^{-2/3}\phi^i_{k-1}W^{i-1}_k\partial_yf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

the last inequality following from (75). As for the first term in the last inequality we have, repeating the arguments in the proof of (75),

$$\begin{aligned}
&\big\|\Lambda^{-2/3}\phi^i_{k-1}W^{i-1}_k\partial_yf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &\big\|\Lambda^{-1}\phi^i_{k-1}W^i_{k-1}\partial_yf_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ &C\big\|\partial_y\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\big\|\phi^i_{k-1}W^i_{k-1}f_{k-1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Thus combining these inequalities we obtain the desired (76), and thus the proof of Lemma 3.5 is complete. □

4. Crucial estimate

We will use L^2 method to prove the Gevrey regularity. To do so we use induction on k to estimate the weighted L^2 norms of $\partial_x^k u$, and in view of the conclusion in the previous section it suffices to estimate its equivalent form f_m instead. Recall f_m is defined in (58), i.e.,

$$f_m=\partial_x^m\omega-\frac{\partial_y\omega}{\omega}\partial_x^mu,\quad m\ge1.$$

We can verify that, seeing the appendix in Section 6, the above f_m satisfies the boundary problem:

$$\partial_tf_m+u\partial_xf_m+v\partial_yf_m-\partial_y^2f_m=\mathcal{Z}_m$$

and

$$\partial_y f_m\big|_{y=0}=0,\quad \left(\partial_t f_m-\partial_y^2 f_m\right)\big|_{y=0}=-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0}, \tag{78}$$

where

$$\begin{aligned}\mathcal{Z}_m=&-\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j}-\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)\partial_y f_{m-j}\\&-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u)-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.\end{aligned} \tag{79}$$

Thus, recalling W_m^ℓ is defined in the previous section, $W_m^\ell f_m$ satisfies

$$\left(\partial_t+u\partial_x+v\partial_y-\partial_y^2\right)W_m^\ell f_m=\mathcal{Z}_{m,\ell}, \tag{80}$$

where

$$\begin{aligned}\mathcal{Z}_{m,\ell}=&-\sum_{j=1}^{m}\binom{m}{j}W_m^\ell(\partial_x^j u)f_{m+1-j}-\sum_{j=1}^{m-1}\binom{m}{j}W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\\&-W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u)-2W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\\&+\left[u\partial_x+v\partial_y-\partial_y^2,\ W_m^\ell\right]f_m.\end{aligned} \tag{81}$$

Moreover, if $\partial_y^2 W_m^i f_m\in L^2(\mathbb{R}^2_+)$, then

$$\partial_y W_m^i f_m\big|_{y=0}=0 \tag{82}$$

in the sense of trace, due to the fact (78) and that

$$\partial_y\left(e^{2cy}\left(1+\frac{2cy}{(3m+i)\sigma}\right)^{-(3m+i)\sigma/2}(1+cy)^{-1}\right)\Big|_{y=0}=0.$$

Assumption 4.1. *Let k_0 be the integer given in (18). Suppose that*

$$\forall\ 0\le\alpha\le k_0+1,\quad \max_{1\le m\le 6}\left\|e^{2cy}\partial_x^\alpha f_m\right\|_{L^\infty\left([0,T]\times\mathbb{R}^2_+\right)}+\max_{1\le m\le 6}\left\|e^{2cy}\partial_y\partial_x^\alpha f_m\right\|_{L^\infty\left([0,T]\times\mathbb{R}^2_+\right)}<\tilde{C}_\alpha,$$

where $\tilde{C}_\alpha$ is a constant depending on α.

Let's first introduce the following induction assumption.

Definition 4.2 (Assumption $H_{k,i}(A)$)**.** *Let A be a given constant and let $k\ge 7$ and $i\ge 1$ be two given integers. We say the sequence $\{f_m\}_{m\ge1}$, which defined by (58), satisfies Assumption $H_{k,i}(A)$ if the following conditions are fulfilled: using the notation $s=3+3\sigma$,*

(i) for any $6\le m\le k-1$, for any $0\le\ell\le 3$ and any $0\le\tilde{\ell}\le i-1$, we have

$$\partial_y^3\Lambda^{-1}\phi_m^\ell W_m^\ell f_m,\quad \partial_y^3\Lambda^{-1}\phi_k^{\tilde{\ell}}W_k^{\tilde{\ell}}f_k\in L^2([0,T]\times\mathbb{R}^2_+);$$

(ii) for any $6\le m\le k-1$ and for any $0\le\ell\le 3$, we have

$$\begin{aligned}&\|\phi_m^\ell W_m^\ell f_m\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}+\|\partial_y\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\&\le A^{m-5+\frac{\ell}{6}}\left((m-5)!\right)^s(m-4)^{\frac{\ell s}{3}};\end{aligned}$$

(iii) for any $0\le\tilde{\ell}\le i-1$

$$\begin{aligned}&\|\phi_k^{\tilde{\ell}}W_k^{\tilde{\ell}}f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}+\|\partial_y\phi_k^{\tilde{\ell}}W_k^{\tilde{\ell}}f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi_k^{\tilde{\ell}}W_k^{\tilde{\ell}}f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\&\le A^{k-5+\frac{\tilde{\ell}}{6}}\left((k-5)!\right)^s(k-4)^{\frac{\tilde{\ell}s}{3}}.\end{aligned}$$

Now we state the main result of this section.

Theorem 4.3. *Let u, v satisfy Assumption 3.1, and let the sequence $\{f_m\}_{m\geq 1}$, defined by (58), satisfy Assumption 4.1 and the above Assumption $H_{k,i}(A)$. Then*

$$\partial_y^3 \Lambda^{-1} \phi_k^i W_k^i f_k \in L^2([0,T]\times\mathbb{R}^2_+)$$

and moreover, there exists a constant L_2, depending only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but independent of k and A, such that

$$\begin{aligned} &\left\|\phi_k^i W_k^i f_k\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \left\|\partial_y \phi_k^i W_k^i f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_k^i W_k^i f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\leq\ L_2 A^{k-5+\frac{i-1}{6}} \left((k-5)!\right)^s (k-4)^{\frac{is}{3}}, \end{aligned} \tag{83}$$

where $s = 3 + 3\sigma$.

4.1. **Estimates on nonlinear terms.** In this subsection we estimate the nonlinear terms $\mathcal{Z}_{m,\ell}$ defined in (81), seeing Proposition 4.4 and Proposition 4.9 below.

Proposition 4.4. *Let $\Lambda^{\tau_1}, \Lambda_\delta^{\tau_2}$ be the Fourier multipliers associated to respectively the symbols $\langle\xi\rangle^{\tau_1}$ and $\langle\delta\xi\rangle^{\tau_2}$ with $\tau_1 \leq 0$ and $\tau_2 \leq 0$, let u, v satisfy Assumption 3.1, and let the sequence $\{f_m\}_{m\geq 1}$, defined by (58), satisfy Assumption 4.1 and the above Assumption $H_{k,i}(A)$ (see Definition 4.2). Then there exists a constant C, such that for any $6 \leq m \leq k$ and any $0 \leq \ell \leq i$ we have*

$$\begin{aligned} &\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell \mathcal{Z}_{m,\ell}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \leq\ & mC\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}. \end{aligned}$$

Remark 4.5. *The constant C in the above proposition depends only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but doesn't depend on m and δ.*

In view of (81), we see

$$\begin{aligned} &\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell \mathcal{Z}_{m,\ell}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \leq& \sum_{j=1}^{m}\binom{m}{j}\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ &+ \sum_{j=1}^{m-1}\binom{m}{j}\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ &+ \sum_{j=1}^{m-1}\binom{m}{j}\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial_x^j v)(\partial_x^{m-j}u)\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ &+ 2\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right] f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ &+ \left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell\left[u\partial_x + v\partial_y - \partial_y^2,\ W_m^\ell\right] f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}. \end{aligned} \tag{84}$$

So in order to prove Proposition 4.4, it suffices to estimate the terms on the right hand side. We will proceed through the following lemmas. *In the following discussions, to simplify the notation we will use C to denote different constants which depend only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but don't depend on m, δ and the constant A in Hypothesis $H_{k,i}(A)$.*

Lemma 4.6. *Under the same assumption as in Proposition 4.4, we have*

$$\begin{aligned} &\sum_{j=1}^{m-1}\binom{m}{j}\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \leq\ & mC\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}. \end{aligned}$$

Proof. We first split the summation as follows:

$$\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=&\sum_{j=1}^{3}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=4}^{m-4}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Moreover as for last term on the right hand side, we use (63) to compute, observing $\tau_1,\tau_2\le 0$,

$$\begin{aligned}
&\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\big\|\phi_m^\ell W_m^0\Lambda^{\ell/3}(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\big\|\phi_m^\ell W_m^0\Lambda^{1}(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\big\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\phi_m^\ell W_m^0\partial_x(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\big\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y\partial_x f_{m-j})\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\sum_{j=1}^{3}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=4}^{m-4}\binom{m}{j}\big\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=4}^{m-4}\binom{m}{j}\big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=4}^{m-4}\binom{m}{j}\big\|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y\partial_x f_{m-j})\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}\tag{85}$$

Next we estimate step by step the terms on the right side of (85).

(a) We treat in this step the first and the second terms on the right hand side of (85), and prove that

$$\begin{aligned}
&\sum_{j=1}^{3}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le& mC\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}\tag{86}$$

To do so we have, observing $\tau_1, \tau_2 \le 0$,

$$\begin{aligned}
&\sum_{j=1}^{3}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\sum_{j=1}^{3}\binom{m}{j}\big\|(\partial_x^j v)\phi_m^\ell W_m^\ell\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\sum_{j=1}^{3}\binom{m}{j}\big\|\left[W_m^\ell,\ (\partial_x^j v)\right]\phi_m^\ell\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_m^\ell W_m^\ell\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell W_{m-j}^\ell\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}
\end{aligned}$$

the third inequality using Corollary 2.5, while the last inequality using (61) and (62). Moreover using (64) gives

$$\begin{aligned}
&\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell W_{m-j}^\ell\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\sum_{j=1}^{3}\binom{m}{j}\big\|\partial_y\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell\left[\partial_y,\ W_{m-j}^\ell\right]f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\sum_{j=1}^{3}\binom{m}{j}\big\|\partial_y\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Thus, we obtain, combining these inequalities,

$$\begin{aligned}
&\sum_{j=1}^{3}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\sum_{j=1}^{3}\binom{m}{j}\big\|\partial_y\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\sum_{j=1}^{3}\binom{m}{j}\big\|\partial_y\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}.
\end{aligned}$$

This, along with the estimate

$$\begin{aligned}
&\sum_{j=1}^{3}\binom{m}{j}\big\|\partial_y\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\sum_{j=1}^{3}\binom{m}{j}\big\|\phi_{m-j}^\ell W_{m-j}^\ell f_{m-j}\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
\le\ &\sum_{j=1}^{3}\frac{m!}{j!(m-j)!}A^{m-j-5+\frac{\ell}{6}}\left((m-j-5)!\right)^s(m-j)^{\frac{\ell s}{3}}\\
\le\ &CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}
\end{aligned}$$

due to Hypothesis $H_{k,i}(A)$ and the fact $s \ge 3$, gives

$$\sum_{j=1}^{3}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\le CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}. \tag{87}$$

Direct computation gives

$$\begin{aligned}
&\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\Lambda^{\ell/3}e^{2cy}\,(1+cy)^{-1}\,\phi_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|e^{2cy}(\partial_y f_{m-j})\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\Lambda^{\ell/3}\,(1+cy)^{-1}\,\phi_m^\ell\partial_x^j v\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\big[e^{2cy}(\partial_y f_{m-j}),\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\Lambda^{\ell/3}\big]\,(1+cy)^{-1}\,\phi_m^\ell\partial_x^j v\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\overset{\text{def}}{=}\ &J_1+J_2.
\end{aligned}$$

On the other hand, in view of Assumption 4.1, we have

$$\begin{aligned}
J_1\ &\le\ C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|(1+cy)^{-1}\big\|_{L^2(\mathbb{R}_+;\ L^\infty([0,T]\times\mathbb{R}_x))}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\le\ C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_{j+1}^\ell\Lambda^{\ell/3}\partial_x^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))},
\end{aligned}$$

since $\phi_m^\ell\le\phi_{j+1}^\ell$ for $j\le m-1$. Moreover Assumption 4.1 enables us to use Lemma 2.4 with $\lambda(D_x)=\Lambda^{\ell/3}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}$ and $q=e^{2cy}\partial_y f_{m-j}, m-3\le j\le m-1$, to obtain

$$\begin{aligned}
J_2\ &\le\ C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\,(1+cy)^{-1}\,\phi_m^\ell\partial_x^j v\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le\ C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\,(1+cy)^{-1}\,\big\|_{L^2(\mathbb{R}_+;\ L^\infty([0,T]\times\mathbb{R}_x))}\big\|\phi_m^\ell\partial_x^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\le\ C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\phi_m^\ell\partial_x^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\le\ C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\phi_{j+1}^\ell\Lambda^{\ell/3}\partial_x^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))},
\end{aligned}$$

the last inequality using (61). Thus combining these inequalities, we obtain

$$\begin{aligned}
&\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\sum_{j=m-3}^{m-1}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_{j+1}^\ell\Lambda^{\ell/3}\partial_x^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
\le\ &C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_{j+1}^\ell W_{j+2}^0\Lambda^{\ell/3}f_{j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\sum_{j=m-3}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_{j+1}^\ell W_{j+1}^\ell f_{j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

the second inequality use (68) and the last inequality following from (63). Moreover we use Hypothesis $H_{k,i}(A)$ (see Definition 4.2), to calculate, observing $s \geq 3$ and $\tau \leq 0$,

$$\begin{aligned}
&\sum_{j=m-3}^{m-1} \binom{m}{j} \big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_{j+1}^\ell W_{j+1}^\ell f_{j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq\ & m\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\sum_{j=m-3}^{m-2}\binom{m}{j}\big\|\phi_{j+1}^\ell W_{j+1}^\ell f_{j+1}\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} \\
\leq\ & m\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \sum_{j=m-3}^{m-2}\frac{m!}{j!(m-j)!}A^{j-4+\frac{\ell}{6}}\left((j-4)!\right)^s(j+1)^{\frac{\ell s}{3}} \\
\leq\ & m\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

Thus we have, combining the above inequalities,

$$\begin{aligned}
&\sum_{j=m-3}^{m-1} \binom{m}{j} \big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq\ & mC\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

This along with (87) gives the desired estimate (86).

(b) We will estimate in this step the third and the fourth terms on the right hand side of (85), and prove that

$$\begin{aligned}
&\sum_{j=4}^{m-4} \binom{m}{j} \big\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&+ \sum_{j=4}^{m-4} \binom{m}{j} \big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\leq CA^{m-6}\left((m-5)!\right)^s.
\end{aligned} \tag{88}$$

For this purpose we write, denoting by $[m/2]$ the largest integer less than or equal to $m/2$,

$$\begin{aligned}
&\sum_{j=4}^{m-4} \binom{m}{j} \big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\leq \sum_{j=4}^{\left[m/2\right]} \binom{m}{j} \big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)(\partial_y f_{m-j})\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + \sum_{j=\left[m/2\right]+1}^{m-4} \binom{m}{j} \big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= S_1 + S_2.
\end{aligned} \tag{89}$$

We first treat S_1. Using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+3}^0\phi_{m-j}^0, \quad W_m^0 \leq W_{m-j}^0 \text{ for } j \geq 1,$$

gives

$$\begin{aligned}
S_1 &= \sum_{j=4}^{\left[m/2\right]} \binom{m}{j} \big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\leq \sum_{j=4}^{\left[m/2\right]} \binom{m}{j} \big\|\phi_{j+3}^0\partial_x^{j+1} v\big\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} \big\|\phi_{m-j}^0 W_{m-j}^0\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned} \tag{90}$$

By Sobolev inequality, we have

$$\begin{aligned}
&\left\|\phi^0_{j+3}\partial_x^{j+1}v\right\|_{L^\infty\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & C\left\|\phi^0_{j+3}\partial_x^{j+1}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}+C\left\|\phi^0_{j+3}\partial_x^{j+2}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}\\
\leq\ & C\left\|\phi^0_{j+2}\partial_x^{j+1}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}+C\left\|\phi^0_{j+3}\partial_x^{j+2}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}\\
\leq\ & C\left\|\phi^0_{j+2}W^0_{j+2}f_{j+2}\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}+C\left\|\phi^0_{j+3}W^0_{j+3}f_{j+3}\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)},
\end{aligned}$$

the secomd inequality using (61) and the last inequlaity following from (68). As a result, we use Hypothesis $H_{k,i}(A)$ (see Definition 4.2) to compute, for any $4\leq j\leq[m/2]$,

$$\left\|\phi^0_{j+3}\partial_x^{j+1}v\right\|_{L^\infty\left([0,T]\times\mathbb{R}^2_+\right)}\leq C\left(A^{j-3}\left((j-3)!\right)^s+A^{j-2}\left((j-2)!\right)^s\right)\leq CA^{j-2}\left((j-2)!\right)^s.$$

Moreover, using (64) and $H_{k,i}(A)$ (see Definition 4.2) we calculate, for any $4\leq j\leq[m/2]$,

$$\begin{aligned}
&\left\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & \left\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\phi^0_{m-j}\left[\partial_y,\ W^0_{m-j}\right]f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & \left\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & \left\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^\infty\left([0,T],\ L^2(\mathbb{R}^2_+)\right)}\\
\leq\ & CA^{m-j-5}\left((m-j-5)!\right)^s.
\end{aligned}$$

Putting these estimate into (90) gives

$$\begin{aligned}
S_1&\leq C\sum_{j=4}^{\left[m/2\right]}\frac{m!}{j!(m-j)!}A^{j-2}\left((j-2)!\right)^s\left(A^{m-j-5}\left((m-j-5)!\right)^s\right)\\
&\leq C\sum_{j=4}^{\left[m/2\right]}\frac{m!}{j^2(m-j)^5}A^{m-7}\left((j-2)!\right)^{s-1}\left((m-j-5)!\right)^{s-1}\\
&\leq C\sum_{j=4}^{\left[m/2\right]}\frac{m!}{j^2(m/2)^5}A^{m-7}\left((m-7)!\right)^{s-1}\\
&\leq C(m-5)!A^{m-7}\left((m-7)!\right)^{s-1}\\
&\leq CA^{m-6}\left((m-5)!\right)^s.
\end{aligned}\tag{91}$$

We now treat S_2. Using the inequality

$$\phi^\ell_m\leq\phi^0_m\leq\phi^0_{j+2}\phi^0_{m-j+1},\quad W^0_m\leq W^0_{m-j+1}\text{ for }j\geq1,$$

and thus

$$\begin{aligned}
S_2&=\sum_{j=\left[m/2\right]+1}^{m-4}\binom{m}{j}\left\|\phi^\ell_m W^0_m(\partial_x^{j+1}v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq\sum_{j=\left[\frac{m}{2}\right]+1}^{m-4}\binom{m}{j}\left\|\phi^0_{j+2}\partial_x^{j+1}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}\\
&\qquad\qquad\times\left\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\\
&\leq\sum_{j=\left[m/2\right]+1}^{m-4}\binom{m}{j}\left\|\phi^0_{j+2}W^0_{j+2}f_{j+2}\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}\\
&\qquad\qquad\times\left\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))},
\end{aligned}\tag{92}$$

the last inequality using (68). As for the last factor in the above inequality, we use Sobolev inequality, (61) and (62) to compute

$$\begin{aligned}
&\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\\
\leq\ & C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y\partial_x f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & C\big\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y\partial_x f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

On the other hand, in view of the definition (58) of f_m, we have

$$\begin{aligned}
&\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y\partial_x f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & \big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y(\partial_x^{m-j}u)\partial_x\left((\partial_y\omega)/\omega\right)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & \big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\phi^0_{m-j+1}W^0_{m-j+1}(\partial_x^{m-j}\omega)\partial_x\left((\partial_y\omega)/\omega\right)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|\phi^0_{m-j+1}W^0_{m-j+1}(\partial_x^{m-j}u)\partial_x\partial_y\left((\partial_y\omega)/\omega\right)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\langle y\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}\omega\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\langle y\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}u\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

the last inequality using Assumption 3.1, (61) and (62). Combining these inequalities, we conclude

$$\begin{aligned}
&\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\\
\leq\ & C\big\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\langle y\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}\omega\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\langle y\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}u\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & C\big\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\partial_y\phi^0_{m-j+1}W^0_{m-j+1}f_{m-j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\phi^0_{m-j}W^0_{m-j}f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\phi^0_{m-j+1}W^0_{m-j+1}f_{m-j+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

where the last inequality follows from (69), (64). This, along with Hypothesis $H_{k,i}(A)$ (see Definition 4.2), yields, for any $\big[m/2\big]+1\leq j\leq m-4$,

$$\begin{aligned}
&\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\\
\leq\ & CA^{m-j-5}\left((m-j-5)!\right)^s+CA^{m-j-4}\left((m-j-4)!\right)^s\\
\leq\ & CA^{m-j-4}\left((m-j-4)!\right)^s.
\end{aligned}$$

On the other hand, Hypothesis $H_{k,i}(A)$ (see Definition 4.2) yields, for any $\big[m/2\big]+1\leq j\leq m-4$,

$$\big\|\phi^0_{j+2}W^0_{j+2}f_{j+2}\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\leq A^{j-3}\left((j-3)!\right)^s.$$

Putting these estimate into (92), we have

$$\begin{aligned}
S_2\ \leq\ & C\sum_{j=\big[m/2\big]+1}^{m-4}\frac{m!}{j!(m-j)!}A^{j-3}\big((j-3)!\big)^s\left(A^{m-j-4}\left((m-j-4)!\right)^s\right)\\
\leq\ & C\sum_{j=\big[m/2\big]+1}^{m-4}\frac{m!}{j^3(m-j)^4}A^{m-7}\big((j-3)!\big)^{s-1}\left((m-j-4)!\right)^{s-1}\\
\leq\ & C\sum_{j=\big[m/2\big]+1}^{m-4}\frac{m!}{(m/2)^3(m-j)^4}A^{m-7}\left((m-7)!\right)^{s-1}\\
\leq\ & C(m-3)!A^{m-7}\left((m-7)!\right)^{s-1}\\
\leq\ & CA^{m-6}\left((m-5)!\right)^s,
\end{aligned}$$

the last inequality using the fact that $s \geq 3$. This along with (91) and (89) yields

$$\sum_{j=4}^{m-4} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j} \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq C A^{m-6} \left((m-5)!\right)^s .$$

Similarly, we have

$$\sum_{j=4}^{m-4} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j} v) \partial_y f_{m-j} \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq C A^{m-6} \left((m-5)!\right)^s .$$

Then the desired estimate (88) follows.

(c) It remains to prove that

$$\sum_{j=4}^{m-4} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j} v) (\partial_y \partial_x f_{m-j}) \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq C A^{m-6} \left((m-5)!\right)^s . \tag{93}$$

The proof is quite similar as in the previous step. To do so we first write

$$\begin{aligned}
&\sum_{j=4}^{m-4} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j} v) (\partial_y \partial_x f_{m-j}) \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
= \quad &\sum_{j=4}^{\lceil m/2 \rceil} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j} v) (\partial_y \partial_x f_{m-j}) \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+ \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j} v) (\partial_y \partial_x f_{m-j}) \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
= \quad &\tilde{S}_1 + \tilde{S}_2.
\end{aligned}$$

For the term $\tilde{S}_1$, we use

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \leq W_{m-j+1}^0 \text{ for } j \geq 2,$$

to obtain

$$\tilde{S}_1 \quad \leq \quad \sum_{j=4}^{\lceil m/2 \rceil} \binom{m}{j} \left\| \phi_{j+2}^0 \partial_x^j v \right\|_{L^\infty\left([0,T]\times\mathbb{R}^2_+\right)} \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j} \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.$$

Then repeating the arguments used to estimate S_1 and S_2 in the previous step, we can deduce that

$$\tilde{S}_1 \leq C(m-5)! A^{m-7} \left((m-7)!\right)^{s-1} \leq C A^{m-6} \left((m-6)!\right)^s .$$

As for $\tilde{S}_2$, using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+1}^0 \phi_{m-j+2}^0, \quad W_m^0 \leq W_{m-j+2}^0 \text{ for } j \geq 2,$$

gives

$$\tilde{S}_2 \leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \left\| \phi_{j+1}^0 \partial_x^j v \right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))} \left\| \phi_{m-j+2}^0 W_{m-j+2}^0 \partial_y \partial_x f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}.$$

Then repeating the arguments used to estimate S_2 in the previous step, we have

$$\tilde{S}_2 \leq C(m-3)! A^{m-7} \left((m-7)!\right)^{s-1} \leq C A^{m-6} \left((m-5)!\right)^s$$

since $s \geq 3$. This along with the estimate on $\tilde{S}_1$ yields (93). Finally, combining (85), (86), (88) and (93) gives the desired estimate in Lemma 4.6, and thus the proof is complete. □

Lemma 4.7. *Under the same assumption as in Proposition 4.4, we have*

$$\begin{aligned}
&\sum_{j=1}^{m}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j u)f_{m+1-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad+\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\;& m\,C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

Proof. The proof is quite similar as in Lemma 4.6. In fact, repeating the arguments used in Lemma 4.6, we can deduce that

$$\begin{aligned}
&\sum_{j=1}^{m}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j u)f_{m+1-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\;& m\,C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

Moreover, Assumption 3.1 enables us to use Lemma 2.4 with $\lambda(D_x)=\Lambda^{\ell/3}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}$ and $q=\langle y\rangle\,\partial_y\left(\frac{\partial_y\omega}{\omega}\right)$, to obtain

$$\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\;&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\left[\partial_y\left(\partial_y\omega/\omega\right)\right]\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=1}^{m-1}\binom{m}{j}\big\|\left[\partial_y\left(\partial_y\omega/\omega\right),\;\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell(\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\;&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\left[\partial_y\left(\partial_y\omega/\omega\right)\right]\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=1}^{m-1}\binom{m}{j}\big\|\left[\langle y\rangle\,\partial_y\left(\partial_y\omega/\omega\right),\;\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\langle y\rangle^{-1}\phi_m^\ell(\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\;&C\sum_{j=1}^{m-1}\binom{m}{j}\big\|\langle y\rangle^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell(\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

the last inequality following from Assumption 3.1. As a result, in view of Lemma 3.3 we can repeat the arguments used in Lemma 4.6, to conclude

$$\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\;& m\,C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

Then the desired estimate follows and the proof of Lemma 4.7 is thus complete. □

Lemma 4.8. *Under the same assumption as in Proposition 4.4, we have*

$$\begin{aligned}
&2\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\left(\partial_y\omega\right)/\omega\right)\right]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell\left[u\partial_x+v\partial_y-\partial_y^2,\;W_m^\ell\right]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\;&C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Proof. This is a just direct verification. Indeed, Assumption 3.1 enables us to use Lemma 2.4 with $\lambda(D_x) = \Lambda^{\ell/3}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}$ and $q = \partial_y\left(\frac{\partial_y\omega}{\omega}\right)$, to obtain

$$\begin{aligned}
&\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\left[\partial_y\left((\partial_y\omega)/\omega\right)\right]f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &\left\|\left[\partial_y\left((\partial_y\omega)/\omega\right)\right]\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\|\left[\partial_y\left((\partial_y\omega)/\omega\right),\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Moreover, using (19) gives

$$\begin{aligned}
&\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell\left[u\partial_x + v\partial_y - \partial_y^2,\ W_m^\ell\right]f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &\left\|\left[u\partial_x + v\partial_y - \partial_y^2,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\|\left[u\partial_x + v\partial_y - \partial_y^2,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

By Corollary 2.5 we have

$$\begin{aligned}
&\left\|\left[u\partial_x + v\partial_y - \partial_y^2,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} = \left\|\left[u\partial_x + v\partial_y,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

and

$$\left\|\left[u\partial_x,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.$$

In view of (65) we have

$$\left\|\left[-\partial_y^2,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.$$

Finally using (64) and Corollary 2.5, it then follows that

$$\begin{aligned}
&\left\|\left[v\partial_y,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &\left\|\left[v,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\partial_y\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\|v\left[\partial_y,\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell\partial_y f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Combing these inequalities we conclude

$$\begin{aligned}
&\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell\left[u\partial_x + v\partial_y - \partial_y^2,\ W_m^\ell\right]f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Then the desired estimate follows and thus the proof of Lemma 4.8 is complete. □

Combing the estimates in Lemma 4.6-Lemma 4.8, we get the desired estimate in Proposition 4.4, completing the proof of Proposition 4.4.

Proposition 4.9. *Let u, v satisfy Assumption 3.1, and let the sequence $\{f_m\}_{m\geq 1}$, defined by (58), satisfy Assumption 4.1 and the Assumption $H_{k,i}(A)$ (see Definition 4.2). Let $\tau \leq 0$ be given and let $\chi(t)$ be a given function of t such that $0 \leq \chi \leq \phi_m^\ell$. Then there exists a constant C, such that for any $6 \leq m \leq k$ and any $0 \leq \ell \leq i$ we have*

$$\begin{aligned}
&\left\|\chi(t)\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\mathcal{Z}_{m,\ell}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\langle y\rangle^{-\sigma}\Lambda^{1/3}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\left\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+m\,C\left(\left\|\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\|\Lambda^{-2/3}\partial_y\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\
&+CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

Proof. In the proof we will use C to denote different constants which depend only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but don't depend on m and δ. Recall the definition of $\mathcal{Z}_{m,\ell}$,

$$\begin{aligned}\mathcal{Z}_{m,\ell} &= -\sum_{j=1}^{m}\binom{m}{j}W_m^\ell(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\\ &\quad -W_m^\ell\Big[\partial_y\Big(\frac{\partial_y\omega}{\omega}\Big)\Big]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) - 2W_m^\ell\Big[\partial_y\Big(\frac{\partial_y\omega}{\omega}\Big)\Big]f_m\\ &\quad +\big[u\partial_x + v\partial_y - \partial_y^2,\ W_m^\ell\big]f_m.\end{aligned}$$

Therefore

$$\begin{aligned}\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\mathcal{Z}_{m,\ell} &= -\sum_{j=1}^{m}\binom{m}{j}\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\\ &\quad -\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\Big[\partial_y\Big(\frac{\partial_y\omega}{\omega}\Big)\Big]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) - 2\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\Big[\partial_y\Big(\frac{\partial_y\omega}{\omega}\Big)\Big]f_m\\ &\quad +\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\big[u\partial_x + v\partial_y - \partial_y^2,\ W_m^\ell\big]f_m,\end{aligned}$$

which yields, observing $0 \le \chi \le \phi_m^\ell$ and $\tau \le 0$,

$$\|\chi(t)\Lambda^{-\frac{2}{3}}\partial_y\mathcal{Z}_{m,\ell}\|_{L^2([0,T]\times\mathbb{R}^2_+)} \le Q_1 + Q_2 + Q_3 + Q_4 + Q_5,$$

where

$$\begin{aligned}Q_1 &= \sum_{j=1}^{m}\binom{m}{j}\big\|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y W_m^\ell(\partial_x^j u)f_{m+1-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\\ Q_2 &= \sum_{j=1}^{m-1}\binom{m}{j}\big\|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\\ Q_3 &= \sum_{j=1}^{m-1}\binom{m}{j}\big\|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y W_m^\ell\big[\partial_y\left((\partial_y\omega)/\omega\right)\big](\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\\ Q_4 &= 2\big\|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y W_m^\ell\big[\partial_y\left((\partial_y\omega)/\omega\right)\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},\\ Q_5 &= \big\|\chi(t)\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\big[u\partial_x + v\partial_y - \partial_y^2,\ W_m^\ell\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.\end{aligned}$$

The terms Q_1-Q_3 can be treated quite similarly as in Lemma 4.6 and Lemma 4.7, and the main difference is that we will use here additionally the induction estimates on the terms of the following form

$$\big\|\partial_y^2\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}$$

in Hypothesis $H_{k,i}(A)$ (see Definition 4.2), while in the proof of Lemma 4.6 and Lemma 4.7, we only used the estimates on the following two forms

$$\big\|\phi_m^\ell W_m^\ell f_m\big\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}, \quad \big\|\partial_y\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}$$

in Hypothesis $H_{k,i}(A)$. Thus we obtain

$$\begin{aligned}\sum_{j=1}^{3}Q_j &\le mC\left(\big\|\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\Lambda^{-2/3}\partial_y\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\ &\quad +CA^{m-6+\frac{\ell}{6}}\left((m-5)!\right)^s m^{\frac{\ell s}{3}}.\end{aligned}$$

Moreover, as in the proof of Lemma 4.8, we can deduce that

$$Q_4 \le C\left(\big\|\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\Lambda^{-2/3}\partial_y\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right).$$

So it remains to estimate Q_5. We use (19) to write

$$\begin{aligned} Q_5 &= \|\chi(t)\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\big[u\partial_x+v\partial_y-\partial_y^2,\ W_m^\ell\big]f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le \|\big[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\big]\chi(t)f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+\|\big[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\big]\chi(t)W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\overset{\text{def}}{=} Q_{5,1}+Q_{5,2}. \end{aligned}$$

We first estimate $Q_{5,1}$. Observe

$$\begin{aligned} &\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\partial_y\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\big[\partial_y,\ W_m^\ell\big]\ \Big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

On the other hand, by virtue of (19), (64) and Corollary 2.5 we compute

$$\begin{aligned} &\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\partial_y\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\big]\partial_y\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\big[u\partial_x,\ \partial_y\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\partial_y\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell(\partial_y u)\partial_x\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &+\big\|(\partial_y u)\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\partial_x\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\big[(\partial_y u),\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\big]\partial_x\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &+\big\|(\partial_y u)\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\partial_x\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\big[(\partial_y u)\langle y\rangle^\sigma,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau W_m^\ell\big]\langle y\rangle^{-\sigma}\partial_x\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &+C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}, \end{aligned}$$

where the last inequality using Assumption 3.1 and Lemma 2.4 with $\lambda(D_x)=\Lambda^{(\ell-2)/3}$ and $q=(\partial_y u)\langle y\rangle^\sigma$. Moreover using again Assumption 3.1 and Lemma 2.4 with $\lambda(D_x)=\Lambda^{(\ell-2)/3}\Lambda_\tau^\delta$ and $q=u\langle y\rangle^\sigma$, yields

$$\begin{aligned} &\big\|\Big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\big[\partial_y,\ W_m^\ell\big]\ \Big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ =\ &\big\|\Big[u\langle y\rangle^\sigma,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\big[\partial_y,\ W_m^\ell\big]\ \Big]\langle y\rangle^{-\sigma}\partial_x\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

As a result, combining these inequalities, we have

$$\begin{aligned} &\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &+C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

Similarly, repeating the above arguments with $u\partial_x$ replaced by $v\partial_y$ and ∂_y^2 respectively, one has

$$\begin{aligned} &\big\|\big[v\partial_y,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \le\ &C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\Lambda^{-1}\partial_y^2\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}, \end{aligned}$$

and

$$\begin{aligned}
&\big\|\big[\partial_y^2,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y^2\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

As a result, we conclude, combining these inequalities,

$$\begin{aligned}
Q_{5,1} &= \big\|\big[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y W_m^\ell\big]\chi(t)f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y^2\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Similarly as the proof of $Q_{5,1}$, we have

$$\begin{aligned}
Q_{5,2} &= \big\|\big[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\big]\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y^2\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Thus

$$\begin{aligned}
Q_5 &\leq C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y^2\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\partial_y\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\langle y\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\Lambda_\delta^\tau\chi(t)W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

the last inequality using the fact that $\tau\leq 0$ and $0\leq\chi(t)\leq\phi_m^i$. This along with the estimates on Q_1-Q_4 gives the desired estimate, completing the proof of Proposition 4.9. □

4.2. **Upper bound of the terms with highest order.** In this subsection we treat the terms with $W_k^i f_k$ involved, we will make use of subelliptic to estimate these terms. Precisely we will prove the following proposition.

Proposition 4.10. *Let u,v satisfy Assumption 3.1, and let the sequence $\{f_m\}_{m\geq 1}$, defined by (58), satisfy Assumption 4.1 and the Assumption $H_{k,i}(A)$ (see Definition 4.2). Then we have, for any $\varepsilon>0$,*

$$\begin{aligned}
&k^{1/2}\big\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq\varepsilon\left(\big\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\quad+C_\varepsilon k^{1+\sigma}\left(A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}\right)
\end{aligned}\tag{94}$$

and

$$\begin{aligned}
&\big\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\left(\big\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\quad+C\,k^{1+\sigma}\left(A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}\right),
\end{aligned}\tag{95}$$

where C_ε, C are two constants depending only on ε,σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but independent of k and δ.

Proof. In the proof we will use C to denote different constants which don't depend on k and δ.

(a) We will prove in this step the following two estimates:

$$\begin{aligned}
&\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le Ck^{\sigma/2}\big\|(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad + C\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad + CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}
\end{aligned}\tag{96}$$

and

$$\begin{aligned}
&\big\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le& C\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+ C\big\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+ CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}.
\end{aligned}\tag{97}$$

To do so, observe

$$\begin{aligned}
(1+y)^{-\frac{\sigma}{2}} &= \left(\frac{(3k+i-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}}\left(\frac{2c}{(3k+i-1)\sigma}+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{\sigma}{2}}\\
&\ge C\left(\frac{(3k+i-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{\sigma}{2}}\\
&\ge Ck^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{\sigma}{2}}.
\end{aligned}$$

Then

$$(1+y)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}} \ge Ck^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}.\tag{98}$$

Moreover we find

$$\begin{aligned}
\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} &= \left(\frac{1}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}\left((3k+i-1)\sigma+2cy\right)^{-\frac{(3k+i)\sigma}{2}}\\
&= \left(\frac{(3k+i)\sigma}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}\left(\frac{(3k+i-1)}{(3k+i)}+\frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}\\
&\ge \left(\frac{3k+i}{3k+i-1}\right)^{-\frac{(3k+i)\sigma}{2}}\left(1+\frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}\\
&\ge C\left(1+\frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}},
\end{aligned}$$

which along with (98) gives

$$\left(1+\frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} \le Ck^{\frac{\sigma}{2}}(1+y)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}}.$$

As a result, recalling

$$(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}W_k^{i-1} = (1+y)^{-\frac{\sigma}{2}}e^{2cy}\left(1+\frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{i}{3}},$$

we have

$$\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \le Ck^{\sigma/2}\big\|(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Moreover, using (72) gives

$$\begin{aligned}
&\left\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\quad & C\left\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\left\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\left\|\partial_y\phi_k^{i-1} W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\left\|\phi_k^{i-1} W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\quad & C\left\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\left\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}},
\end{aligned}$$

the last inequality using Hypothesis $H_{k,i}(A)$ (see Definition 4.2) and the fact that

$$\phi_k^i \leq \phi^{3(k-3)+i-\frac{1}{2}}.$$

Similarly,

$$\begin{aligned}
&\left\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\quad & C\left\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}.
\end{aligned}$$

Thus, combining these inequalities, the desired estimate (96) and (97) follows.

(b) Now we will use Proposition 2.6, with $f = \Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k$, to estimate the terms on the right hand sides of (96).

First we prove that all the conditions in 2.6 are fulfilled. Observe

$$\Lambda_\delta^{-2}:\ L^2(\mathbb{R}_x) \longrightarrow H^2(\mathbb{R}_x)$$

is a bounded operator, then

$$\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \in L^2\left([0,T];\ H^2\left(\mathbb{R}^2_+\right)\right) \quad \text{and} \quad \partial_y^3\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \in L^2([0,T]\times\mathbb{R}^2_+)$$

due to Hypotheisi $H_{k,i}(A)$ (see Defintion 4.2), since

$$\left\|\partial_y^3\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq C_\delta\left\|\partial_y^3\Lambda^{-1}\phi_k^{i-1} W_k^{i-1} f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}$$

and

$$\begin{aligned}
&\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\right\|_{L^2\left([0,T];\ H^2\left(\mathbb{R}^2_+\right)\right)}\\
\leq\quad & C_\delta\left\|\phi_k^{i-1} W_k^{i-1} f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_\delta\left\|\partial_y\phi_k^{i-1} W_k^{i-1} f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C_\delta\left\|\partial_y^2\Lambda^{-2/3}\phi_k^{i-1} W_k^{i-1} f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

with C_δ a constant depending only δ. Here we used the fact that $\phi^{3(k-3)+i-\frac{1}{2}} \leq \phi_k^{i-1}$.

Next we check that the boundary conditions in Proposition 2.6 are fulfilled. By (82), we have

$$\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\big|_{y=0} = 0.$$

And observe $\phi|_{t=0} = \phi|_{t=T} = 0$, thus

$$\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\big|_{t=0} = \Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k\big|_{t=T} = 0.$$

Now we compute, using (78),

$$\begin{aligned}
&\partial_t\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big|_{y=0}\\
=\;&\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)W_k^{i-1}f_k\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}\partial_t f_k\big|_{y=0}\\
=\;&\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)W_k^{i-1}f_k\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}\partial_y^2 f_k\big|_{y=0}\\
&-2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_k\Big|_{y=0}\\
=\;&\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)\Lambda^{(i-1)/3}f_k\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}\partial_y^2 f_k\big|_{y=0}\\
&-2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\Lambda^{(i-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_k\Big|_{y=0}.
\end{aligned}$$

This, along with the fact that

$$\begin{aligned}
&\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}\partial_y^2 f_k\big|_{y=0}\\
=\;&\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big|_{y=0}-\big[\partial_y^2,\ W_k^{i-1}\big]\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}f_k\big|_{y=0}\\
=\;&\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big|_{y=0}-\left(\frac{2c^2}{(3k+i-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\Lambda^{(i-1)/3}f_k\big|_{y=0}
\end{aligned}$$

due to the fact that $\partial_y\Lambda_\delta^{-2}f_k|_{y=0}=0$, gives

$$\begin{aligned}
&\partial_t\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big|_{y=0}-\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big|_{y=0}\\
=\;&\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)\Lambda^{(i-1)/3}f_k\big|_{y=0}-\left(\frac{2c^2}{(3k+i-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\Lambda^{(i-1)/3}f_k\big|_{y=0}\\
&-2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\Lambda^{(i-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_k\Big|_{y=0}\\
\overset{\text{def}}{=}\;&\varphi\big|_{y=0}
\end{aligned}$$

with

$$\begin{aligned}
\varphi=&\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)\Lambda^{(i-1)/3}f_k-\left(\frac{2c^2}{(3k+i-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\Lambda^{(i-1)/3}f_k\\
&-2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\Lambda^{(i-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_k.
\end{aligned}\tag{99}$$

Moreover, in view of (80) we see $\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k$ satisfies the following equation

$$\begin{aligned}
&\big(\partial_t+u\partial_x+v\partial_y-\partial_y^2\big)\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\\
=\;&\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\mathcal{Z}_{k,i-1}+\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)W_k^{i-1}f_k+\big[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k.
\end{aligned}$$

Then by Proposition 2.6 we conclude, for any $\varepsilon_1>0$,

$$\begin{aligned}
&\big\|\langle y\rangle^{-\frac{\sigma}{2}}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\qquad+\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\le\varepsilon_1\Big(R_2+R_4+R_6\Big)+C_{\varepsilon_1}\Big(R_1+R_3+R_5+R_7+R_8+R_9+R_{10}\Big)
\end{aligned}\tag{100}$$

and

$$\big\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\le\sum_{j=1}^{10}R_j,\tag{101}$$

where the above $R_j, 1 \le j \le 10$, are defined by

$$
\begin{aligned}
R_1 &= \big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}\mathcal{Z}_{k,i-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_2 &= \big\|\Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}\mathcal{Z}_{k,i-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_3 &= \big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac{1}{2}}\right)W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_4 &= \big\|\Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac{1}{2}}\right)W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_5 &= \big\|\Lambda^{-1/3}\big[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_6 &= \big\|\Lambda^{-2/3}\partial_y\big[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_7 &= \big\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_8 &= \big\|\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_9 &= \big\|\langle y\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
R_{10} &= \big\|\langle y\rangle\Lambda^{-1/3}\varphi\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
\tag{102}
$$

where φ is given in (99).

(c) In this step we treat R_j defined above. Using the estimate

$$\phi^{3(k-3)+i-\frac{1}{2}} \le \phi_k^{i-1},$$

we have,

$$R_1 = \big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}\mathcal{Z}_{k,i-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \le \big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^{i-1}\mathcal{Z}_{k,i-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Moreover, using Proposition 4.4 yields

$$
\begin{aligned}
&\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^{i-1}\mathcal{Z}_{k,i-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & kC\left(\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+CA^{k-6+\frac{(i-1)}{6}}((k-5)!)^s\,k^{\frac{(i-1)s}{3}}\\
\le\ & kC\left(\big\|\Lambda^{-1/3}\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+CA^{k-6+\frac{i-1}{6}}((k-5)!)^s\,k^{\frac{(i-1)s}{3}}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
R_1 \le\ & kC\left(\big\|\Lambda^{-1/3}\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+CA^{k-6+\frac{i-1}{6}}((k-5)!)^s\,k^{\frac{(i-1)s}{3}}.
\end{aligned}
$$

On the other hand, observing $\phi_k^{i-1} \le \phi_{k-1}^i$,

$$
\begin{aligned}
&\big\|\Lambda^{-1/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &\left(\Lambda^{-2/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k,\ \partial_y\phi_k^{i-1}W_k^{i-1}f_k\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\big\|\Lambda^{-2/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\big\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\big\|\Lambda^{-2/3}\partial_y\phi_{k-1}^{i}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\big\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &C\big\|\partial_y\phi_{k-1}^{i}W_{k-1}^{i}f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\big\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\phi_{k-1}^{i}W_{k-1}^{i}f_{k-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\big\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
$$

where the last inequality follows from (76). As a result, we use Hypothesis $H_{k,i}(A)$ (see Definition 4.2) to compute, observing $s \geq 3$,

$$\begin{aligned} & k\big\|\Lambda^{-1/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ & \leq Ck\left(A^{k-6+\frac{i}{6}}\left((k-6)!\right)^s k^{\frac{is}{3}}\right)^{\frac12}\left(A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}\right)^{\frac12} \\ & \leq Ck^{1-\frac{s}{3}}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}} \\ & \leq CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \end{aligned} \tag{103}$$

Similarly, by virtue of (75),

$$k\big\|\Lambda^{-1/3}\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \leq CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \tag{104}$$

Then, combining these inequalities, we conclude

$$R_1 \leq CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \tag{105}$$

Now using Proposition 4.9 with $\chi = \phi^{3(k-3)+i-\frac12} \leq \phi_k^{i-1}$, we have

$$\begin{aligned} R_2 &= \big\|\Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}\mathcal{Z}_{k,i-1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\leq C\big\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad +C\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad +Ck\left(\big\|\Lambda^{-2/3}\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\Lambda^{-2/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\ &\quad +CA^{k-6+\frac{(i-1)}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \end{aligned}$$

which, together with (103) and (104), yields

$$\begin{aligned} R_2 &\leq C\big\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad + C\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad + CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \end{aligned} \tag{106}$$

In order to estimate R_3, we use the inequality

$$\left|\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right| \leq 3k\phi^{3(k-3)+i-\frac32} \leq 3k\phi_k^{i-2}, \quad k \geq 6,$$

to conclude that

$$R_3 = \big\|\Lambda^{-\frac13}\Lambda_\delta^{-2}\left(\frac{d}{dt}\phi^{3(k-3)+i-\frac12}\right)W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \leq 3k\big\|\Lambda^{-\frac13}\phi_k^{i-2}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}$$

and

$$R_4 \leq 3k\big\|\Lambda^{-\frac23}\partial_y\phi_k^{i-2}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Moreover, we use the relation

$$\phi_k^{i-2}\phi_k^{i-2} = \phi^{6(k-3)+2(i-2)} = \phi^{3((k-1)-3)+i}\phi^{3(k-3)+(i-1)} = \phi_{k-1}^i\phi_k^{i-1} \leq \phi_{k-1}^i\phi_{k-1}^i,$$

to write

$$\begin{aligned} \big\|\Lambda^{-1/3}\phi_k^{i-2}W_k^{i-1}f_k\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} &= \left(\Lambda^{-2/3}\phi_{k-1}^iW_k^{i-1}f_k,\ \phi_k^{i-1}W_k^{i-1}f_k\right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\leq \big\|\Lambda^{-2/3}\phi_{k-1}^iW_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\big\|\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \end{aligned}$$

and

$$\big\|\Lambda^{-2/3}\partial_y\phi_k^{i-2}W_k^{i-1}f_k\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \leq \big\|\Lambda^{-2/3}\partial_y\phi_{k-1}^iW_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.$$

Then, repeating the arguments used in the proof of (103) and (104), we have

$$\begin{aligned} & k\big\|\Lambda^{-1/3}\phi_k^{i-2}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+k\big\|\Lambda^{-2/3}\partial_y\phi_k^{i-2}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}, \end{aligned}$$

and thus

$$R_3+R_4\leq CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \tag{107}$$

As for R_5, using (19) gives

$$\begin{aligned} R_5 &= \big\|\Lambda^{-1/3}\big[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq \big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-1/3}\Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-1/3}\big]\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}, \end{aligned}$$

which, along with Corollary 2.5 the fact that $\phi^{3(k-3)+i-\frac12}\leq\phi_k^{i-1}$, yields

$$\begin{aligned} R_5 &\leq C\big\|\Lambda^{-\frac13}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac13}\partial_y\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq C\big\|\Lambda^{-1/3}\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac13}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

As a result, it follows from (103) and (103) that

$$R_5\leq Ck^{-\frac{s}{3}}A^{k-5+\frac{i-1}{3}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{6}}\leq CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \tag{108}$$

Now we estimate R_6, and use (19) to write

$$\begin{aligned} R_6 &= \big\|\Lambda^{-2/3}\partial_y\big[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq \big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\big]\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\partial_y\big]\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq \big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\Lambda_\delta^{-2}\big]\partial_y\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\big[u\partial_x+v\partial_y,\ \partial_y\big]\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad+\big\|\Lambda^{-2/3}\big[u\partial_x+v\partial_y,\ \partial_y\big]\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

On the other hand, by Corollary 2.5 we calculate

$$\begin{aligned} & \big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\Lambda_\delta^{-2}\big]\partial_y\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\qquad+\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & C\big\|\Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-2/3}\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & C\big\|\Lambda^{-1/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-1/3}\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}+C\big\|\Lambda^{-1/3}\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac12}W_k^{i-1}f_k\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}, \end{aligned}$$

the last inequality holding due to (103). Similarly,

$$\begin{aligned}
&\left\|\Lambda^{-2/3}\Lambda_\delta^{-2}\left[u\partial_x+v\partial_y,\ \partial_y\right]\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &\left\|\Lambda^{-2/3}\Lambda_\delta^{-2}\left(\partial_y u\right)\partial_x\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\left\|\Lambda^{-2/3}\Lambda_\delta^{-2}\left(\partial_y v\right)\partial_y\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &\left\|\left(\partial_y u\right)\Lambda^{-2/3}\Lambda_\delta^{-2}\partial_x\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\left\|\left[\left(\partial_y u\right),\ \Lambda^{-2/3}\Lambda_\delta^{-2}\right]\partial_x\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\left\|\left(\partial_y v\right)\Lambda^{-2/3}\Lambda_\delta^{-2}\partial_y\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\left\|\left[\left(\partial_y v\right),\ \Lambda^{-2/3}\Lambda_\delta^{-2}\right]\partial_y\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\left\langle y\right\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\left\|\Lambda^{-1/3}\partial_y\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\Lambda^{-1/3}\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\left\langle y\right\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\Lambda^{-2/3}\left[u\partial_x+v\partial_y,\ \partial_y\right]\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ &C\left\|\left\langle y\right\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}.
\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}
R_6\leq\ &C\left\|\left\langle y\right\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\left\|\Lambda^{-1/3}\partial_y^2\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+Ck^{1-\frac{s}{3}}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}.
\end{aligned}\tag{109}$$

Using Hypothesis $H_{k,i}(A)$ (see Definition 4.2) and the fact that $\phi^{3(k-3)+i-\frac{1}{2}}\leq\phi_k^{i-1}$, yields

$$\begin{aligned}
R_7+R_8=&\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq&\left\|\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq&CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}.
\end{aligned}\tag{110}$$

As for R_9, we have, using $\phi^{3(k-3)+i-1}=\phi_k^{i-1}$,

$$\begin{aligned}
R_9^2&=\left\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda_\delta^{-2}\Lambda^{1/6}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&=\left(\left\langle y\right\rangle^{-\sigma}\partial_y\Lambda_\delta^{-2}\Lambda^{1/6}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k,\ \partial_y\Lambda_\delta^{-2}\Lambda^{1/6}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&=\left(\left\langle y\right\rangle^{-\sigma}\partial_y\Lambda_\delta^{-2}\Lambda^{1/3}\phi^{3(k-3)+i}W_k^{i-1}f_k,\ \partial_y\Lambda_\delta^{-2}\phi^{3(k-3)+i-1}W_k^{i-1}f_k\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq\left\|\left\langle y\right\rangle^{-\sigma}\partial_y\Lambda_\delta^{-2}\Lambda^{1/3}\phi_k^iW_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq C\left(\left\|\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\left\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

the last inequality following from Lemma 3.4. This along with Hypotheisis $H_{k,i}(A)$ (see Definition 4.2) implies, for any $\varepsilon > 0$,

$$\begin{aligned} R_9 \le\ & \varepsilon k^{-(1+\sigma)/2} \left(\left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right) \\ & + C_\varepsilon k^{(1+\sigma)/2} \left\| \partial_y \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \le\ & \varepsilon k^{-(1+\sigma)/2} \left(\left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right) \\ & + C_\varepsilon k^{(1+\sigma)/2} A^{k-5+\frac{i-1}{6}} \left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \end{aligned} \tag{111}$$

It remains to estimate R_{10}. Recall

$$\begin{aligned} \varphi \ =\ & \Lambda_\delta^{-2} \left(\frac{d}{dt} \phi^{3(k-3)+i-\frac{1}{2}} \right) \Lambda^{(i-1)/3} f_k - \left(\frac{2c^2}{(3k+i-1)\sigma} + 3c^2 \right) \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} \Lambda^{(i-1)/3} f_k \\ & -2\Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} \Lambda^{(i-1)/3} \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_k. \end{aligned}$$

Then observing

$$\left(\frac{d}{dt} \phi^{3(k-3)+i-\frac{1}{2}} \right) \le 3k \phi_k^{i-2}$$

and

$$\langle y \rangle \le e^{2cy} \left(1 + \frac{2cy}{(3k+i-1)\sigma} \right)^{-(3k+i-1)\sigma/2} (1+cy)^{-1},$$

we have, using the arguments used in the proof of (107),

$$\begin{aligned} R_{10} = \left\| \langle y \rangle \Lambda^{-1/3} \varphi \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \le & Ck \left\| \Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \le & C A^{k-5+\frac{i-1}{6}} \left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \end{aligned} \tag{112}$$

This along with (100), (105)-(111) gives, for any $\varepsilon, \varepsilon_1 > 0$,

$$\begin{aligned} & \left\| \langle y \rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ & \qquad + \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \le\ & \varepsilon_1 \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ & + \varepsilon_1 \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ & + \varepsilon C_{\varepsilon_1} k^{-(1+\sigma)/2} \left(\left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right) \\ & + C_{\varepsilon,\varepsilon_1} k^{(1+\sigma)/2} A^{k-5+\frac{i-1}{6}} \left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}, \end{aligned}$$

with $C_{\varepsilon,\varepsilon_1}$ a constant depending on ε and ε_1. As a result, letting ε_1 be small enough, we obtain

$$\begin{aligned} & \left\| \langle y \rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ & \qquad + \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\ \le & \varepsilon C k^{-(1+\sigma)/2} \left(\left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right) \\ & + C_\varepsilon k^{(1+\sigma)/2} A^{k-5+\frac{i-1}{6}} \left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}, \end{aligned} \tag{113}$$

which, along with (96), yields

$$\begin{aligned} & \left\| \Lambda_\delta^{-2} \phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le\ & \varepsilon C k^{-1/2} \left(\left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right) \\ & + C_\varepsilon k^{\frac{1}{2}+\sigma} A^{k-5+\frac{i-1}{6}} \left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}. \end{aligned}$$

This gives the first estimate (94) in Proposition 4.10. The proof of the second one is similar. In fact, combining (101) and (105)-(112), we have

$$\begin{aligned}
&\left\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & C\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\left\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\varepsilon k^{-(1+\sigma)/2}\left(\left\|\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\
&+C_\varepsilon k^{(1+\sigma)/2}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}}\\
\leq\ & Ck^{-(1+\sigma)/2}\left(\left\|\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\left\|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\
&+Ck^{(1+\sigma)/2}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}},
\end{aligned}$$

the last inequality following from (113). This along with (97) gives the second estimate (95) in Proposition 4.10. The proof of Proposition 4.10 is thus complete. □

4.3. **Proof of Theorem 4.3.** Now we prove Theorem 4.3, the main result of section 4. To simplify the notation, we will use C to denote different constants depending only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but independent of k and δ. We will proceed through the following steps.

Step 1) In view of (80) we see the function $\Lambda_\delta^{-2}\phi_k^iW_k^if_k$ satisfies the equation

$$\begin{aligned}
&\left(\partial_t+u\partial_x+v\partial_y-\partial_y^2\right)\Lambda_\delta^{-2}\phi_k^iW_k^if_k\\
=&\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i}+\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^if_k+\left[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\right]\phi_k^iW_k^if_k
\end{aligned}\tag{114}$$

in the sense of distribution. In this step we will show that all the terms in the above equation belong to $L^2\left([0,T]\times\mathbb{R}^2_+\right)$. To do so, observe

$$\Lambda_\delta^{-2}:\ L^2(\mathbb{R}_x)\longrightarrow H^2(\mathbb{R}_x)$$

is a bounded operator and then

$$\Lambda_\delta^{-2}\phi_k^iW_k^if_k\in L^2\left([0,T];\ H^1\left(\mathbb{R}^2_+\right)\right)\quad\text{and}\quad\partial_y^2\Lambda_\delta^{-2}\phi_k^iW_k^if_k\in L^2([0,T]\times\mathbb{R}^2_+)$$

due to Hypotheisi $H_{k,i}(A)$ (see Defintion 4.2), since

$$\left\|\partial_y^2\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\leq C_\delta\left\|\partial_y^2\Lambda^{-2/3}\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}$$

and

$$\begin{aligned}
&\left\|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\right\|_{L^2\left([0,T];\ H^1\left(\mathbb{R}^2_+\right)\right)}\\
\leq\ & C_\delta\left\|\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C_\delta\left\|\partial_y\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

with C_δ a constant depending only δ. Here we used (61) and (63). As a result we conclude, along with Proposition 4.4, Corollary 2.5 and (82),

$$u\partial_x\Lambda_\delta^{-2}\phi_k^iW_k^if_k,\quad v\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k,\quad\partial_y^2\Lambda_\delta^{-2}\phi_k^iW_k^if_k,\quad\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i}\in L^2\left([0,T]\times\mathbb{R}^2_+\right),$$

$$\left[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\right]\phi_k^iW_k^if_k\in L^2\left([0,T]\times\mathbb{R}^2_+\right)$$

and

$$\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k\big|_{y=0}=0.\tag{115}$$

Similarly, observe

$$\left|\frac{d\phi_k^i}{dt}\right|\leq 3k\phi^{3(k-3)+i-1}=3k\phi_k^{i-1},$$

and thus

$$\left\|\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \le kC_\delta\left\|\phi_k^{i-1}W_k^{i-1}f_k\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} < +\infty,$$

which gives

$$\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^i f_k \in L^2\left([0,T]\times\mathbb{R}^2_+\right).$$

Finally, the above conclusions along with the equation (114) yields that $\partial_t\Lambda_\delta^{-2}\phi_k^i W_k^i f_k \in L^2\left([0,T]\times\mathbb{R}^2_+\right)$.

Step 2) Let $t\in[0,T]$ be an arbitrary number. The conclusion in the previous step enables us to take $L^2\left([0,t]\times\mathbb{R}^2_+\right)$ inner product with $\Lambda_\delta^{-2}\phi_k^i W_k^i f_k$ on both sides; this gives

$$\begin{aligned}
&\mathrm{Re}\ \left(\partial_t\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\\
&\quad+\mathrm{Re}\ \left(\left(u\partial_x+v\partial_y\right)\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\\
&\quad-\mathrm{Re}\ \left(\partial_y^2\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\\
=\ &\mathrm{Re}\ \left(\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i},\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}+\mathrm{Re}\ \left(\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\\
&\quad+\mathrm{Re}\ \left(\left[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\right]\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Moreover observe

$$\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\big|_{t=0}=0$$

due to the fact that $\phi_k^i(0)=0$, and thus integrating by parts gives

$$\mathrm{Re}\ \left(\partial_t\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}=\frac{1}{2}\left\|\Lambda_\delta^{-2}\phi_k^i(t)W_k^i f_k(t)\right\|^2_{L^2\left(\mathbb{R}^2_+\right)}.$$

Similarly observing $u|_{y=0}=v|_{y=0}=0$, we integrate by parts to get

$$\begin{aligned}
&\mathrm{Re}\ \left(\left(u\partial_x+v\partial_y\right)\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\\
=\ &-\frac{1}{2}\left(\left(\partial_x u+\partial_y v\right)\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\\
=\ &0,
\end{aligned}$$

the last inequality using the fact that $\partial_x u+\partial_y v=0$. In view of (115) we have,

$$-\mathrm{Re}\ \left(\partial_y^2\Lambda_\delta^{-2}\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}=\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|^2_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}.$$

Combining these equations we conclude, for any $t\in[0,T]$,

$$\begin{aligned}
&\frac{1}{2}\|\Lambda_\delta^{-2}\phi_k^i(t)W_k^i f_k(t)\|^2_{L^2(\mathbb{R}^2_+)}+\int_0^t\|\partial_y\Lambda_\delta^{-2}\phi_k^i(t)W_k^i f_k(t)\|^2_{L^2(\mathbb{R}^2_+)}dt\\
\le\ &\left|\left(\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i},\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\right|+\left|\left(\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\right|\\
&+\left|\left(\left[u\partial_x+v\partial_y,\ \Lambda_\delta^{-2}\right]\phi_k^i W_k^i f_k,\ \Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right)_{L^2\left([0,t]\times\mathbb{R}^2_+\right)}\right|\\
\le\ &\left\|\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+3k\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}}W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

the last inequality using Corollary 2.5 and the fact that

$$\left|\frac{d\phi_k^i}{dt}\right|\le 3k\phi^{3(k-3)+i-1}.$$

This implies

$$\begin{aligned}
&\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|^2_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & \left\|\Lambda_\delta^{-2}\phi_k^i \mathcal{Z}_{k,i}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+Ck\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Moreover using Proposition 4.4 gives

$$\begin{aligned}
&\left\|\Lambda_\delta^{-2}\phi_k^i \mathcal{Z}_{k,i}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & Ck\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + CA^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}
\end{aligned}$$

and thus

$$\begin{aligned}
&\left\|\Lambda_\delta^{-2}\phi_k^i \mathcal{Z}_{k,i}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & Ck\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left(A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\right)\\
\leq\ & Ck\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + Ck^{-1}\left(A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\right)^2\\
&+C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

As a result we conclude, observing $\phi_k^i = \phi^{3(k-3)+i} \leq \phi^{3(k-3)+i-\frac{1}{2}}$,

$$\begin{aligned}
&\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|^2_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\Lambda_\delta^{-2}\partial_y\phi_k^i W_k^i f_k\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & Ck\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + Ck^{-1}\left(A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\right)^2\\
&+C\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & Ck\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + Ck^{-1}\left(A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\right)^2\\
&+C\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \frac{1}{2}\left\|\Lambda_\delta^{-2}\partial_y\phi_k^i W_k^i f_k\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\Lambda_\delta^{-2}\partial_y\phi_k^i W_k^i f_k\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & Ck^{1/2}\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + Ck^{-1/2}A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}.
\end{aligned}$$

As a result, combining (94) in Proposition 4.10,

$$\begin{aligned}
&\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\Lambda_\delta^{-2}\partial_y\phi_k^i W_k^i f_k\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\leq\ & Ck^{1/2}\left\|\Lambda_\delta^{-2}\phi^{3(k-3)+i-\frac{1}{2}} W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+Ck^{-1/2}A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\\
\leq\ & \varepsilon\left(\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\
&+C_\varepsilon k^{1+\sigma}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}} + Ck^{-1/2}A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\\
\leq\ & \varepsilon\left(\left\|\partial_y\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + T\left\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}\right)\\
&+C_\varepsilon k^{1+\sigma}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}} + Ck^{-1/2}A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}},
\end{aligned}$$

and thus letting ε be small sufficiently, we have

$$\begin{aligned}
&\|\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + \|\Lambda_\delta^{-2}\partial_y\phi_k^i W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_k^i W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\le Ck^{1+\sigma}A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{(i-1)s}{3}} + Ck^{-1/2}A^{k-6+\frac{i}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\\
&\le CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}},
\end{aligned}$$

the last inequality holding because $s = 3+3\sigma$. Observe the above constant C depends only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but doesn't dependent on k and δ. Then letting the δ in the above inequality trend to 0, we infer $\phi_k^i W_k^i f_k \in L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)$ and

$$\partial_y\phi_k^i W_k^i f_k,\ \partial_y^2\Lambda^{-2/3}\phi_k^i W_k^i f_k \in L^2\left([0,T]\times\mathbb{R}^2_+\right),$$

and

$$\begin{aligned}
&\|\phi_k^i W_k^i f_k\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + \|\partial_y\phi_k^i W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2\Lambda^{-2/3}\phi_k^i W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & CA^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s k^{\frac{is}{3}}\\
\le\ & L_2A^{k-5+\frac{i-1}{6}}\left((k-5)!\right)^s (k-4)^{\frac{is}{3}},
\end{aligned}$$

where L_2 is a constant depending only on σ, c, and the constants $C_\alpha, C_*, \tilde{C}_\alpha$ in Assumption 3.1 and Assumption 4.1, but independent of k and the constant A in Hypothesis $H_{k,i}(A)$. The is just the desired estimate (83). Moreover by virtue of (95) we conclude

$$\partial_y^3\Lambda^{-1}\phi_k^i W_k^i f_k \in L^2\left([0,T]\times\mathbb{R}^2_+\right).$$

Thus the proof of Theorem 4.3 is complete.

5. Proof of the main result

This section is devoted to the proof of Theorem 1.1, the main result on Gevrey smoothing effect. Here we will use the same notations as in Section 3.

Proposition 5.1. *Under the same assumptions as in Theorem 1.1, we have that*

$$\forall\ 0\le i\le 3,\quad \partial_y^3\Lambda^{-1}\phi_k^i W_k^i f_k \in L^2([0,T]\times\mathbb{R}^2_+)$$

for all integers k with $k\ge 6$, and that the following estimate: for any integer $0\le i\le 3$,

$$\begin{aligned}
&\|\phi_k^i W_k^i f_k\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + \|\partial_y\phi_k^i W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2\Lambda^{-\frac{2}{3}}\phi_k^i W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & A^{k-5+\frac{i}{6}}\left((k-5)!\right)^s (k-4)^{\frac{is}{3}}
\end{aligned}$$

holds for all integers k with $k\ge 6$, where $s = 3+3\sigma$.

Remark that Theorem 1.1 is just a consequence of the above proposition. Indeed, if $k\ge 6$ then

$$\left\|t^{3k}(T-t)^{3k}e^{\tilde{c}y}\partial_x^k u\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} \le C\left\|\phi_k^0 W_k^0 f_k\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}$$

due to (67), which along with Proposition 5.1 yields, for any $k\ge 6$,

$$\left\|t^{3k}(T-t)^{3k}e^{\tilde{c}y}\partial_x^k u\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} \le CA^{k-5}\left((k-5)!\right)^s \le CA^k(k!)^s.$$

Now we choose A such that

$$A \ge C+1+\max_{1\le k\le 5}\left\|e^{2cy}\partial_x^k u\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}.$$

Then

$$\forall\ k\ge 1,\quad \left\|t^{3k}(T-t)^{3k}e^{\tilde{c}y}\partial_x^k u\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} \le A^{k+1}(k!)^s.$$

Thus Theorem 1.1 follows.

The rest of this section is devoted to

Proof of Proposition 5.1. We we use induction on k to prove the conclusion. First observe the conditions in Theorem 1.1 together with Sobolev inequality guarantees that Assumption 3.1 and Assumption 4.1 hold, with the constants $C_\alpha, \tilde{C}_\alpha$ there depending only on the integer N_0 in Theorem 1.1. In particular the conclusion in Proposition 5.1 is valid for $k = 6$, since Assumption 4.1 guarantees that

$$\forall\ 0 \le i \le 3, \quad \partial_y^3 \Lambda^{-1} \phi_6^i W_6^i f_6 \in L^2([0,T] \times \mathbb{R}^2_+)$$

and moreover enables us to choose A such that

$$A \ge \sum_{1 \le m \le 6} \left\| e^{2cy} \Lambda^1 f_m \right\|_{L^\infty\left(0,T];\ L^2(\mathbb{R}^2_+)\right)}. \tag{116}$$

Now assume that

$$\forall\ 0 \le i \le 3, \quad \partial_y^3 \Lambda^{-1} \phi_m^i W_m^i f_m \in L^2([0,T] \times \mathbb{R}^2_+) \tag{117}$$

is true for all $6 \le m \le k-1$, and that the following estimate: for any $0 \le \ell \le 3$,

$$\begin{aligned} &\|\phi_m^\ell W_m^\ell f_m\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\le A^{m-5+\frac{\ell}{6}} \left((m-5)!\right)^s (m-4)^{\frac{\ell s}{3}} \end{aligned} \tag{118}$$

holds for any $6 \le m \le k-1$. We will show that

$$\forall\ 0 \le i \le 3, \quad \partial_y^3 \Lambda^{-1} \phi_k^i W_k^i f_k \in L^2([0,T] \times \mathbb{R}^2_+) \tag{119}$$

and that

$$\begin{aligned} &\|\phi_k^\ell W_k^\ell f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y \phi_k^\ell W_k^\ell f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_k^\ell W_k^\ell f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le& A^{k-5+\frac{\ell}{6}} \left((k-5)!\right)^s (k-4)^{\frac{\ell s}{3}} \end{aligned} \tag{120}$$

holds for all integers $0 \le \ell \le 3$. This gives the conclusion in Proposition 5.1.

Now under the induction assumption (118), we prove (119) and (120). First observe

$$\begin{aligned} &\left\|\phi_k^0 W_k^0 f_k\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + \left\|\partial_y \phi_k^0 W_k^i f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2 \Lambda^{-2/3} \phi_k^0 W_k^0 f_k\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le\ & L_1 \left\|\phi_{k-1}^3 W_{k-1}^3 f_{k-1}\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + L_1 \left\|\partial_y \phi_{k-1}^3 W_{k-1}^3 f_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &+ L_1 \left\|\partial_y^2 \Lambda^{-2/3} \phi_{k-1}^3 W_{k-1}^3 f_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \end{aligned}$$

due to (73) in Lemma 3.5, with L_1 the constant, independent of k, given in Lemma 3.5. Then using (118) gives

$$\begin{aligned} &\|\phi_k^0 W_k^0 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y \phi_k^0 W_k^0 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_k^0 W_k^0 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le\ & L_1 A^{k-5-\frac{1}{2}} \left((k-5)!\right)^s. \end{aligned}$$

Observing the above constant L_1 is independent of A, and thus we can choose A such that

$$A \ge L_1^2. \tag{121}$$

Then we obtain

$$\begin{aligned} &\|\phi_k^0 W_k^0 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y \phi_k^0 W_k^0 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_k^0 W_k^0 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le& A^{k-5} \left((k-5)!\right)^s. \end{aligned} \tag{122}$$

Moreover by (74), we conclude, in view of (117) and (118),

$$\partial_y^3 \Lambda^{-1} \phi_k^0 W_k^0 f_k \in L^2([0,T] \times \mathbb{R}^2_+).$$

This along with (117), (118) and (122) yields that the sequence $f_m, m \ge 1$, satisfies Hypothesis $H_{k,1}(A)$ (see Definition 4.2). Then by Theorem 4.3 we conclude $\partial_y^3 \Lambda^{-1} \phi_k^1 W_k^1 f_k \in L^2([0,T] \times \mathbb{R}^2_+)$ and

$$\begin{aligned} &\|\phi_k^1 W_k^1 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y \phi_k^1 W_k^1 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_k^1 W_k^1 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \le\ & L_2 A^{k-5} \left((k-5)!\right)^s (k-4)^{\frac{s}{3}}, \end{aligned}$$

where L_2 is a constant depending only on σ, c and the constants C_*, N_0 in Theorem 1.1, but doesn't depend on A and k. This enables us to choose A such that

$$A \ge L_2^6, \tag{123}$$

and thus we have

$$\begin{aligned}&\|\phi_k^1 W_k^1 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y\phi_k^1 W_k^1 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2\Lambda^{-\frac{2}{3}}\phi_k^1 W_k^1 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & A^{k-5+\frac{1}{6}}\left((k-5)!\right)^s(k-4)^{\frac{s}{3}},\end{aligned}$$

As a result, we see the sequence $f_m, m\geq 1$, satisfies Hypothesis $H_{k,2}(A)$ (see Definition 4.2). Then using again Theorem 4.3 we conclude $\partial_y^3\Lambda^{-1}\phi_k^2 W_k^2 f_k \in L^2([0,T]\times\mathbb{R}^2_+)$ and, observing (123),

$$\begin{aligned}&\|\phi_k^2 W_k^2 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y\phi_k^2 W_k^2 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2\Lambda^{-\frac{2}{3}}\phi_k^2 W_k^2 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & L_2 A^{k-5+\frac{1}{6}}\left((k-5)!\right)^s(k-4)^{\frac{2s}{3}}\\ \leq\ & A^{1/6}A^{k-5+\frac{1}{6}}\left((k-5)!\right)^s(k-4)^{\frac{2s}{3}} \leq A^{k-5+\frac{2}{6}}\left((k-5)!\right)^s(k-4)^{\frac{2s}{3}}.\end{aligned}$$

This gives that sequence $f_m, m\geq 1$, satisfies Hypothesis $H_{k,3}(A)$ (see Definition 4.2). Then using again Theorem 4.3 we conclude $\partial_y^3\Lambda^{-1}\phi_k^3 W_k^3 f_k \in L^2([0,T]\times\mathbb{R}^2_+)$ and , observing (123),

$$\begin{aligned}&\|\phi_k^3 W_k^3 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \|\partial_y\phi_k^3 W_k^3 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2\Lambda^{-\frac{2}{3}}\phi_k^3 W_k^3 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ & L_2 A^{k-5+\frac{2}{6}}\left((k-5)!\right)^s(k-4)^s \leq A^{k-5+\frac{3}{6}}\left((k-5)!\right)^s(k-4)^s.\end{aligned}$$

Then we obtain (119) and that (120) holds for all $0\leq\ell\leq 3$ if we choose A satisfying (116), (121) and (123). Thus the proof of Proposition 5.1 is complete. □

6. Appendix

Here we deduce the equation fulfilled by f_m (cf. [22]). Recall that

$$f_m = \partial_x^m\omega - \frac{\partial_y\omega}{\omega}\partial_x^m u, \quad m\geq 1,$$

where u is a smooth solution to Prandtl equation (2) and $\omega = \partial_y u$. We will verify that

$$\partial_t f_m + u\partial_x f_m + v\partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m,$$

where

$$\begin{aligned}\mathcal{Z}_m \ =\ & -\sum_{j=1}^m\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-j})\\ & -\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) - 2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.\end{aligned}$$

To do so, we firstly notice that

$$u_t + uu_x + vu_y - u_{yy} = 0, \tag{124}$$

and

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

Thus by Leibniz's formula, $\partial_x^m u, \partial_x^\omega$ satisfy, respectively, the following equation

$$\begin{aligned}&\partial_t\partial_x^m u + u\partial_x\partial_x^m u + v\partial_y\partial_x^m u - \partial_y^2\partial_x^m u\\ =& -\sum_{j=1}^m\binom{m}{j}(\partial_x^j u)(\partial_x^{m-j+1}u) - \sum_{j=1}^m(\partial_x^j v)(\partial_y\partial_x^{m-j}u)\\ =& -\sum_{j=1}^m\binom{m}{j}(\partial_x^j u)(\partial_x^{m-j+1}u) - \sum_{j=1}^{m-1}(\partial_x^j v)(\partial_y\partial_x^{m-j}u) - (\partial_x^m v)(\partial_y u)\end{aligned} \tag{125}$$

and

$$
\begin{aligned}
&\partial_t\partial_x^m\omega + u\partial_x\partial_x^m\omega + v\partial_y\partial_x^m\omega - \partial_y^2\partial_x^m\omega \\
=& -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)(\partial_x^{m-j+1}\omega) - \sum_{j=1}^{m}(\partial_x^j v)(\partial_y\partial_x^{m-j}\omega) \\
=& -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)(\partial_x^{m-j+1}\omega) - \sum_{j=1}^{m-1}(\partial_x^j v)(\partial_y\partial_x^{m-j}\omega) - (\partial_x^m v)(\partial_y\omega).
\end{aligned} \tag{126}
$$

In order to eliminate the last terms on the right sides of the above two equations, we observe $\partial_y u = \omega > 0$ and thus multiply (125) by $-\frac{\partial_y\omega}{\omega}$, and then add the resulting equation to (126); this gives

$$
\partial_t f_m + u\partial_x f_m + v\partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m
$$

where

$$
\begin{aligned}
\mathcal{Z}_m \;=\;& -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) + (\partial_x^m u)f_1 \\
&+\left(\partial_t\left(\frac{\partial_y\omega}{\omega}\right) + u\partial_x\left(\frac{\partial_y\omega}{\omega}\right) + v\partial_y\left(\frac{\partial_y\omega}{\omega}\right) - \partial_y^2\left(\frac{\partial_y\omega}{\omega}\right)\right)\partial_x^m u \\
&-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_y\partial_x^m u.
\end{aligned}
$$

On the other hand we notice that

$$
\begin{aligned}
&\partial_t\left(\frac{\partial_y\omega}{\omega}\right) + u\partial_x\left(\frac{\partial_y\omega}{\omega}\right) + v\partial_y\left(\frac{\partial_y\omega}{\omega}\right) - \partial_y^2\left(\frac{\partial_y\omega}{\omega}\right) \\
=\;& \frac{1}{\omega}\left(\partial_t\partial_y\omega + u\partial_x\partial_y\omega + v\partial_y\partial_y\omega - \partial_y^2\partial_y\omega\right) \\
&-\frac{\partial_y\omega}{\omega^2}\left(\partial_t\omega + u\partial_x\omega + v\partial_y\omega - \partial_y^2\omega\right) + 2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right) \\
=\;& -\partial_x\omega + \frac{(\partial_x u)(\partial_y\omega)}{\omega} + 2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right)
\end{aligned}
$$

Therefore we have

$$\begin{aligned}
\mathcal{Z}_m &= -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&\quad - \left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) + (\partial_x^m u)f_1 \\
&\quad + \left(\partial_x\omega - \frac{(\partial_x u)(\partial_y\omega)}{\omega}\right)\partial_x^m u + 2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\partial_x^m u - 2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_y\partial_x^m u \\
&= -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&\quad - \left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) + \left[\partial_y\Big(\frac{\partial_y\omega}{\omega}\Big)^2\right]\partial_x^m u \\
&\quad -2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_x^m\omega \\
&= -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&\quad - \left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) - 2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.
\end{aligned}$$

Next we will give the boundary value of $\partial_y f_m$ and $\partial_t f_m - \partial_y^2 f_m$. In view of (124), we infer, recalling $u|_{y=0} = v|_{y=0} = 0$,

$$\partial_y\omega\big|_{y=0} = \partial_y^2 u\big|_{y=0} = 0.$$

As a result, observing

$$\partial_y f_m = \partial_y\partial_x^m\omega - \left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_x^m u - \left(\frac{\partial_y\omega}{\omega}\right)\partial_y\partial_x^m u,$$

we have

$$\partial_y f_m|_{y=0} = 0.$$

Direct verification shows

$$\mathcal{Z}_m|_{y=0} = -2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0},$$

and thus

$$\left(\partial_t f_m - \partial_y^2 f_m\right)|_{y=0} = \mathcal{Z}_m|_{y=0} = -2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0},$$

due to the equation fulfilled by f_m.

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